Master Thesis
Factorisation homology and skein algebras

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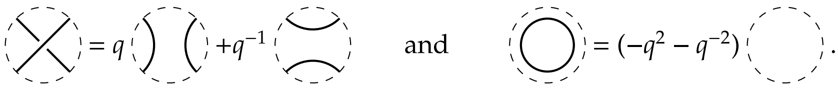
1 Introduction

The goal of this master thesis is to understand the links between factorisation homology, which is a generalisation of singular homology one could call tailor-cut for n-manifolds, and skein algebras, which are classical surface/link invariants derived from the Kauffman bracket.

The first part of this master thesis exposes skein categories, which are a general way to encode tangle invariants obtained by local – skein – relations.

The skein algebra of an oriented surface Σ is obtained as linear combinations of isotopy classes of framed links on the thickened surface $\Sigma \times (0,1)$ modulo the Kauffman bracket relationsx:



where the links depicted here coincide outside the little ball.

In Section 2, we study skein algebras and stated skein algebras, a version for marked surfaces with boundary, which behaves nicely under the action of cutting along an arc.

These skein relations are an example of a more general construction, based on the same idea but where the local relations take place in any ribbon category. We study in Section 3 and 5 ribbon categories which arise as categories of finite dimensional comodules over a quantum group.

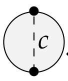
A coalgebra C gives rise to an abelian category $C\text{–}comod^{fin}$ of finite dimensional comodules over C.

A bialgebra A gives a monoidal category $A\text{–}comod^{fin}$. Namely A coacts on tensor products of vector spaces.

A Hopf algebra H gives a rigid category $H\text{–}comod^{fin}$, with duals given in $Vect_k$.

A co-quasi-triangular Hopf algebra gives a braided category $H\text{–}comod^{fin}$.

Finally, a coribbon Hopf algebra gives a ribbon category $H\text{–}comod^{fin}$.

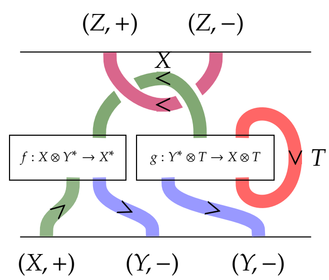
The stated skein algebra $\mathcal{S}(B)$ of the bigon is a quantum group with coproduct given by cutting along the arc 

. It is isomorphic to the well-known quantum group $\mathcal{O}_{q^2}(SL_2)$, and is coribbon with a geometric definition of braiding and twist.

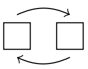
Ribbon categories will play an important role as they provide link invariants through the Reshetikhin–Turaev functor, which we present in Section 5. To any framed tangle in $\mathbb{R}^2 \times [0,1]$ whose strands are coloured with objects of a ribbon category $\mathcal{V}$, we can associate a morphism in $\mathcal{V}$ determining an isotopy invariant of the tangle. Formally, we get a functor $RT : Tan^{fr}_{\mathcal{V}} \to \mathcal{V}$ from $\mathcal{V}$-coloured framed tangles to $\mathcal{V}$, preserving the structures of ribbon categories.

The category $Ribbon_{\mathcal{V}}$ is obtained by also inserting coupons in the definition, coloured with morphisms of $\mathcal{V}$, replacing framed tangles by ribbon tangles, and allowing linear combinations of those. This construction can be extended to any oriented surface Σ, not just $\mathbb{R}^2$ here, and one obtains the category $Ribbon_{\mathcal{V}}(\Sigma)$. There is no canonical functor $Ribbon_{\mathcal{V}}(\Sigma) \to \mathcal{V}$ but we can always evaluate RT locally on a disk of Σ. The skein category $Sk_{\mathcal{V}}(\Sigma)$, studied in Section 6, is the quotient of $Ribbon_{\mathcal{V}}(\Sigma)$ where two tangles are identified if they are equal outside a little cube of $\Sigma \times [0,1]$, and coincide after evaluation under the Reshetikhin–Turaev functor on this cube. We obtain Kauffman-bracket-like local relations: the ones satisfied in $\mathcal{V}$ after evaluation under RT.

A $\mathcal{V}$-coloured ribbon graph :



In particular, the usual skein algebra is the algebra of endomorphisms of $\emptyset$ in the skein category with colours in the ribbon category $\mathcal{O}_{q^2}(SL_2)\text{–}comod^{fin}$, which satisfies the demanded relations between braiding, identity and duality morphisms coloured by a generating object V.

The second part of this master thesis studies the theory of factorisation homology for (oriented, framed) n-manifolds. Factorisation homologies are a generalisation of classical homology theories with coefficients. They satisfy a generalised $\otimes$-excision property, and the same monoidal and invariance axioms. They are defined on the category $Mfld_n$ of topological n-manifolds and embeddings, but not on Top. They take values in a symmetric monoidal ∞-category (or topological category) $\underline{C}^{\otimes}$ and have as coefficients an E_n^{fr}-algebra in $\underline{C}^{\otimes}$.
A topological category is a category whose Hom-sets are topological spaces, such that the composition is continuous. We define in Section 7 the notion of an E_n-algebra in a symmetric monoidal topological category: an object X equipped with a family of products $X^{\otimes k} \to X$ parametrised by the smooth framed embeddings $\sqcup_k \mathbb{R}^n \to \mathbb{R}^n$. If $n = 1$, this is the notion of a homotopy-associative algebra object, and as n increases it gets more and more commutative. For $n = 2$ one could say it is a braided algebra object, with braiding the homotopy  between the product and its opposite. When all the embeddings of $\mathbb{R}^n$ are supposed to be oriented instead of framed, one obtains the notion of an E_n^{or}-algebra. If we consider all topological embeddings, we obtain the notion of an E_n^{fr}-algebra.
These notions are, however, best defined in the context of ∞-categories, presented in Section 8. An ∞-category is a simplicial set whose points are thought of as objects of the category, and edges morphisms. It must satisfy an extension condition, which makes composition of morphisms well-defined up to homotopy. Homotopy here is to be reinterpreted as ≥ 2-morphisms, i.e. ≥ 2 cells of the simplicial set.
The data of an E_n^{fr}-algebra $\mathcal{A}$ is encoded in a symmetric monoidal ∞-functor $Disk_n^{\sqcup} \to \underline{C}^{\otimes}$. Factorisation homology gives a way of extending it to any n-manifold M. The resulting object $\int_M \mathcal{A} \in \underline{C}$ provides both a geometric invariant of M, and an algebraic invariant of $\mathcal{A}$. We construct it in Section 9. The same constructions can be made in the oriented or framed cases.
As for ordinary homologies with coefficients, factorisation homologies can be described by an axiomatic point of view, as symmetric monoidal ∞-functors satisfying a $\otimes$-excision property, called homologies. An Eilenberg–Steenrod-like theorem states that such homologies are determined by their value at $\mathbb{R}^n$, their coefficients, and are obtained as factorisation homology of these coefficients.
A special case that will be of particular interest is when $n = 2$ and $\underline{C}^{\otimes} = Cat^{\times}$ seen as a 2-category with natural isomorphisms. Indeed, E_1-algebras in Cat corresponds to monoidal categories, E_2-algebras to braided categories and E_2^{or}-algebras to balanced categories, an example of which is ribbon categories. Hence the factorisation homology of a ribbon category over a surface is well-defined. In Section 10, using the Eilenberg–Steenrod theorem for factorisation homology and an excision property of skein categories, we show:

> **Theorem 1.1 (Cooke):** *Let $\mathcal{V}$ be a ribbon category and $Cat_k^{\times}$ the ∞-category of small k-linear categories. The factorisation homology $\int_{-}^{Cat_k^{\times}} \mathcal{V} : Mfld_2^{or} \to Cat_k^{\times}$ is equivalent to the skein category functor $Sk_{\mathcal{V}}(-)$.*

To prove this theorem, we introduce the notion of module category and of Tambara relative tensor product. In particular, the skein category $Sk_{\mathcal{V}}(\Sigma)$ of a *punctured* surface Σ with coefficients in a ribbon category $\mathcal{V}$ is a $\mathcal{V}$-module category. For $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$–$comod^{fin}$, a theorem in [BBJ18] shows that it is represented by an algebra object A_{Σ}. This object, however, does not live in $\mathcal{V}$ but in its free cocompletion $Free(\mathcal{V})$ which is an E_2^{or}-algebra in LFP_k. Then, the factorisation homology over Σ of this E_2^{or}-algebra is isomorphic to both the free cocompletion of $Sk_{\mathcal{V}}(\Sigma)$ and to the category of A_{Σ}-modules in $Free(\mathcal{V})$. Juliet Cooke proves that the algebra of $\mathcal{O}_{q^2}(SL_2)$-invariants of A_{Σ} is the skein algebra of Σ, and a theorem of Thang Lê and Tao Yu states that A_{Σ} can be obtained as the stated skein algebra of the surface punctured by a disk instead of a point and marked by a single point on the boundary of the removed disk.

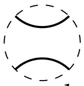
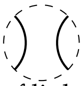
Part I

The skein Category of a surface

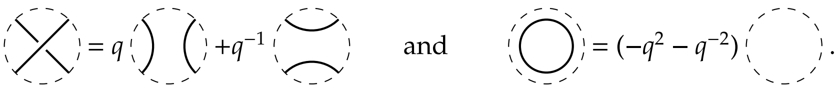
Skein algebras and their stated generalisation are presented in [CL19], with emphasis on the example of the bigon and the quantum group $\mathcal{O}_{q^2}(SL_2)$. Quantum groups and the way they provide monoidal, rigid, braided and ribbon categories are studied in [Kas95] or [Maj95]. Links invariants obtained from ribbon categories were introduced in [Tur10]. Finally, the skein category of a surface is defined in [Coo19].

2 Skein Algebras

The Kauffman bracket is a well-known invariant for framed links obtained by local relations. Skein algebras generalise it in any thickened oriented surfaces, which are locally $\mathbb{R}^3$ so where the local relations make sense. It can then be extended to marked surfaces with boundary, with the notion of stated skein algebra. In the case of the bigon B, we obtain the quantum group $\mathcal{O}_{q^2}(SL_2)$.

A framed link in $\mathbb{R}^3$ is a possibly empty unoriented closed 1-submanifold equipped with a framing, i.e. a continuous choice of a transverse vector at each point. A framed link invariant is a map $\langle . \rangle$: {framed links in $\mathbb{R}^3$}/isotopy $\to \mathcal{R}$ where $\mathcal{R}$ is a commutative ring. The Kauffman bracket depends on an invertible parameter $q \in \mathcal{R}$. For example, $\mathcal{R} = \mathbb{C}$ or $\mathcal{R} = \mathbb{C}[q^{\pm}]$. It can be used to define the Jones polynomial for unframed links. A local relation is one that takes place in any little cube of $\mathbb{R}^3$, namely any embedding $\phi : \mathbb{D}^3 \to \mathbb{R}^3$. Consider a framed link L which is exactly  inside a little cube, with framing coming out of the paper, called blackboard framing. Then we can decompose it into two other links L_h and L_v which are exactly L outside the little cube, and respectively  and  outside. Namely, we add the relation $L = qL_v + q^{-1}L_h$. This relation is actually between isotopy classes of links.

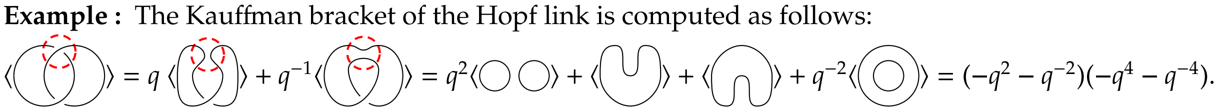
Definition 2.1: The Kauffman bracket of a link is its image in the $\mathcal{R}$-module generated by isotopy classes of framed links in $\mathbb{R}^3$ modulo the following Kauffman (or skein) relations. Let L be a framed link and $\phi : \mathbb{D}^3 \to \mathbb{R}^3$ a little cube in $\mathbb{R}^3$, we ask :



Where each drawing represents the isotopy class of a link which is exactly like the others outside the little cube, and as depicted inside.

If one chooses a link diagram for a framed link L with blackboard framing, all the crossings can be removed using the first relation, and then all circles using the second. Hence the Kauffman bracket has values in $\mathcal{R}$. One can show that the value we obtain is well-defined and to not depend on how one eliminates each crossing and circle. Moreover, it is multiplicative, namely $\langle L \sqcup L' \rangle = \langle L \rangle . \langle L' \rangle$, with a well-separated disjoint union. For example, one can take push L into $\mathbb{R}^2 \times (0, \frac{1}{2})$ and L' into $\mathbb{R}^2 \times (\frac{1}{2}, 1)$ and then take their disjoint union.

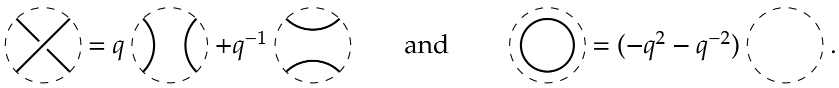
Example : The Kauffman bracket of the Hopf link is computed as follows:



2.1 The skein algebra of a surface One can easily generalise this idea by taking the quotient of isotopy classes of framed links in any 3-manifold M (or thickened oriented surface $\mathfrak{S} \times (0, 1)$ here) by the skein relation where

ϕ is a little cube of M instead of $\mathbb{R}^3$. However, every link cannot be killed down to an element of $\mathcal{R}$ any longer: there are non-empty links without crossings nor small circles, like the core of a cylinder.

⌜ **Definition 2.2:** Let $\mathfrak{S}$ be an oriented surface. The skein algebra $\mathring{\mathcal{S}}(\mathfrak{S})$ is the $\mathcal{R}$-module generated by isotopy classes of framed links in $\mathfrak{S} \times (0,1)$ modulo the skein relations in a little cube of $\mathfrak{S} \times (0,1)$:



where the links here coincide outside the little cube and are as depicted inside, with blackboard framing.
It is an algebra with product given by superposition $\mathfrak{S}\times(0,1)\sqcup\mathfrak{S}\times(0,1) \to \mathfrak{S}\times(0,1)$ induced by $(0,1)\sqcup(0,1) \simeq (\frac{1}{2},1) \sqcup (0,\frac{1}{2}) \hookrightarrow (0,1)$. ⌟

Examples : • The skein algebra of the disk is $\mathcal{R}$.
• The skein algebra of the cylinder $S^1 \times (0,1)$ is isomorphic to $\mathcal{R}[X]$. The element X is the core of the cylinder.

⌜ **Definition 2.3:** A framed link on $\mathfrak{S}$ is simple if it has no double points (two points with same $\mathfrak{S}$-coordinate) nor trivial circles (bounding a disk). ⌟

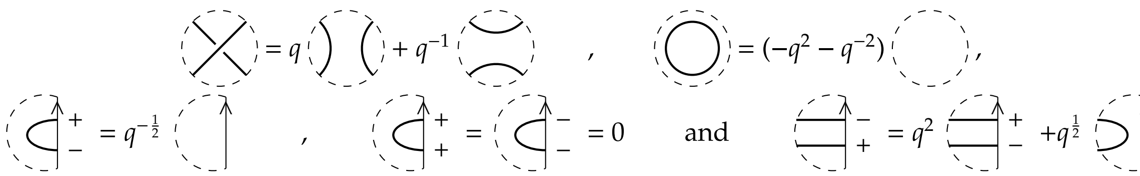
Proposition 2.4: *The set of isotopy classes of simple framed links forms an $\mathcal{R}$-basis for $\mathring{\mathcal{S}}(\mathfrak{S})$.*

2.2 Stated skein algebras The skein algebra does not take into account the boundary of $\mathfrak{S}$: $\mathring{\mathcal{S}}(\mathfrak{S}) = \mathring{\mathcal{S}}(\mathring{\mathfrak{S}})$. Using only closed links, one cannot cut a surface along an arc. Stated skein algebras generalise skein algebras for marked surfaces with boundary, and make cutting along an arc well-defined. This cutting/gluing provides stated skein algebras with an excision property.

⌜ **Definition 2.5:** A marked surface is a compact oriented surface with boundary $\overline{\mathfrak{S}}$ with a finite set $P \subseteq \partial\overline{\mathfrak{S}}$ of boundary points. We note $\mathfrak{S} = \overline{\mathfrak{S}} \setminus P$ and call this the marked surface. We note $\partial_P\overline{\mathfrak{S}}$ the boundary components of $\overline{\mathfrak{S}}$ that contains a point of P and $\partial\mathfrak{S} := \partial_P\overline{\mathfrak{S}} \setminus P$.
A stated tangle α on $\mathfrak{S}$ is an unoriented, framed, compact, properly embedded 1-submanifold of $\mathfrak{S} \times (0,1)$ whose boundary $\partial\alpha \subseteq \partial\mathfrak{S}\times(0,1)$ has vertical framing and comes equipped with a state $st : \partial\alpha \to \{+,-\}$. We call height the $(0,1)$-coordinate of a point, and require that all boundary points of α laying over a same boundary component b of $\partial\mathfrak{S}$ have distinct heights. An isotopy of stated tangles is an isotopy with values in stated tangles, in particular preserving the height order over a same boundary component. ⌟

Given a square root $q^{\frac{1}{2}} \in \mathcal{R}^\times$ of q, we can define stated skein algebras of marked surfaces.

⌜ **Definition 2.6 ([CL19]):** The stated skein algebra $\mathcal{S}(\mathfrak{S})$ of a marked surface $\mathfrak{S}$ is the $\mathcal{R}$-module generated by isotopy classes of stated tangles on $\mathfrak{S}$ modulo the stated skein relations



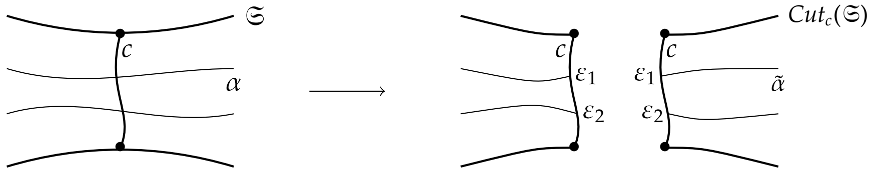
where the arrows on the boundary edges represent the relative height order of the two points. ⌟

Remark 2.7: It is easy to check that $\mathcal{S}(\mathfrak{S} \sqcup \mathfrak{S}') \simeq \mathcal{S}(\mathfrak{S}) \otimes \mathcal{S}(\mathfrak{S}')$ since all relations happen in a connected disk.

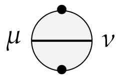
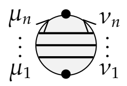
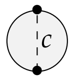
⌜ **Definition 2.8:** A stated tangle on $\mathfrak{S}$ is simple if it has no double points nor trivial circles. Given an orientation $\mathfrak{o}$ on $\partial\mathfrak{S}$. It is called $\mathfrak{o}$-ordered if the height order of boundary points on a same boundary component is increasing in the direction of $\mathfrak{o}$. It is called increasingly stated if + signs are always above − signs on a same boundary component, namely the states are also increasing in the direction of $\mathfrak{o}$. ⌟

Proposition 2.9: *The set of isotopy classes of increasingly stated $\mathfrak{o}$-ordered simple framed tangles forms an $\mathcal{R}$-basis for $\mathcal{S}(\mathfrak{S})$.*

Let $\mathfrak{S}$ be a marked surface and c an ideal arc on $\mathfrak{S}$, i.e. an embedding $(0,1) \to \mathfrak{S}$ joining two points of P. Note $Cut_c(\mathfrak{S})$ the marked surface obtaining by cutting $\mathfrak{S}$ along c. Given a stated tangle α on $\mathfrak{S}$ we can cut it along c and get a tangle on $Cut_c(\mathfrak{S})$. To get a stated tangle of $Cut_c(\mathfrak{S})$ we only need states on $\partial_c\alpha := \alpha \cap (c \times (0,1))$, which we assume is finite, transversal with different heights and vertical framing. We say such a choice is compatible if the two different copies of $x \in \partial_c\alpha$ in $Cut_c(\mathfrak{S})$ have same states. The resulting stated tangle is called a lift of α.



Theorem 2.1 ([CL19]): *The map $\rho_c : \mathcal{S}(\Sigma) \to \mathcal{S}(Cut_c(\Sigma))$, $\alpha \mapsto \sum_{lifts\ \tilde{\alpha}} \tilde{\alpha}$ is well-defined (it only depends on the isotopy class of α) and is an injective algebra morphism.*

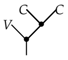
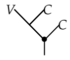
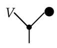
Fundamental example : The bigon B is the marked surface $(\mathbb{D}, \{\pm i\})$, the disk with two marked points. The algebra $\mathcal{S}(B)$ is generated by $\alpha_{\mu,\nu} =$  , $\mu, \nu \in \{\pm\}$, and has $\mathcal{R}$-module basis the $\alpha_{\bar{\mu},\bar{\nu}} =$  where $\bar{\mu} = (\mu_1, \dots, \mu_n)$ and $\bar{\nu} = (\nu_1, \dots, \nu_n)$ are ascending sequences of signs. Cutting along the "unique" arc joining the two marked points  gives a morphism $\Delta = \rho_c : \mathcal{S}(B) \to \mathcal{S}(B \sqcup B) \simeq \mathcal{S}(B) \otimes \mathcal{S}(B)$. Along with the counit $\varepsilon : \mathcal{S}(B) \to \mathcal{R}$ defined on the basis by $\varepsilon(\alpha_{\bar{\mu},\bar{\nu}}) = \delta_{\bar{\mu},\bar{\nu}}$, it turns $\mathcal{S}(B)$ into a bialgebra, a notion we define and study in the next section.

3 Quantum groups and braided categories

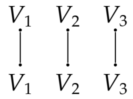
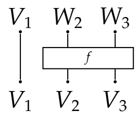
Quantum groups, or Hopf algebras, arise naturally from the finite-comodules-category point of view. Coalgebras give abelian categories, bialgebras give monoidal categories, Hopf algebras give rigid categories and coquasitriangular Hopf algebras give rigid braided categories. There is a sort of converse, see [EGNO15]. This exposition is made from the finite-modules-category point of view in [Maj95].

⌜ **Definition 3.1:** Let k be a field. A k-linear category is a category enriched over $Vect_k$ (see Appendix B), i.e. whose Hom-sets have k-vector space structures, which are preserved by composition.
The category Cat_k has objects small k-linear categories and morphisms functors preserving the k-linear structure. It has usual natural transformations. We work in Cat_k throughout this section. ⌟

⌜ **Definition 3.2:** A coalgebra is a k-vector space C equipped with a coproduct $\Delta : C \to C \otimes C$, denoted $\Delta =$ or algebraically $\Delta(c) = c_{(1)} \otimes c_{(2)}$ where the sum is implicit, and a counit $\varepsilon : C \to k$, denoted $\varepsilon =$, such that:
coassociativity: $(Id_C \otimes \Delta) \circ \Delta = \quad = \quad = (\Delta \otimes Id_C) \circ \Delta$ and
counit: $(Id_C \otimes \varepsilon) \circ \Delta = \quad = | = Id_C = \quad = (\varepsilon \otimes Id_C) \circ \Delta$.
Algebraically, coassociativity reads $c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$ which we note $c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$.
A right C-comodule is a k-vector space V equipped with a coaction $\Delta_V : V \to V \otimes C$ denoted $\Delta_V =$ or

$\Delta(v) = v_{(1)} \otimes c_{(2)}$, such that  $=$  (coassociativity) and  $=$  $= Id_V$ (counit).
Note C–*comod* the category of right C-comodules, with morphisms the linear maps $f : V \to W$ preserving the coaction and the counit, namely $\Delta_W \circ f = (f \otimes Id_C) \circ \Delta_V$ and $\varepsilon_W \circ f = \varepsilon_V$, and C–$comod^{fin}$ the full subcategory spanned by finite dimensional comodules. ⌟

The category C–*comod* of comodules over a coalgebra form a k-linear abelian category.

Remark 3.3: The graphical depiction of the coproduct and counit above are an example of the graphical calculus, see for example [Tur10]. We denote objects by points and tensor product of objects by juxtaposition of points $\dot{V}_1\ \dot{V}_2\ \dot{V}_3$. A bunch of straight lines  denote the identity. We denote morphisms by drawings linking two sets of points, which read bottom to top. There may be inserted coupons  , coloured by corresponding morphisms, here $f : V_2 \otimes V_3 \to W_2 \otimes W_3$. Composition is given by vertical superposition.

3.1 Bialgebras and monoidal categories

⌜ **Definition 3.4:** A monoidal category $C^{\otimes}$ is a category C equipped with a bifunctor $\otimes : C \times C \to C$, an object $1_C \in C$ and three natural isomorphisms $\alpha : (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)$, $l : 1 \otimes - \Rightarrow -$ and $r : - \otimes 1 \Rightarrow -$ satisfying the pentagon

$$\begin{tikzcd}[column sep=small] & (X\otimes(Y\otimes Z))\otimes T \arrow[dr] & \\ ((X\otimes Y)\otimes Z)\otimes T \arrow[ur] \arrow[d] & \circlearrowright & X\otimes((Y\otimes Z)\otimes T) \arrow[dl] \\ (X\otimes Y)\otimes(Z\otimes T) \arrow[r] & X\otimes(Y\otimes(Z\otimes T)) & \end{tikzcd}$$

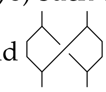
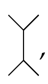
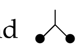
and the triangle

$$\begin{tikzcd} (X\otimes 1)\otimes Y \arrow[rr, "\alpha"] \arrow[dr, "r"'] & \circlearrowleft & X\otimes(1\otimes Y) \arrow[dl, "l"] \\ & X\otimes Y & \end{tikzcd}$$

, $X, Y, Z, T \in C$.

In other words, two parenthesised products of objects $X_1, \ldots, X_n$ (preserving the order) with arbitrary insertions of the object 1_C are canonically isomorphic.
A monoidal functor $F : C^{\otimes} \to \mathcal{D}^{\otimes}$ is one endowed with a natural isomorphism $F(-) \otimes F(-) \overset{\sim}{\Rightarrow} F(- \otimes -)$ and an isomorphism $F(1_C) \overset{\sim}{\to} 1_{\mathcal{D}}$ coherent with associativity and unit natural isomorphisms. ⌟

There are many well-known examples of monoidal categories, such as $Top^{\sqcup}$, $Top^{\times}$, $Cat^{\times}$ or $Vect^{\otimes}$. The category Cat_k is monoidal with $\times$ the product on objects and k-linear tensor product on morphisms.

⌜ **Definition 3.5:** A bialgebra is an algebra $(A, m, 1)$ equipped with a coalgebra structure (A, Δ, ε) such that Δ and ε are algebra morphisms. We denote $m =$  and $1 =$ . The compatibility conditions read  $=$ , where the crossing here simply represents the flip of tensors, and  $=$. ⌟

Fundamental example : The stated skein algebra of the bigon $\mathcal{S}(B)$ is a bialgebra with Δ the splitting morphism. It can be described as follows. Let $q \in k \setminus \{0\}$, $O_{q^2}(SL_2)$ is the free non-commutative k-algebra generated by a, b, c, d modulo the relations $ca = q^2 ac$, $\ db = q^2 bd$, $\ ba = q^2 ab$, $\ dc = q^2 cd$, $\ bc = cb$, $\ ad - q^{-2} bc = 1$ and $da - q^2 cb = 1$. The coproduct can be written $\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the counit $\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on generators. The isomorphism $\mathcal{S}(B) \to O_{q^2}(SL_2)$ is given by $\alpha_{++} \mapsto a$, $\alpha_{--} \mapsto d$, $\alpha_{+-} \mapsto b$ and $\alpha_{-+} \mapsto c$.

Proposition 3.6: *The category A–comod of comodules over a bialgebra A is monoidal with $\otimes = \otimes_k$ with coaction on tensor product given by* $V \otimes_k W \overset{\Delta_V \otimes \Delta_W}{\longrightarrow} (V \otimes A) \otimes (W \otimes A) \overset{Id_V \otimes \tau_{A,W} \otimes Id_A}{\longrightarrow} V \otimes W \otimes A \otimes A \overset{Id_V \otimes Id_W \otimes m}{\longrightarrow} V \otimes W \otimes A.$

3.2 Hopf algebras and rigid categories Hopf algebras are the preferred candidate for quantum groups. They generalise both groups and Lie algebras, and allow "quantum" variants of these.

Definition 3.7: Let $\mathcal{C}^{\otimes}$ be a monoidal category. A left dual for an object $V \in \mathcal{C}$ is an object V^* together with two morphisms $ev =$: $V^* \otimes V \to k$ and $coev =$: $k \to V \otimes V^*$ such that $(ev \otimes Id_{V^*})(Id_{V^*} \otimes coev) =$ $= Id_{V^*}$ and $(Id_V \otimes ev)(coev \otimes Id_V) =$ $= Id_V$.

These identities give that $Hom_{\mathcal{C}}(X \otimes V, Y) \overset{-\otimes Id_{V^*}}{\longrightarrow} Hom_{\mathcal{C}}(X \otimes V \otimes V^*, Y \otimes V^*) \overset{-\circ(Id_X \otimes coev)}{\longrightarrow} Hom_{\mathcal{C}}(X, Y \otimes V^*)$ and $Hom_{\mathcal{C}}(X, Y \otimes V^*) \overset{-\otimes Id_V}{\longrightarrow} Hom_{\mathcal{C}}(X \otimes V, Y \otimes V^* \otimes V) \overset{(Id_Y \otimes ev)\circ -}{\longrightarrow} Hom_{\mathcal{C}}(X \otimes V, Y)$ are mutual inverses. Namely, $- \otimes V$ is left adjoint to $- \otimes V^*$. Similarly, $V^* \otimes -$ is left adjoint to $V \otimes -$.

The left dual is unique up to canonical isomorphism. It is actually functorial with $f^* :=$ f .

There is a similar notion of right dual ${}^*V$ with $ev =$: $V \otimes {}^*V \to k$ and $coev =$: $k \to {}^*V \otimes V$ satisfying $=$ and $=$. From the definitions, the right dual of a left dual V^* is ${}^*(V^*) = V$.

A category with both left and right duals for every object is called rigid.

Example : In $Vect$, only finite dimensional vector spaces have duals. For V with basis $(e_i)_i$, its dual is $V^* = Hom(V, k)$ with the usual evaluation and the coevaluation given by $coev(1) = \sum_i e_i \otimes e^i$. The tensor product is symmetric so right and left duals coincide. Hence $Vect^{fin}$ is rigid.

Definition 3.8: A Hopf algebra is a bialgebra H equipped with an antipode $S : H \to H$ denoted S such that S $=$ $=$ S . Though unusual, we, moreover, assume that S is invertible, which is true in general as soon as H is finite dimensional or (co)-quasi-triangular.

Examples : • A group G gives rise to a Hopf algebra kG which is the k-vector space with basis G, product the product in G, unit the unit of G, coproduct given by $\Delta g = g \otimes g$, counit $\varepsilon g = 1$ and antipode $Sg = g^{-1}$.
• A Lie algebra $\mathfrak{g}$ gives rise to a Hopf algebra $U(\mathfrak{g})$ which is the enveloping algebra of $\mathfrak{g}$ with coproduct $\Delta\xi = \xi \otimes 1 + 1 \otimes \xi$, counit $\varepsilon\xi = 0$ and antipode $S\xi = -\xi$.

Fundamental example : The bialgebra $\mathcal{O}_{q^2}(SL_2)$ is a Hopf algebra with $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^2 b \\ -q^{-2}c & a \end{pmatrix}$ on generators, which extends on all $\mathcal{O}_{q^2}(SL_2)$ with the fact that S is an anti-algebra morphism. It has a geometric depiction on $\mathcal{S}(B)$ with $S\left(\mu_m \vdots \mu_1 \; \beta \; \nu_n \vdots \nu_1 \right) = -\nu_1 \vdots -\nu_n \; g \; -\mu_1 \vdots -\mu_m \cdot \prod_{i=1}^{m} (C_{\mu_i}^{-\mu_i})^{-1} \cdot \prod_{i=1}^{n} C_{\nu_i}^{-\nu_i}$, where $C_+^- = -q^{-\frac{5}{2}}$ and $C_-^+ = q^{-\frac{1}{2}}$.

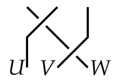
Proposition 3.9: *For H a Hopf algebra, the category H–comod^{fin} is rigid, with duals given by duals in $Vect$. Duality in $Vect$ gives $V^* \otimes H \simeq Hom(V, H)$, hence the H-coaction on a form $f : V \to k$ should be an element $\Delta f \in Hom(V, H)$. It is given by $\Delta f(v) = f(v_{(1)}) \otimes Sh_{(2)}$ on left duals, and by $\Delta f(v) = f(v_{(1)}) \otimes S^{-1}h_{(2)}$ on right duals, $v \in V$.*

Definition 3.10: Let H be a Hopf algebra. An H-comodule algebra is an H-comodule A endowed with a product $m : A \otimes A \to A$ and a unit $1_A : k \to A$ which are morphisms of H-comodules. Algebraically, for $a, b \in A$, we get $\Delta(a.b) = \Delta(a).\Delta(b) = a_{(1)}.b_{(1)} \otimes a_{(2)}.b_{(2)}$, with the first product in A and the second in H (given by the monoidal structure), and $\Delta(1_A) = 1_A \otimes 1$.

A right module over A is an H-comodule V equipped with an associative map of H-comodules $\triangleleft : V \otimes A \to V$. Algebraically, $(v \triangleleft a) \triangleleft b = v \triangleleft (a.b)$ and $\Delta(v \triangleleft a) = \Delta(v) \triangleleft \Delta(a) = v_{(1)} \triangleleft a_{(1)} \otimes v_{(2)}.a_{(2)}$. A morphism of A-modules is a morphism of H-comodules that commutes with the actions $\triangleleft$.

3.3 Co-quasi-triangular Hopf algebras and braided categories

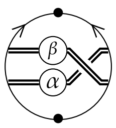
⌜ **Definition 3.11:** A braided category is a monoidal category $C^{\otimes}$ equipped with a braiding, i.e. a natural isomorphism $c : \otimes \to \otimes^{op}$ denoted $\times$ such that: $c_{U\otimes V,W}$ = 

= 

$= (c_{U,W} \otimes Id_V) \circ (Id_U \otimes c_{V,W})$.
It is called symmetric if $c_{X,Y} \circ c_{Y,X} = Id_{X\otimes Y}$ for all objects $X, Y \in C$. ⌟

⌜ **Definition 3.12:** A co-quasi-triangular Hopf algebra H is a Hopf algebra equipped with a co-R-matrix, or R-form, $R : H \otimes H \to k$ which is invertible by convolution, i.e. there exists $R^{-1} : H \otimes H \to k$ such that $\forall a, b \in H$, $R(a_{(1)} \otimes b_{(1)})R^{-1}(a_{(2)} \otimes b_{(2)}) = \varepsilon(a)\varepsilon(b)$, and satisfying $b_{(1)}.a_{(1)}.R(a_{(2)} \otimes b_{(2)}) = R(a_{(1)} \otimes b_{(1)}).a_{(2)}.b_{(2)}$, $R(ab \otimes c) = R(a \otimes c_{(1)}).R(b \otimes c_{(2)})$ and $R(a \otimes bc) = R(a_{(1)} \otimes c).R(a_{(2)} \otimes b)$. ⌟

Fundamental example : The Hopf algebra $O_{q^2}(SL_2)$ admits a co-R-matrix defined geometrically on $\mathcal{S}(B)$ by

$R(\alpha \otimes \beta) = \varepsilon$ 

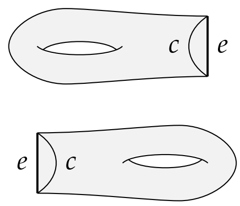
. On the generators, $R\begin{pmatrix} a\otimes a & b\otimes b & a\otimes b & b\otimes a \\ c\otimes c & d\otimes d & c\otimes d & d\otimes c \\ c\otimes a & d\otimes b & c\otimes b & d\otimes a \\ a\otimes c & b\otimes d & a\otimes d & b\otimes c \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & q^{-1} & q-q^{-3} \end{pmatrix}$.

Proposition 3.13: *The category H–comod of comodules over a co-quasi-triangular Hopf algebra H is braided with braiding $c_{V,W} : V \otimes W \to W \otimes V$ given by $c_{V,W}(v \otimes w) = w_1 \otimes v_1.R(v_2 \otimes w_2)$.*

4 Excision for stated skein algebras

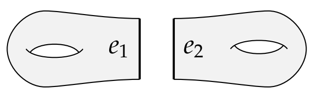
Stated skein algebras are actually a great example of $O_{q^2}(SL_2)$-comodules. This structure moreover gives a nice algebraic formulation of excision for stated skein algebras. This is essentially Section 4 of [CL19].

Given a marked surface $\mathfrak{S}$ and a boundary edge e of $\mathfrak{S}$, we can consider an ideal arc c going along e but inside $\mathring{\mathfrak{S}}$. The piece between e and c is a bigon, and the splitting morphism along c gives a morphism $\Delta = \rho_c : \mathcal{S}(\mathfrak{S}) \to \mathcal{S}(\mathfrak{S} \sqcup B) \simeq \mathcal{S}(\mathfrak{S}) \otimes \mathcal{S}(B)$. This gives $\mathcal{S}(\mathfrak{S})$ the structure of a right $O_{q^2}(SL_2)$-comodule. The algebra structure on $\mathcal{S}(\mathfrak{S})$ is compatible, namely $\mathcal{S}(\mathfrak{S})$ is a $O_{q^2}(SL_2)$-comodule algebra. If one sees the edge e at the left instead of the right of the surface, one gets a structure of left $O_{q^2}(SL_2)$-comodule.



Remark 4.1: Actually, we get such a structure for each boundary edge of $\mathfrak{S}$, and if $\mathfrak{S}$ has n boundary edges, $\mathcal{S}(\mathfrak{S})$ is an $O_{q^2}(SL_2)^{\otimes n}$-comodule.

Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be two marked surfaces and e_1 and e_2 two boundary edges of respectively $\mathfrak{S}_1$ and $\mathfrak{S}_2$. We see $\mathcal{S}(\mathfrak{S}_1)$ as a right $O_{q^2}(SL_2)$-comodule and $\mathcal{S}(\mathfrak{S}_2)$ as a left $O_{q^2}(SL_2)$-comodule.

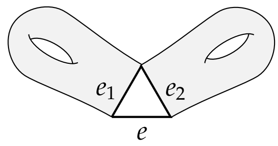


We note $\mathfrak{S} = \mathfrak{S}_1 \cup_{e_1=e_2} \mathfrak{S}_2$ the surface obtained by glueing $\mathfrak{S}_1$ and $\mathfrak{S}_2$ along e_1 and e_2, and e the common image of e_1 and e_2 in $\mathfrak{S}$, which is an ideal arc. We have $Cut_e(\mathfrak{S}) = \mathfrak{S}_1 \sqcup \mathfrak{S}_2$, and Theorem 2.1 gives an injective algebra morphism $\rho_e : \mathcal{S}(\mathfrak{S}) \to \mathcal{S}(\mathfrak{S}_1 \sqcup \mathfrak{S}_2) \simeq \mathcal{S}(\mathfrak{S}_1)\otimes\mathcal{S}(\mathfrak{S}_2)$. Note that by definition, the splitting morphism has values in lifts of tangles on $\mathfrak{S}$, namely tangles on $\mathfrak{S}_1 \sqcup \mathfrak{S}_2$ which have all possible signs on e_1 and e_2. In particular, if we note Δ_1 and Δ_2 the coproducts of $\mathcal{S}(\mathfrak{S}_1)$ and $\mathcal{S}(\mathfrak{S}_2)$, such a lift $\tilde{\alpha} \in \mathcal{S}(\mathfrak{S}_1 \sqcup \mathfrak{S}_2) \simeq \mathcal{S}(\mathfrak{S}_1) \otimes \mathcal{S}(\mathfrak{S}_2)$ verifies $\Delta_1 \otimes Id_2(\tilde{\alpha} = Id_1 \otimes \Delta_2(\tilde{\alpha})$.

⌜ **Definition 4.2:** Let H be a Hopf algebra, M_1 a right H-comodule and M_2 a left H-comodule, with coproducts denoted respectively Δ_1 and Δ_2. The cotensor product of M_1 and M_2 over H is the subalgebra of $M_1 \otimes M_2$ defined as $M_1 \square_H M_2 := \{x \in M_1 \otimes M_2 \ / \ \Delta_1 \otimes Id_2(x) = Id_1 \otimes \Delta_2(x)\}$. ⌟

Theorem 4.1 ([CL19]): *The stated skein algebra of a gluing $\mathfrak{S} = \mathfrak{S}_1 \cup_{e_1=e_2} \mathfrak{S}_2$ maps isomorphically on the cotensor product of $\mathcal{S}(\mathfrak{S}_1)$ and $\mathcal{S}(\mathfrak{S}_2)$ over $\mathcal{S}(B)$.*

There is another form of excision which seems more appropriate to parallel Section 10. Consider the same $\mathfrak{S}_1$ and $\mathfrak{S}_2$, which we both see as right $\mathcal{O}_{q^2}(SL_2)$-comodules, but this time instead of gluing e_1 on e_2 we glue them on two edges of a triangle. This way, the resulting surface $\mathfrak{S}$ still has a boundary component where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ had one, and we only have to deal with right comodules.



⌜ **Definition 4.3:** Let H be a co-quasi-triangular Hopf algebra and M_1 and M_2 two right H-comodule algebras. The category of H-comodules is braided, so we have an isomorphism $c_{M_1M_2} : M_1 \otimes M_2 \to M_2 \otimes M_1$. The braided tensor product of M_1 and M_2 is the H-comodule algebra $M_1 \underline{\otimes}_H M_2$ which is the H-comodule $M_1 \otimes M_2$ endowed with the product $*$ defined by $(x_1 \otimes x_2) * (y_1 \otimes y_2) = (x_1 \otimes 1).c_{M_1M_2}(x_2 \otimes y_1).(1 \otimes y_2)$. ⌟

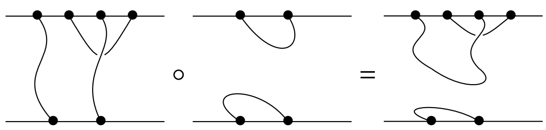
Theorem 4.2 ([CL19]): *The stated skein algebra of a gluing along a triangle $\mathfrak{S} = \mathfrak{S}_1 \cup_{e_1=t_1} T \cup_{t_2=e_2} \mathfrak{S}_2$ maps isomorphically on the braided tensor product of $\mathcal{S}(\mathfrak{S}_1)$ and $\mathcal{S}(\mathfrak{S}_2)$.*

5 Ribbon categories and link invariants

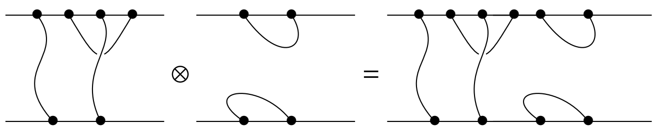
In [Sel09], the author proves coherence theorems which state that the drawings (graphical calculus) are coherent, and represent one and only one morphism. This means in particular that rigid categories provide invariants of tangles without crossings, braided categories of braids, braided rigid categories of tangles without the first Reidemeister move, and finally ribbon categories of framed tangles.

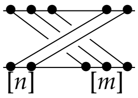
5.1 The category of tangles

⌜ **Definition 5.1:** A tangle α in a 3-manifold with boundary M is an unoriented, framed, compact, properly embedded 1-sub-manifold of M. Its boundary $\partial\alpha \subseteq \partial M$ is a finite set of points.
The category Tan of tangles in $\mathbb{R}^2 \times [0,1]$ has objects finite sets of points $[n] = \{1, \dots, n\} \times \{0\} \subseteq \mathbb{R}^2$ and morphisms from $[m]$ to $[n]$ isotopy classes of tangles in $\mathbb{R}^2 \times [0,1]$ with boundary the union of $[n] \subseteq \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \{1\}$ and $[m] \subseteq \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \{0\}$. Composition is given by vertical juxtaposition, and contraction $[0,1] \sqcup [0,1] \xrightarrow{[\frac{1}{2},1] \sqcup [0,\frac{1}{2}]} [0,1]$. Identities are given by n straight lines $[n] \times [0,1]$. ⌟

Example of composition in Tan :  .

The category Tan is monoidal with tensor product given by horizontal juxtaposition. On objects, we get $[n] \otimes [m] = [n+m]$. The unit object is the empty set.

Example of tensor product in Tan :  .

The category Tan is braided with $c_{[n],[m]} =$ 

$: [n] \otimes [m] \to [m] \otimes [n]$, which is natural by sliding tangles across the intersection.

The category *Tan* is rigid with duals $[n]^* = [n]$, evaluation and coevaluation . The identities $=$ and $=$ are obvious since we take isotopy classes of tangles. Note that these work perfectly fine for right duals too. There is actually a general feature in this.

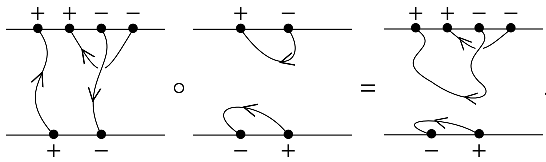
Proposition 5.2: *In a braided category $\mathcal{C}^{\otimes}$, left duals are right duals. Namely, if V^* is a left dual of V, then it is also a right dual with* $= \; = \; \circ\, c_{V,V^*}$ *and* $= \; = c^{-1}_{V,V^*} \circ$.

We want to construct isotopy invariants of tangles, namely monoidal functors $F : Tan \to \mathcal{C}^{\otimes}$. Since *Tan* is braided, $F(\;)$ induces a braiding on (its image in) $\mathcal{C}^{\otimes}$. Conversely, if $\mathcal{C}^{\otimes}$ is braided, we would like to define a functor $F : Tan \to \mathcal{C}$. We already know how to represent , with the braiding in $\mathcal{C}$. Of course, not every tangle can be obtained by composition and juxtaposition of identities and braidings (the evaluation above for example), but we do get a functor from the wide subcategory $Braid \subseteq Tan$ of braids. For any choice of object $X \in \mathcal{C}$, $[1] \mapsto X$ extends to a unique braided functor $F : Braid \to \mathcal{C}^{\otimes}$.

Seemingly, for a functor $F : Tan \to \mathcal{C}^{\otimes}$, the evaluation and coevaluation of $[1]$ in *Tan* would map to evaluation and coevaluation maps of $X = F([1])$, making it self-dual. This is too restrictive, and we will remove this condition by changing the category *Tan*.

Definition 5.3: The category Tan^{or} of oriented tangles has objects finite sequences $\bar{\eta}$ of $+$ or $-$, which we still see as a set of (now oriented) points in $\mathbb{R}^2$, and morphisms isotopy classes of oriented tangles. We impose that a strand coming out of a positively oriented point should be oriented upward, and of a negatively oriented downward. Hence composition (gluing) preserves the orientation.

Example of composition in Tan^{or} :



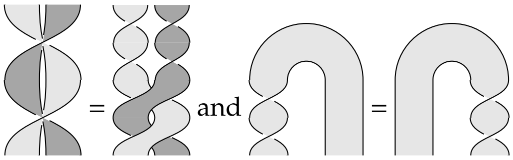
The category Tan^{or} has essentially the same monoidal and braided structure than *Tan*. Duality changes, however, because if we orient the evaluation described above, we get a morphism $\bar{\eta} \otimes -mirror(\bar{\eta}) \to \emptyset$. In the construction of tangles invariants, we now get $F(+) = X$ and $F(-) = X^*$. There is, however, a problem of coherence. The twist is equal to the identity in Tan^{or} and (in this representation) will be sent to a composition of left evaluation, braiding and right coevaluation that is not necessarily the identity in $\mathcal{C}$. Again, we prefer to change *Tan* rather than $\mathcal{C}$.

Definition 5.4: The category Tan^{fr} of framed oriented tangles has objects finite sets of oriented points $\bar{\eta}$, and morphisms isotopy classes of framed oriented tangles, with blackboard framing on the boundary. Hence composition preserves the framing.

Now the twist with blackboard framing is non-trivial in Tan^{fr} because it induces a twist in the framing if straightened. Equivalently, we can consider ribbon tangles instead of framed tangles and see the twist as .
There is a little problem now with our definition of and given in Proposition 5.2: they differ from the "good" definition in Tan^{fr} by a twist. Concretely, in Tan^{fr} the double loop is trivial, though the morphism it represents in $\mathcal{C}$ (a composition of braidings and left and right duality morphisms) may not be. To correct this, we need an action of the twist on $\mathcal{C}$.

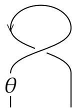
⌜ **Definition 5.5:** A twist on a braided category $\mathcal{C}^{\otimes}$ is a natural isomorphism $\theta : Id_{\mathcal{C}} \Rightarrow Id_{\mathcal{C}}$ compatible with monoidal structure : $\theta_{V\otimes W} = \theta_V \otimes \theta_W \circ c_{W,V} \circ c_{V,W}$. A braided category endowed with a twist is called balanced. A ribbon category is a rigid balanced category $\mathcal{V}$ whose twist is compatible with duality : $\theta_{V^*} = (\theta_V)^*$. ⌟

Remark 5.6: The category Tan^{fr} with twist is ribbon, and actually motivates the definition. The compatibility conditions are

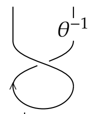


Theorem 5.1 ([Tur10]): *Let $\mathcal{V}^{\otimes}$ be a ribbon category and V an object of $\mathcal{V}$. There exists a unique monoidal functor, called the Reshetikhin–Turaev functor, $RT_V : Tan^{fr} \to \mathcal{V}$ such that $RT_V(+) = V$, $RT_V(-) = V^*$, $RT_V(\times) = c_{V,V}$, $RT_V(\frown) = ev$, $RT_V(\smile) = coev$ and $RT_V(\ ) = \theta_V$.*

Remark 5.7: Here, the right duality morphisms have been corrected with twists to untwist the ribbon: $\frown =$



$= \frown \circ c_{V,V^*} \circ (\theta_V \otimes Id_{V^*})$ and $\smile =$



$= (Id_{V^*} \otimes \theta_V^{-1}) \circ c_{V,V^*}^{-1} \circ \smile$. Concretely, θ_V provides a square root for the image of the double loop with definitions of Proposition 5.2, and adding a θ^{-1} to each $\smile$ appearing in it makes it trivial.

5.2 Coribbon Hopf algebras and ribbon categories Following the line of Section 3, we want to know when the category of finite dimensional comodules over a co-quasi-triangular Hopf algebra is ribbon.

⌜ **Definition 5.8:** A coribbon Hopf algebra is a co-quasi-triangular Hopf algebra H equipped with a coribbon functional, i.e. a map $\theta : H \to k$ such that :

(1) θ is invertible by convolution: there exists $\theta^{-1} : H \to k$ such that $\theta(a_{(1)})\theta^{-1}(a_{(2)}) = \theta^{-1}(a_{(1)})\theta(a_{(2)}) = \varepsilon(a)$,

(2) θ is central: $\theta(a_{(1)})a_{(2)} = a_{(1)}\theta(a_{(2)})$,

(3) compatibility with product: $\theta(ab) = R(b_{(1)} \otimes a_{(1)})\theta(b_{(2)})\theta(a_{(2)})R(a_{(3)} \otimes b_{(3)})$ and

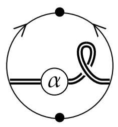
(4) compatibility with antipode: $\theta \circ S = \theta$. ⌟

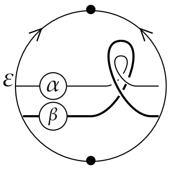
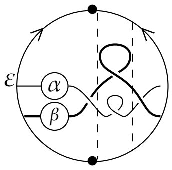
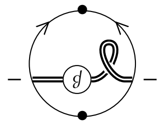
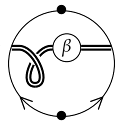
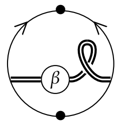
Proposition 5.9: *The category H–comod of comodules over a coribbon Hopf algebra H is balanced with twist on a H-comodule V given by $\theta_V : V \xrightarrow{\Delta_V} V \otimes H \xrightarrow{Id_V \otimes \theta} V$. The category H–comod^{fin} is ribbon.*

Idea of proof : It is a map of H-comodules because θ is central, and an isomorphism because θ is invertible. The compatibilities with monoidal structure and duality are respectively given by compatibility of θ with product and antipode, as monoidal structure and duality are defined via product and antipode. □

Remark 5.10: In the literature (see [Wak03] for the coribbon case, or [Kas95] for the ribbon case), the coribbon functional is defined to be θ^{-1} in our definition. The compatibility condition are hence deformed. In particular, if H is finite dimensional, the usual ribbon element in the dual Hopf algebra H^* is given by the inverse of our coribbon functional on H.

Fundamental example : The stated skein algebra of the bigon $\mathcal{S}(B)$, and hence the quantum group $\mathcal{O}_{q^2}(SL_2)$, is a coribbon Hopf algebra with $\theta(\alpha) = \varepsilon$



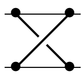
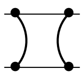
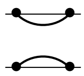
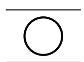
. On the generators, $\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -q^3\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It is central because the splitting morphism Δ is well-defined and does not depend on whether we cut right or left from the loop. It is invertible with θ^{-1} obtained the same way but with the inverse twist. Compatibility with product comes from $\theta(\alpha.\beta) = \varepsilon$  $= \varepsilon$  $= R(\beta_{(1)} \otimes \alpha_{(1)})\theta(\beta_{(2)})\theta(\alpha_{(2)})R(\alpha_{(3)} \otimes \beta_{(3)})$ and compatibility with antipode from $\theta(S\beta) = \varepsilon$  $\times$ coeff $= \varepsilon S$  $= \varepsilon S$  $= \theta(\beta)$, because $\varepsilon \circ S = \varepsilon$.

5.3 Skein relations It is remarkable to notice that we do re-obtain the skein relations, that were used (in this paper) to define the quantum group $\mathcal{O}_{q^2}(SL_2)$, after their interpretation under the Reshetikhin–Turaev functor $Tan^{fr} \to \mathcal{O}_{q^2}(SL_2)$*–comod*^{fin} for a particular self-dual object V.

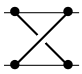
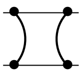
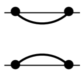
⌜ **Definition 5.11:** The standard co-representation of the quantum group $\mathcal{O}_{q^2}(SL_2)$ is the 2-dimensional vector space $V = k\langle v_0, v_1\rangle$ with coaction $\Delta_V\begin{pmatrix} v_0 & v_1 \end{pmatrix} = \begin{pmatrix} v_0 & v_1 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. ⌟

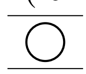
The dual V^* of V has basis (v_0^*, v_1^*) and coaction $\Delta_{V^*}(v_0^*, v_1^*) = (v_0^* \otimes d - q^2 v_1^* \otimes b, -q^{-2}v_0^* \otimes c + v_1^* \otimes a)$. It is isomorphic to V with $\varphi : \begin{cases} V & \to & V^* \\ (v_0, v_1) & \mapsto & (-qv_1^*, q^{-1}v_0^*) \end{cases}$. Hence V is self-dual with  $= ev \circ (\varphi \otimes Id_V)$ and  $= (Id_V \otimes \varphi^{-1}) \circ coev$, with ev and $coev$ the usual in $Vect$.

Proposition 5.12: *Let $RT_V : Tan^{fr} \to \mathcal{O}_{q^2}(SL_2)$–comod^{fin} be the functor with $RT_V(+) = V$ given in Theorem 5.1. Then,*

RT_V  $= q.RT_V$  $+q^{-1}.RT_V$  *and* RT_V  $= (-q^2 - q^{-2}).RT_V$  .

Proof : The first three are morphisms $V \otimes V \to V \otimes V$. In the basis $(v_0 \otimes v_0, v_1 \otimes v_1, v_0 \otimes v_1, v_1 \otimes v_0)$,

RT_V  $= \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & q^{-1} & q - q^{-3} \end{pmatrix}$, RT_V  $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and RT_V  $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q^2 & 1 \\ 0 & 0 & 1 & -q^{-2} \end{pmatrix}$,

and the first relation holds. The second is between morphisms $k \to k$, and indeed RT_V  $= -q^2 - q^{-2}$. □

6 The skein Category

The skein category is a way to construct invariants of tangles on an oriented surface Σ with local relations, in the spirit of skein algebras, those which hold after evaluation under the Reshetikhin–Turaev functor. These relations take place in a ribbon category $\mathcal{V}$, called colours, or coefficients.

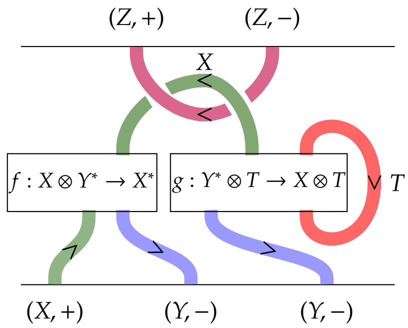
6.1 $\mathcal{V}$-coloured ribbon graphs In the preceding Section, we had a different functor $RT_X : Tan^{fr} \to \mathcal{V}$ for each object X of a ribbon category $\mathcal{V}$. They can be patched together if we think that RT_X is defined on tangles coloured by the object X.

Definition 6.1: The category $Tan^{fr}_{\mathcal{V}}$ of $\mathcal{V}$-coloured framed tangles has objects finite sets of coloured points, i.e. finite sequences $(\bar{X}, \bar{\eta})$ of pairs $(X, \pm)$, $X \in \mathcal{V}$. It has morphisms from $(\bar{X}, \bar{\eta})$ to $(\bar{Y}, \bar{\mu})$ isotopy classes of oriented framed (or ribbon) tangles from $\bar{\eta}$ to $\bar{\mu}$ with each strand coloured by an object of $\mathcal{V}$, and such that the two extremity points of a (non-circular) strand have same colour as the strand.

There are two little improvements to make. First, the skein relation (or, say, its form in Proposition 5.12) contains sums with coefficients of morphisms. We assume $\mathcal{V}$ is a k-linear category, and all its structure morphisms $(\otimes, c, \theta, ev, coev)$ are k-linear. Second, we want to consider all morphisms of $\mathcal{V}$ (not just the ones obtained from the structure morphisms) and to identify $(X, -)$ and $(X^*, +)$.

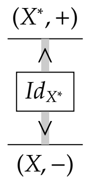
Definition 6.2: A $\mathcal{V}$-coloured ribbon graph between coloured points is a coloured oriented ribbon tangle with coupons. A coupon is an embedding of a little square $[0,1]^2$ in $\mathbb{R}^2 \times (0,1)$, and the ends of a ribbon strand may be glued to either a point $(X, \pm)$ with blackboard framing or to a part of the top or bottom edge of a coupon. We mark the end of a ribbon strand glued to a coupon with $+$ if it is going upward, and $-$ if it is going downward in the coupon. Each ribbon strand is coloured by an object of $\mathcal{V}$ and each coupon with ribbon strands coming from the bottom face $((X_1, \pm), \dots, (X_n, \pm))$, in this order, and from the top face $((Y_1, \pm), \dots, (Y_m, \pm))$ is coloured by a morphism $f : X_1^{\pm} \otimes \cdots \otimes X_n^{\pm} \to Y_1^{\pm} \otimes \cdots \otimes Y_m^{\pm}$, where $X^+ = X$ and $X^- = X^*$.

A $\mathcal{V}$-coloured ribbon graph :

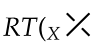
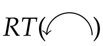
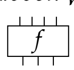


Definition 6.3: The category $Ribbon_{\mathcal{V}}$ of $\mathcal{V}$-coloured ribbon graphs has objects finite sets of coloured points and morphisms the k-vector space generated by isotopy classes of $\mathcal{V}$-coloured ribbon graphs. It is monoidal by juxtaposition, and rigid, braided and ribbon by the usual morphisms (without use of coupons).

Remark 6.4: By taking k-sums of ribbon graphs we obtain a k-linear category. Coupons provide representations for all morphisms of $\mathcal{V}$. The identification $(X, -) \simeq (X^*, +)$ is made by the identity coupon $Id_{X^*} : X^* \to X^*$ with entry a downward oriented X-coloured ribbon and output an upward oriented X^*-coloured ribbon.



6.2 Skein categories There are still some criticisms to be made. We would like to merge coupons from top to bottom and obtain the composition of maps, and merge them side by side to obtain tensor product of maps. We would like to delete identity coupons, identify ev and $coev$ coupons with simply cap and cup ribbons, and so on. Namely, all those things that coincide after the following functor.

Theorem 6.1 (Turaev): *Let $\mathcal{V}$ be a k-linear ribbon category, then there is a unique functor $RT : Ribbon_{\mathcal{V}} \to \mathcal{V}$ such that $RT((X, \pm)) = X^{\pm}$, $RT($$) = c_{X,Y}$, $RT($$) = ev$, $RT($$) = coev$, $RT($$) = \theta$ and $RT($$) = f$.*

Definition 6.5: The skein category $Sk_{\mathcal{V}}(\mathbb{R}^2)$ for a k-linear ribbon category $\mathcal{V}$ is the quotient of $Ribbon_{\mathcal{V}}$ where two linear sums of ribbon graphs are identified if they give the same morphism under RT in $\mathcal{V}$.

Remark 6.6: Since RT is full, via coupons, it induces an equivalence of categories $Sk_{\mathcal{V}}(\mathbb{R}^2) \to \mathcal{V}$.

Definition 6.7: Let Σ be an oriented surface with boundary. Replacing $\mathbb{R}^2$ by Σ in every definition:
The category $Tan(\Sigma)$ has objects finite sets of points of Σ and morphisms tangles in $\Sigma \times [0,1]$ joining a set in $\Sigma \times \{0\}$ to the other in $\Sigma \times \{1\}$.
$Tan^{or}(\Sigma)$ has objects finite sets of oriented points of Σ and morphisms oriented tangles.

$Tan^{fr}(\Sigma)$ has objects finite sets of oriented framed points of Σ and morphisms oriented framed tangles.
$Tan^{fr}_{\mathcal{V}}(\Sigma)$ has objects finite sets of coloured points of Σ and morphisms coloured framed tangles.
$Ribbon_{\mathcal{V}}(\Sigma)$ has objects finite sets of coloured points of Σ and morphisms coloured ribbon graphs. ⌟

Remark 6.8: In our definition for $\mathbb{R}^2$, we did not allow all finite sets of (framed) points, but only the ones of the form $[n] = \{1, \dots, n\} \times \{0\}$ (with blackboard framing). The categories obtained are equivalent.

As in $\mathbb{R}^2$, we would like to identify two (linear sums of) ribbon graphs which give the same morphism in $\mathcal{V}$ under the Reshetikhin–Turaev functor. This functor is not well-defined on $Ribbon_{\mathcal{V}}(\Sigma)$, it is, however, on a little cube $\phi : [0,1]^3 \to \Sigma \times [0,1]$. Given a ribbon graph F on Σ which intersect $\phi(\partial[0,1]^3)$ on either the top or the bottom face, transversally, it induces by restriction a ribbon graph on this little cube, which can be evaluated as $RT(\phi^{-1}(F|_{im\ \phi}))$ in $\mathcal{V}$.

⌜ **Definition 6.9:** The skein category $Sk_{\mathcal{V}}(\Sigma)$ with coefficients in a k-linear ribbon category $\mathcal{V}$ is the quotient of $Ribbon_{\mathcal{V}}(\Sigma)$ by the local relation $\sum \lambda_i F_i = 0$ if all of the F_i's coincide outside a little cube $\phi : [0,1]^3 \to \Sigma \times [0,1]$, intersect $\phi(\partial[0,1]^3)$ on either the top or the bottom face, transversally, and give the zero morphism in $\mathcal{V}$ after evaluation of the functor RT on this little cube, namely $\sum \lambda_i RT(\phi^{-1}(F_i|_{im\ \phi})) = 0$. ⌟

Remark 6.10: For a general surface Σ, the categories defined above are not monoidal because there is no notion of horizontal juxtaposition, which we used in $\mathbb{R}^2$. However, if $\Sigma = C \times [0,1]$ for a 1-manifold C, the category $Sk_{\mathcal{V}}(C \times [0,1])$ is monoidal with tensor product induced by $C \times [0,1] \sqcup C \times [0,1] \overset{[0,\frac{1}{3}] \sqcup [\frac{2}{3},1]}{\hookrightarrow} C \times [0,1]$.

Remark 6.11: The construction of skein categories with coefficients in a ribbon category $\mathcal{V}$ is functorial with respects to embeddings of surfaces $f : \Sigma \to \Sigma'$, which induces a functor $Sk_{\mathcal{V}}(f) : Sk_{\mathcal{V}}(\Sigma) \to Sk_{\mathcal{V}}(\Sigma')$ by mapping framed points and ribbon graphs to their images under f. They remain ribbon graphs because f is an embedding. We have a functor $Sk_{\mathcal{V}} : \begin{cases} Mfld_2^{or} & \to & Cat_k \\ \Sigma & \mapsto & Sk_{\mathcal{V}}(\Sigma) \end{cases}$, which will be further studied in Section 10.

The philosophy of skein categories is very close to the one of skein algebras. We consider framed tangles over a surface Σ (but with different "kind" of tangles for different colours) and quotient by some local relations in a little cube (but the relations can be anything taking place in $\mathcal{V}$, and not necessarily the skein relations). It turns out that skein algebras do appear as a special case of skein categories. But first, in the category $Sk_{\mathcal{V}}(\Sigma)$ we have tangles with boundaries instead of links. To avoid them, we consider the endomorphisms of the empty set $SkAlg_{\mathcal{V}}(\Sigma) := End_{\emptyset}(Sk_{\mathcal{V}}(\Sigma))$. It is an algebra with composition, which is vertical stacking here. If the k-linear ribbon category is $O_{q^2}(SL_2)$–$comod^{fin}$ and all ribbon graphs are coloured by the object V introduced in Definition 5.11 and have no coupons, then we showed in Proposition 5.12 that we quotient by the skein relations.

Proposition 6.12: *Suppose $k = \mathbb{C}$ and $q \in \mathbb{C}^\times$ is generic, the skein algebra $\mathring{\mathcal{S}}(\Sigma)$ of a surface Σ is isomorphic to the endomorphism algebra $SkAlg_{\mathcal{V}}(\Sigma)$ of the skein category $Sk_{\mathcal{V}}(\Sigma)$ for $\mathcal{V} = O_{q^2}(SL_2)$–$comod^{fin}$.*

Idea of proof : Every finite dimensional $O_{q^2}(SL_2)$-comodule is semi-simple and every finite dimensional simple $O_{q^2}(SL_2)$-comodule is a direct summand in some tensor product $V^{\otimes n}$ of V. It is hence enough to consider tangles coloured by $V^{\otimes n}$, which are n strands coloured by V. Moreover, every coupon can be removed because the morphisms $V^{\otimes n} \to V^{\otimes m}$ are generated by the image under RT of tangles without coupons. □

Part II

Factorisation homology

E_n-algebras were introduced as topological operads in [May72] for his n-fold loop space recognition principle. They can be extended in any topological category, or in the handiest notion of ∞-category presented in [Lur09]. The notions of monoidal, braided and balanced categories arise nicely as E_1-, E_2- and E_2^{or}-algebras in *Cat*. Factorisation homology is a way to extend an E_n-algebra, seen as a functor on n-disks, to any n-manifold. It is presented in [AF15], or in [Gin14]. Factorisation homologies prove to be quite analogous to the usual singular homologies with coefficients, and satisfy an Eilenberg–Steenrod theorem. Finally, the thesis [Coo19] shows how skein categories defined in Section 6 compute factorisation homologies.

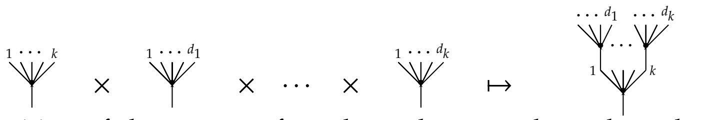
7 E_n-algebras

An E_n-algebra is an algebra object which is homotopy-associative (E_1-algebra) and homotopy commutative with a homotopy which is coherent up to order n.

7.1 Little n-cubes operad An operad is a way to represent continuous families of (abstract) products and relations between them.

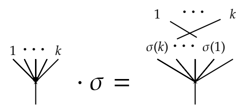
⌜ **Definition 7.1:** A symmetric topological operad is a family of weak Hausdorff compactly generated spaces $\mathcal{P}(k) = \{$  $\} \in \mathcal{CG}, k \in \mathbb{N}$, which we think as families of products of arity k, together with:

A continuous composition map $c : \left\{ \begin{array}{ccccccccc} \mathcal{P}(k) & \times & \mathcal{P}(d_1) & \times & \cdots & \times & \mathcal{P}(d_k) & \to & \mathcal{P}(d_1 + \cdots + d_k) \end{array} \right.$ which



is associative, i.e. such that the composition of three rows of products does not depend on the order of composition.

A unit for the composition $1 \in \mathcal{P}(1)$ represented by a straight line $|$.

A continuous right action $\mathcal{P}(k) \curvearrowleft \mathfrak{S}_k$ which intuitively consists in reversing the inputs: 
compatible with the composition, such that the last drawing makes sense.

Namely, $c(m_k \cdot \sigma; n_{d_1}, ..., n_{d_k}) = c(m_k; n_{d_{\sigma^{-1}(1)}}, ..., n_{d_{\sigma^{-1}(k)}}) \cdot \sigma(d_1, ..., d_k)$, where $\sigma(d_1, ..., d_k)$ acts on blocks of d_i elements, and $c(m_k; n_{d_1} \cdot \sigma_1, ..., n_{d_k} \cdot \sigma_k) = c(m_k; n_{d_1}, ..., n_{d_k}) \cdot (\sigma_1 \oplus ... \oplus \sigma_k)$ ⌟

⌜ **Definition 7.2:** A morphism of operad $f : \mathcal{P} \to \mathcal{Q}$ is a family of $\mathfrak{S}_k$-equivariant continuous maps $f_k : \mathcal{P}(k) \to \mathcal{Q}(k)$ commuting with composition.

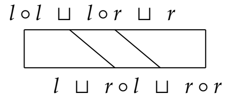
Let X be a topological space, the operad End_X of endomorphisms of X is defined by $End_X(k) = Map(X^k, X)$, composition is the composition and product of maps, namely $c(f, g_1, \dots, g_k) = f \circ (g_1 \times \cdots \times g_k)$, and $\mathfrak{S}_k$ acts by interchanging the inputs. It is the operad of all possible products $X^k \to X$.

A $\mathcal{P}$-algebra over an operad $\mathcal{P}$ is a topological space X endowed with a morphism of operads $\mathcal{P} \to End_X$. It is a family of products in X parametrised by $\mathcal{P}$. ⌟

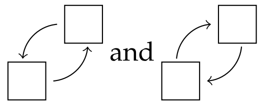
⌜ **Definition 7.3:** The little n-cubes operad $\mathcal{E}_n$ is the operad of rectilinear embeddings $\sqcup_k (0,1)^n \to (0,1)^n$. Namely, $\mathcal{E}_n(k) = Emb^{rect}(\sqcup_k (0,1)^n, (0,1)^n) = \left\{ \sqcup_k f_i \Big/ f_i : \begin{array}{ccc} (0,1)^n & \to & (0,1)^n \\ (x_1, \dots, x_n) & \mapsto & (a_1^i x_1 + b_1^i, \dots, a_n^i x_n + b_n^i) \end{array}, a_j^i > 0 \right\}$.

Composition is given by composition and disjoint union of maps, and $\mathfrak{S}_k$ acts by interchanging the little cubes.⌟

There are inclusions of operads $\mathcal{E}_{n-1} \hookrightarrow \mathcal{E}_n$ by taking the identity on the n^{th} coordinate ($a_n^i = 1$, $b_n^i = 0$ above). Hence an $\mathcal{E}_n$-algebra is always a $\mathcal{E}_k$-algebra for $k \leq n$. The colimit of $\mathcal{E}_1(k) \hookrightarrow \mathcal{E}_2(k) \hookrightarrow \mathcal{E}_3(k) \hookrightarrow \cdots$ is denoted $\mathcal{E}_\infty(k)$. Their family form an operad $\mathcal{E}_\infty$ with composition and $\mathfrak{S}_k$-action induced from the ones of the $\mathcal{E}_n$'s.

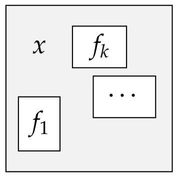
Examples : • An $\mathcal{E}_1$-algebra is a space endowed with a homotopy-associative product $m : X \times X \to X$, the image of $l \sqcup r : (0,1) \sqcup (0,1) \overset{(0,\frac{1}{2})\sqcup(\frac{1}{2},1)}{\longrightarrow} (0,1)$. The associativity is given by the homotopy 

.
There are higher associativity homotopies, like the well-known pentagon for 4-term associativity.
• An $\mathcal{E}_2$-algebra is a space endowed with a homotopy-associative product m and with two homotopies $h, (h^{op})^{-1} : m \overset{h}{\sim} m^{op}$ which are not demanded to be homotopic, plus some higher coherence conditions.



• An $\mathcal{E}_3$-algebra is an $\mathcal{E}_2$-algebra where h and $(h^{op})^{-1}$ are homotopic by two non-homotopic homotopies.
• An $\mathcal{E}_\infty$-algebra is a space endowed with a homotopy commutative product. Each space $\mathcal{E}_\infty(k)$ of arity k products is contractible.

There is a canonical example of $\mathcal{E}_n$-algebra for a based topological space X: the n-fold loop space $\Omega^n X$ of maps $(I^n, \partial I^n) \to (X, x)$. The product on $\Omega^n X$ induced by an embedding $m : \sqcup_k I^n \to I^n$ is obtained by doing the k maps on the images of the k little cubes, with basepoint elsewhere. Namely, $m(f_1, \ldots, f_k)$ for $f_1, \ldots, f_k \in \Omega^n X$ is the map $I^n \to X$ depicted hereby.



> **Theorem 7.1 (Recognition, [May72]):** *Any connected $\mathcal{E}_n$-algebra is weakly homotopy equivalent to some $\Omega^n X$.*

7.2 E_n**-algebras in topological categories** Now, the notion of an algebra over an operad can be generalised in other contexts. Intuitively, operads are just a way of parametrising products $X^k \to X$, here $X \in Top$, and this is done by a continuous map $\mathcal{P}(k) \to Map(X^k, X)$. For this to make sense in another category C, there must be (1) a topological structure on the sets $Hom_C(X, Y)$ and (2) a notion of product $X \times Y$.

Remark 7.4: We demanded compactly generated spaces in Definition 7.1 in order to have internal Hom, namely a power space X^Y right adjoint to the product. This is a reasonable assumption to add.

⌜ **Definition 7.5:** A topological category C is a category enriched over Top, i.e. whose Hom-sets are (compactly generated) topological spaces : $Hom_C(X, Y) = Map_C(X, Y) \in Top$, and such that the composition is continuous. A functor of topological categories is a functor of the underlying categories (forgetting the topological structures) that induces continuous maps on the Hom-sets (called mapping spaces). Their category is denoted by Cat_{Top}.⌟

The notion of symmetric monoidal topological category is not so easy to define (some ideas will be given in Section 8). However, we can accept that a symmetric monoidal structure on its underlying category such that all involved functors are functors of topological categories should be a symmetric monoidal topological category, and we will call this a 2-strict symmetric monoidal topological category. In general, we shouldn't require for the pentagon to commute "on the nose", but only up to homotopy, and require some higher coherence conditions on this homotopy.
Given a 2-strict symmetric monoidal topological category $C^\otimes$, an $\mathcal{E}_n$-algebra would be a family of maps $\mathcal{E}_n(k) = Map_{Top}(\sqcup_k (0,1)^n, (0,1)^n) \to Map_C(X^{\otimes k}, X)$ that commutes with composition and $\mathfrak{S}_k$-action. However, written in this form, it is much handier to define it as a functor of topological categories.

⌜ **Definition 7.6:** The topological category $Disk_n^{rect}$ of little n-cubes has objects finite disjoint unions $\sqcup_k (0,1)^n$

and mapping spaces $Map_{Disk_n^{rect}}(\sqcup_k(0,1)^n, \sqcup_r(0,1)^n) = Emb^{rect}(\sqcup_k(0,1)^n, \sqcup_r(0,1)^n)$.
It is more conventional, and equivalent, to consider the topological category $Disk_n^{fr}$ of smooth framed n-disks with objects finite disjoint unions of $\mathbb{R}^n$ ($\simeq (0,1)^n$) and mapping spaces $Emb^{fr}(\sqcup_k\mathbb{R}^n, \sqcup_r\mathbb{R}^n)$, smooth embeddings respecting the canonical framing of $\mathbb{R}^n$. The injection $Emb^{rect}(\sqcup_k(0,1)^n, \sqcup_r(0,1)^n) \hookrightarrow Emb^{fr}(\sqcup_k\mathbb{R}^n, \sqcup_r\mathbb{R}^n)$ is a deformation retract.
They are both 2-strict symmetric monoidal topological categories with disjoint unions. ⌟

It is easy to verify that an $\mathcal{E}_n$-algebra (as sketched above, or more formally in *Top*) corresponds to a symmetric monoidal functor of topological categories $Disk_n^{rect,\sqcup} \to C^{\otimes}$. The only difference is that in the latter we consider maps in $Emb^{rect}(\sqcup_k(0,1)^n, \sqcup_r(0,1)^n)$, but these are obtained as disjoint unions of maps in $Emb^{rect}(\sqcup_k(0,1)^n, (0,1)^n)$ since $(0,1)^n$ is connected. The condition monoidal is important for "abstract products" in $Emb^{rect}(\sqcup_k(0,1)^n, (0,1)^n)$ to map to products $X^{\otimes k} \to X$.

⌜ **Definition 7.7:** An E_n-algebra in a 2-strict symmetric monoidal topological category $C^{\otimes}$ is a symmetric monoidal functor of topological categories $Disk_n^{fr,\sqcup} \to C^{\otimes}$. Their category is $Alg_{Disk_n^{fr}}(C) := Fun^{\otimes}(Disk_n^{fr,\sqcup}, C^{\otimes})$.
We will often call E_n-algebra the image of $\mathbb{R}^n$ under the actual E_n-algebra $Disk_n^{fr} \to C$. ⌟

The notion of E_n-algebra in *Top* coincides with the notion of $\mathcal{E}_n$-algebra in *Top* up to homotopy. From an $\mathcal{E}_n$-algebra seen as a functor $Disk_n^{rect} \to Top$ one induces an E_n-algebra using the retraction $Emb^{fr}(\sqcup_k\mathbb{R}^n, \sqcup_r\mathbb{R}^n) \twoheadrightarrow Emb^{rect}(\sqcup_k(0,1)^n, \sqcup_r(0,1)^n)$, and from an E_n-algebra one induces an $\mathcal{E}_n$-algebra by restriction on the subspace $Emb^{rect}(\sqcup_k(0,1)^n, \sqcup_r(0,1)^n) \hookrightarrow Emb^{fr}(\sqcup_k\mathbb{R}^n, \sqcup_r\mathbb{R}^n)$. The two constructions are homotopy inverses, and we won't make any difference between E_n and $\mathcal{E}_n$ in *Top*. In particular, the Recognition Theorem 7.1 still holds for E_n-algebras.
There are other (and this time, non-equivalent) variants of E_n-algebras.

⌜ **Definition 7.8:** The topological category $Disk_n^{or}$ of smooth oriented n-disks has objects finite disjoint unions $\sqcup_k\mathbb{R}^n$ and mapping spaces $Map_{Disk_n^{or}}(\sqcup_k\mathbb{R}^n, \sqcup_r\mathbb{R}^n) = Emb^{or}(\sqcup_k\mathbb{R}^n, \sqcup_r\mathbb{R}^n)$.
An E_n^{or}-algebra is a symmetric monoidal functor of topological categories $Disk_n^{or,\sqcup} \to C^{\otimes}$.
The topological category $Disk_n$ of topological n-disks has objects finite disjoint unions $\sqcup_k\mathbb{R}^n$ and mapping spaces $Map_{Disk_n}(\sqcup_k\mathbb{R}^n, \sqcup_r\mathbb{R}^n) = Emb(\sqcup_k\mathbb{R}^n, \sqcup_r\mathbb{R}^n)$.
An E_n^{fr}-algebra is a symmetric monoidal functor of topological categories $Disk_n^{\sqcup} \to C^{\otimes}$.
Note the inclusions $Disk_n^{fr} \hookrightarrow Disk_n^{or} \hookrightarrow Disk_n$, and the unexpected association $Disk_n^{fr} \rightsquigarrow E_n$ and $Disk_n \rightsquigarrow E_n^{fr}$. ⌟

7.3 E_n-algebras in *Cat* The example that interests us the most, for Section 10, is the notion of E_n-algebras (actually, E_2^{or}-algebras) in the category *Cat* of small categories. It is not, however, a topological category, but a strict (2,1)-category. Namely, there are "morphisms between morphisms", the natural isomorphisms, which are all invertible.

⌜ **Definition 7.9:** A strict (2,1)-category C is a category enriched over groupoids, i.e. whose *Hom*-sets are the objects set of a groupoid : $Hom_C(X,Y) = Ob(Mor_C(X,Y))$ where $Mor_C(X,Y) \in \mathcal{G}rpd$ is a small category whose arrows (which we call 2-morphisms) are all invertible. Moreover, the composition should be given as functors $Mor_C(Y,Z) \times Mor_C(X,Y) \to Mor_C(X,Z)$. See Appendix B for more details.
A functor of strict (2,1)-categories is a functor of the underlying categories (forgetting the groupoid structures) that induces (or more precisely is the restriction on the set of objects of) functors on the *Hom*-groupoids. ⌟

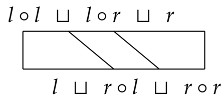
The 2-morphisms here play the role of homotopies in topological categories. However, they are indivisible cells, unlike homotopies which are continuums of the objects they deform. But the two worlds speak:

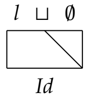
⌜ **Definition 7.10:** Let X be a topological space, its fundamental groupoid $\pi_{\leq 1}X$ has objects points on X and

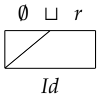
morphisms homotopy classes of paths between those points, with composition the concatenation of path. It is, of course, functorial in X.
Let C be a topological category, it induces a strict (2,1)-category $\pi_{\leq 1}C$ with same objects and by taking the fundamental groupoid of each mapping space. ⌟

⌜ **Definition 7.11:** An E_n-algebra in *Cat* is a symmetric monoidal functor $\pi_{\leq 1}Disk_n^{fr,\sqcup} \to Cat^{\times}$, and similarly for E_n^{or}- and E_n^{fr}-algebras. ⌟

Proposition 7.12: *An E_1-algebra in Cat (actually, the image of $\mathbb{R}$) is a monoidal category C.*

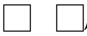
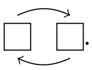
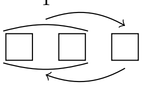
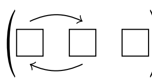
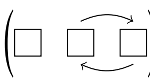
Idea of proof : The embedding $l \sqcup r : (0,1) \sqcup (0,1) \overset{(0,\frac{1}{2})\sqcup(\frac{1}{2},1)}{\longrightarrow} (0,1)$ maps to a functor $\otimes : C \times C \to C$ which is a tensor product. Indeed, the unit $\emptyset$ maps to a unit 1_C. The isotopy, i.e. path in $Emb^{fr}(\sqcup_3\mathbb{R}, \mathbb{R})$, 

maps to a natural isomorphism $\alpha : (-\otimes-)\otimes- \Rightarrow -\otimes(-\otimes-)$. Similarly 

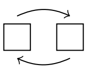
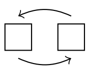
and 

map to left and right unit natural isomorphisms $l : -\otimes 1_C \Rightarrow -$ and $r : 1_C \otimes - \Rightarrow -$. It is easy to check that the paths compared in the pentagon or in the triangle in Definition 3.4 are homotopic. Hence they coincide as elements of $\pi_{\leq 1}Disk_n^{fr}$, and map to the same natural isomorphism, so the pentagon and the triangle commute in C. □

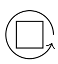
Proposition 7.13: *An E_2-algebra in Cat (the image of $\mathbb{R}^2$) is a braided category C.*

Idea of proof : An E_2-algebra is in particular an E_1-algebra, so a monoidal category with tensor product induced by , the image of $l \sqcup r$ in $Disk_1^{fr} \hookrightarrow Disk_2^{fr}$. Moreover, we get a natural isomorphism $c : \otimes \to \otimes^{op}$ from the isotopy . The isotopies  and $\left(\right.$$\left.\right) \circ \left(\right.$$\left.\right)$ are homotopic, hence induce the same natural isomorphisms. Namely, $c_{U\otimes V,W} = (c_{U,W} \otimes Id_V) \circ (Id_U \otimes c_{V,W})$. □

Proposition 7.14: *An E_3-algebra in Cat (the image of $\mathbb{R}^3$) is a symmetric monoidal category C.*

Idea of proof : The braiding  and its inverse  are homotopic in $\mathbb{R}^3$, so $c_{U,V} = c_{V,U}{}^{-1}$. □

Proposition 7.15: *An E_2^{or}-algebra in Cat (the image of $\mathbb{R}^2$) is a balanced category C.*

Idea of proof : An E_2^{or}-algebra is in particular an E_2-algebra. The isotopy , which consists of turning the disk around itself, induces a natural isomorphism $\theta : Id_C \Rightarrow Id_C$, which is a twist. □

This presentation using strict $(2,1)$-categories is but a trick to make Section 10 understandable without ∞-categories, which we will introduce briefly now.

8 ∞-categories

Higher category theory is a tool to study homotopy theory in a categorical context. The central idea is to add higher morphisms that should be thought of as homotopies between morphisms, then homotopies between homotopies, and so on. This doesn't describe all higher categories, because homotopies are always invertible,

but the notion of $(\infty, 1)$-categories, where all $\geq$ 2-morphisms are invertible.
A simple way to obtain $(\infty, 1)$-categories is the notion of topological categories described above. However, some more discrete, or cellular, higher categories hardly fit in this setting, such as the category *Cat* with 2-morphisms the natural isomorphisms, and our description of strict (2,1)-categories is not really sustainable. The notion of ∞-categories described by [Lur09] is an elegant solution, where ∞-categories are simplicial sets satisfying some extension property. It was first introduced by Joyal under the name of quasi-categories.

8.1 Definition of ∞-categories As described above, topological spaces provide groupoids: the fundamental groupoid $\pi_{\leq 1}X$. This is the truncated version of the ∞-groupoid associated with a topological space.

⌜ **Definition 8.1:** A topological space X induces a simplicial set $Sing(X)$ whose points are points of X, edges are paths on X, and more generally n-cells are continuous maps $\Delta_n \to X$ with obvious face maps obtained by restriction. We define this to be an ∞-groupoid, noted $\pi_{\leq\infty}X$. ⌟

The simplicial set $Sing(X)$ satisfies nice extension properties, namely it is a Kan complex:

⌜ **Definition 8.2:** The horn $\Lambda_n^i \subseteq \Delta_n$, $0 \leq i \leq n$ in the sub-simplicial set of Δ_n obtained by removing the interior (the n-cell) and the interior of the face opposite to the vertex i. It is called an inner horn if $0 < i < n$.

A Kan complex is a simplicial set K such that every horn $\Lambda_n^i \to K$ can be extended to $\Delta_n \to K$: $\begin{tikzcd} \Lambda_n^i \arrow[r] \arrow[d] & K \\ \Delta_n \arrow[ur, dashed, "\exists"'] \end{tikzcd}$. ⌟

We want to define ∞-groupoids to be Kan complexes, which coincide essentially with the image of $Sing(-)$. This definition is actually a relaxation of the following presentation of groupoids.

⌜ **Definition 8.3:** Let C be a small category, its nerve $N(C)$ is the simplicial set whose n-simplices are the set of functors from the poset $\langle n \rangle = \{0 < \cdots < n\}$ to C, or more concretely of sequences of n composable arrows. Face maps derive from face maps in posets, and consist in dropping the first or last arrow, or composing two arrows. Degeneracy maps consist in adding an identity. ⌟

> **Proposition 8.4:** *A simplicial complex K is the nerve of a small category $C = K_1$ if and only if every* inner *horn $\Lambda_n^i \to K$ can be extended* uniquely *to $\Delta_n \to K$. Here, K_1 is the 1-skeleton of K.* $\begin{tikzcd} \Lambda_n^i \arrow[r] \arrow[d] & K \\ \Delta_n \arrow[ur, dashed, "\exists!"'] \end{tikzcd}$

Proof : The oriented graph K_1 becomes a category if it has composition and identities. Composition is given by the (unique) extension of $\Lambda_2^1 \to K$: $\begin{tikzcd} x \arrow[rr, dashed] \arrow[dr, "f"'] & \exists! & z \\ & y \arrow[ur, "g"'] & \end{tikzcd}$. Identity is the degenerated 1-simplex over a point. Associativity is given by the extension of either Λ_3^1 or Λ_3^2. □

> **Proposition 8.5:** *A simplicial complex K is the nerve of a groupoid $\mathcal{G} = K_1$ if and only if every horn $\Lambda_n^i \to K$ can be extended* uniquely *to $\Delta_n \to K$.*

Proof : The extension of $\Lambda_2^0 \to K$: $\begin{tikzcd} x \arrow[rr, "Id_x"] \arrow[dr, "f"'] & & x \\ & y \arrow[ur, dashed, "f^{-1}"'] & \end{tikzcd}$ provides a left inverse, and the extension of Λ_2^2 a right inverse.□

We know how to switch from groupoids to ∞-groupoids, and from groupoids to categories.

⌜ **Definition 8.6:** An ∞-category is a simplicial set $\underline{C}$ such that every inner horn $\Lambda_n^i \to \underline{C}$ can be extended to $\Delta_n \to \underline{C}$. The points of $\underline{C}$ are called its objects, and its edges its morphisms. The extension of Λ_2^1 gives a notion of composition between morphisms, which is well-defined only up to homotopy, i.e. 2-morphism, i.e. 2-simplex.

An ∞-functor is a map of simplicial sets. It maps 2-simplices to 2-simplices, and hence preserves composition, at least up to homotopy. The category of ∞-categories is denoted by Cat_∞. ⌟

The category Cat_∞ has higher structure. A "natural transformation" between ∞-functors $F, G : \underline{C} \to \underline{\mathcal{D}}$ should be a collection of elements of $Map_{\underline{C}}(F(x), G(x))$, and is encoded in an ∞-functor $\Delta_1 \times \underline{C} \to \underline{\mathcal{D}}$. There are higher transformation encoded by ∞-functors $\Delta_n \times \underline{C} \to \underline{\mathcal{D}}$. To get an ∞-category Cat_∞, we should recall only natural isomorphisms, i.e. natural transformations that are invertible up to higher transformations.

Remark 8.7: There is a notion of mapping space $Map_{\underline{C}}(x, y)$ between two objects x and y of an ∞-category $\underline{C}$, whose n-simplices are morphisms $\Delta_1 \times \Delta_n \to \underline{C}$ mapping the face $\{0\} \times \Delta_n$ to the n-simplex degenerated on x and the face $\{1\} \times \Delta_n$ to the n-simplex degenerated on y. This simplicial set is a "space", i.e. a Kan complex. Hence every ≥ 2-morphism in $\underline{C}$ is invertible, and ∞-categories are indeed a model for $(\infty, 1)$-categories.

Topological and ∞-categories are equivalent in a suitable sense (they are Quillen equivalent). An ∞-category induces a topological category with the remark above. A topological category induces an ∞-category via the topological nerve. It looks like the usual nerve but encodes moreover the notions of homotopy between morphisms. The topological nerve $N(C)$ of a topological category C is the simplicial set with points the objects of C and edges the morphisms of C. It has 2-simplices $\begin{tikzcd}[sep=small] 0 \arrow[rr, "f_{02}"] \arrow[dr, "f_{01}"'] & \arrow[d, phantom, "H"] & 2 \\ & 1 \arrow[ur, "f_{12}"'] & \end{tikzcd}$ homotopies H between f_{02} and $f_{12} \circ f_{01}$. Intuitively, its n-simplices are homotopies that fill the n-simplex. For a more precise definition see [Lur09].

Examples : • The topological category $Disk_n^{fr}$ gives an ∞-category also noted $Disk_n^{fr}$ by taking its topological nerve. To define E_n-algebras in the ∞-category setting, we still need to define the notion of symmetric monoidal ∞-category.

• The strict (2,1)-category Cat "extends" to an ∞-category by taking all ≥ 3 morphisms to be identities.

Concretely, 0-simplices are small categories,

1-simplices are functors $0 \xrightarrow{F_{01}} 1$,

2-simplices are natural isomorphisms $\eta_{012} : F_{12} \circ F_{01} \Rightarrow F_{02}$, $\begin{tikzcd}[sep=small] 0 \arrow[rr, "F_{02}"] \arrow[dr, "F_{01}"'] & \arrow[d, phantom, "\Uparrow"] & 2 \\ & 1 \arrow[ur, "F_{12}"'] & \end{tikzcd}$.

The boundary of a 3-simplex $\begin{tikzcd}[sep=small] & 3 & 2 \arrow[l, "F_{23}"'] \\ 0 \arrow[ur] \arrow[urr] \arrow[r, "F_{01}"'] & 1 \arrow[u] \arrow[ur, "F_{12}"'] & \end{tikzcd}$ gives a square of natural isomorphisms $\begin{tikzcd}[sep=small] F_{23} \circ F_{02} \arrow[r, Rightarrow, "\eta_{023}"] & F_{03} \\ F_{23} \circ F_{12} \circ F_{01} \arrow[u, Rightarrow, "\eta_{012}"] \arrow[r, Rightarrow, "\eta_{123}"] & F_{13} \circ F_{01} \arrow[u, Rightarrow, "\eta_{013}"'] \end{tikzcd}$,

and a 3-simplex is a commutative square of this form. Intuitively, the homotopy inside a 3-simplex can only be an equality.

The boundary of a 4-simplex is a cube of natural isomorphisms, and fills if and only if the cube commutes, which is automatic as all its faces do (they are 3-simplices). Homotopies between equalities are indeed simple.

⌜ **Definition 8.8:** A full sub-∞-category of an ∞-category $\underline{C}$ spanned by a subset of objects $V \subseteq \underline{C}_0$ is the largest sub-simplicial set of $\underline{C}$ containing only the vertices of V. It is easy to check that it is indeed an ∞-category, with extensions given in $\underline{C}$. ⌟

8.2 Weak equivalences We define weak equivalences here for topological categories, but one can also define them for ∞-categories. Topological and ∞-categories are (Quillen) equivalent only up to weak equivalence.

⌜ **Definition 8.9:** A morphism $f : x \to y$ in a topological category is called an equivalence if it has a homotopy inverse, i.e. if there exists $g : y \to x$ and homotopies (paths in the mapping spaces) $g \circ f \overset{h}{\sim} Id_x$ and $f \circ g \overset{h}{\sim} Id_y$.

A functor of topological categories $F : C \to \mathcal{D}$ is a weak equivalence if:

(1) it is weakly essentially surjective, every object of $\mathcal{D}$ is equivalent to an object of the image of F

(2) for all objects x, y of C, $Map_C(x,y) \overset{F(-)}{\to} Map_{\mathcal{D}}(F(x), F(y))$ is a weak homotopy equivalence ⌟

Example : The topological categories $Disk_n^{rect}$ and $Disk_n^{fr}$ are weakly equivalent.

8.3 Symmetric monoidal ∞-categories In order to generalise them to the ∞-category setting, we need to see symmetric monoidal categories as pseudofunctors $Fin_* \to Cat$.

⌜ **Definition 8.10:** A pseudofunctor $C \to Cat$, where C is a category, is an assignment on objects and morphisms as a functor but that preserves identities and composition only up to natural isomorphism, and satisfies some identity and associativity conditions.

More precisely, it is an ∞-functor $p : N(C) \to Cat$. Indeed, such an ∞-functor maps a 2-simplex in $N(C)$ $\sigma :$
$$\begin{tikzcd} 0 \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & \arrow[d, phantom, "\shortparallel"] & 2 \\ & 1 \arrow[ur, "g"'] & \end{tikzcd}$$
to a 2-simplex in Cat
$$\begin{tikzcd} 0 \arrow[rr, "p(g \circ f)"] \arrow[dr, "p(f)"'] & \arrow[d, phantom, "\Uparrow"] & 2 \\ & 1 \arrow[ur, "p(g)"'] & \end{tikzcd}$$
, namely a natural isomorphism $p(\sigma) : p(g) \circ p(f) \Rightarrow p(g \circ f)$.
It satisfies some higher relations encoded in the fact that the boundaries of 3-simplices commute. ⌟

⌜ **Definition 8.11:** The category Fin_* has objects finite pointed sets (represented by $\langle n \rangle = \{*, 1, \ldots, n\}$, $n \in \mathbb{N}$) and morphisms pointed maps.

For a pseudofunctor $p : Fin_* \to Cat$, we note $C_{\langle n \rangle} = p(\langle n \rangle)$. The morphisms $\rho_i : \begin{array}{ccc} \langle n \rangle & \to & \langle 1 \rangle \\ i & \mapsto & 1 \\ 0, \ldots, i-1, i+1, \ldots, n & \mapsto & * \end{array}$
induce functors $p(\rho_i) : C_{\langle n \rangle} \to C_{\langle 1 \rangle}$. The Segal map is the functor $\Phi_n := \prod_{i=1}^n p(\rho_i) : C_{\langle n \rangle} \to (C_{\langle 1 \rangle})^n$. ⌟

Proposition 8.12: *A symmetric monoidal category corresponds to an ∞-functor $p : N(Fin_*) \to Cat$ such that all Segal maps Φ_n, $n \in \mathbb{N}$, are equivalences in Cat, i.e. have a homotopy inverse (which is an inverse up to natural isomorphisms).*

Proof : Given such an ∞-functor $p : N(Fin_*) \to Cat$, the category $C = C_{\langle 1 \rangle}$ has a monoidal structure induced by $\otimes = p\left(\begin{array}{ccc} \langle 2 \rangle & \to & \langle 1 \rangle \\ 1,2 & \mapsto & 1 \end{array} \right) : C_{\langle 2 \rangle} \overset{\Phi_2}{\simeq} (C)^2 \to C$. The functor $(- \otimes -) \otimes -$ is the composition $C_{\langle 3 \rangle} \overset{1,2 \mapsto 1}{\to} C_{\langle 2 \rangle} \overset{1,2 \mapsto 1}{\to} C$, hence is homotopic (natural isomorphic) to $p(1,2,3 \mapsto 1)$. Similarly, $- \otimes (- \otimes -)$ is $C_{\langle 3 \rangle} \overset{2,3 \mapsto 2}{\to} C_{\langle 2 \rangle} \overset{1,2 \mapsto 1}{\to} C$ and is homotopic to $p(1,2,3 \mapsto 1)$ and hence to $(- \otimes -) \otimes -$. The identity is given by $p(\langle 0 \rangle = \{*\} \to \langle 1 \rangle) : C_{\langle 0 \rangle} \overset{\Phi_0}{\simeq} * \to C$, and identity natural isomorphisms by $p(1,2 \mapsto 1) \circ p(1 \mapsto 1 \text{ or } 2) \Rightarrow p(1 \mapsto 1) \Rightarrow Id_C$. The pentagon and triangle conditions arise as boundary of (bunches of) 3-simplices, hence commutative diagrams of natural isomorphisms. The symmetry natural isomorphism is given by $\otimes^{op} \simeq p(1,2 \mapsto 1) \circ p(1,2 \mapsto 2,1) \Rightarrow p((1,2 \mapsto 1) \circ (1,2 \mapsto 2,1)) = \otimes$. Conversely, a symmetric monoidal category gives an ∞-functor $p : N(Fin_*) \to Cat$ with the same definitions. □

⌜ **Definition 8.13:** A symmetric monoidal ∞-category is an ∞-functor $p : N(Fin_*) \to Cat_\infty$ such that all Segal maps Φ_n, $n \in \mathbb{N}$, are equivalences in Cat_∞. Again, we say that $\underline{C} = p(\langle 1 \rangle)$ is a symmetric monoidal ∞-category.⌟

Here, the pentagon and triangle conditions still appear as boundaries of 3-simplices, and hence should commute in $\underline{C}$ up to 3-morphisms. However, 3-morphisms in Cat_∞ may not be equalities, and have non-trivial behaviours. Hence we also require higher coherence relations between these 3-morphisms, all encoded in the ∞-functor p.

Examples : • The ∞-category $Cat^\times$ is symmetric monoidal.
• Any symmetric monoidal category $C^\otimes$ gives a symmetric monoidal ∞-category $N(C)^\otimes$.
• The topological nerve of a 2-strict symmetric monoidal topological category is a symmetric monoidal ∞-category.

⌜ **Definition 8.14:** Let $\underline{C}$ and $\underline{\mathcal{D}}$ be two symmetric monoidal ∞-categories with monoidal structures $p, q : N(Fin_*) \to Cat_\infty$. A symmetric monoidal ∞-functor $\mathcal{F} : p \to q$ is a 2-morphism in Cat_∞, namely a homotopy invertible ∞-functor $\mathcal{F} : \Delta_1 \times N(Fin_*) \to Cat_\infty$ such that $\mathcal{F}|_{\{0\}\times N(Fin_*)} = p$ and $\mathcal{F}|_{\{1\}\times N(Fin_*)} = q$. By abuse, we say that $F = \mathcal{F}(\Delta_1 \times \langle 1\rangle)$, which is a 1-simplex in Cat_∞, namely an ∞-functor $F : \underline{C} \to \underline{\mathcal{D}}$, is a symmetric monoidal ∞-functor.

The tensor product is preserved up to natural isomorphism:
$$\begin{tikzcd} \underline{C}^2 \arrow[d, "F^2"] & \underline{C}_{\langle 2\rangle} \arrow[l, "\rho_1,\rho_2"', "\sim"] \arrow[r, "1,2\mapsto 1"] \arrow[d, "\mathcal{F}(\Delta_1\times\langle 2\rangle)"] & \underline{C} \arrow[d, "F"] \\ \underline{\mathcal{D}}^2 & \underline{\mathcal{D}}_{\langle 2\rangle} \arrow[l, "\sim", "\rho_1,\rho_2"'] \arrow[r, "1,2\mapsto 1"'] & \underline{\mathcal{D}} \end{tikzcd}.$$ ⌟

Remark 8.15: The Grothendieck construction classifies ∞-functors $N(Fin_*) \to Cat$ or Cat_∞ by some kind of fibrations $C^\otimes \to Fin_*$ where $C^\otimes \in Cat$ or Cat_∞. This is the standard definition for symmetric monoidal ∞-categories.

⌜ **Definition 8.16:** An E_n^{fr}- (resp E_n^{or}-, E_n-) algebra in a symmetric monoidal ∞-category $\underline{C}$ is a symmetric monoidal ∞-functor $\mathcal{F} : Disk_n \to \underline{C}$ (resp $Disk_n^{or} \to \underline{C}$, $Disk_n^{fr} \to \underline{C}$). ⌟

Examples : • An E_n-algebra in a topological category $F : Disk_n \to C$ induces an E_n-algebra $N(F) : N(Disk_n) \to N(C)$ via the topological nerve.
• Monoidal, braided, balanced and symmetric monoidal categories give respectively E_1^{fr}-, E_2^{fr}-, E_2- and E_3^{fr}-algebras in the ∞-category Cat.

9 Factorisation homology

Factorisation homology takes a topological input, a manifold M of dimension n, and an algebraic input, an E_n-algebra $\mathcal{A}$, in a symmetric monoidal ∞-category $\underline{C}$. It produces an object $\int_M \mathcal{A} \in \underline{C}$ which is both a topological (though not always homotopy) invariant of M and an algebraic invariant of $\mathcal{A}$. It is relative to a tangential structure on M, for example framed or oriented, which are developed in Appendix C. Factorisation homology is very similar to singular homology with coefficients, and satisfies an axiomatic description and an Eilenberg–Steenrod theorem.
Most definitions are given using categorical tools, such as colimits and Kan extensions, which are defined for ∞-categories in Appendix B. The intuitions, however, differ very little.

9.1 Definition of factorisation homology An E_n^{fr}-algebra $\mathcal{A}$ in a symmetric monoidal ∞-category $\underline{C}$ is a symmetric monoidal ∞-functor $\mathcal{A} : Disk_n^{\sqcup} \to \underline{C}^{\otimes}$. Factorisation homology is simply a tool to extend this from n-disks to all n-manifolds.

⌜ **Definition 9.1:** The topological category $Mfld_n$ has objects topological manifolds M of dimension n which admit a finite good cover, i.e. a finite cover $M = \cup_i U_i$ such that each U_i is homeomorphic to $\mathbb{R}^n$, and, moreover, any intersection $\cap_j U_j$ is either empty or homeomorphic to $\mathbb{R}^n$. It has morphisms embeddings of topological manifolds. It is symmetric monoidal with disjoint union.
The topological category $Disk_n$ is precisely its full subcategory spanned by disjoint unions of $\mathbb{R}^n$. ⌟

We are in the situation $\begin{tikzcd} Mfld_n \arrow[dr, dashed, "?"] & \\ Disk_n \arrow[u, "i"] \arrow[r, "\mathcal{A}"'] & \underline{C} \end{tikzcd}$ where we want to extend some functor $\mathcal{A}$ along an inclusion functor i.
This inclusion induces a functor $i^* : Fun(Mfld_n, \underline{C}) \to Fun(Disk_n, \underline{C})$ by restriction, and we want to find the best lift of $\mathcal{A}$ through it.

⌜ **Definition 9.2:** The left Kan extension of $\mathcal{A}$ along i is an initial object of the slice category $(\mathcal{A} \downarrow i^*)$, see Appendix A, namely a functor $L = Lan_i\mathcal{A} : Mfld_n \to \underline{C}$ together with a natural transformation $\eta : \mathcal{A} \to i^*L = L \circ i$ which is initial among such pairs. In particular, $Nat_{Mfld_n \to \underline{C}}(L, S) \overset{i^*(-)\circ\eta}{\longrightarrow} Nat_{Disk_n \to \underline{C}}(\mathcal{A}, S \circ i)$ is an isomorphism for all $S : Mfld_n \to \underline{C}$. ⌟

Remark 9.3: If it exists for all $\mathcal{A} : Disk_n \to \underline{C}$, the functor Lan_i- provides a left adjoint to i^*.

We would ideally like the Kan extension L to actually extend $\mathcal{A}$, so we know where to send disks. Now for any manifold M, each embedding $\mathbb{D} \hookrightarrow M$ should induce a map $L(\mathbb{D}) = \mathcal{A}(\mathbb{D}) \to L(M)$. A good candidate for $L(M)$ is the colimit of all these $\mathcal{A}(\mathbb{D})$ for $\mathbb{D} \hookrightarrow M$, with "relations" the intersections of disks.

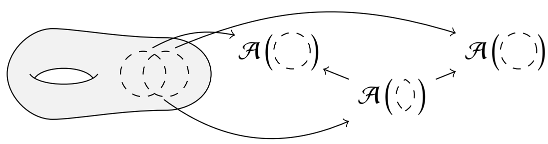
Proposition 9.4 ([McL98]): *If the colimits $L(M) := colim\Big((i \downarrow M) \overset{p_1}{\to} Disk_n \overset{\mathcal{A}}{\to} \underline{C}\Big)$ displayed below exist for all $M \in Mfld_n$, they provide the left Kan extension of $\mathcal{A}$ along i. A left Kan extension obtained this way is called pointed.*

$$\begin{tikzcd}
 & M & & & & & & & & & & & & L(M) & \\
i\mathbb{D} \arrow[ur, "f"] \arrow[r, "ih"'] & i\mathbb{D}' \arrow[u, "f'"] \arrow[r, "ih'"'] & i\mathbb{D}'' \arrow[ul, "f''"'] & \overset{p_1}{\mapsto} & \mathbb{D} \arrow[r, "h"'] & \mathbb{D}' \arrow[r, "h'"'] & \mathbb{D}'' & \overset{\mathcal{A}}{\mapsto} & \mathcal{A}(\mathbb{D}) \arrow[r, "\mathcal{A}(h)"'] & \mathcal{A}(\mathbb{D}') \arrow[r, "\mathcal{A}(h')"'] & \mathcal{A}(\mathbb{D}'') & \overset{colim}{\mapsto} & \mathcal{A}(\mathbb{D}) \arrow[ur] \arrow[r, "\mathcal{A}(h)"'] & \mathcal{A}(\mathbb{D}') \arrow[u] \arrow[r, "\mathcal{A}(h')"'] & \mathcal{A}(\mathbb{D}'') \arrow[ul]
\end{tikzcd}$$

In particular, if $\underline{C}$ is cocomplete the left Kan extension always exists.

Remark 9.5: Here $Disk_n$ is a full subcategory of $Mfld_n$, hence this colimit for $M \in Disk_n$ is trivial (Id_M is terminal in $(i \downarrow M)$) and therefore $L|_{Disk_n} = \mathcal{A}$ and $\eta = Id$.

⌜ **Definition 9.6:** Let $\mathcal{A} : Disk_n \to \underline{C}$ be an E_n^{fr}-algebra and $M \in Mfld_n$ a n-manifold. The factorisation homology over M with coefficients in $\mathcal{A}$ is the colimit $\int_M \mathcal{A} := colim\Big((i \downarrow M) \overset{p_1}{\to} Disk_n \overset{\mathcal{A}}{\to} \underline{C}\Big)$. ⌟

Namely, we reconstruct M from its disks by taking the colimit of  .

⌜ **Definition 9.7:** A symmetric monoidal ∞-category $\underline{C}^{\otimes}$ is $\otimes$-presentable if

(1) $\underline{C}$ is locally presentable i.e. cocomplete and generated by λ-presentable objects, see Appendix A.

(2) For every object $V \in \underline{C}$, the ∞-functor $V \otimes -$ preserves small colimits. ⌟

Theorem 9.1 (Ayala–Francis): *Let $\underline{C}$ be a $\otimes$-presentable symmetric monoidal ∞-category.*
There is an adjunction $\int : Alg_{Disk_n}(\underline{C}) = Fun^{\otimes}(Disk_n, \underline{C}) \rightleftarrows Fun^{\otimes}(Mfld_n, \underline{C}) : i^*$.

Idea of proof : The factorisation homology exists because $\underline{C}$ is cocomplete, and is the left adjoint of i^* by Remark 9.3. It takes values in symmetric monoidal ∞-functors because tensor product in $\underline{C}$ preserves colimits. □

From now on we fix a $\otimes$-presentable symmetric monoidal ∞-category $\underline{C}$.
These definitions and results transpose easily to the framed case, with the categories $Mfld_n^{fr}$ and $Disk_n^{fr}$ of smooth framed manifolds and disks, or to the oriented case with $Mfld_n^{or}$ and $Disk_n^{or}$.

9.2 Manifolds with boundary In order to define excision for factorisation homology, we need to extend it to the context of manifolds with boundaries.

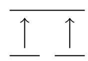
⌜ **Definition 9.8:** The topological category $Mfld_n^{\partial}$ has objects topological manifolds with boundary $(M, \partial M)$ of dimension n which admit a finite good cover, i.e. a finite cover $M = \cup_i U_i$ such that each U_i is homeomorphic to

$\mathbb{R}^n$ or $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, and, moreover, any intersection $\cap_j U_j$ is either empty or homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. It has morphisms embeddings of topological manifolds with boundary, sending the boundary in the boundary. The topological category $Disk_n^\partial$ is its full subcategory spanned by disjoint unions of $\mathbb{R}^n$ and of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. They are both symmetric monoidal with disjoint union. ⌟

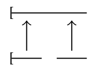
As before, a $Disk_n^\partial$-algebra in $\underline{C}^\otimes$ is a symmetric monoidal functor $Disk_n^{\partial,\sqcup} \to \underline{C}^\otimes$. The factorisation homology for manifolds with boundary is defined as its left Kan extension along $Disk_n^\partial \hookrightarrow Mfld_n^\partial$.
There are oriented and framed variants of these categories, introduced in Appendix C. For example, $Mfld_n^{\partial,*\to *}$ is the category of framed (topological) manifolds with framed boundary, where the framing on the boundary is induced from the framing in the interior with inwards normal.

Example : The ∞-category $Mfld_1^{\partial,**\to *}$ is the ∞-category of topological 1-manifolds with boundary where both the interior and the boundary are framed, but where there are two possible choices of framing on the boundary, namely inwards or outwards.

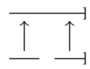
The subcategory $Disk_1^{\partial,**\to *}$ contains $Disk_1^{fr}$, so objects of the form $(-1,1)$ their disjoint unions and their framed embeddings, for objects without boundary.



It also contains a copy of $Disk_1^{\partial,*\to *}$ which has objects disjoint unions of $[-1,1)$ and of $(-1,1)$ and framed embeddings between them, so $Disk_1^{fr}$ is a part of $Disk_1^{\partial,*\to *}$ and we think of it as acting from the right.



Finally, it contains another copy of $Disk_1^{\partial,*\to *}$ which has objects disjoint unions of $(-1,1]$ and of $(-1,1)$ and framed embeddings between them, so $Disk_1^{fr}$ is a part of $Disk_1^{\partial,*\to *}$ and we think of it as acting from the left.



A $Disk_1^{\partial,**\to *}$-algebra $\mathcal{A} : Disk_1^{\partial,**\to *} \to \underline{C}$ hence induces by restriction a $Disk_1$-algebra $A = \mathcal{A}((-1,1))$, and two $Disk_1^{\partial,*\to *}$-algebras $M_- = \mathcal{A}([-1,1))$ and $M_+ = \mathcal{A}((-1,1])$ which come with respectively a right and a left A-action induced by the figures above.
The category $Mfld_1^{\partial,**\to *}$ has, moreover, an object $[-1,1]$ which has two boundaries with different orientations. It is obtained from $Disk_1^{\partial,**\to *}$ by gluing $[-1,1)$ and $(-1,1]$ along $(-1,1)$.

Proposition 9.9: *The factorisation homology* $\int_{[-1,1]} \mathcal{A}$ *of a* $Disk_1^{\partial,**\to *}$*-algebra* $\mathcal{A}$ *is the relative tensor product* $M_- \otimes_A M_+$.

Idea of proof : Such a relative tensor product has not been defined here yet, it can be defined as the colimit of a two-sided bar construction. The main idea is to identify the action of A from the right on M_- and from the left on M_+ up to homotopy, which is exactly what the gluing of factorisation homology does. □

In this paper, we may rather see this proposition as a definition. A concrete construction of such a relative tensor product is given in Section 10 in the particular case of $\underline{C}^\otimes = Cat_k^\times$.

9.3 Axioms for factorisation homology Factorisation homology, as singular homology with coefficients, is characterised by a few axioms, similar to Eilenberg–Steenrod's.

⌜ **Definition 9.10:** A collar gluing is a continuous map $f : M \to [-1,1]$ such that $f|_{(-1,1)}$ is a fibre bundle with fibre a manifold $M_0 = f^{-1}(\{0\})$ over $(-1,1)$. We say it is the collar gluing of $M_- = f^{-1}([-1,1))$ and $M_+ = f^{-1}((-1,1])$ along $A = f^{-1}((-1,1)) \simeq M_0 \times (-1,1)$ and denote it $M = M_- \cup_A M_+$. ⌟

More generally, taking pre-images of f gives an ∞-functor $f^{-1} : Disk_1^{\partial,**\to *}{}_{/[-1,1]} \to Mfld_n^{fr}{}_{/M}$ from little disks (with up to one boundary) in $[-1,1]$ to submanifolds of M.
Given a symmetric monoidal ∞-functor $F : Mfld_n^{fr} \to \underline{C}$, one gets a symmetric monoidal ∞-functor $F \circ f^{-1}$:

$Disk_{1}^{\partial,**\to *}{}_{/[-1,1]} \xrightarrow{f^{-1}} Mfld_{n}^{fr}{}_{/M} \xrightarrow{p_1} Mfld_{n}^{fr} \xrightarrow{F} \underline{C}$ whose colimit we note inaccurately $\int_{[-1,1]} F\circ f^{-1}$. Proposition 9.9 still gives an equivalence $\int_{[-1,1]} F\circ f^{-1} \simeq F(M_-)\otimes_{F(A)} F(M_+)$.

Dropping the arrow p_1 above, one gets a cocone $F\circ f^{-1} \Rightarrow F(M)$, arising from each $(\mathbb{D}' \hookrightarrow \mathbb{D}) \xrightarrow{f^{-1}} (M' \hookrightarrow M) \xrightarrow{F} (F(M') \to F(M))$. Hence there is a canonical morphism $F(M_-)\otimes_{F(A)} F(M_+) \to F(M)$ by universality of the colimit.

⌜ **Definition 9.11:** A symmetric monoidal ∞-functor $F : Mfld_n^{fr} \to \underline{C}$ satisfies the $\otimes$-excision if for any collar gluing f, the canonical morphism $F(M_-)\otimes_{F(A)} F(M_+) \to F(M)$ is an equivalence.
Such an ∞-functor is called a homology theory. We note $\mathbb{H}(Mfld_n, \underline{C})$ the full sub-∞-category of $Fun^{\otimes}(Mfld_n, \underline{C})$ of ∞-functors satisfying the $\otimes$-excision. ⌟

Example : Factorisation homology with coefficients in an E_n^{fr}-algebra $\mathcal{A} : Disk_n \to \underline{C}$, where $\underline{C}$ is $\otimes$-presentable, provides a symmetric monoidal ∞-functor $\int_{\_} \mathcal{A} : Mfld_n \to \underline{C}$.

Proposition 9.12: *Factorisation homology satisfies the $\otimes$-excision, namely $\int_{M_-} \mathcal{A} \otimes_{\int_A \mathcal{A}} \int_{M_+} \mathcal{A} \simeq \int_M \mathcal{A}$.*

Idea of proof : The colimit in the construction of $\int_{[-1,1]} F\circ f^{-1}$ above completes the colimit for $\int_M \mathcal{A}$. □

Finally, we have an Eilenberg–Steenrod theorem for those homologies.

Theorem 9.2 ([AF15]): *There is a weak equivalence of ∞-categories $\int : Alg_{Disk_n}(\underline{C}) \leftrightarrows \mathbb{H}(Mfld_n, \underline{C}) : i^*$ between $Disk_n$-algebras on $\underline{C}$ and homology theories, implemented by factorisation homology and restriction to $Disk_n$.*

Here restriction to $Disk_n$ gives the coefficients of the homology theory by Remark 9.5. It is sometimes called evaluation at $\mathbb{R}^n$, identifying a $Disk_n$-algebra $\mathcal{A} : Disk_n \to \underline{C}$ with the algebra object $A = \mathcal{A}(\mathbb{R}^n) \in \underline{C}$.

Remark 9.13: This theorem holds in the oriented and framed cases as well.

Corollary 9.14: *A symmetric monoidal ∞-functor $F : Mfld_n \to \underline{C}$ which satisfies the $\otimes$-excision is equivalent to the factorisation homology $\int_{\_} F(\mathbb{R}^n)$ with coefficients in $F(\mathbb{R}^n)$.*

9.4 Computations Finally, we compute some classical examples of factorisation homology, namely for $\underline{C} = Ch_{\geq 0}(\mathbb{Z})^{\oplus}$, $\underline{C} = Top^{\times}$ and for the case $n = 1$. Section 10 is devoted to $n = 2$ and $\underline{C} = Cat$, in the oriented case.

⌜ **Definition 9.15:** The category $Ch_{\geq 0}(\mathbb{Z})$ has objects positively graded chain complexes of abelian groups and morphisms morphisms of chain complexes. It is monoidal with direct sum of chain complexes. It is enriched over *sSet* with morphisms between two chain complexes P_* and Q_* the simplicial set $(Mor(P_*, Q_*))_n = Hom_{Ch_{\geq 0}(\mathbb{Z})}(P_* \otimes C_*(\Delta^n), Q_*)$, where $C_*(\Delta^n)$ are the singular chains of the n-simplex. Taking geometric realisation, it is enriched over *Top*, and taking the topological nerve it becomes an ∞-category, which we still denote $Ch_{\geq 0}(\mathbb{Z})$. ⌟

Proposition 9.16 ([AF15, example 3.28]): *The ∞-category $Alg_{Disk_n^{fr}}(Ch_{\geq 0}(\mathbb{Z})^{\oplus})$ of E_n-algebra in $Ch_{\geq 0}(\mathbb{Z})^{\oplus}$ is equivalent to $Ch_{\geq 0}(\mathbb{Z})$. The equivalence $Ch_{\geq 0}(\mathbb{Z}) \to Alg_{Disk_n^{fr}}(Ch_{\geq 0}(\mathbb{Z})^{\oplus})$ is given by taking the abelian group structure on a chain complex, which is a commutative, and hence E_n-, algebra structure. By abuse, we say that an E_n-algebra in $Ch_{\geq 0}(\mathbb{Z})^{\oplus}$ is simply a chain complex.*

Proposition 9.17: *Factorization homology with coefficients in a chain complex is given by singular homology with same coefficients.*

Idea of proof : The $\otimes$-excision property in $Ch_{\geq 0}(\mathbb{Z})^{\oplus}$ is equivalent to the Mayer-Vietoris long exact sequence, hence singular homology is a homology theory and is equivalent to factorisation homology with same coefficients. □

Proposition 9.18: *Let A be an E_1-algebra in $\underline{C}$, namely a homotopy-associative algebra. The only interesting 1-manifold is the circle S^1. The factorisation homology $\int_{S^1} A$ is equivalent to HC_*A, the Hochschild complex of A.*

Proof : The circle is the collar gluing of two intervals along shorter intervals on their extremities. The $\otimes$-excision gives $\int_{S^1} A \simeq A \otimes_{A \otimes A^{op}} A = HC_*A$. □

This example is well-known when $\underline{C}^{\otimes}$ is (the nerve of) $Vect_k^{\otimes}$, so that A is a usual algebra.

Proposition 9.19 (Non-abelian Poincaré Duality): *Let $\Omega^n X$ be an E_n-algebra in $Top^{\times}$ where X is $n-1$ connective and $M \in Mfld_n^{fr}$ a framed n-manifold. The factorisation homology $\int_M \Omega^n X$ is equivalent to the space of compactly supported sections of X over M, $\Gamma_c(M, X) = \{f : M \to X$ such that outside a compact $K \subseteq M$, f is constant equal to the basepoint of $X\}$.*

Proof : For $M = \mathbb{R}^n$, $\Gamma_c(\mathbb{R}^n, X) \simeq \Omega^n X$. Moreover, one can prove that the symmetric monoidal ∞-functor $\Gamma_c(-, X)$ satisfies $\otimes$-excision. By Corollary 9.14, it is equivalent to factorisation homology with coefficients in $\Gamma_c(\mathbb{R}^n, X) \simeq \Omega^n X$. Lurie outlines a more direct proof in [Lur14a]. □

10 The skein category and factorisation homology

To conclude, we compute skein categories via factorisation homology, and skein algebras via skein categories and factorisation homology with coefficients in categories of representations of quantum groups. This section is based on [Coo19].

10.1 Tambara relative tensor product We introduce a better suited notion of relative tensor product, called Tambara relative tensor product, which happens to be equivalent to the one of Proposition 9.9.

⌜ **Definition 10.1:** A right $\mathcal{A}$-module over a k-linear monoidal category $\mathcal{A}$ is a k-linear category $\mathcal{M}$ endowed with a bilinear action functor $\triangleleft : \mathcal{M} \times \mathcal{A} \to \mathcal{M}$ with a natural associativity isomorphism $\beta : (- \triangleleft -) \triangleleft - \Rightarrow - \triangleleft (- \otimes -)$ and a natural unity isomorphism $\eta : - \triangleleft 1_{\mathcal{A}} \Rightarrow -$ satisfying the expected pentagon and triangle conditions.
A left $\mathcal{A}$-module is defined similarly. Given a right $\mathcal{A}$-module $\mathcal{M}$ and a left $\mathcal{A}$-module $\mathcal{N}$, a bilinear functor $F : \mathcal{M} \times \mathcal{N} \to C$ is called $\mathcal{A}$-balanced if it is endowed with a natural isomorphism $i : F(m \triangleleft a, n) \xrightarrow{\sim} F(m, a \triangleright n)$ coherent with the associativity isomorphisms β and η. An $\mathcal{A}$-balanced natural transformation between $\mathcal{A}$-balanced functors is one that preserves the balancings i. ⌟

⌜ **Definition 10.2:** The Tambara relative tensor product of $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{A}$ is given by the category $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ with same objects than $\mathcal{M} \times \mathcal{N}$ and morphisms generated by the ones of $\mathcal{M} \times \mathcal{N}$ and a natural isomorphism $i_{m,a,n} : (m \triangleleft a, n) \xrightarrow{\sim} (m, a \triangleright n)$ satisfying the pentagon and triangle conditions. It comes with a projection functor $P : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$. ⌟

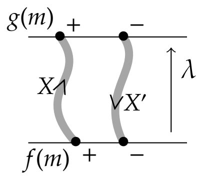
Proposition 10.3: *The Tambara relative tensor product $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{A}$ with $P : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}$ gives an equivalence of categories $Fun_{\mathcal{A}\text{-balanced}}(\mathcal{M} \times \mathcal{N}, C) \simeq Fun_k(\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}, C)$.*

Proposition 10.4: *Let $\mathcal{M}$ be a right $\mathcal{A}$-module and $\mathcal{N}$ a left $\mathcal{A}$-module. They define a $Disk_1^{\partial,**\to *}$-algebra A in $Cat_k^{\times}$ by setting $A((-1,1)) = \mathcal{A}$, $A([-1,1)) = \mathcal{M}$, $A((-1,1]) = \mathcal{N}$, $A(\uparrow\ \uparrow) = \otimes$, $A(\uparrow\ \uparrow) = \triangleleft$ and $A(\uparrow\ \uparrow) = \triangleright$. Moreover, there is an equivalence of categories $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \simeq \int_{[-1,1]} A$ between the Tambara relative tensor product and the one of Proposition 9.9.*

10.2 Skein categories are k-linear factorisation homology We now want to prove that skein categories with coefficients in a (fixed) k-linear ribbon category $\mathcal{V}$, namely the functor $Sk_{\mathcal{V}} : \begin{cases} Mfld_2^{or} & \to & Cat_k \\ \Sigma & \mapsto & Sk_{\mathcal{V}}(\Sigma) \end{cases}$, is the factorisation homology with coefficients in the E_2^{or}-algebra in Cat_k, i.e. balanced category, $\mathcal{V}$. The main part is that it satisfies $\otimes$-excision, which we will prove using Tambara relative tensor product here.
First, $Sk_{\mathcal{V}} : Mfld_2^{or,\sqcup} \to Cat_k^{\times}$ is monoidal since points or ribbon graphs on a disjoint union of surfaces is a pair of such on each surface.

Proposition 10.5: *The functor $Sk_{\mathcal{V}} : Mfld_2^{or} \to Cat_k$ induces an ∞-functor between the associated ∞-categories.*

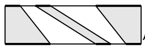
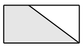
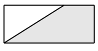
Proof : $Sk_{\mathcal{V}}$ is already defined on the 1-skeleton. The 2-skeleton of $Mfld_2^{or}$ corresponds to homotopies of embeddings, i.e. isotopies, whereas the 2-skeleton of Cat_k corresponds to natural isomorphisms.
An isotopy $\lambda : M \times [0,1] \to N$ between embeddings f and g induces a natural isomorphism $rib_\lambda : Sk_{\mathcal{V}}(f) \Rightarrow Sk_{\mathcal{V}}(g)$. For an object $m = \{(m_i, \eta_i, X_i)\}$, $m_i \in M$, $X_i \in \mathcal{V}$, the ribbon graph $rib_{\lambda,m} : f(m) \to g(m)$ is the one drawn by $(\lambda(\{m_i\} \times -), -)$ in $\Sigma \times [0,1]$ with strands oriented according to η_i and coloured by X_i. Its inverse is given by $rib_{\bar{\lambda}}$.



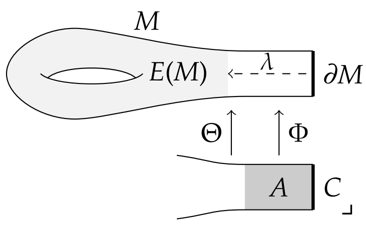
Homotopic isotopies give isotopic ribbon graphs, so $Sk_{\mathcal{V}}$ extends by equalities on the $\geq$ 3-skeleta. □

It is actually a symmetric monoidal ∞-functor $Mfld_2^{or,\sqcup} \to Cat_k^{\times}$. This is very handy: for example $Sk_{\mathcal{V}}|_{Disk_2^{or}}$ is an E_2^{or}-algebra in Cat_k, it is the E_2^{or}-algebra $\mathcal{V} \simeq Sk_{\mathcal{V}}(\mathbb{R}^2)$.

Proposition 10.6: *If $A = C \times [0,1]$ for a 1-manifold C, the category $Sk_{\mathcal{V}}(A)$ is monoidal with tensor product induced by $C \times [0,1] \sqcup C \times [0,1] \overset{[0,\frac{1}{3}]\sqcup[\frac{2}{3},1]}{\hookrightarrow} C \times [0,1]$.*

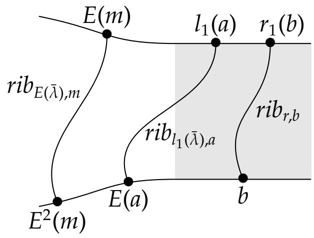
Proof : The associativity natural isomorphism of $Sk_{\mathcal{V}}(C \times [0,1])$ is rib_α, where $\alpha =$ , the left unit is rib_l, $l =$  and the right unit is rib_r, $r =$ . (These isotopies read top to bottom) □

To define the Tambara relative tensor product of skein categories, we still need to see $Sk_{\mathcal{V}}(M)$ as a module category over $Sk_{\mathcal{V}}(C \times [0,1])$ for C a well-embedded part of ∂M.
⌜ **Definition 10.7:** A thick right embedding of a 1-manifold C in a surface with boundary M is an embedding $\Theta : \big((-1,1] \times C, \{1\} \times C\big) \to \big(M, \partial M\big)$. It induces a self-embedding $E : M \to M$ isotopic to Id_M by pushing $(-1,1]$ into $(-1,-\frac{1}{2}]$, an isotopy denoted λ. It induces too an embedding $\Phi : A = C \times [0,1] \to M$ by restriction, which we think of as an inclusion, whose image is disjoint from the image of E. ⌟



Proposition 10.8: *A thick right embedding $C \to M$ induces a right $Sk_{\mathcal{V}}(A)$-module structure on $Sk_{\mathcal{V}}(M)$.*

Proof : Since E and Φ have disjoint images, $E \sqcup \Phi : M \sqcup A \to M$ is an embedding. The action functor $\triangleleft : Sk_{\mathcal{V}}(M) \times Sk_{\mathcal{V}}(A) \simeq Sk_{\mathcal{V}}(M \sqcup A) \to Sk_{\mathcal{V}}(M)$ is the functor induced by $E \sqcup \Phi$.



The associativity natural isomorphism is $\beta = rib_{E(\bar{\lambda})} \sqcup rib_{l_1(\bar{\lambda})} \sqcup rib_r$, where l_1 is extended on M by the identity outside $\Phi(A)$. The component $\beta_{m,a,b} : (m \triangleleft a) \triangleleft b = E^2(m) \sqcup E(\Phi(a)) \sqcup \Phi(b) \to m \triangleleft (a \otimes b) = E(m) \sqcup \Phi(l_1(a)) \sqcup \Phi(r_1(b))$ is depicted hereby. The unity natural isomorphism is $\eta = rib_{\bar{\lambda}}$. $\square$

Similarly, a thick left embedding $C \to N$ is an embedding $[0,2) \times C \to N$ and induces a self-embedding $E_N : N \to N$ isotopic by λ_N to Id_N and an embedding $\Phi_N : A = C \times [0,1] \to N$.

Given thick right and thick left embeddings $M \leftarrow C \to N$, the surface $M \cup_A N$ is the gluing :

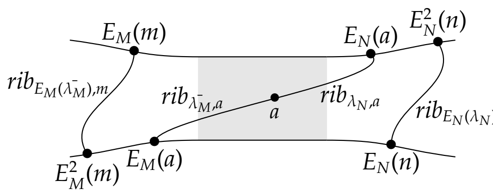
$$\begin{tikzcd} M \cup_A N \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[l] \\ M \arrow[u] & A \arrow[l, "\Phi_M"] \arrow[u, "\Phi_N"'] \end{tikzcd}$$

It is a collar gluing, and all collar gluings can be obtained this way.

Theorem 10.1: *Let $M \cup_A N$ be the collar gluing of two thick embeddings $M \leftarrow C \to N$. The skein category $Sk_{\mathcal{V}}(M \cup_A N)$, together with the projection $Sk_{\mathcal{V}}(M) \times Sk_{\mathcal{V}}(N) \to Sk_{\mathcal{V}}(M \cup_A N)$ induced by the two embeddings $M \hookrightarrow M \cup_A N$ and $N \hookrightarrow M \cup_A N$, gives the Tambara relative tensor product of the right $Sk_{\mathcal{V}}(A)$-module $Sk_{\mathcal{V}}(M)$ and the left $Sk_{\mathcal{V}}(A)$-module $Sk_{\mathcal{V}}(N)$.*

Idea of proof : We describe an isomorphism $F : Sk_{\mathcal{V}}(M) \otimes_{Sk_{\mathcal{V}}(A)} Sk_{\mathcal{V}}(N) \to Sk_{\mathcal{V}}(M \cup_A N)$.

On $Sk_{\mathcal{V}}(M) \times Sk_{\mathcal{V}}(N) \simeq Sk_{\mathcal{V}}(M \sqcup N)$ it is induced by $E_M \sqcup E_N$. Finally, $F(i) = rib_{E(\bar{\lambda}_M)} \sqcup rib_{\lambda_N} \circ rib_{\bar{\lambda}_M} \sqcup rib_{\lambda_M}$. The component $i_{m,a,n} : (m \triangleleft a, n) = E_M^2(m) \sqcup E_M(\Phi_M(a)) \sqcup E_N(n) \to (m, a \triangleright n) = E_M(m) \sqcup E_N(\Phi_N(a)) \sqcup E_N^2(n)$ is depicted hereby.



The functor F is essentially surjective because any point in the middle region A can be pushed in, say, $E_N(N)$. It is full because any ribbon graph in $E_M(M)$ or $E_N(N)$ can be obtained from a ribbon graph in M or N, and any ribbon graph in the middle region can be pushed in, say, $E_N(N)$ leaving only straight lines crossing the middle region, that are of the type $F(i)$.

It is faithful because any isotopy can be cut to pieces corresponding to isotopies in $E_M(M)$ or in $E_N(N)$, naturality of i, isomorphicity of i or pentagon and triangle conditions. The rather technical proof is done in [Coo19]. $\square$

Theorem 10.2 ([Coo19]): *Let $\mathcal{V}$ be k-linear ribbon category, and hence an E_2^{or}-algebra in Cat_k. The skein category ∞-functor $Sk_{\mathcal{V}} : Mfld_2^{or,\sqcup} \to Cat_k^{\times}$ is equivalent to the k-linear factorisation homology $\int_{-}^{Cat_k^{\times}} \mathcal{V} : Mfld_2^{or,\sqcup} \to Cat_k^{\times}$ of surfaces with coefficients in $\mathcal{V}$.*

Proof : The skein category ∞-functor is a symmetric monoidal ∞-functor $Sk_{\mathcal{V}} : Mfld_2^{or,\sqcup} \to Cat_k^{\times}$. It satisfies $\otimes$-excision by Theorem 10.1 and Proposition 10.4. Hence it is equivalent to the factorisation homology with coefficients in $Sk_{\mathcal{V}}(\mathbb{R}^2) \simeq \mathcal{V}$ by Corollary 9.14. $\square$

10.3 The algebra A_Σ For a punctured surface Σ, $Sk_{\mathcal{V}}(\Sigma)$ is a $\mathcal{V}$-module category and, for suitable choices of $\mathcal{V}$, a theorem of [BBJ18] shows that it is even represented by an algebra object A_Σ. This object, however, does not live in the ribbon category $\mathcal{V}$, but in its free cocompletion *Free*($\mathcal{V}$), and one should consider factorisation homology of this category, which is an E_2^{or}-algebra in LFP_k.

⌜ **Definition 10.9:** The strict (2,1)-category LFP_k has objects locally finitely presentable k-linear categories and morphisms functors which are both cocotinuous (which preserve small colimits) and compact (which maps compact objects to compact objects). Its 2-morphisms are natural isomorphisms.
It is equipped with the Kelly–Deligne tensor product $\boxtimes$ characterised by $Cocont(\mathcal{A} \boxtimes \mathcal{B}, C) \simeq Cocont(\mathcal{A}, \mathcal{B}; C) \simeq Cocont(\mathcal{A}, Cocont(\mathcal{B}, C))$ where $Cocont$ is the category of cocontinuous functors between cocomplete categories.⌟

This makes $LFP_k^{\boxtimes}$ a $\otimes$-presentable symmetric monoidal ∞-category.

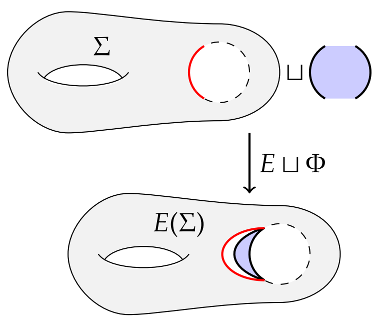
⌜ **Definition 10.10:** Let C be a k-linear category, its cocompletion is a cocomplete k-linear category $Free(C)$ with a functor $i : C \to Free(C)$ which is initial (in the 2-categorical sense) among such pairs. Namely, each functor $F : C \to \mathcal{D}$, where $\mathcal{D}$ is k-linear and cocomplete, factorises up to natural isomorphism as $F \overset{\sim}{\Rightarrow} \hat{F} \circ i$ with $\hat{F}$ cocontinuous, and this factorisation is unique up to unique natural isomorphism. ⌟

This cocompletion is given by the Yoneda embedding $C \to Fun(C^{op}, Vect_k) = Free(C)$.

Proposition 10.11 (Kelly): *The free cocompletion $Free(C)$ of a small k-linear category C is locally finitely presentable.*

Example : We take $k = \mathbb{C}$ and $q \in \mathbb{C}^{\times}$ generic. The free cocompletion of the ribbon category $\mathcal{O}_{q^2}(SL_2)$–$comod^{fin}$ is the category $\mathcal{O}_{q^2}(SL_2)$–$comod$ of all comodules on $\mathcal{O}_{q^2}(SL_2)$. It is still monoidal, braided and balanced, and gives an E_2^{or}-algebra in LFP_k. For the following, we take $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$–$comod^{fin}$ and $\mathcal{E} = Free(\mathcal{V}) \simeq \mathcal{O}_{q^2}(SL_2)$–$comod$. The following holds more generally when one replaces $SL_2(\mathbb{C})$ by another simply connected Lie group.

Proposition 10.12 (Cooke): *There are equivalences of categories $\int_{\Sigma}^{LFP_k} \mathcal{E} = \int_{\Sigma}^{LFP_k} Free(\mathcal{V}) \simeq Free\left(\int_{\Sigma}^{Cat_k} \mathcal{V}\right) \simeq Free\left(Sk_{\mathcal{V}}(\Sigma)\right)$. This actually holds for any small k-linear ribbon category $\mathcal{V}$.*



Let Σ be a punctured surface, one can always enlarge the puncture to be a removed small disk, and choose an arc on its boundary, coloured in red in the figure hereby. This arc can be thickened in the surface, locally, which gives a thickened embedding $(0,1) \to \Sigma$. As we have seen in Proposition 10.8, this gives a structure of right $Sk_{\mathcal{V}}((0,1)\times[0,1]) \simeq \mathcal{V}$-module category on $Sk_{\mathcal{V}}(\Sigma) = \int_{\Sigma}^{Cat_k} \mathcal{V}$. This works exactly the same in the LFP_k-context, namely the embedding $E \sqcup \Phi : \Sigma \sqcup (0,1) \times [0,1] \to \Sigma$ induces a functor $\int_{\Sigma}^{LFP_k} \mathcal{E} \times \int_{(0,1)\times[0,1]}^{LFP_k} \mathcal{E} \to \int_{\Sigma}^{LFP_k} \mathcal{E}$, where $\int_{(0,1)\times[0,1]}^{LFP_k} \mathcal{E} \simeq \mathcal{E}$, which gives the $\mathcal{E}$-module category structure.
The algebra A_{Σ} is defined as the internal endomorphism of a distinguished object on this $\mathcal{E}$-module category.

⌜ **Definition 10.13:** Let $\mathcal{A}$ be a k-linear monoidal category and $\mathcal{M}$ a right $\mathcal{A}$-module category. Let M_1 and M_2 be two objects of $\mathcal{M}$. The internal Hom of M_1 and M_2 with respects to the $\mathcal{A}$-module structure is an object $\underline{Hom}(M_1, M_2) \in \mathcal{A}$ together with a natural isomorphism $\eta_X : Hom_{\mathcal{M}}(M_1 \triangleleft X, M_2) \simeq Hom_{\mathcal{A}}(X, \underline{Hom}(M_1, M_2))$ for every object X of $\mathcal{A}$. ⌟

Remark 10.14: The internal Hom of two objects of $\mathcal{M}$ lives in $\mathcal{A}$, so it is not really internal.

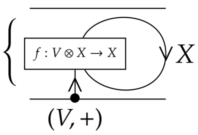
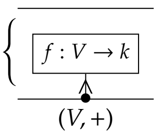
⌜ **Definition 10.15:** The evaluation map is $ev_{M_1,M_2} = \eta^{-1}_{\underline{Hom}(M_1,M_2)}(Id_{\underline{Hom}(M_1,M_2)}) : M_1 \triangleleft \underline{Hom}(M_1, M_2) \to M_2$.
The composition map is $c : \underline{Hom}(M_1, M_2) \otimes \underline{Hom}(M_2, M_3) \to \underline{Hom}(M_1, M_3)$ associated by η with the morphism $ev_{M_2,M_3} \circ (ev_{M_1,M_2} \triangleleft Id_{\underline{Hom}(M_2,M_3)}) : M_1 \triangleleft \underline{Hom}(M_1, M_2) \otimes \underline{Hom}(M_2, M_3) \to M_2 \triangleleft \underline{Hom}(M_2, M_3) \to M_3$.
In particular, an internal endomorphism $\underline{End}(M) := \underline{Hom}(M, M)$ is an algebra object in $\mathcal{A}$. ⌟

⌜ **Definition 10.16:** The embedding $\emptyset \hookrightarrow \Sigma$ induces a functor $\int_{\emptyset}^{LFP_k} \mathcal{E} = Free(Sk_{\mathcal{V}}(\emptyset)) = Free(\{k\}) = Vect_k \to \int_{\Sigma}^{LFP_k} \mathcal{E}$. The image of k under this functor is called the distinguished object of $\int_{\Sigma}^{LFP_k} \mathcal{E}$ and is denoted $O_{\mathcal{E},\Sigma}$. It is actually the image of the empty set under the inclusion functor $Sk_{\mathcal{V}}(\Sigma) \to Free(Sk_{\mathcal{V}}(\Sigma)) \simeq \int_{\Sigma}^{LFP_k} \mathcal{E}$. ⌟

⌜ **Definition 10.17:** The moduli algebra of Σ with coefficients in $\mathcal{E}$ is $A_\Sigma := \underline{End}(O_{\mathcal{E},\Sigma}) \in \mathcal{E}$. It is an $O_{q^2}(SL_2)$-comodule algebra, namely the algebra structure $m : A_\Sigma \otimes A_\Sigma \to A_\Sigma$ is a morphism of $O_{q^2}(SL_2)$-comodules.
It is an $O_{q^2}(SL_2)$-comodule, and we can define the algebra of invariants $\mathscr{A}_\Sigma := A_\Sigma^{inv} = \{x \in A_\Sigma,\ \Delta x = x \otimes 1\}$. ⌟

Theorem 10.3 ([BBJ18]): *Let Σ be a punctured surface, the factorisation homology $\int_{\Sigma}^{LFP_k} \mathcal{E}$ in LFP_k of the E_2^{or}-algebra $\mathcal{E}$ over Σ is equivalent to the category of right modules over A_Σ.*

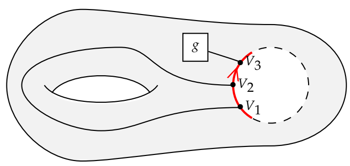
Example : For $\Sigma = \mathbb{R}^2$, we consider $Hom_{Free(Sk_{\mathcal{V}}(\mathbb{R}^2))}(\emptyset \triangleleft V, \emptyset) \simeq Hom_{Free(\mathcal{V})=\mathcal{E}}(k \otimes V, k) \simeq Hom_{\mathcal{E}}(V, A_{\mathbb{R}^2})$, and indeed $A_{\mathbb{R}^2} = k$. A right k-module is simply an $O_{q^2}(SL_2)$-comodule, and $\int_{\mathbb{R}^2}^{LFP_k} \mathcal{E} \simeq \mathcal{E}$. Graphically on $Sk_{\mathcal{V}}(\mathbb{R}^2)$, it comes down to the fact that every ribbon graph on $\mathbb{R}^2$ can be evaluated by the Reshetikhin–Turaev functor to give a single coupon:

$Hom_{Sk_{\mathcal{V}}(\mathbb{R}^2)}(\emptyset \triangleleft V, \emptyset) = Hom_{Sk_{\mathcal{V}}(\mathbb{R}^2)}((V,+), \emptyset) = \Bigg\{$  $\Bigg\} \underset{Sk}{\equiv} \Bigg\{$  $\Bigg\} = Hom_{\mathcal{E}}(V, k)$.

Theorem 10.4 (Cooke): *Let Σ be a punctured surface, the algebra of invariants $\mathscr{A}_\Sigma$ of the surface is isomorphic to its skein algebra $SkAlg_{\mathcal{V}}(\Sigma) = End_{\emptyset}(Sk_{\mathcal{V}}(\Sigma)) \simeq \mathring{\mathcal{S}}(\Sigma)$.*

Idea of proof : First, $O_{\mathcal{E},\Sigma}$ is compact and $Hom_{\int_{\Sigma}^{LFP_k} \mathcal{E}}(O_{\mathcal{E},\Sigma}, O_{\mathcal{E},\Sigma}) = Hom_{Free(Sk_{\mathcal{V}}(\Sigma))}(\emptyset, \emptyset) = Hom_{Sk_{\mathcal{V}}(\Sigma)}(\emptyset, \emptyset) = SkAlg_{\mathcal{V}}(\Sigma)$, see [Coo19]. Then, by definition of internal Hom, $Hom_{\int_{\Sigma}^{LFP_k} \mathcal{E}}(O_{\mathcal{E},\Sigma} \triangleleft k, O_{\mathcal{E},\Sigma}) \simeq Hom_{\mathcal{E}}(k, A_\Sigma) = A_\Sigma^{inv} = \mathscr{A}_\Sigma$. □

Graphically, a morphism in $Sk_{\mathcal{V}}(\Sigma)$ between $\emptyset \triangleleft \emptyset = \emptyset$ and $\emptyset$ is just a ribbon graph without free ends, and corresponds to a link on Σ modulo the skein relations. In general, we must consider morphisms between $\emptyset \triangleleft V$, a point $(V,+)$ near the red arc, and $\emptyset$. Considering the monoidal structure, we may see it as an ordered sequence of points near the red arc, and by analogy with stated skein algebras we will talk of height order (the actual order is given by distance to the red arc, but as for stated skein algebras we will represent points on the arc and the order by an orientation of the arc).

We are looking for a correspondence  $\leftrightsquigarrow f : V_1 \otimes V_2 \otimes V_3 \to A_\Sigma$.

When each boundary point is colored by the standard co-representation V, which is always possible up to adding a projection, and when there are no coupons, the colored ribbon graph is simply a tangle. The vector space V is generated by two vectors v_+ and v_-, and the value of f on $v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_k}$ is the same tangle with states $(\varepsilon_1, \ldots, \varepsilon_k)$.

Theorem 10.5 (Thang T.Q. Lê and Tao Yu): *Let Σ be a punctured surface and $\mathfrak{S}$ the marked surface obtained from Σ by removing an open disk instead of the puncture, and marking a single point on the newly created boundary. The moduli algebra A_Σ with coefficients $\mathcal{E} = O_{q^2}(SL_2)$–comod is isomorphic as an $O_{q^2}(SL_2)$-comodule algebra to the stated skein algebra $\mathcal{S}(\mathfrak{S})$.*

Appendices

A Category theory

In this appendix we define some well-known notions of category theory that have been used in this paper. Most definitions come from [McL98]. Presentable categories are studied in [AR94].

A.1 Slice category First we give the definition of the category of arrows over an object, called slice category.

⌜ **Definition A.1:** Let C be a category and $X \in C$ an object. The slice category of objects over X is the category $C_{/X}$ whose objects are pairs (Y, f) where Y is an object of C and f is a morphism $Y \to X$. Its morphisms are commutative triangles $\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$, namely a morphism $(Y, f) \to (Z, g)$ is a morphism $h : Y \to Z$ over X in C.
Dually, the category $C_{X/}$ of objects under X has objects pairs $(Y, f : X \to Y)$ and morphisms similar commutative triangles. ⌟

This definition can be extended for maps over or under more than one object.

⌜ **Definition A.2:** Let $p : \mathcal{D} \to C$ be a functor, the slice category $C_{/p}$ of objects over p has objects are pairs (Y, f) where Y is an object of C and f is a cone $\delta(Y) \Rightarrow p$, where $\delta(Y)$ is the constant functor $\mathcal{D} \to C$. Its morphisms are commutative triangles $\begin{tikzcd} \delta(Y) \arrow[rr, "\delta(h)"] \arrow[dr, "f"'] & & \delta(Z) \arrow[dl, "g"] \\ & p & \end{tikzcd}$, namely a morphism $(Y, f) \to (Z, g)$ is a morphism $h : Y \to Z$ in C such that $g \circ \delta(h) = f$.
Dually, the category $C_{p/}$ of objects under p has objects pairs $(Y, f : p \Rightarrow \delta(Y))$ of an object and a cocone of p on this object, and morphisms similar commutative triangles. ⌟

Remark A.3: As objects of $C_{/p}$ are cones on p, the terminal object of $C_{/p}$ is precisely the limit of p. Dually, the initial object of $C_{p/}$ is the colimit of p.

We would like to write this definition as a special case of the first where $C = Fun(\mathcal{D}, C)$ and $X = p$ but we cannot because the object Y and morphism h live at the level of C and not in $Fun(\mathcal{D}, C)$.

⌜ **Definition A.4:** Let $S : \mathcal{D} \to C$ be a functor and $X \in C$ an object. The comma category $(S \downarrow X)$ of objects S-over X has objects pairs $(A \in \mathcal{D}, f : S(A) \to X)$ and morphisms commutative triangles $\begin{tikzcd} S(A) \arrow[rr, "S(h)"] \arrow[dr, "f"'] & & S(B) \arrow[dl, "g"] \\ & X & \end{tikzcd}$, where $h : A \to B$ is a morphism in $\mathcal{D}$.
There is a canonical projection $(S \downarrow X) \to \mathcal{D}$ by forgetting the arrow f, and keeping only the object A of $\mathcal{D}$ and the morphisms h between them.
Dually, the category $(X \downarrow S)$ of objects S-under X has objects $(A, f : X \to S(A))$ and morphisms $h : A \to B$, $S(h) \circ f = g$. ⌟

An initial (dually terminal) object in such a comma category is called a universal arrow. They recover the notions of colimits (dually limits) and of left (dually right) Kan extension.

A.2 Locally presentable categories A finite set can be characterised by the fact that when it maps to a (possibly infinite) union of space, it always factors through a finite union. This is the notion of compact, or finitely presentable, object.

⌜ **Definition A.5:** Let λ be a regular cardinal. A category $\mathcal{D}$ is said λ-filtered if any subcategory of size (objects and morphisms) strictly smaller than λ admits a cocone in $\mathcal{D}$.
A λ-filtered colimit is a colimit over a λ-filtered diagram category. It is hence trivial on every subcategory of size $< \lambda$. ⌟

⌜ **Definition A.6:** An object K of a category C is said λ-presentable if the functor $Hom_C(K, -) : C \to Set$ preserves λ-filtered colimits. We note $Pres_\lambda C$ their full subcategory. When $\lambda = \aleph_0$, we talk about finitely presentable objects, or compact objects. ⌟

Examples : • An object of *Set* is λ-presentable if and only if it is of cardinal $< \lambda$. It is finitely presentable if and only if it is finite.
• A group is finitely presentable if and only if it is presentable in the usual sense.
• A small category in *Cat* is finitely presentable if and only if it has finitely many objects and finitely many morphisms between any two objects.

⌜ **Definition A.7:** A category C is locally λ-presentable if:
(1) C is cocomplete,
(2) $Pres_\lambda C$ is essentially small, and
(3) any object of C is a λ-filtered colimit of λ-presentable objects. ⌟

Examples : • There is only a set of finite sets up to isomorphism, namely $\mathbb{N}$, and any set is obtained as the colimit of its finite subsets. Hence *Set* is locally finitely presentable.
• Seemingly, a small category is obtained as the colimit of its finite subcategories, and the category *Cat* is locally finitely presentable.
• A vector space is locally presentable if and only if it is finite dimensional, and again $Vect_k$ is locally finitely presentable.
• The category *Top* is not locally λ-presentable for any cardinal λ.

B Higher category theory

In Section 9, we used many notions that were defined for categories, but not for ∞-categories, yet. This Appendix gives the missing definitions of slice ∞-categories, limits and colimits in ∞-categories, and Kan extensions. All and more is detailed in [Lur09].

B.1 Enriched categories First, we formalise a notion we have used implicitly throughout this paper.

⌜ **Definition B.1:** Let $\mathcal{V}^\otimes$ be a monoidal category, such as $Set^\times$ for ordinary categories, $Vect_k^\otimes$ for k-linear categories, $Top^\times$ for topological categories or $\mathcal{G}rpd^\times$ for strict (2,1)-categories.
A $\mathcal{V}$-enriched category C is :
— a collection of objects of C
— an object $Mor_C(x, y)$ of $\mathcal{V}$ for each pair of objects $x, y \in C$, called morphisms between x and y
— an arrow $c_{x,y,z} : Mor_C(y, z) \otimes Mor_C(x, y) \to Mor_C(x, z)$ in $\mathcal{V}$ for each triple of objects, called composition
— an arrow $I_x : 1_\mathcal{V} \to Mor_C(x, x)$ for each object $x \in C$, called identities
which satisfy the usual associativity and unity relations, namely:

$$\begin{tikzcd}
(Mor_C(z,t) \otimes Mor_C(y,z)) \otimes Mor_C(x,y) \arrow[rr, "\alpha"] \arrow[d, "c_{y,z,t} \otimes Id"'] & & Mor_C(z,t) \otimes (Mor_C(y,z) \otimes Mor_C(x,y)) \arrow[d, "Id \otimes c_{x,y,z}"] \\
Mor_C(y,t) \otimes Mor_C(x,y) \arrow[r, "c_{x,y,t}"'] & Mor_C(x,t) & Mor_C(z,t) \otimes Mor_C(x,z) \arrow[l, "c_{x,z,t}"]
\end{tikzcd}$$ and

$$\begin{tikzcd}
Mor_C(y,y)\otimes Mor_C(x,y) \arrow[dr, "c_{x,y,y}"] & & Mor_C(x,y)\otimes Mor_C(x,x) \arrow[dl, "c_{x,x,y}"'] \\
1_{\mathcal{V}}\otimes Mor_C(x,y) \arrow[u, "Id\otimes I_y"] \arrow[r, "l"'] & Mor_C(x,y) & Mor_C(x,y)\otimes 1_{\mathcal{V}} \arrow[l, "r"] \arrow[u, "I_x\otimes Id"']
\end{tikzcd} \quad \text{commute.}$$

A $\mathcal{V}$-enriched functor F between $\mathcal{V}$-enriches categories C and $\mathcal{D}$ is an assignment $x \mapsto Fx$ from objects of C to objects of $\mathcal{D}$ and a collection of arrows $Mor_C(x,y) \overset{F(-)}{\to} Mor_{\mathcal{D}}(Fx,Fy)$ in $\mathcal{V}$, $x,y \in C$, which maps identities to identities and which commutes with composition, namely:

$$\begin{tikzcd}
Mor_C(y,z)\otimes Mor_C(x,y) \arrow[r, "c_{x,y,z}"] \arrow[d, "F(-)\otimes F(-)"'] & Mor_C(x,z) \arrow[d, "F(-)"] \\
Mor_{\mathcal{D}}(Fy,Fz)\otimes Mor_{\mathcal{D}}(Fx,Fy) \arrow[r, "c_{Fx,Fy,Fz}"] & Mor_{\mathcal{D}}(Fx,Fz)
\end{tikzcd} \quad \text{and} \quad
\begin{tikzcd}
 & Mor_C(x,x) \arrow[d, "F(-)"] \\
1_{\mathcal{V}} \arrow[ur, "I_x"] \arrow[r, "I_{Fx}"'] & Mor_{\mathcal{D}}(Fx,Fx)
\end{tikzcd} \quad \text{commute.}$$

A $\mathcal{V}$-enriched natural transformation η between $\mathcal{V}$-enriched functors $F,G : C \to \mathcal{D}$ is a collection of components $\eta_x : 1_{\mathcal{V}} \to Mor_{\mathcal{D}}(F(x),G(x))$, $x \in C$, which are natural in the usual sense, namely:

$$\begin{tikzcd}
 & 1_{\mathcal{V}}\otimes Mor_C(x,y) \arrow[r, "\eta_y\otimes F(-)"] & Mor_{\mathcal{D}}(Fy,Gy)\otimes Mor_{\mathcal{D}}(Fx,Fy) \arrow[dr, "c_{Fx,Fy,Gy}"'] & \\
Mor_C(x,y) \arrow[ur, "l^{-1}"] \arrow[dr, "r^{-1}"'] & & & Mor_{\mathcal{D}}(Fx,Gy) \\
 & Mor_C(x,y)\otimes 1_{\mathcal{V}} \arrow[r, "G(-)\otimes\eta_x"] & Mor_{\mathcal{D}}(Gx,Gy)\otimes Mor_{\mathcal{D}}(Fx,Gx) \arrow[ur, "c_{Fx,Gx,Gy}"] &
\end{tikzcd} \quad \text{commutes.}$$ ⌟

B.2 Slice ∞-category Arrows in ∞-categories are already defined: they are 1-simplices, and morphisms between arrows too: they are 2-simplices, i.e. triangles as in Definition A.1 that are not commutative but "filled". Similarly, cones over an ∞-functors are easy to define, as well as morphisms between cones. However, we do not want a category of arrows or cones, but an ∞-category, namely we also want higher morphisms between them. To encode easily the shape of all these higher morphism, we introduce the join construction.

⌜ **Definition B.2:** The join $S \star T$ of two simplicial sets S and T is the simplicial set defined by: $(S\star T)(0<1<...<n) = \coprod_{i=-1}^{n} S(0<...<i)\times T(i+1<...<n)$, where $S(\emptyset)=*$. It comes with inclusions $S \hookrightarrow S\star T$ given by the term $S(0<1<...<n)\times *$ above, and $T \hookrightarrow S\star T$ given by $*\times T(0<1<...<n)$. ⌟

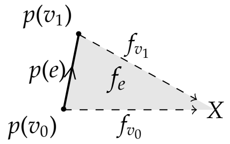
It should be thought of as the juxtaposition $P<Q$ of posets, and the above formula computes the nerves of the posets, namely $N(P<Q) = N(P)\star N(Q)$. In particular, $\Delta_n\star\Delta_m = \Delta_{n+m+1}$. Topologically, it corresponds to the join construction: if we note $X=|S|$ and $Y=|T|$ then $|S\star T| = |S|\star|T| = X\times[0,1]\times Y\big/ (x,0,y)\sim(x,0,y'),\ (x,1,y)\sim(x',1,y)$.

Example : For $S=\Delta_0=*$, the join $*\star T$ is the shape of a cone on T, namely it has a $n+1$-simplex for each n-simplex of T doing a cone of base this simplex and tip S. Topologically, $*\star X = CX$ is a cone.
In the simplicial world, left and right cone are different because the edges have different orientations, and $T\star *$ is the shape of a cocone on T.

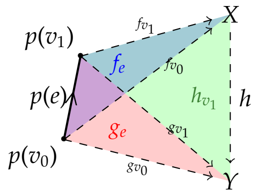
⌜ **Definition B.3:** Let $p : K \to \underline{C}$ be an ∞-functor. The slice ∞-category $\underline{C}_{/p}$ of objects over p is given by: $(\underline{C}_{/p})_n = Hom_{sSet}(\Delta^n, \underline{C}_{/p}) := Hom_p(\Delta^n\star K, \underline{C})$, where Hom_p here represents the morphisms $\Delta^n\star K \to \underline{C}$ which coincide with p on $K \hookrightarrow \Delta^n\star K \to \underline{C}$.
Dually, the slice ∞-category $\underline{C}_{p/}$ of objects under p is given by: $(\underline{C}_{p/})_n = Hom_{sSet}(\Delta^n, \underline{C}_{p/}) := Hom_p(K\star\Delta^n, \underline{C})$, again the morphisms $K\star\Delta^n \to \underline{C}$ which coincide with p on $K \hookrightarrow K\star\Delta^n \to \underline{C}$. ⌟

Proposition B.4 (Lurie): *The simplicial sets $\underline{C}_{/p}$ and $\underline{C}_{p/}$ are ∞-categories as soon as $\underline{C}$ is one. Moreover, their weak equivalence classes only depend on the one of $\underline{C}$.*

An object of $\underline{C}_{p/}$ is a morphism from the cocone shape $K \star \Delta^0 \to \underline{C}$ which coincides with p on K. If we note X its value on Δ^0, it is exactly a cocone $f : p \Rightarrow X$, namely a morphism $f_v : p(v) \to X$ for each vertex v of K, a 2-morphism f_e between $p(e)$, f_{v_0} and f_{v_1} for each edge e joining v_0 and v_1 in K, and so on...



A morphism of $\underline{C}_{p/}$ is a morphism $K * \Delta^1 \to \underline{C}$ which coincides with p on K. It restricts to two cocones $f : p \Rightarrow X$ and $g : p \Rightarrow Y$on the vertices of Δ^1, and gives, moreover, a morphism h between them. Precisely, it gives a morphism $h : X \to Y$, a 2-morphism h_v between f_v, g_v and h for each vertex v of K, a 3-morphism between f_e, g_e, h_{v_0} and h_{v_1} (which fill the tetrahedra hereby) for each edge e , and so on...



Example : For $K = *$ and $p(*) = X$ is an object of $\underline{C}$, we note $\underline{C}_{/p} = \underline{C}_{/X}$. An object of $\underline{C}_{/X}$ is a 1-simplex f of target X, namely a pair (Y, f) where $Y \in \underline{C}_0$ is an object of $\underline{C}$ and $f : Y \to X$ is a morphism of $\underline{C}$. Higher morphisms are 2-simplices $\sigma :$ $\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$, and so on...

Remark B.5: For $i : \underline{\mathcal{D}} \subseteq \underline{C}$ a full sub-∞-category and $X \in \underline{C}_0$ an object of $\underline{C}$, the comma-category $(i \downarrow X)$ is simply the full sub-∞-category of $\underline{C}_{/X}$ spanned by pairs $(Y, f : Y \to X)$ where $Y \in \underline{\mathcal{D}}_0$.
So we don't need to introduce general comma-categories for Section 9 to make sense.

B.3 Limits in ∞-categories Following Remark A.3, we define the limit or colimit of an ∞-functor as a terminal or initial object of the slice category.

⌜ **Definition B.6:** An object X of an ∞-category $\underline{C}$ is initial if for every object Z the space $Map_{\underline{C}}(X, Z)$ is contractible. If it exists, the initial object of $\underline{C}$ is unique up to essentially unique equivalence.
Dually, X is terminal if for every object Z the space $Map_{\underline{C}}(Z, X)$ is contractible. ⌟

⌜ **Definition B.7:** The limit of an ∞-functor $p : K \to \underline{C}$ is the terminal object of $\underline{C}_{/p}$, and its colimit is the initial object of $\underline{C}_{p/}$, if they exist. ⌟

B.4 Kan extensions We define left Kan extensions for the easier particular case of an inclusion of a full sub-∞-category, which is what we used with $Disk_n \hookrightarrow Mfld_n$.

⌜ **Definition B.8:** Let $i : \underline{\mathcal{D}} \subseteq \underline{\mathcal{M}}$ be a full sub-∞-category and $\mathcal{A} : \underline{\mathcal{D}} \to \underline{C}$ an ∞-functor. We want to extend $\mathcal{A}$ along the inclusion functor i. Consider a commutative diagram $\begin{tikzcd} \underline{\mathcal{M}} \arrow[dr, "L"] & \\ \underline{\mathcal{D}} \arrow[u, "i"] \arrow[r, "\mathcal{A}"'] & \underline{C} \end{tikzcd}$ in Cat_∞. For $M \in \underline{\mathcal{M}}_0$ an object, the ∞-functor $L : \underline{\mathcal{M}} \to \underline{C}$ induces a cocone on $\mathcal{A}_M$: $\begin{array}{ccccc} (\underline{\mathcal{D}} \downarrow M) & \to & \underline{\mathcal{D}} & \overset{\mathcal{A}}{\to} & \underline{C} \\ (D, f : D \to M) & \mapsto & D & \mapsto & \mathcal{A}(D) \end{array}$ by $L_M : \begin{array}{ccc} (\underline{\mathcal{D}} \downarrow M) & \overset{L}{\to} & \underline{C}_{/L(M)} \\ (D, f : D \to M) & \mapsto & (L(D) = \mathcal{A}(D), L(f) : L(D) \to L(M)) \end{array}$.
We say that L is the pointed left Kan extension of $\mathcal{A}$ along i if this cocone L_M exhibits $L(M)$ as the colimit of $\mathcal{A}_M$ in $\underline{C}$ for each object $M \in \underline{\mathcal{M}}_0$. ⌟

C Tangential structure

We want to define tangential structures for topological manifolds, such as framed or oriented, and more, namely the B-framing of [AF15]. We need both a homotopy-theoretic classification of such structures (see for example [Coh02]) and a notion of tangent bundle for topological manifolds (see [Lur14b]).

C.1 Classification of fibre bundles Structures on tangent bundles arise as properties of their underlying fibre bundle.

⌜ **Definition C.1:** Let B and F be topological spaces. A fibre bundle of base B and fibre F is a continuous map $p : E \to B$, where E is called the total space, together with a local trivialisation atlas, namely an open cover $\mathcal{U} = (U_i)_i$ of B with trivialisation maps $\varphi_{U_i} : p^{-1}(U_i) \to U_i \times F$ such that the
$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "\varphi_{U_i}"] \arrow[dr, "p"'] & & U_i \times F \arrow[dl, "p_1"] \\ & U_i & \end{tikzcd}$$
commute. For naturality, we suppose the atlas maximal.

A morphism of fibre bundles $(p : E \to B) \to (p' : E' \to B')$ is a commutative square
$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$
. A morphism of fibre bundles above a same base B is one with $f = Id_B$.
A section of a fibre bundle $p : E \to B$ is a map $s : B \to E$ such that $p \circ s = Id_B$. ⌟

For a fibre bundle $p : E \to B$, the trivialisation maps on two trivial open sets U and V of B give transition functions $\psi_{U,V} : (U \cap V) \times F \xrightarrow{\varphi_U^{-1}} p^{-1}(U \cap V) \xrightarrow{\varphi_V} (U \cap V) \times F$. This transition map induce the identity on B, and hence (as soon as B and F are compactly generated) is equivalent to a map $\Psi_{U,V} : U \cap V \to Homeo(F)$.
These maps determine the fibre bundle up to isomorphism, by gluing together trivial bundles $U_i \times F$ for each trivial open $U_i \in \mathcal{U}$ along the transition functions.
For G a subgroup of $Homeo(F)$, we wonder if we may take $\Psi_{U,V} : U \cap V \to G$, at least for a subcover of $\mathcal{U}$. If this is the case, we say that the fibre bundle p reduces on G. We say that p is reduced on G if we have chosen a maximal open cover $\mathcal{U}' \subseteq \mathcal{U}$ with transition functions taking values in G.

⌜ **Definition C.2:** Let G be a topological group. A principal G-bundle is a fibre bundle with fibre G reduced on G, where $G \subseteq Homeo(G)$ by translation. Consequently, the action of G on the total space is well defined, because translations are G-equivariant.
Equivalently, it is a fibre bundle $G \hookrightarrow P \xrightarrow{p} B$ together with a free fibrewise continuous right action $P \curvearrowleft G$ which acts transitively on the fibres, and such that the trivialisation maps (again, on a given maximal subatlas) are G-equivariant.
A morphism of principal G-bundles is a G-equivariant morphism of fibre bundles.
We note $\mathcal{P}_G B$ the set of isomorphism classes of principal G-bundles. ⌟

These principal G-bundles are simply a general way to describe fibre bundles with transitions functions taking values in G. For example, if G is a subgroup of some $Homeo(F)$, then a principal G-bundle is the same data as a fibre bundle with fibre F reduced on G. Indeed, one can construct one from the other by gluing trivial bundles (with fibre either F or G) along the transitions functions, which take values in G in both cases.

Example : A vector bundle of rank n is the same data as a principal $GL_n(\mathbb{R})$-bundle.

⌜ **Definition C.3:** Let G be a topological group. A universal G-bundle is a principal G-bundle $G \hookrightarrow EG \xrightarrow{p} BG$ where EG is a contractible topological space. The space BG is called the classifying space of G. ⌟

Proposition C.4: *Let X be a topological space and $p : P \to B$ a principal G-bundle. A continuous map $f : X \to B$ induces a principal G-bundle f^*p on X by the pullback*
$$\begin{tikzcd} f^*P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$
. This pullback bundle depends only on the homotopy

class of f, namely we have a map $[X, B] \to \mathcal{P}_G X$.
If $p : EG \to BG$ is a universal G-bundle, this map $[X, BG] \to \mathcal{P}_G X$ is a bijection.

We have an explicit construction for the classifying space of a topological group G. The topological category $\mathcal{B}G$ is the category with one object $*$ and morphisms $Hom(*, *) = G$, with composition given by multiplication in G. The simplicial set $\mathbf{B}G$ is the topological nerve of $\mathcal{B}G$. Finally, the classifying space BG of G is the geometric realisation of $\mathbf{B}G$. This construction is functorial, and in particular for $H < G$ a subgroup, the inclusion gives a map $\iota : BH \to BG$.

Proposition C.5: *Let X be a topological space, $p : P \to X$ a principal G-bundle, classified by a map $f : X \to BG$, and $H < G$ a subgroup. Then p reduces on H if and only if f factorises through ι up to homotopy. Namely, we have a homotopy commutative triangle* $\begin{tikzcd} & BH \arrow[d, "\iota"] \\ X \arrow[ur, dashed, "\exists \tilde{f}"] \arrow[r, "f"'] & BG \end{tikzcd}$. *A choice of a homotopy class of such a lift $\tilde{f}$ corresponds to a reduction of p on H.*

Example : A principal G-bundle $p : P \to X$ is trivial is and only if its classifying map f can be chosen to be constant, i.e. if it factorises up to homotopy through $* = B1 \to BG$.

C.2 Tangent microbundles The tangent bundle of a manifold is usually defined with respects to a differential structure. We present here the notion of tangent microbundle, which generalises it for topological manifold. The general idea is to consider as "tangent space" of a point $x \in M$ a small neighbourhood of this point, which is homeomorphic to $\mathbb{R}^n$. The transition functions will take values in $Top(n) := Homeo(\mathbb{R}^n)$.

⌜ **Definition C.6:** A micro bundle is a continuous map $p : E \to B$ together with a section $s : B \to E$, such that p is a fibre bundle of fibre $\mathbb{R}^n$ locally around $s(B)$. Namely, for all $b \in B$ in the base, there exists an open $U \ni b$ containing b and a subopen $V \subseteq p^{-1}(U)$ in E containing $s(U)$, such that there is a homeomorphism $\varphi_U : V \tilde{\to} U \times \mathbb{R}^n$ preserving both p and s, i.e. with $p = proj_U \circ \varphi_U$ and $\varphi_U \circ s = (i_0 : U \overset{U \times \{0\}}{\hookrightarrow} U \times \mathbb{R}^n)$.
An equivalence of microbundles $p : E \to B$ and $p' : E \to B$ is a homeomorphism $f : U \to U'$ between open neighbourhoods of the section $U \supseteq s(B)$ and $U' \supseteq s'(B)$ which preserves p and s, namely $p' \circ f = p$ and $f \circ s = s'$. ⌟

Example : Microbundles generalise fibre bundles of fibre $\mathbb{R}^n$ reduced on $Top_*(n) := Homeo_*(\mathbb{R}^n)$, the subgroup of $Top(n)$ of homeomorphisms that fix the origin. The section is then given by the origin. In particular, a vector bundle induces a microbundle. The novelty is that in microbundles we only demand a bundle behaviour near the section, and not globally.
The inclusion $Top_*(n) \to Top(n)$ is a deformation retract, and one can show that a $Top(n)$-bundle (a fibre bundle with fibre $\mathbb{R}^n$) reduces canonically on $Top_*(n)$, and hence induces a microbundle.

⌜ **Definition C.7:** Let $M \in Mfld_n$ be a topological manifold. Its tangent microbundle is the projection $p = p_1 : M \times M \to M$, with diagonal section $s = \Delta : M \to M \times M$. ⌟

If M is a smooth manifold, then its tangent microbundle is equivalent to its tangent vector bundle.

Theorem C.1 (Kister-Mazur): *Any microbundle on a paracompact base is equivalent to a $Top(n)$-bundle.*

Thus, the tangent microbundle of a topological manifold is equivalent to a (unique up to isomorphism) $Top(n)$-bundle which is classified by a (homotopy class of) map $\tau_M : M \to BTop(n)$. This map is called the tangent classifier of M. It can be obtained in a maybe more transparent way, by:

$\tau: \begin{array}{ccccccc} Mfld_n & \overset{Yoneda}{\to} & Pshv(Mfld_n) & \overset{restr}{\to} & Pshv(Euc_n) & \simeq & Spaces_{/BTop(n)} \\ M & \mapsto & Emb(-,M) & \mapsto & Emb(\mathbb{R}^n, M) & \mapsto & \tau_M \end{array}$, where Euc_n is the full subcategory of $Mfld_n$ containing only $\mathbb{R}^n$. The last equivalence is, however, not so easy to describe.

Written in this form, it is easier to see that τ is an ∞-functor. It is monoidal symmetric because $\mathbb{R}^n$ is connected.

C.3 ***B*-framing** We can now define a very general notion of tangential structure on a topological manifold, called B-framing.

⌜ **Definition C.8:** Let $B \to BTop(n)$ be a continuous map and $M \in Mfld_n$ a topological manifold. A B-framing on M is a factorisation (up to homotopy) through B of its tangent classifier τ. Namely, it is a homotopy commutative diagram

$$\begin{tikzcd} & B \arrow[d] \\ M \arrow[ur, "g"] \arrow[r, "\tau_M"'] & BTop(n) \end{tikzcd}.$$ ⌟

⌜ **Definition C.9:** The ∞-categorie $Mfld_n^B$ of B-framed manifolds is the pullback

$$\begin{tikzcd} Mfld_n^B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Spaces_{/B} \arrow[d] \\ Mfld_n \arrow[r] & Spaces_{/BTop(n)} \end{tikzcd}$$

in $sSet$. The full subcategory $Disk_n^B$ is the one spanned by disjoint unions of $\mathbb{R}^n$. They are both monoidal symmetric with disjoint union. ⌟

Examples : • The ∞-category $Disk_n^*$ is equivalent to the ∞-category $Disk_n^{fr}$ used to define E_n-algebras.
• The ∞-category $Disk_n^{BO(n)}$ is equivalent to the ∞-category of smooth embeddings of smooth disks.
• The ∞-category $Disk_n^{BSO(n)}$ is equivalent to the ∞-category of $Disk_n^{or}$ used to define E_n^{or}-algebras.

These definitions extend to manifolds with boundary. There is an inclusion map $Top(n-1) \to Top(n)$ by extending with the identity on the n^{th} coordinate, but it does not take into consideration a boundary. Actually, there are homotopy equivalences $Top(n-1) \tilde{\to} Emb(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}) \tilde{\to} Emb^{\partial}(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}, \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1})$. There is a tangent classifier for manifolds with boundaries $(M, \partial M)$ which restricts to $\tau_{\partial M}$ on the boundary and to τ_M on the interior. On the boundary, we actually consider embeddings $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} \hookrightarrow M$, which are homotopy equivalent to their restriction on the boundary.

⌜ **Definition C.10:** Let $(B_\partial \to B) \to (BTop(n-1) \to BTop(n))$ be a commutative square. A B-framing of a manifold with boundary $(M, \partial M)$ is a factorisation of its tangent classifier

$$\begin{tikzcd} & (B_\partial \to B) \arrow[d] \\ (\partial M \to M) \arrow[ur] \arrow[r, "\tau_{\partial M}, \tau_M"'] & (BTop(n-1) \to BTop(n)) \end{tikzcd}.$$

We note $Mfld_n^{\partial, B_\partial \to B}$ the ∞-category of B-framed manifolds with boundary. Its full subcategory $Disk_n^{\partial, B_\partial \to B}$ is the one spanned by disjoint unions of $\mathbb{R}^n$ and of $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. ⌟

Examples : • For $B_\partial = \emptyset$, we obtain B-framed manifolds without boundary.
• The category of framed manifolds with framed boundary is $Mfld_n^{\partial, * \to *}$. Its subcategory $Disk_n^{\partial, * \to *}$ is used to define the Swiss Cheese operad. Concretely, it contains framed embeddings $\sqcup \mathbb{R}^n \to \mathbb{R}^n$ (the E_n operad) as well as framed embeddings with boundary $(\sqcup \mathbb{R}^n) \sqcup (\sqcup \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}) \to \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$.
• The ∞-category $Mfld_1^{\partial, ** \to *}$ used in Section 9 is $Mfld_1^{\partial, \{-1,1\} \to [-1,1]}$, namely with B-framing associated with the tangent classifier of the manifold with boundary $[-1, 1]$. More concretely, the interior of $[-1, 1]$ is framed so $B = *$. The two boundaries may are framed too, but with different framings. Indeed, for one boundary the manifold is at the right, and for the other at the left. Hence B_∂ is two points $**$.

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