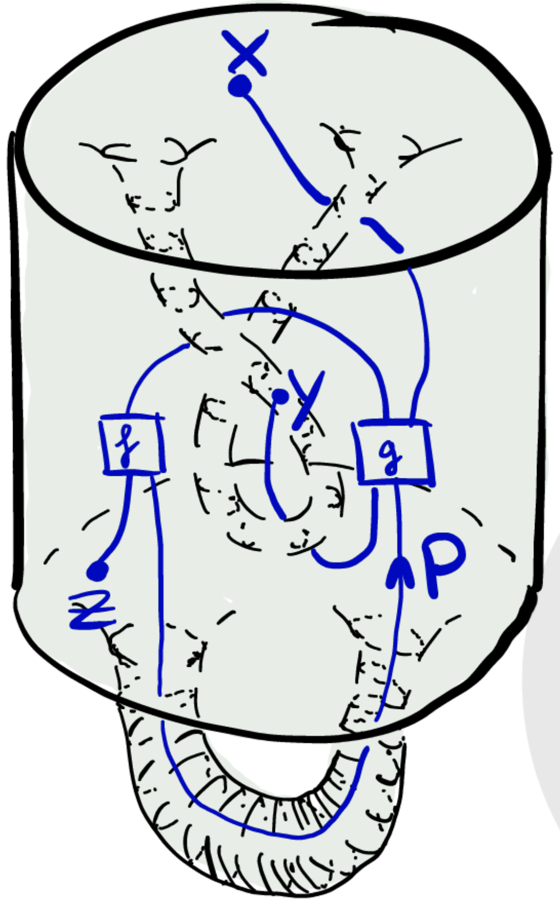
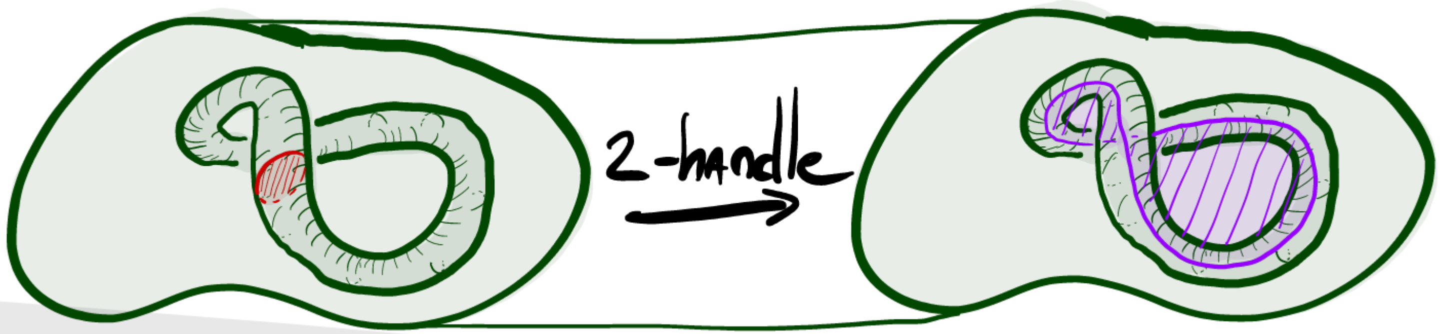
Non-semisimple (fully extended) skein 4-TQFTs

for the Quantum Symetry Seminar





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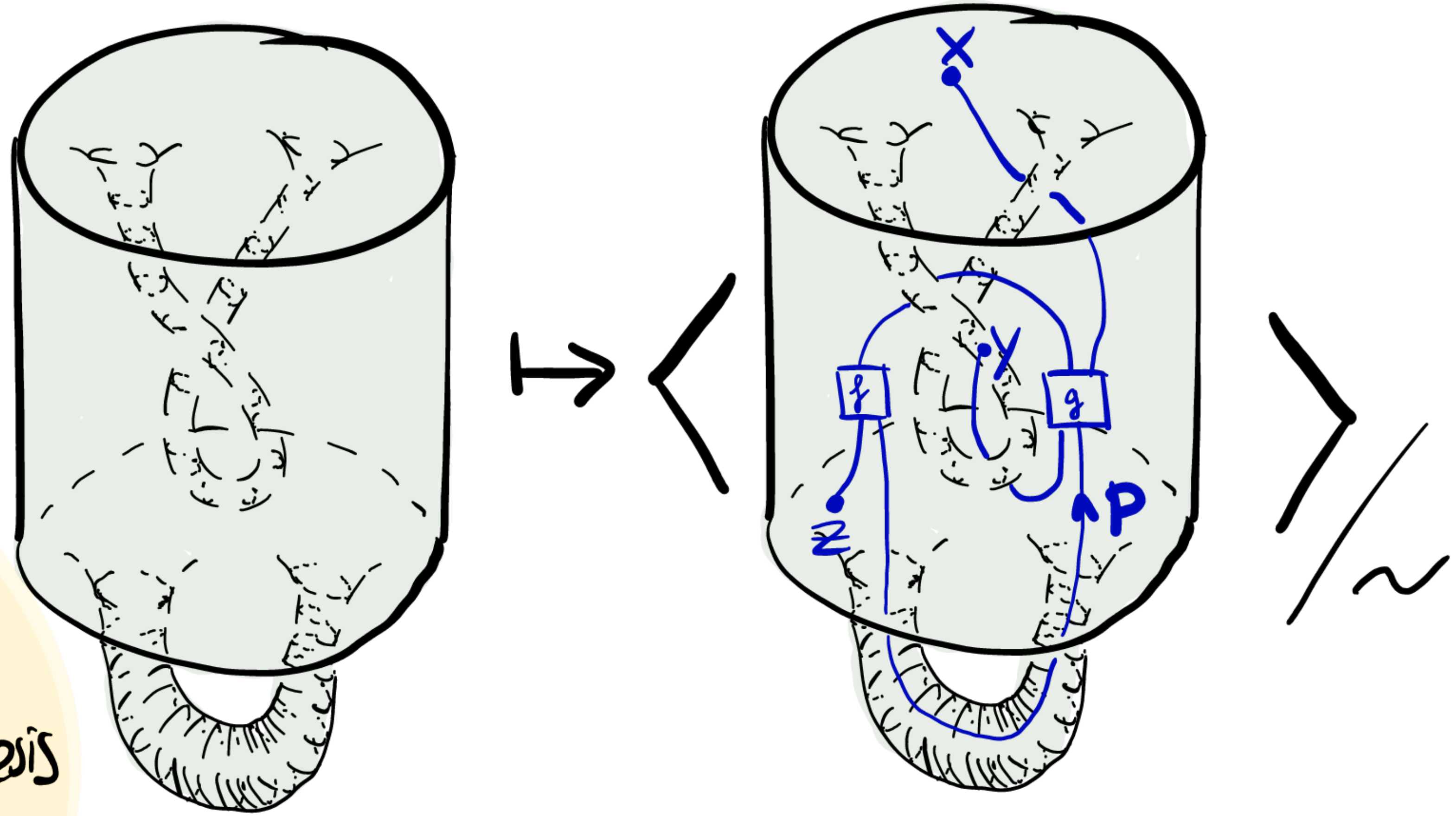
Goal: Workable example of a fully extended 4-TQFT

$$Z : \mathrm{Bord}_4 \longrightarrow \mathrm{BrTens}$$



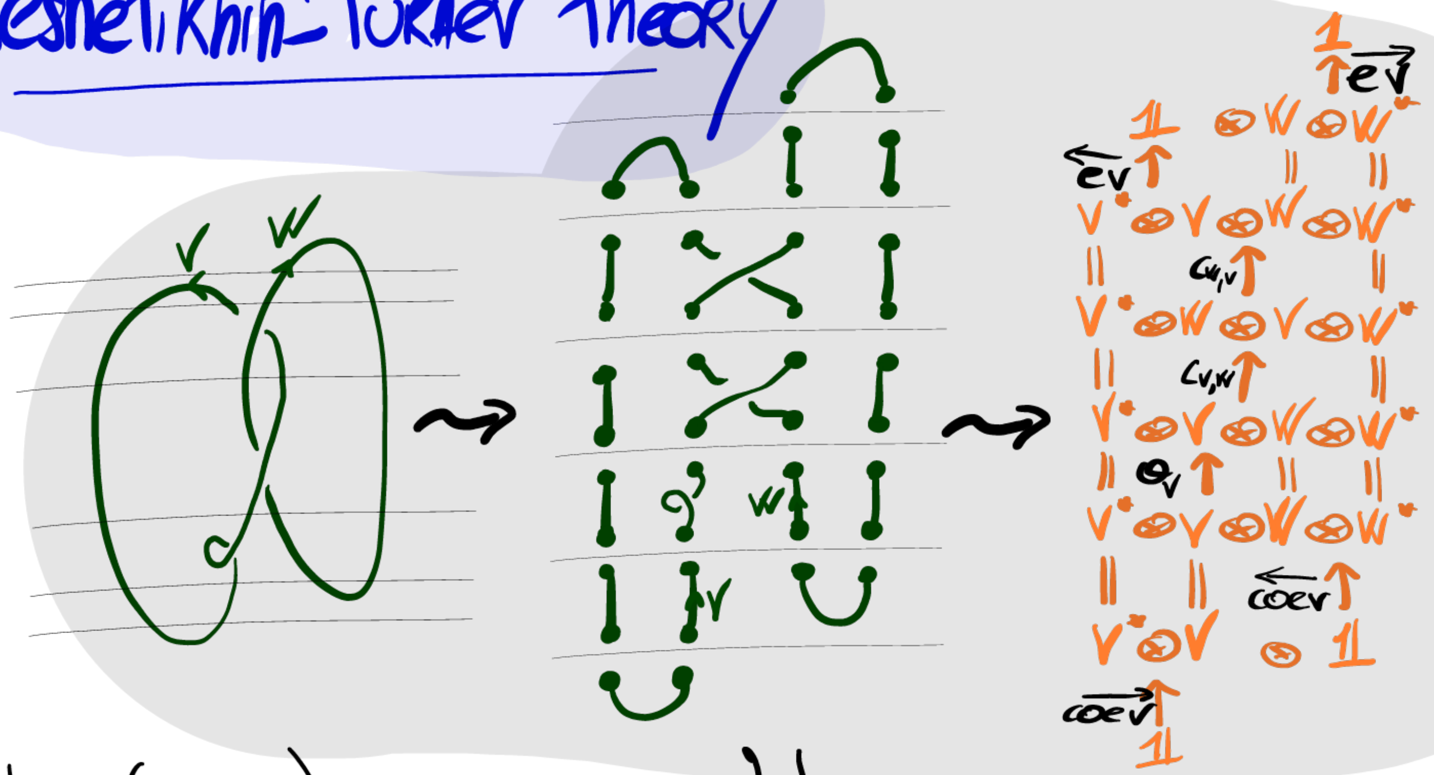
Plan: I. Skeins

II. 3,4 part

III. Cobordism Hypothesis

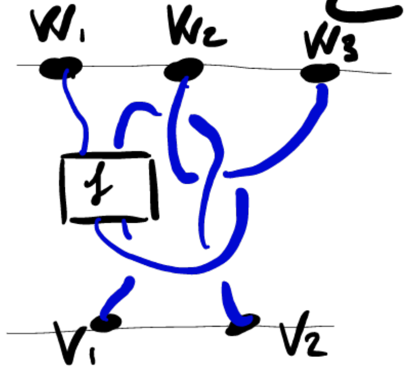
IV. 0,1,2 part

Reshetikhin-Turaev theory



Thm (R-T) $\exists!$ ~~symmetric~~ ribbon ~~monoidal~~ functor

$RT_{\mathcal{C}} : Rib_{\mathcal{C}} \longrightarrow \mathcal{C}$



$W_1 \otimes W_2 \otimes W_3$

$\longmapsto T\uparrow$

$V_1 \otimes V_2$

Algebraic requirements

$\mathcal{C}$ linear category

$V \otimes W \rightsquigarrow$ monoidal tensor — EGNO: abelian, $\mathbb{1}$ is simple

$V^* \rightsquigarrow$ Rigid

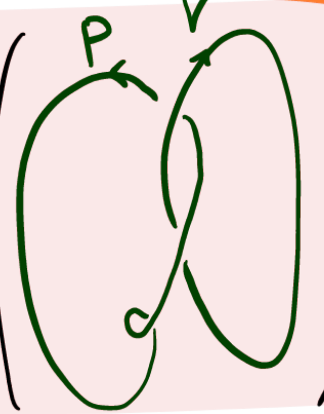
$c_{V,W} \rightsquigarrow$ braided

$\theta \rightsquigarrow$ ribbon

Non-semisimple issues

$\mathcal{C}$ which is NOT semisimple

$\mathbb{1}$ not projective

P projective $\Rightarrow RT\Big($  $\Big) = 0$

Kill projectives "semisimplification"

Restrict to projectives and renormalize

Idea:

$\neq 0$

$\in End(P)$

modified trace

$mtr_P : End(P) \to k$

Thm (Geer-Patureau-Mirand et al.) $\exists!$ up to scalar modified trace

$\rightsquigarrow$ renormalized invariant of closed links with projective $L =$ 

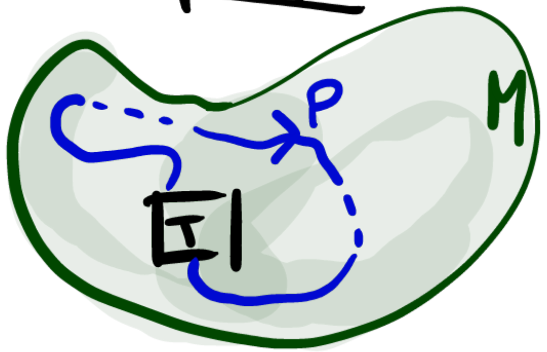
$RT'(L) = mtr_P\Big($  $\Big)$

Admissible Skein modules
(our state spaces)

$\mathcal{C}$ ribbon tensor category

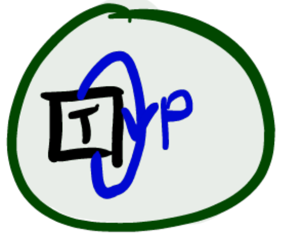
M closed oriented 3-manifold

$$Sk^{adm}_{\mathcal{C}}(M) := \Big\langle \text{Ribbon graphs in } M \text{ with } \underline{\text{projectives}} \Big\rangle \Big/ \Big/$$



- isotopy
- two skeins that agree outside a ball
  - * their RT evaluation inside the ball agree
  - * there is projective outside the ball

Ex: $Sk^{adm}_{\mathcal{C}}(S^3) = \Big\langle$



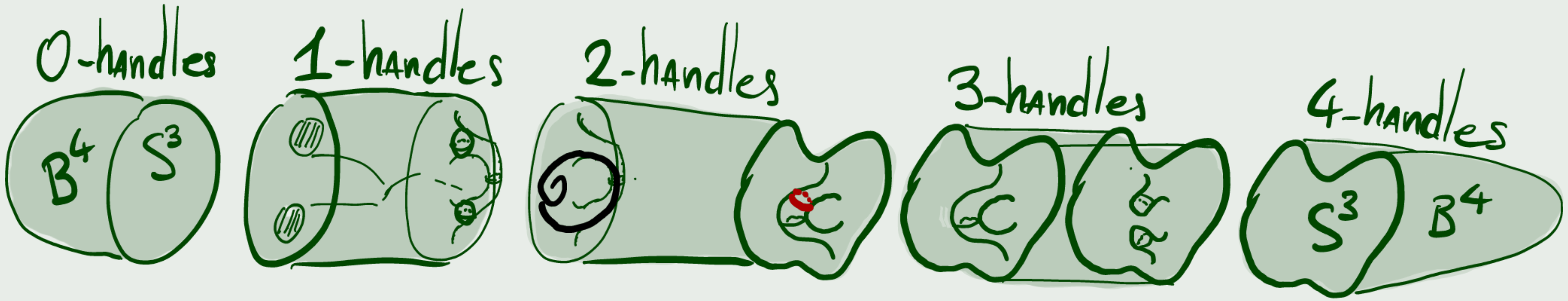
$\Big\rangle \Big/ \sim \;\simeq \bigoplus_{P \text{ proj}} \mathrm{End}(P) \Big/ \sim$

{ modified traces on $\mathcal{C}$ $\xleftrightarrow{1:1}$ linear forms on $Sk^{adm}_{\mathcal{C}}(S^3)$

Created with Notewise

Presentation of the cobordism category

W oriented smooth cobordism

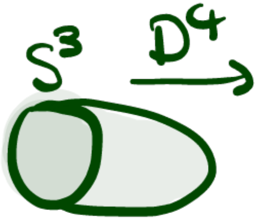


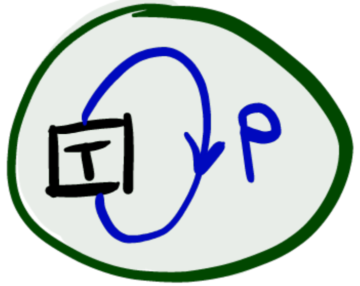
k-Handle $\quad H_k : S^{k-1} \times D^{4-k} \xrightarrow{D^k \times D^{4-k}} D^4 \times S^{4-k-1}$

Thm (Juhász) The category $Cob = \begin{cases} \text{obj: closed or. 3-mfld} \\ \text{mor: or. 4-cobordisms} \end{cases}$

has generators $\begin{cases} \text{handle attachments} \\ \text{diffeomorphisms} \end{cases}$ and relations $\begin{cases} \text{handle cancellation} \\ \text{handle slides} \\ \vdots \end{cases}$

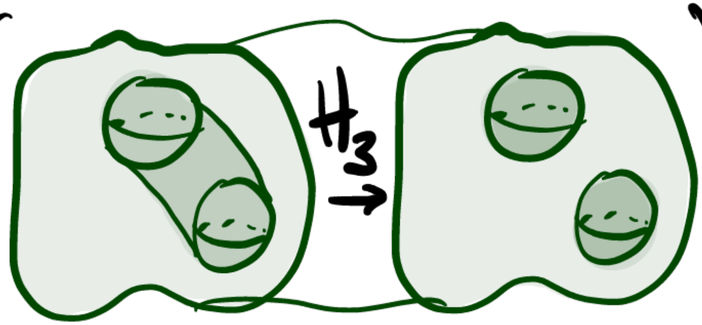
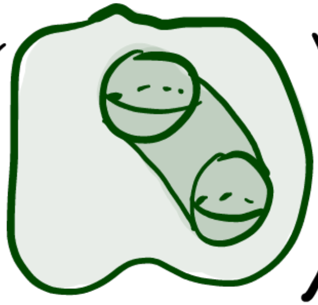
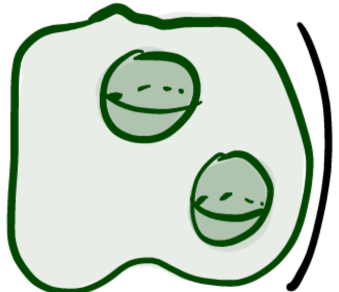
4-handle

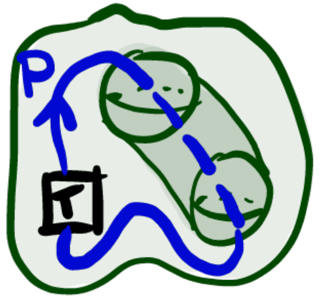
$Z\big($$\big)$: $Sk_{\mathcal{C}}^{adm}(S^3) \longrightarrow k$

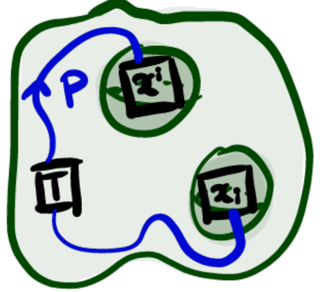


$\longmapsto mtr_{R_P}\big($$\big)$

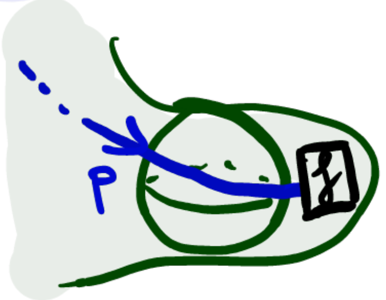
3-handle

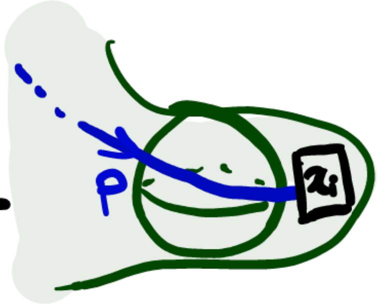
$Z\big($$\big)$: $Sk_{\mathcal{C}}^{adm}\big($$\big) \longrightarrow Sk_{\mathcal{C}}^{adm}\big($$\big)$



$\longmapsto \sum_i$ 

3-4 cancellation



$= \sum_i$ 

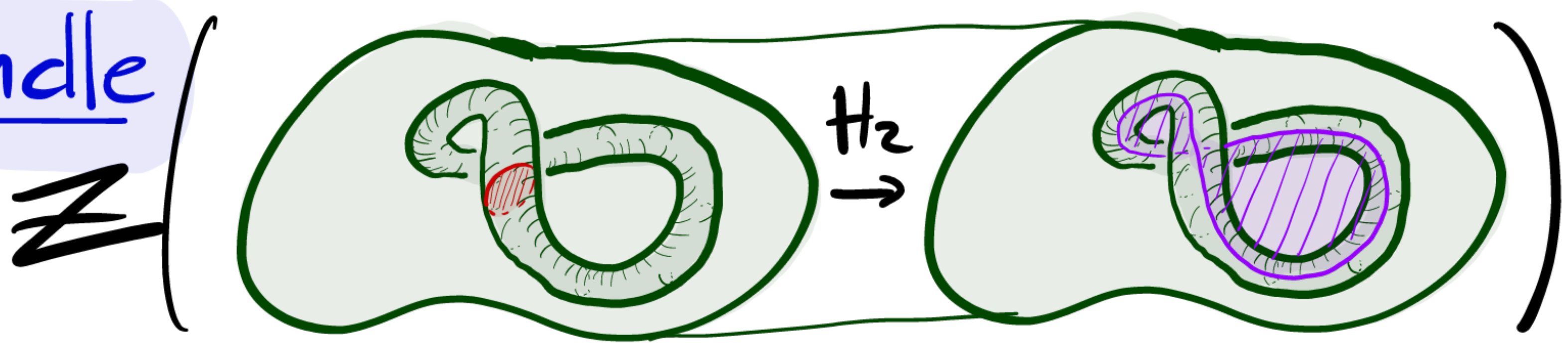
$. \, mtr_{R_P}\big($$\big)$

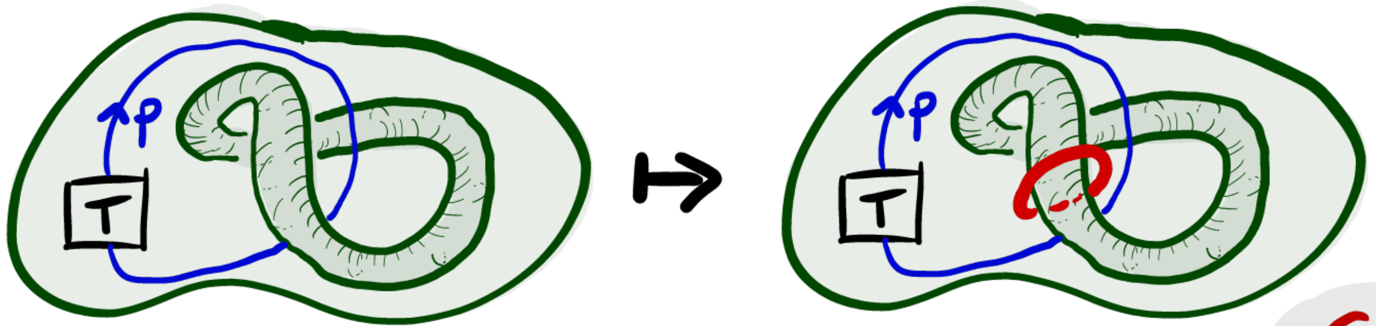
$\rightsquigarrow \Omega_P = \sum x_i \otimes x^i \in Hom(\mathbb{1}, P) \otimes Hom(P, \mathbb{1})$

copairing for mtr_R

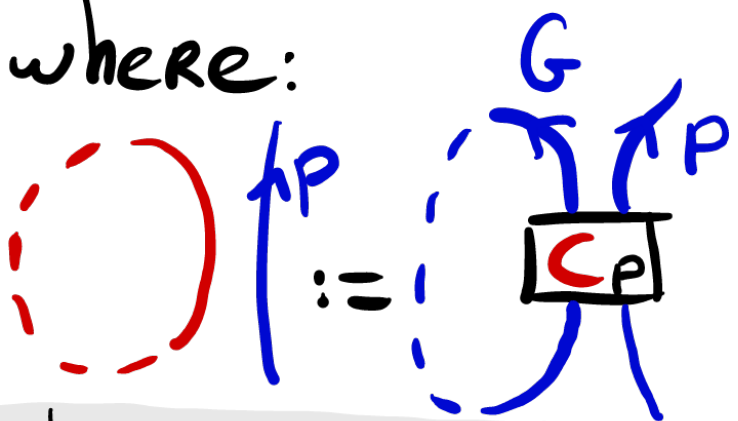
need $\mathcal{C}$ unimodular

2-handle



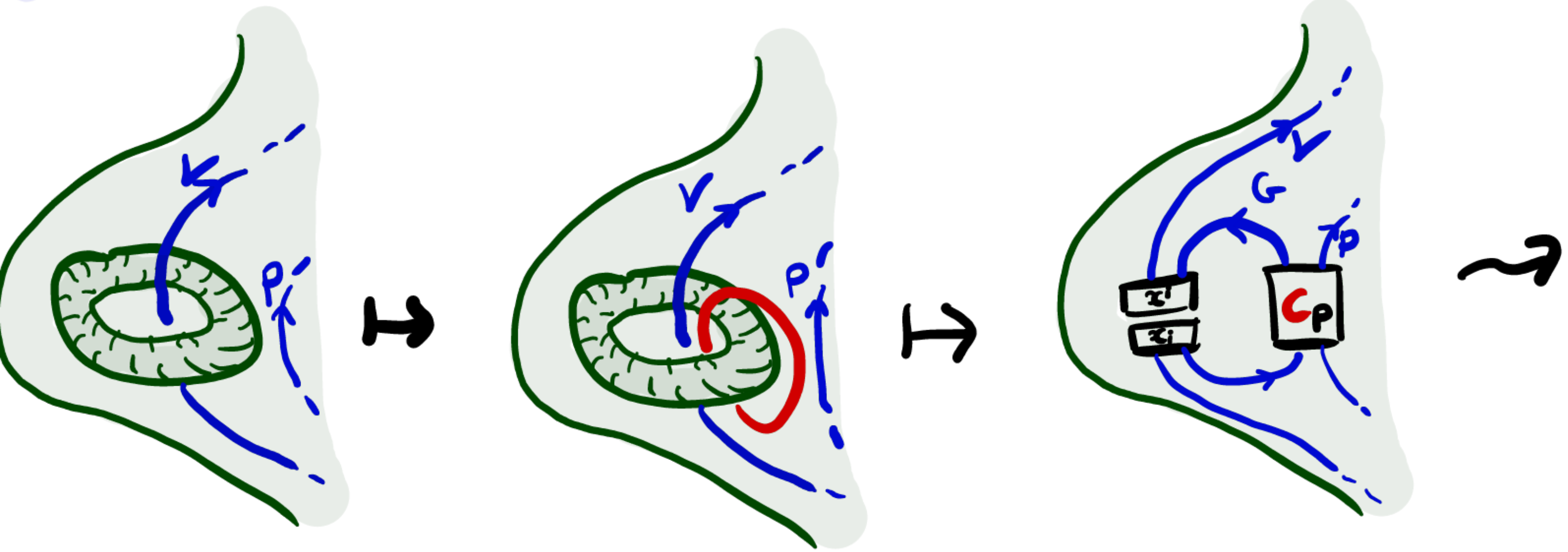


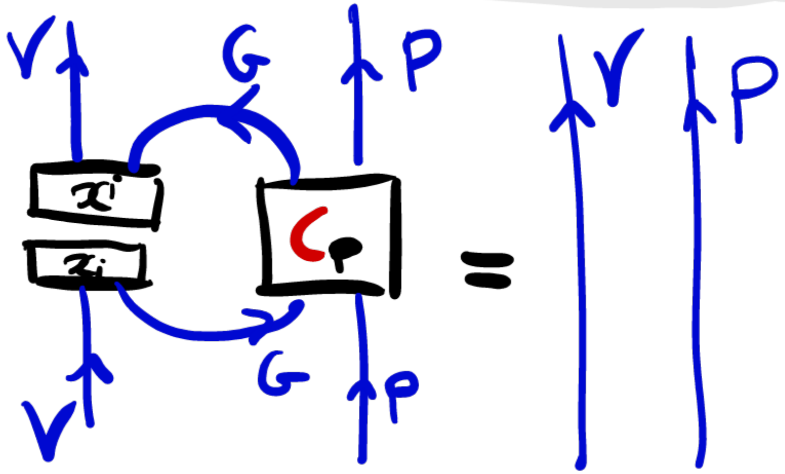
where:



c chromatic morphism

2-3 cancellation





↳ need e finite

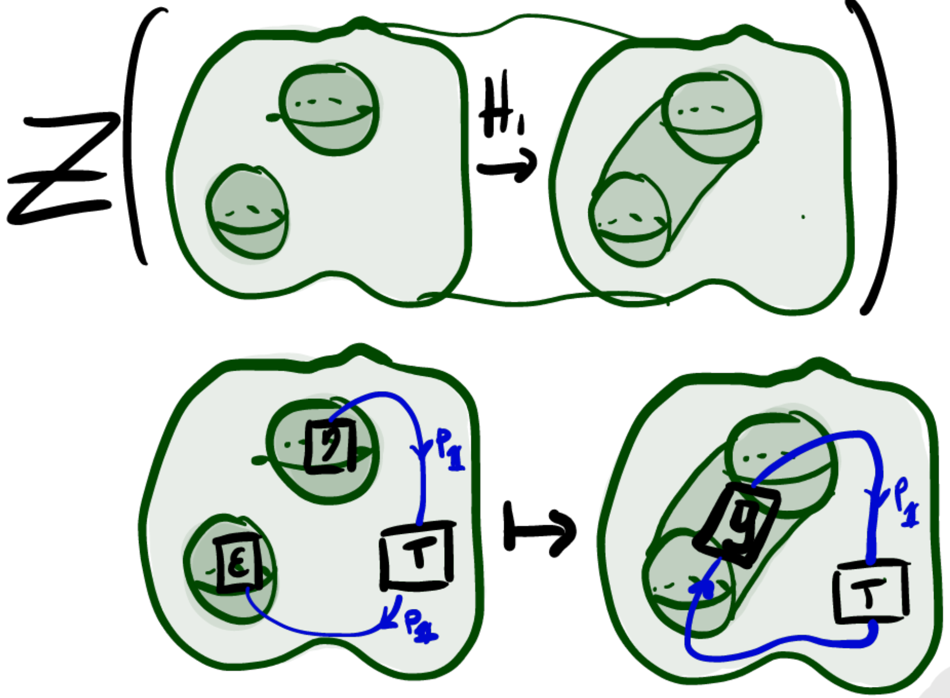
$G = \bigoplus_{p \text{ irr's}} P$

1-handle

Projective cover of $\mathbb{1}$:

$$P_{\mathbb{1}} \xrightarrow{\varepsilon} \mathbb{1}$$

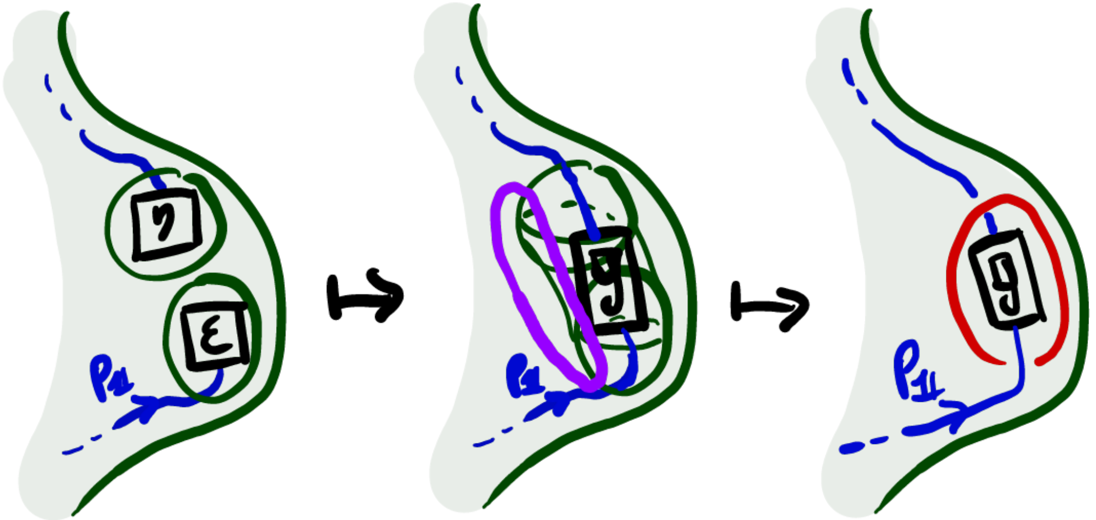
unimodular $\rightsquigarrow$ $\mathbb{1} \xrightarrow{\eta} P_{\mathbb{1}}$



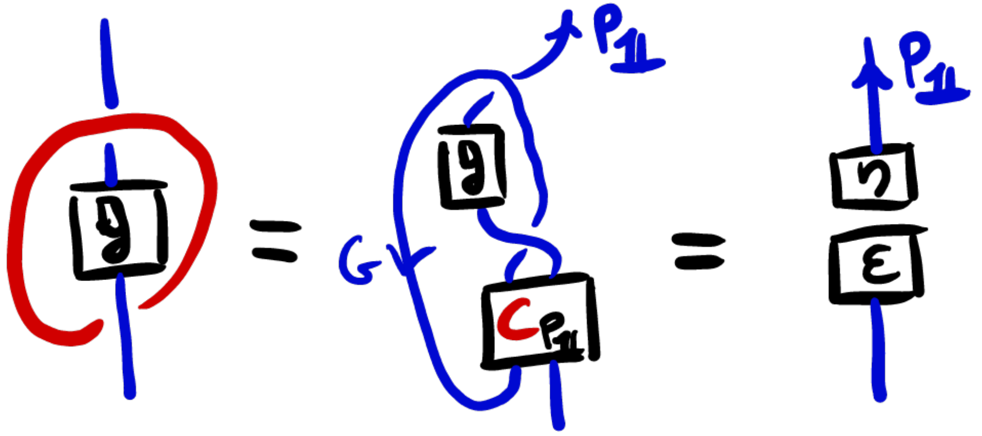
, $g : P_{\mathbb{1}} \to P_{\mathbb{1}}$

gluing morphism

1-2 cancelation



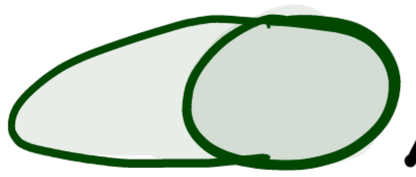
$\rightsquigarrow$



$\Lsh$ $\mathcal{C}$ chromatic non-degenerate

0-handle

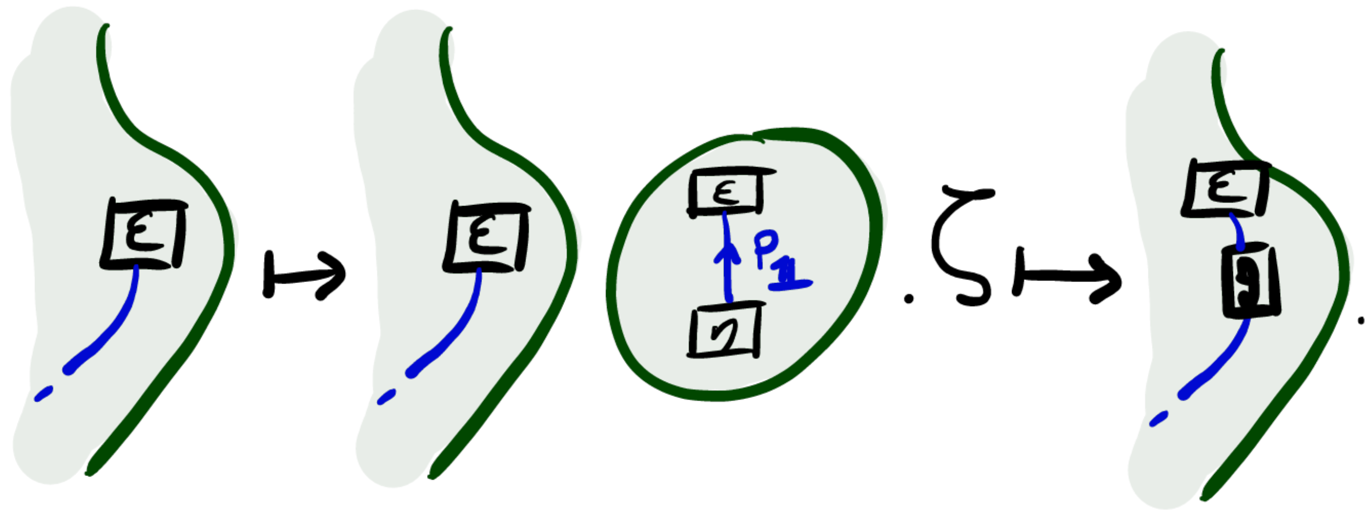
$$Z\left(\emptyset \xrightarrow{D^4} S^3\right):$$



$$k \longrightarrow Sk_e^{Adm}(S^3)$$

$$1 \longmapsto \zeta$$

0-1 cancelation



$$\rightsquigarrow \varepsilon \circ g = \zeta \cdot \varepsilon$$

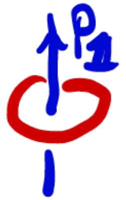
↳ g invertible

e chromatic compact

Main result

Theorem (Costantino – Geer – H. – Patureau-Mirand)

Given $\mathcal{C}$ a ribbon unimodular finite tensor category ($\Rightarrow \Omega, c$ exist)

which is chromatic non-degenerate :  $\neq 0$ ($\Rightarrow g$ exists)

and chromatic compact : g invertible ($\Rightarrow \zeta$ exists)

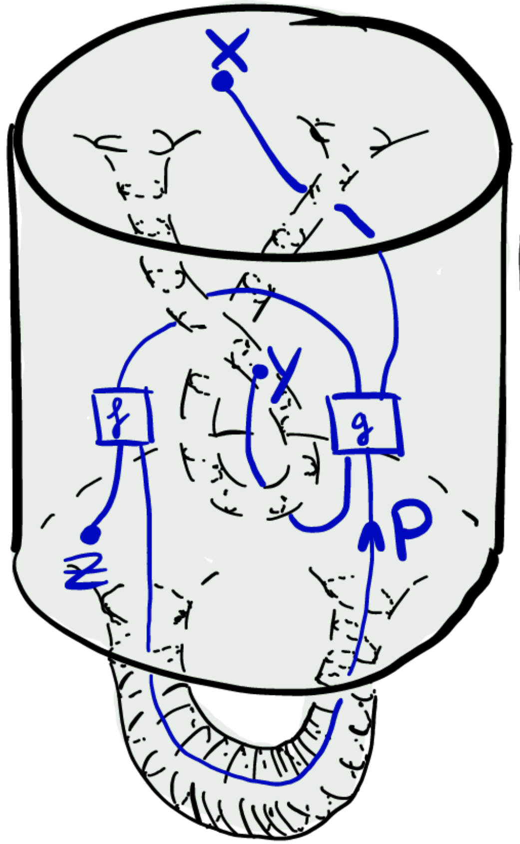
There exists a TQFT

$$Z_{\mathcal{C}} : \mathrm{Cob}_{3+1} \to \mathrm{Vect}_{\Bbbk}$$

with $Z_{\mathcal{C}}(M) = Sk^{adm}_{\mathcal{C}}(M)$ and the description above on handle attachments.

Extending down

M 3-manifold with boundary with corners

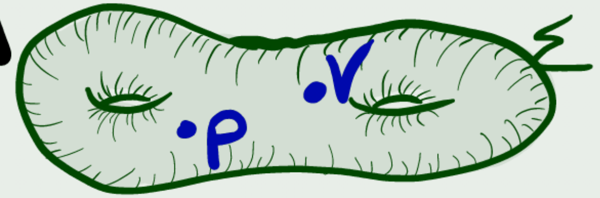
$\longmapsto \quad Sk_e^{adm}(M; X, Y, Z) = \Big\langle$ 

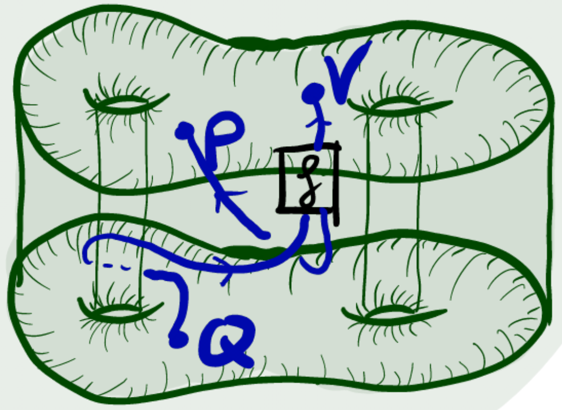
$\Big\rangle / \sim$

Σ surface

$\leadsto \quad SkCat_e^{adm}(\Sigma) = \begin{cases} \text{obj: } \\ \text{hom: } \end{cases}$

with proj.

obj: 

hom: 

Claim (unproven):

This gives a once extended TQFT

Fully extended, fully dualizable

The Cobordism Hypothesis:

$$\left\{\begin{array}{c}\text{fully extended}\\ \text{TQFTs}\\ Z: \mathrm{Bord}_n \to \mathcal{C}\end{array}\right\} \simeq \left\{\begin{array}{c}\text{fully dualizable}\\ \text{object}\\ Z(\cdot^+) \in \mathcal{C}\end{array}\right\}$$

Thm (Brochier-Jordan-Snyder-Safronov)

A fusion / modular ribbon category is fully dualizable in BrTens

Claim (unproven):

The 3-4 part of the associated TQFT is what we described

The 0-1-2 part

Thm (Scheimbauer, Ayala-Francis)
The 0-1-2 part has to agree with factorization homology.

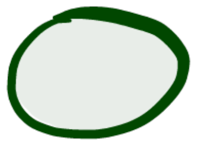
Thm (Cooke) If $\mathcal{V}$ is semisimple, factorization homology agrees with skein categories.

Work in progress (Brown-H.) If $\mathcal{V}$ is non-semisimple, factorization homology agrees with admissible skein categories.
the free cocompletion of

Factorization homology: E_2-Algebras

A ribbon category $\mathcal{V}$ gives a symmetric monoidal 2-functor

$$\mathcal{V}: \mathrm{Disk}_2^{OR} = \begin{cases} \text{obj: } \sqcup^n D^2 \\ \text{1-m: oriented embeddings} \\ \text{2-m: isotopies} \end{cases} \longrightarrow \mathrm{Cat} = \begin{cases} \text{obj: categories} \\ \text{1-m: functors} \\ \text{2-m: nat. iso} \end{cases}$$



$\longmapsto \mathcal{V}$

$\longmapsto \mathcal{V}\otimes\mathcal{V} \xrightarrow{\otimes} \mathcal{V}$

$\longmapsto c: \otimes \overset{\sim}{\Rightarrow} \otimes^{op}$

Factorization homology: Integrating

Definition (Ayala–Francis):

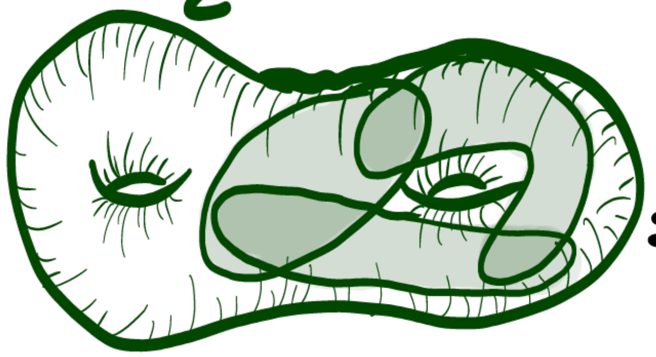
$$\mathrm{Disk}_2^{or} \xrightarrow{\mathcal{V}} \mathrm{Cat}$$

$$\cap\!|$$

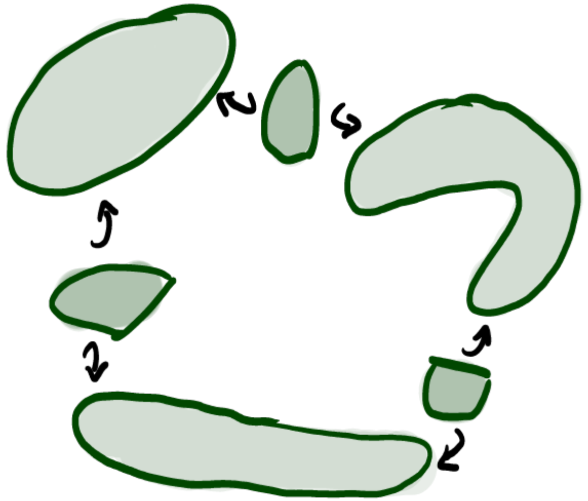
$$\mathrm{Mfld}_2^{or} = \begin{cases} \text{obj: or. closed surfaces} \\ \text{1-m: oriented embeddings} \\ \text{2-m: isotopies} \end{cases} \dashrightarrow \int_{-} \mathcal{V}$$

Left Kan extension

Σ

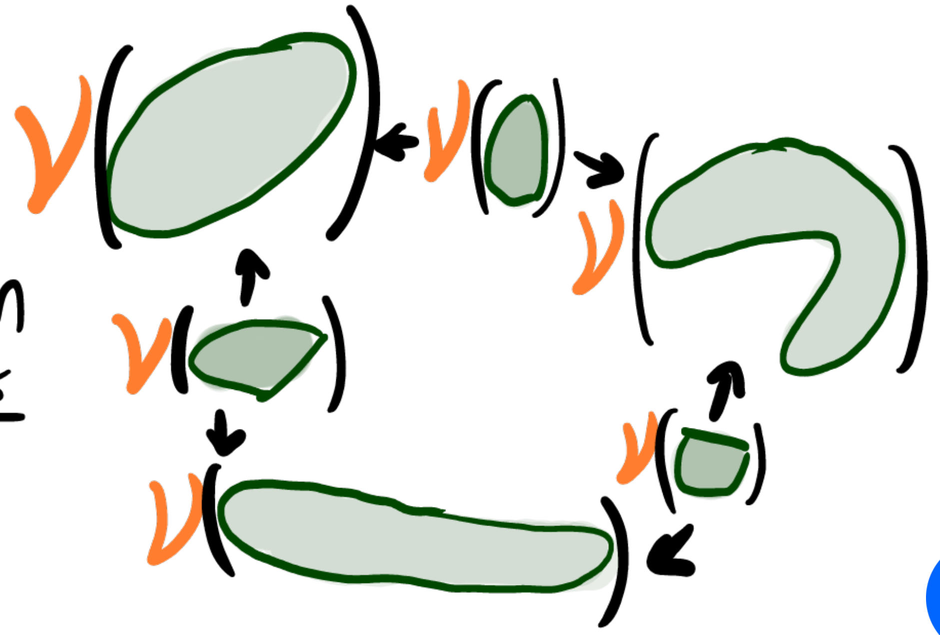


= "colim" $_{\mathrm{Disk}/\Sigma}$



$\downarrow$

$$\int_\Sigma \mathcal{V} := \operatorname{colim}_{\mathrm{Disk}/\Sigma}$$



Fully extended skein 4-TQFT

dim =	0	1	2	3	4
	$\bullet^+$	Γ	Σ	M	W
bottom-up (cobordism hypothesis / factorization homology)	$\mathcal{V}$	$\int_{\Gamma\times\mathbb{R}} \mathcal{V}$	$\int_\Sigma \mathcal{V} \simeq A_\Sigma\text{-mod}_\mathcal{V}$	?	?
				exists for framed manifolds if $\mathcal{V}$ either fusion or modular	
top-down (skein modules / handle formulas)	?	?	?	$Sk_\mathcal{V}^{adm}(M)$	formulas above
	Admissible skein categories?				

Thank you