Internship Report PLR2: Homfly skein invariants and the Deligne interpolation $Rep_q(GL_t)$

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1 Introduction

Quantum invariants as developed in [Tur10] give a way to obtain a link invariant from some structure on a category, namely a ribbon structure. Interesting examples of linear ribbon categories arise as categories of modules over a quantum group, and interesting examples of quantum groups arise as quantization of universal enveloping algebras of Lie algebras, see [KS97]. We will be interested in Lie algebras of type A here, namely in the Lie algebras $\mathfrak{gl}_n$ for $n \in \mathbb{N}$.

The representation theory of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ for generic values of q is very close to the representation theory of the classical $\mathfrak{gl}_n$. This one is very close to the representation theory of the symmetric groups, and is described by a beautiful combinatorics of Young tableaux, see [FH91]. These tableaux describe simple submodules of tensor powers of the standard representation V. The endomorphism algebra of these tensor powers can be described diagrammatically by braids, namely by Hecke algebras. The endomorphism algebras of mixed tensor powers of V and its dual can be described by tangles.

The above diagrammatic description actually holds only for small enough tensor powers, namely smaller than n. There is a natural way to define a category $\mathcal{OS}$ doing as if this description held for arbitrary big tensor powers. The dimension of the generating object usually is a quantum integer, but in this description the integer n can be replaced by any complex parameter t. This way, the category $\mathcal{OS}$ interpolates the categories of representations of $\mathcal{U}_q(\mathfrak{gl}_n)$ at complex values, and is sometimes denoted $Rep_q(GL_t)$. Note that at $t = n$ it is not exactly equivalent to the category of representations of $\mathcal{U}_q(\mathfrak{gl}_n)$ because the diagrammatic description does not hold in general there.

The link invariant obtained from $\mathcal{OS}$ is the (framed) HOMFLY polynomial in two variables q and q^t. The link invariants obtained from the categories of representations of $\mathcal{U}_q(\mathfrak{gl}_n)$ are specializations of the HOMFLY polynomial at $t = n$. The interpolation procedure of the categories corresponds to an interpolation of the link invariants, which adds a parameter that can be evaluated to give the other invariants. This motivates the use of interpolation of categories to find link invariants containing a known family of link invariants.

The same category $\mathcal{OS}$ actually interpolates another family of categories, namely representations of the quantum supergroup $\mathcal{U}_q(\mathfrak{gl}(m|n))$ associated with the Lie superalgebra $\mathfrak{gl}(m|n)$. At the level of link invariants, the one associated with $\mathcal{U}_q(\mathfrak{gl}(m|n))$ coincides with the one of $\mathcal{U}_q(\mathfrak{gl}_{m-n})$, see [QS15], so nothing interesting happens. However, this time when t takes integer values, the interpolation is rather trivial and the category $\mathcal{OS}$ is simply the limit of the categories of representations of $\mathcal{U}_q(\mathfrak{gl}(t+n|n))$ as n goes to infinity. The diagrammatic description holds for larger and larger classes as $n \to \infty$ and the ribbon structures of the categories, as well as morphism spaces, eventually stay fixed.

2 Hecke algebras, Young tableaux and the quantum group $\mathcal{U}_q(\mathfrak{gl}_n)$

In this Section we recall the Schur-Weyl duality linking representations of the quantum group $\mathcal{U}_q(\mathfrak{gl}_n)$ with Hecke algebras. Let k be a field of characteristic 0.

⌜ **Definition 2.1:** The Hecke algebra H_n is the quotient of the braid algebra on n strands kB_n with generators s_i, $1 \leq i < n$ by the quadratic skein relations $s_i^2 = 1 + (q - q^{-1})s_i$. It depends on a parameter $q \in k^\times$. ⌟

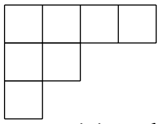
For q generic, i.e. not a root of unity, the Hecke algebras are semisimple with simples indexed by Young diagrams with n boxes.

⌜ **Definition 2.2:** Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n. Its Young diagram is the left-justified arrangement of n boxes with λ_1 boxes in the first row, λ_2 in the second and so on. We call length of the partition the integer $l(\lambda) = k$, i.e. the number of rows in the Young diagram of λ. A standard Young tabloid of shape λ is a filling of the boxes of the Young tableau of λ by the numbers $1, \dots, n$ which are ascending from left to right and top to bottom.

A partition of $n = 7$

$\lambda = (4, 2, 1)$

Its Young tableau



A standard Young tabloid

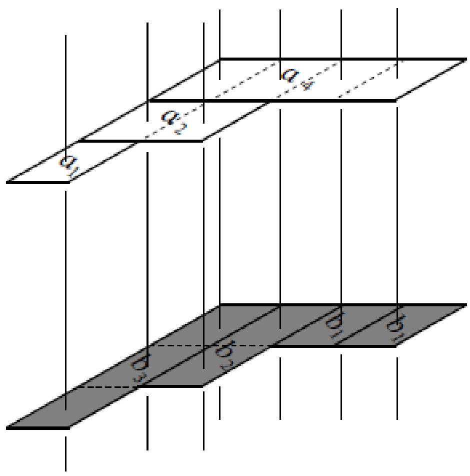
1	2	5	6
3	4		
7			

The transpose of a partition λ is the partition λ' that counts the number of boxes in each column of its Young tableau. ⌟

Idempotents of the Hecke algebras corresponding to Young tableaux are given in [AM97]

⌜ **Definition 2.3:** For each permutation $\sigma \in \mathfrak{S}_n$, there is a unique braid b_σ that realises the permutation σ with the first strand going above every strand, the second strand going above every other and so on. We call length of σ denoted $l(\sigma)$ the writhe of this braid. We denote g_σ the corresponding element of H_n. ⌟

Let $a_n = \sum_{\sigma \in \mathfrak{S}_n} q^{l(\sigma)} g_\sigma$ and $b_n = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)} g_\sigma$. These are two weak idempotents of H_n corresponding to the two partitions (n) and $(1, \dots, 1)$. In general, for T a standard Young tabloid of shape λ, one defines a weak idempotent $\hat{e}_T$ by doing an a_{λ_i} on the strands whose indices lie in the i-th row of the tabloid T and then a $b_{\lambda'_j}$ on the strands whose indices lie in the j-th column. The picture at the right is borrowed from [AM97].



For q generic, these weak idempotents can be renormalized to give idempotents $e_T \in H_n$. They form a complete family of primitive orthogonal idempotents of H_n. They are conjugated if and only if the standard Young tabloids have same shape. In the following we will denote by e_λ the primitive idempotent of H_n corresponding to the standard Young tabloid obtained by filling the boxes in reading order.

The Lie algebra $\mathfrak{gl}_n$ is the vector space of $n \times n$ matrices endowed with the bracket $[A, B] = AB - BA$. Its universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_n)$ is generated by the matrices $h_i = E_{i,i}$, $1 \leq i \leq n$, and $E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$, $1 \leq i \leq n-1$. This algebra admits a deformation quantization as follows, see [KS97, Section 6.1.2]:

⌜ **Definition 2.4:** The quantum group $\mathcal{U}_q(\mathfrak{gl}_n)$ is the $\mathbb{C}(q)$-algebra with generators $k_i^{\pm 1}$, $1 \leq i \leq n$, and E_i, F_i, $1 \leq i \leq n-1$. We denote $K_i = k_i k_{i+1}^{-1}$. It has relations:

$$[-_i, -_j] = 0 \text{ if } \left|i-j\right| \geq 2 \quad , \quad [k_i, k_{i\pm 1}] = 0 \quad ,$$
$$k_i E_j = q^{\delta_{i,j} - \delta_{i-1,j}} E_j k_i \quad , \quad k_i F_j = q^{-\delta_{i,j} + \delta_{i-1,j}} F_j k_i \quad , \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \quad ,$$
$$E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0 \quad \text{and} \quad F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 = 0.$$

It has a Hopf algebra structure with:

$$\Delta(k_i) = k_i \otimes k_i \quad , \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \quad , \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$$
$$\varepsilon(k_i) = 1 \quad , \quad \varepsilon(E_i) = \varepsilon(F_i) = 0$$
$$S(k_i) = k_i^{-1} \quad , \quad S(E_i) = -E_i K_i^{-1} \quad \text{and} \quad S(F_i) = -K_i F_i.$$

Its standard representation is the vector space V of dimension n with basis $v_1, \ldots, v_n$ on which k_i acts as $qE_{i,i} + \sum_{j \neq i} E_{j,j}$, E_i acts as $E_{i,i+1}$ and F_i acts as $E_{i+1,i}$. ⌟

The Hopf algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ has a topological quasi-triangular structure. In particular, the braiding $c_{V,V} =$  $: V \otimes V \to V \otimes V$ is given by :

$$c_{V,V}(v_i \otimes v_j) = \begin{cases} q\, v_i \otimes v_i & \text{if } i = j \\ v_j \otimes v_i & \text{if } i < j \\ v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & \text{if } i > j \end{cases}$$

This braiding induces an action of the braid group B_k on $V^{\otimes k}$. This action factorizes through an action of the Hecke algebra H_k.

Theorem 2.5 (Schur-Weyl duality): *The action $\Psi_k : H_k \to \mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_n)}(V^{\otimes k})$ is an isomorphism for $k \leq n$, and is surjective otherwise.*

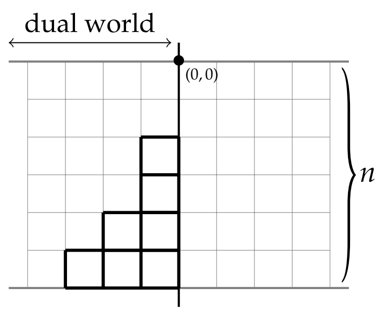
From this result we conclude that Hecke algebras are a good way to interpolate the endomorphism algebra of tensor powers of the standard representation of $\mathcal{U}_q(\mathfrak{gl}_n)$, for n big enough.

Idempotents e_λ of the Hecke algebra will be sent to idempotents of $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_n)}(V^{\otimes k})$, i.e. projections onto direct summands V_λ of $V^{\otimes k}$. The idempotent $\Psi_k(e_\lambda)$ is zero if and only if λ has more than n rows. Standard Young tabloids with k boxes and at most n rows are in bijection with irreducible submodules of $V^{\otimes k}$.
One can explicitly compute the tensor product of two simple representations $V_\lambda \otimes V_\mu$. In the simplest case where $\mu = (1)$, i.e. $V_\mu = V$, one has $V_\lambda \otimes V_\mu = \bigoplus_{\nu \in \lambda + \square} V_\nu$, where $\lambda + \square$ is the set of Young tableaux that can be obtained from λ by adding one box. In general, the rule is still very diagrammatic and is known as the Littlewood-Richardson rule.

⌜ **Definition 2.6:** We denote $Tilt_q(\mathfrak{gl}_n)$ the category of finite dimensional tilting representations of $\mathcal{U}_q(\mathfrak{gl}_n)$, namely of sub-representations of tensor powers of the standard representation V and of its dual. This category is ribbon, and the quantum dimension of the standard representation is $[n]_q$. ⌟

Proposition 2.7: *The category $Tilt_q(\mathfrak{gl}_n)$ is semi-simple if and only if q is generic.*

In this case, we have already described its simple objects arising as submodule of some $V^{\otimes k}$: they are images V_λ of idempotents e_λ of Hecke algebras and are indexed by Young tableaux λ with at most n rows.
Submodules of $(V^*)^{\otimes k}$ are also in bijection with standard Young tabloid with k boxes and at most n rows, as $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_n)}((V^*)^{\otimes k})$ is also isomorphic to the Hecke algebra H_k. Now the submodule V^*_λ of $(V^*)^{\otimes k}$ associated with a Young tableau λ is the dual of the submodule V_λ of $V^{\otimes k}$. To represent this dual module we will rotate the Young tableau by 180 degrees. One should think of these boxes as being removed from the infinite n-row rectangle of negative indices.



$\lambda = (3, 2, 1, 1)$

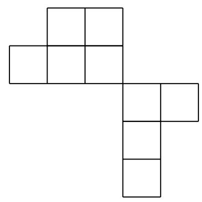
The idempotent associated with the Young diagram $(1)^n$ with n vertical boxes plays a particular role, and is called the determinant representation *det*. It is an invertible object in $Tilt_q(\mathfrak{gl}_n)$, namely there exists an object (its dual) det^{-1} such that $det \otimes det^{-1} \simeq k$. Taking the tensor product of a simple V_λ with *det* gives a simple module $V_\lambda \otimes det = V_\mu$ where μ is obtained by adding one box to each row of λ. If λ has n rows, taking the tensor product with det^{-1} has the effect of removing one box to each row. Applying this to Young tableaux that do not have n rows allows us to talk about Young tableaux with a negative number of boxes in some row. These "negative rows" actually correspond to interactions with tensor powers of the dual V^* of the standard representation.
To obtain all of the objects of $Tilt_q(\mathfrak{gl}_n)$ one needs to consider subobjects of the mixed tensor spaces $T(r,s) = V^{\otimes r} \otimes (V^*)^{\otimes s}$. The algebra $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_n)}(T(r,s))$ contains a tensor product of

two Hecke algebras $H_r \otimes H_s$ by acting separately on $V^{\otimes r}$ and $(V^*)^{\otimes s}$.

Definition 2.8: A bipartition $\boldsymbol{\lambda}$ of type (r, s) is a pair of partitions $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ where λ^1 is a partition of r and λ^2 a partition of s. Its Young diagram is represented as the Young diagram of λ_1 glued at (0,0) with the Young diagram of λ_2 rotated of 180 degrees. The length of a bipartitions $l(\boldsymbol{\lambda})$ is the sum of the lengths of λ_1 and λ_2, i.e. the number of rows in the Young diagram of $\boldsymbol{\lambda}$.

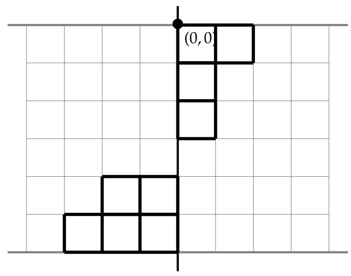
Note that both partitions and dual partitions are special cases of bipartitions with either $s = 0$ or $r = 0$.

$\boldsymbol{\lambda} = ((2,1,1),(3,2))$



Given a bipartition $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ of length at most n, one gets an idempotent $e_{\lambda^1} \otimes e^*_{\lambda^2}$ of $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_n)}(T(r, s))$. This idempotent is a projection on the submodule $V_{\lambda_1} \otimes V^*_{\lambda_2} \subseteq T(r, s)$. This submodule splits as a direct sum of simple modules, all of which are weight module. A unique one has the highest weight, and we call this one $V_{\boldsymbol{\lambda}}$. Again, one should think of the Young diagram of λ_2 as being removed from the infinite n-row rectangle of negative indices.

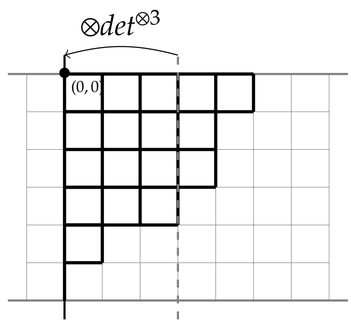
$\boldsymbol{\lambda} = ((2,1,1),(3,2))$ and $n = 6$



The decomposition of a tensor product $V_{\boldsymbol{\lambda}} \otimes V_{\boldsymbol{\mu}}$ of two simple modules indexed by bipartitions can be calculated using Littlewood-Richardson coefficients, see [CW11, Theorem 5.1.2]. In the simplest case where $\boldsymbol{\mu} = ((1), \emptyset)$ is a single box then $V_{\boldsymbol{\lambda}} \otimes V = \bigoplus_{\boldsymbol{\nu} \in \boldsymbol{\lambda} + \square} V_{\boldsymbol{\nu}}$ where $\boldsymbol{\lambda} + \square$ is the set of bipartitions obtained by either adding a box to λ^1 or removing one to λ^2.

> **Proposition 2.9:** *For q generic, bipartitions $\boldsymbol{\lambda}$ of length at most n are in bijection with simple objects $V_{\boldsymbol{\lambda}}$ in $Tilt_q(\mathfrak{gl}_n)$ up to isomorphism.*

Now the precedent trick of tensoring with *det* applies and the simple module $V_{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} = ((2,1,1),(3,2))$ at $n = 6$ can be described as $V_{\boldsymbol{\lambda}} \otimes det^{\otimes 3} = V_\mu$ where $\mu = (5,4,4,3,1)$ is a usual partition, so $V_{\boldsymbol{\lambda}} = V_\mu \otimes det^{\otimes -3}$. We see that all the simple objects of $Tilt_q(\mathfrak{gl}_n)$ can be written uniquely as some $V_\mu \otimes det^k$ where μ is a partition with at most $n-1$ rows and $k \in \mathbb{Z}$.

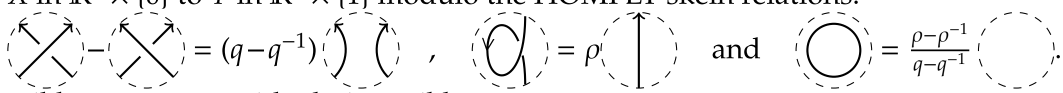


3 The HOMFLY category

We saw that the Hecke algebra H_k interpolates endomorphism algebras of $V^{\otimes k}$ seen as a $\mathfrak{gl}_n$-module for n big enough. We would now like to extend this interpolation to the whole category $Tilt_q(\mathfrak{gl}_n)$. We study the category of framed oriented tangles modulo the HOMFLY skein relations. It has been widely studied in [Bru17], of whom we borrow the notations.

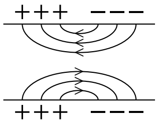
⌜ **Definition 3.1:** The integral HOMFLY category $\mathcal{OS}$ is the $\mathbb{Z}[q, q^{-1}, \rho, \rho^{-1}, \frac{\rho-\rho^{-1}}{q-q^{-1}}]$-linear category with:

- objects finite sets X of framed oriented points in $\mathbb{R}^2$. We will always suppose this set is of the form $[n] = \{1, \dots, n\} \times \{0\}$ with blackboard framing.
- morphisms from X to Y isotopy classes oriented framed tangles in $\mathbb{R}^2 \times [0, 1]$ linking X in $\mathbb{R}^2 \times \{0\}$ to Y in $\mathbb{R}^2 \times \{1\}$ modulo the HOMFLY skein relations:



It is a ribbon category with obvious ribbon structure.
We call the positively oriented point the generating object of $\mathcal{OS}$, and the negatively oriented point is its dual.
We let $\delta := \frac{\rho-\rho^{-1}}{q-q^{-1}}$ denote the dimension of the generating object.
Given a morphism $\mathbb{Z}[q, q^{-1}, \rho, \rho^{-1}, \frac{\rho-\rho^{-1}}{q-q^{-1}}] \to \mathbb{C}$, namely a choice of two parameters q and ρ in $\mathbb{C}$ and eventually of δ if $\rho = \rho^{-1}$ and $q = q^{-1}$, the HOMFLY category, still denoted $\mathcal{OS}$, is the $\mathbb{C}$-linear category obtained from the integral HOMFLY category by base change.
We denote by $\dot{\mathcal{OS}}$ the additive Karoubian envelope of $\mathcal{OS}$. ⌟

For r and s two non-negative integers, we denote by $T(r, s)$ the sequence of r positively oriented points followed by s negatively oriented points, the mixed tensor space of r copies of the generating object and s of its dual. They represent every isomorphism class of objects of $\mathcal{OS}$. The endomorphism algebra of $T(r, 0)$ is isomorphic to the Hecke algebra H_r, and the endomrophism algebra of $T(0, s)$ to the Hecke algebra H_s. The endomorphism algebra of $T(r, s) = T(r, 0) \otimes T(0, s)$ contains a copy of $H_r \otimes H_s$, but it has additional cups and caps linking positively and negatively oriented points. It is referred to as the quantized walled Brauer algebra $B_{r,s}$ and is studied in [DDS13], [RS14a] or [ST15].
In the endomorphisms of a mixed tensor space $T(r, s)$, for q generic, there are in particular idempotents of the form $e_{\lambda^1} \otimes e_{\lambda^2}$ for λ^1 a partition of r and λ^2 a partition of s, i.e. for $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ a bipartition. These idempotents are in general not primitive, but split in a sum of primitive idempotents and there is a unique one $e_{\boldsymbol{\lambda}} \in B_{r,s}$ which will not be mapped to 0 under the quotient $B_{r,s} \twoheadrightarrow H_r \otimes H_s$ that kills the ideal generated by caps and cups, see [CW11, Section 4.3].

Remark 3.2: For $i \leq \min(r, s)$ one can pair the i last positive points with the i first negative points by i caps at the bottom and i cups at the top. This gives an endomorphism $\hat{e}_i =$  $\in \mathrm{End}_{\mathcal{OS}}(T(i,i))$. When $\delta \neq 0$ this gives an idempotent $e_i = \frac{1}{\delta^i}\hat{e}_i$. Then there is an algebra map from the tensor product of Hecke algebras $H_{r-i} \otimes H_{s-i} \to \mathrm{End}_{\mathcal{OS}}(T(r,s))$ which maps a pair of braids $f \otimes g$ to the tangle $f \otimes e_i \otimes g$. Now idempotents of $H_{r-i} \otimes H_{s-i}$, and in particular bipartitions of type $(r-i, s-i)$, give rise to idempotents of $\mathrm{End}_{\mathcal{OS}}(T(r,s))$, i.e. split subobjects of $T(r,s)$. These "lower order" idempotents are necessary to describe all subobjects of $T(r,s)$, they are studied in greater detail in [CW11, Section 4.4] . However, they will be isomorphic to subobjects of $T(r-i, s-i)$ in $\dot{\mathcal{OS}}$ and are therefore not necessary to describe all objects of $\dot{\mathcal{OS}}$.

Theorem 3.3 ([Bru17, Theorem 1.5]): *The category $\dot{\mathcal{OS}}$ is semisimple if and only if q is not a root of unity and ρ is not of the form $\pm q^n$ for some $n \in \mathbb{Z}$. In this case, isomorphism classes of simple objects are in bijection with bipartitions.*

When $\rho = q^n$, $n \in \mathbb{N}$, there is a quotient functor $\dot{\mathcal{OS}} \to Tilt_q(\mathfrak{gl}_n)$ which maps the generating object to the fundamental representation and that preserves the ribbon structure. On idempotents indexed by bipartitions, this functor kills the Young diagrams that have more than n rows.

Theorem 3.4 ([Bru17, Theorem 1.5]): *When q is generic and $\rho = q^n$, the mapping* $\begin{array}{ccc} \dot{\mathcal{OS}} & \to & Tilt_q(\mathfrak{gl}_n) \\ \bullet & \mapsto & V \end{array}$ *extends essentially uniquely to a ribbon functor $F_n : \dot{\mathcal{OS}} \to Tilt_q(\mathfrak{gl}_n)$. This functor is full and essentially surjective, and its kernel consists of the negligible morphisms, which again correspond to the ideal generated by $e_{n+1} = e_{\square_{n+1}}$. On indecomposable objets indexed by bipartitions, it kills the Young diagrams that have more than n rows. In other words, $Tilt_q(\mathfrak{gl}_n)$ is the semisimplification of $\dot{\mathcal{OS}}$ at $\rho = q^n$, see [EO19].*

From this result we conclude that the HOMFLY category is a good way to interpolate the categories $Tilt_q(\mathfrak{gl}_n)$. Note however that unlike endomorphism algebras, the ribbon structure does not stabilize at big n.

Remark 3.5: The decomposition of a tensor product of two indecomposables indexed by bipartitions (at any ρ) is the same as in the $\mathfrak{gl}_n$ case for n big enough, i.e. greater than the sum of the lengths of the bipartitions.

Remark 3.6: We can extend this "integer" case to whenever $\rho = \pm q^n$ for some $n \in \mathbb{Z}$, so that $\delta = [\pm n]_q$ is a quantum integer, either by using a symmetry of the HOMFLY category under $q \leftrightarrow -q^{-1}$ or by introducing a sign in the pivotal structure (which is understandable "at the level of" vector spaces).

Proposition 3.7: *There is an involution of the HOMFLY category $\iota : \dot{\mathcal{OS}} \to \dot{\mathcal{OS}}$ by mapping q to $-q^{-1}$ and ρ to itself. This is well defined because it does not affect the skein relations. Under this map, a symmetrizer of type a_n is sent to one of type b_n, and therefore the idempotent associated with a bipartition is mapped to the idempotent associated with the transpose of the bipartition.*

In general we will still formally denote $\rho = q^t$ and $\delta = [t]_q$ with the subtlety that t is not necessarily an integer. The HOMFLY category is sometimes noted $Rep_q(GL_t)$.

4 Hook-shape Young tableaux and the quantum supergroup $\mathcal{U}_q(\mathfrak{gl}(m|n))$

We now introduce the representation theory of the quantum supergroup $\mathcal{U}_q(\mathfrak{gl}(m|n))$ which is also interpolated by the HOMFLY category, but this time in a more stable way.

The Lie algebra $\mathfrak{gl}(m|n)$ is the supervector space of $(m+n)\times(m+n)$ matrices endowed with the bracket $[A,B] = AB-(-1)^{p(A)p(B)}BA$, where p is the parity and elementary matrices in the diagonal $m\times m$ and $n\times n$ blocks are even, and the others are odd. Its universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(m|n)$ is generated by the matrices $h_i = E_{i,i}$, $1 \leq i \leq n$, and $E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$, $1 \leq i \leq n-1$. All of these matrices are even except for E_m and F_m which are odd. This algebra admits a deformation quantization as follows, see [RW19, Definition A.1] or [Moo01, Definition 1.6]:

Definition 4.1: The quantum group $\mathcal{U}_q(\mathfrak{gl}(m|n))$ is the $\mathbb{C}(q)$-algebra with generators $k_i^{\pm 1}$, $1 \leq i \leq m+n$, and E_i, F_i, $1 \leq i \leq m+n-1$. We denote $K_i = k_i k_{i+1}^{-1}$ for $i \neq m$ and $K_m = k_m k_{m+1}$. All of these generators are even except for E_m and F_m which are odd. We denote $l_i = 1$ if $1 \leq i \leq m$ and $l_i = -1$ if $m+1 \leq i \leq m+n$. It has relations:

$$[-_i, -_j] = 0 \text{ if } \left|i-j\right| \geq 2 \quad , \quad [k_i, k_{i\pm 1}] = 0 \quad ,$$
$$k_i E_j = q^{l_i(\delta_{i,j}-\delta_{i-1,j})} E_j k_i \quad , \quad k_i F_j = q^{-l_i(\delta_{i,j}-\delta_{i-1,j})} F_j k_i \quad , \quad [E_i, F_j] = l_i \delta_{ij} \frac{K_i - K_i^{-1}}{q-q^{-1}} \quad ,$$
$$\text{for } i \neq m: \quad (q+q^{-1}) E_i E_{i\pm 1} E_i = E_i^2 E_{i\pm 1} + E_{i\pm 1} E_i^2$$
$$(q+q^{-1}) F_i F_{i\pm 1} F_i = F_i^2 F_{i\pm 1} + F_{i\pm 1} F_i^2$$
$$E_m^2 = F_m^2 = 0 \quad ,$$
$$(q-q^{-1}) F_m F_{m-1} F_{m+1} F_m = F_m F_{m-1} F_m F_{m+1} + F_m F_{m+1} F_m F_{m-1} + F_{m-1} F_m F_{m+1} F_m + F_{m+1} F_m F_{m-1} F_m$$
$$\text{and} \quad (q-q^{-1}) E_m E_{m-1} E_{m+1} E_m =$$
$$E_m E_{m-1} E_m E_{m+1} + E_m E_{m+1} E_m E_{m-1} + E_{m-1} E_m E_{m+1} E_m + E_{m+1} E_m E_{m-1} E_m.$$

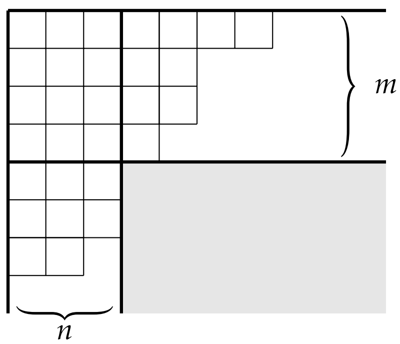
Its standard representation is the vector space V of dimension $m+n$ with basis $v_1, \ldots, v_m$ which are even and $w_{m+1}, \ldots, w_{m+n}$ which are odd, on which k_i acts as $qE_{i,i} + \sum_{j\neq i} E_{j,j}$, E_i acts as $E_{i,i+1}$ and F_i acts as $E_{i+1,i}$.

The Hopf algebra $\mathcal{U}_q(\mathfrak{gl}(m|n))$ has a topological quasi-triangular structure. In particular, the braiding $c_{V,V} = \times : V \otimes V \to V \otimes V$ is given by :

$$\begin{aligned}
c_{V,V}(v_i \otimes v_j) &= \begin{cases} q\, v_i \otimes v_i & \text{if } i = j \\ v_j \otimes v_i & \text{if } i < j \\ v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & \text{if } i > j \end{cases} \\
c_{V,V}(w_i \otimes w_j) &= \begin{cases} -q^{-1}\, v_i \otimes v_i & \text{if } i = j \\ -v_j \otimes v_i + (q - q^{-1}) v_i \otimes v_j & \text{if } i < j \\ -v_j \otimes v_i & \text{if } i > j \end{cases} \\
c_{V,V}(v_i \otimes w_j) &= -w_j \otimes v_i + (q - q^{-1}) v_i \otimes w_j \\
c_{V,V}(w_j \otimes v_i) &= v_i \otimes w_j
\end{aligned}$$

Again, the induced action of the braid group factorises through the Hecke algebra. Idempotents e_λ of the Hecke algebra will be sent to idempotents of $\mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}(m|n))}(V^{\otimes k})$, i.e. projections onto direct summands V_λ. The image idempotent will be zero if and only if $\lambda_{m+1} > n$, i.e. if λ gets out of the (m, n)-hook-shape. Standard Young tabloids with k boxes and fitting in the (m, n)-hook-shape are in bijection with indecomposable summands of $V^{\otimes k}$.

A Young diagram fitting in the (m, n)-hook-shape :
The gray part is forbidden.
The upper left rectangle is called the (m, n)-rectangle.



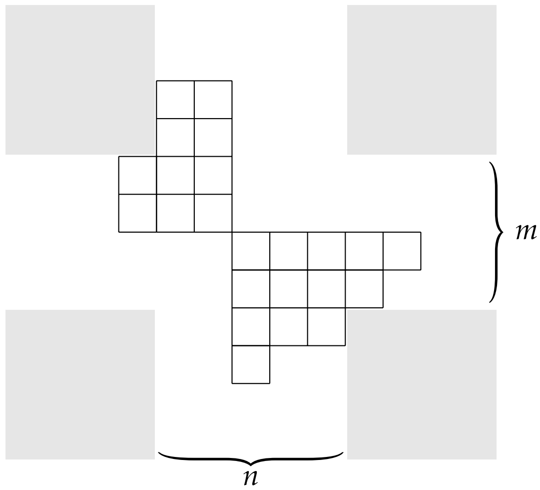
⌜ **Definition 4.2:** We denote $Tilt_q(\mathfrak{gl}(m|n))$ the category of finite dimensional tilting representations of $\mathcal{U}_q(\mathfrak{gl}_n)$, namely of summands of tensor powers of the standard representation V and of its dual. This category is ribbon, and the quantum dimension of the standard representation is $[m-n]_q$. Note that for simplicity we do not allow all subobjects and quotients but only direct summands. ⌟

The action of the Hecke algebra extends to a full and essentially surjective ribbon functor $\dot{\mathcal{OS}} \to Tilt_q(\mathfrak{gl}(m|n))$.

Theorem 4.3 ([CW11, Section 8.3]): *When q is generic and $\rho = q^{m-n}$, the mapping* $\begin{array}{ccc} \dot{\mathcal{OS}} & \to & Tilt_q(\mathfrak{gl}(m|n)) \\ \bullet & \mapsto & V \end{array}$ *extends essentially uniquely to a full and essentially surjective ribbon functor $F_{m|n} : \dot{\mathcal{OS}} \to Tilt_q(\mathfrak{gl}(m|n))$.*

For q generic and m, n big enough, $Tilt_q(\mathfrak{gl}(m|n))$ is not semi-simple. The (m, n)-hook-shape Young diagrams λ correspond to indecomposable summands V_λ of $V^{\otimes k}$ but these will not necessarily be simple injective modules. When they are, the partition λ is called typical. A Young diagram containing the (m, n)-rectangle is always typical.
The functor $F_{m|n}$ sends idempotents $e_{\boldsymbol{\lambda}}$ associated with a bipartition $\boldsymbol{\lambda}$ to indecomposable modules $V_{\boldsymbol{\lambda}}$. This module is non zero 0 if and only if the Young diagram of the bipartition $\boldsymbol{\lambda}$ fits is an (m, n)-cross, i.e. the union of an horizontal band of height m and a vertical band of width n.

The bipartition $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ with $\lambda^1 = (5, 4, 3, 1)$ and $\lambda^2 = (3, 3, 2, 2)$ fits in a (4,5)-cross but in no smaller crosses.
The gray part is forbidden.



Proposition 4.4: *For q generic, bipartitions $\boldsymbol{\lambda}$ fitting in a (m, n)-cross are in bijection with indecomposable objects $V_{\boldsymbol{\lambda}}$ of $Tilt_q(\mathfrak{gl}(m|n))$ up to isomorphism.*

Finally, the HOMFLY category at integer t is the limit of the categories $Tilt_q(\mathfrak{gl}(m|n))$ for $m - n = t$ as a ribbon category. For $m, n \geq 1$, there is a ribbon functor $Tilt_q(\mathfrak{gl}(m|n)) \to Tilt_q(\mathfrak{gl}(m-1|n-1))$. On the standard representation V of dimension $m + n$ it is given by restricting to the first $m + n - 1$ coordinates and quotienting by the first coordinate, see [EHS19, Section 7]. It induces an equivalence on subcategories generated by small enough tensor powers of V and V^*.

Theorem 4.5 ([EHS19, Proposition 8.1.2 and Corollary 8.1.5]): *The HOMFLY category $\dot{\mathcal{OS}}$ at $\rho = q^d$ and q generic is the limit of the categories $Tilt_q(\mathfrak{gl}(n+d|n))$ as $n \to +\infty$ as a ribbon category.*

This completes the description of the HOMFLY category as an interpolation category. On endomorphism spaces of $V^{\otimes k}$ the Hecke algebra H_k is the stabilization of the endomrophism algebras of $V^{\otimes k}$ as a $\mathfrak{gl}_n$-module. On mixed tensor powers, and as a ribbon category, it is properly speaking an interpolation of the categories $Tilt_q(\mathfrak{gl}_n)$. However, at integer values, it can be described again as a stabilisation of the categories $Tilt_q(\mathfrak{gl}(n+d|n))$.

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