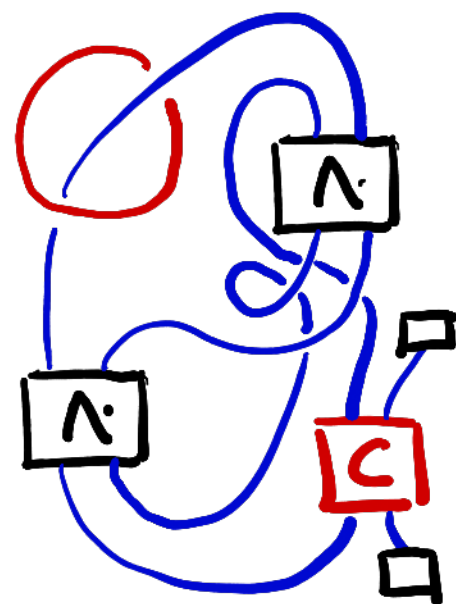


# 4-manifold invariants from TQFTs non-semisimple finite ribbon Hopf algebras Categories

AT TU Dresden



$S^1 \times S^1 \times S^2$

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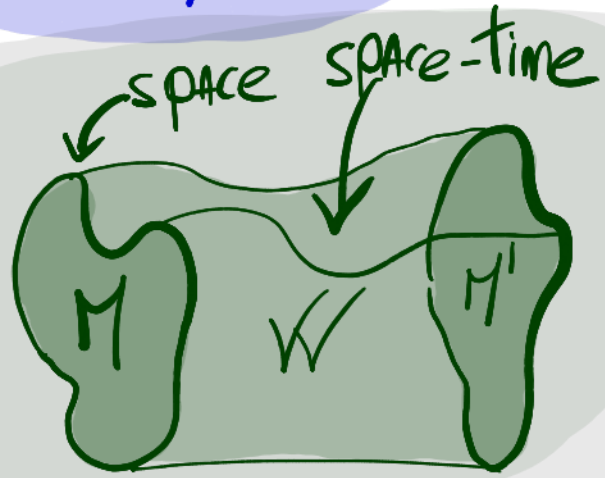
Joint work with

F. Costantino, N. Geer and B. Patureau-Mirand

ARXiv: 2306.03225

# Some motivation (And some physics)

geometric



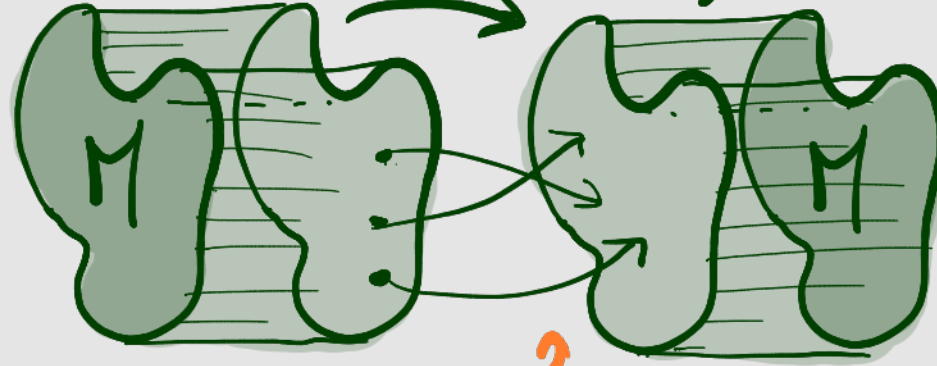
(T)QFT

linear  $Z(M) \xrightarrow{Z(W)} Z(M')$   
 ↑ vector space of states  
 ↑ time evolution operator

$$Z: \text{Cob}_n^L \rightarrow \text{Vect}_k^\otimes$$

~

$f \in \text{Diff}(M)$



$Z(n) \xrightarrow{Z(c_1^2)} Z(n)$   
 RCG representations.

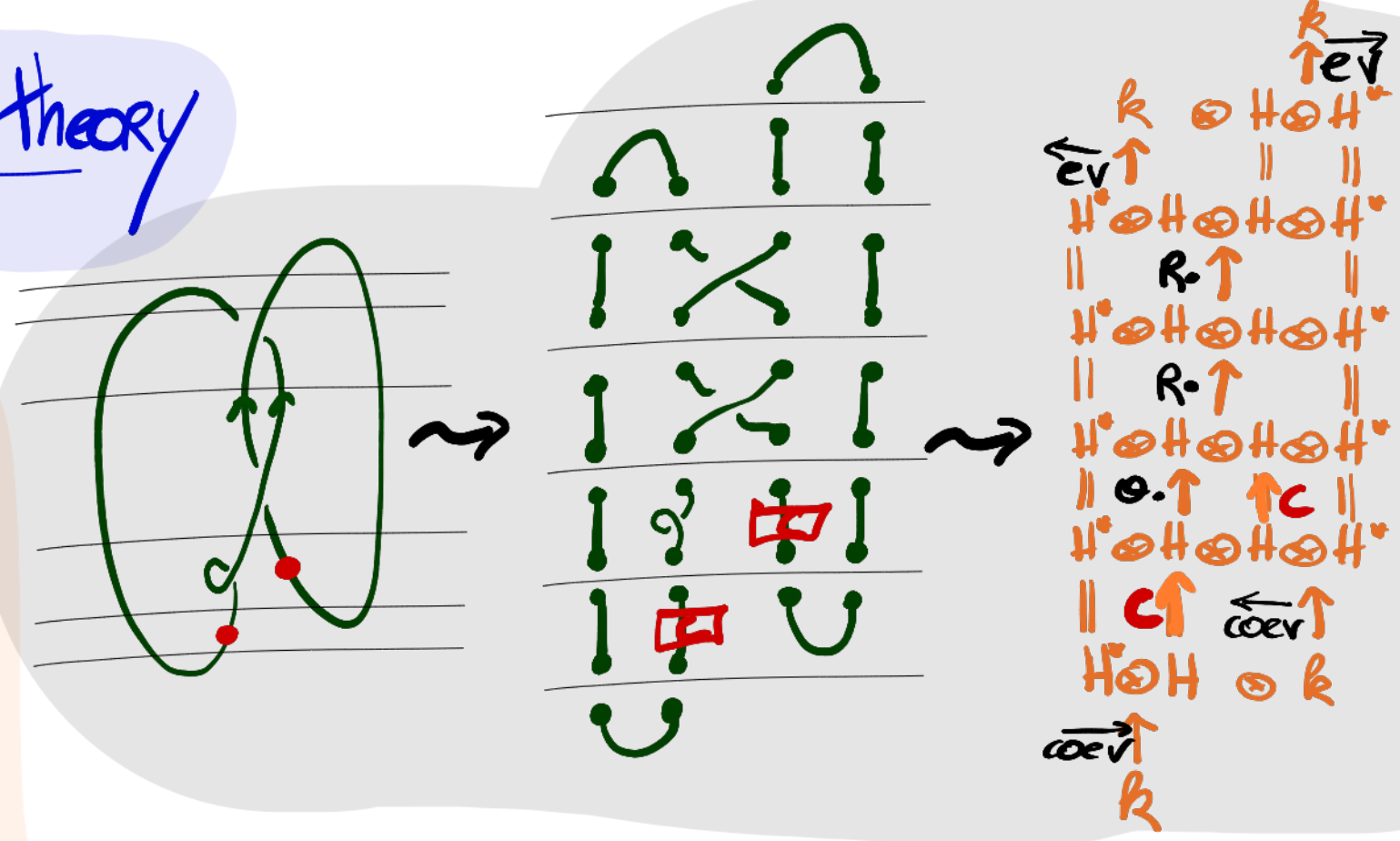
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$Z(0) \xrightarrow{Z(W) \in k} Z(0)$   
 $Z(0) \xrightarrow{Z(W) \in \mathbb{R}} Z(0)$   
 4-nd invariant.

# Reshetikhin-Turaev theory

- $\mathcal{C} = \text{Rep } H$
- Algebraic Requirements
- $H$  Hopf algebra
  - $\text{ev}, \text{coev} \rightsquigarrow$  finite
  - $\vec{\text{ev}}, \overleftarrow{\text{ev}} \rightsquigarrow$  pivotal
  - $R \rightsquigarrow$  quasitriangular
  - $\Theta \rightsquigarrow$  ribbon
  - $\mathcal{C} \rightsquigarrow$  semisimple

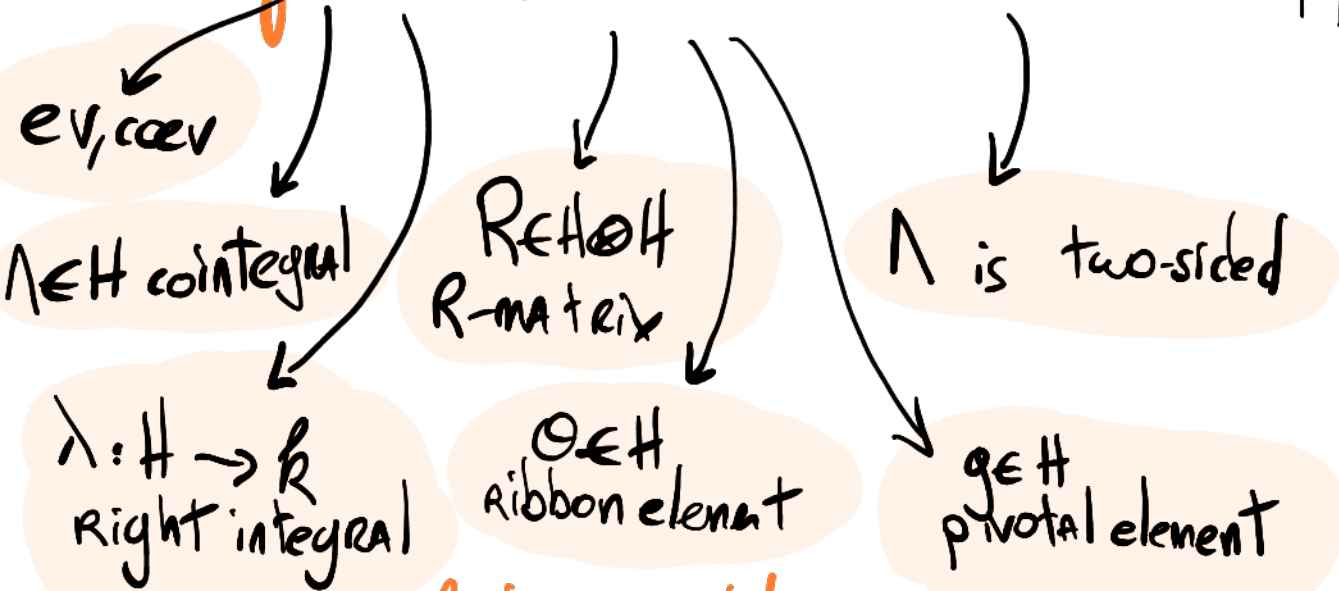


$$M = \text{Surg} \left( \begin{array}{c} S^3 \\ \text{link} \end{array} \right)$$

# Algebraic setting

$$\lambda \in H \text{ st } \forall h \in H \quad \lambda \cdot h = \lambda \cdot \epsilon(h) = h \cdot \lambda$$

$H$  finite ribbon unimodular Hopf algebra



Set  
 $C: H \otimes H \rightarrow H \otimes H$   
 $x \otimes y \mapsto \lambda(S(y_{(1)})g x) y_{(2)} \otimes y_{(3)}$   
"chromatic morphism"

factorizable  
 $\Downarrow$   
 semisimple + char  $k = 0$   
 It is called chromatic non-degenerate if:

$$\exists g \in H, (\lambda \otimes Id_H)(R_{21} R_{12})g = \lambda$$

"giving morphism"

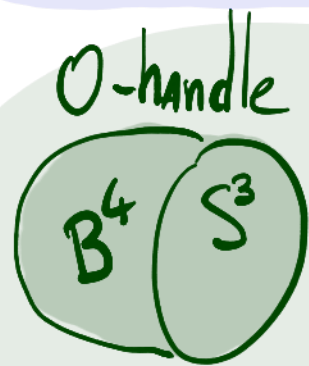
And chromatic compact  
 if moreover:  
 $\epsilon(g) \neq 0$

$C: H \otimes H \rightarrow H \otimes H$ ,  $(C_{12})^2$  is semisimple / transparent / invertible.

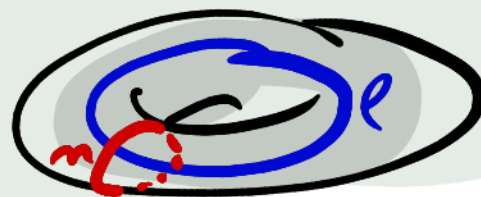


# Geometric Setting

$V$  connected closed smooth 4-manifold



$k$ -Handle  $H_k: S^{k-1} \times D^{4-k} \xrightarrow{D^k \times D^{4-k}} D^4 \times S^{4-k-1}$



Examples:  $S^4 =$

$\mathbb{C}P^2 =$

$S^1 \times S^3 =$

$S^2 \times S^2 =$

$S^4 = \emptyset, \mathbb{C}P^2 =$

$S^1 \times S^3 =$

$S^2 \times S^2 =$

$S^1 \times S^1 \times S^2 =$

# Our construction

0-handle  
 $B^4 \cup S^3$ : start with

$H \uparrow =$   $H \xrightarrow{\sim} H \oplus H$

1-handles

2-handles

3-handles

4-handle  
 $B^4$

$\epsilon \left( \dots \right) =: Z(W)$

# Examples of manifolds

$$S^4 = \emptyset \rightsquigarrow \begin{array}{c} \boxed{\varepsilon} \\ \downarrow \\ \boxed{\Lambda} \end{array} \rightsquigarrow Z(S^4) = \varepsilon \left( \begin{array}{c} \text{blue line} \\ \boxed{1} \end{array} \right) = 1$$

$$\mathbb{C}P^2 = \text{red torus} \rightsquigarrow \begin{array}{c} \boxed{\varepsilon} \\ \downarrow \\ \boxed{\Lambda} \end{array} \rightsquigarrow Z(\mathbb{C}P^2) = \varepsilon \left( \begin{array}{c} \text{blue line with red box } \boxed{c} \\ \boxed{1} \end{array} \right) = \lambda(\theta)$$

$$S^1 \times S^3 = \text{cylinder with green circle} \rightsquigarrow \begin{array}{c} \boxed{\varepsilon} \\ \downarrow \\ \boxed{\Delta} \\ \downarrow \\ \boxed{S} \\ \downarrow \\ \boxed{\Lambda} \end{array} \rightsquigarrow Z(S^1 \times S^3) = \varepsilon \left( \begin{array}{c} \text{blue line with boxes } \boxed{\Delta}, \boxed{S}, \boxed{\Lambda} \\ \boxed{1} \end{array} \right) = \varepsilon(g)$$

$$S^2 \times S^2 = \text{red torus} \rightsquigarrow Z(S^2 \times S^2) = \varepsilon \left( \begin{array}{c} \text{blue line with boxes } \boxed{c}, \boxed{c} \\ \boxed{1} \end{array} \right) = (\lambda \otimes \lambda)(R_{21} R_{12})$$

$$S^1 \times S^1 \times S^2 = \text{red torus with green circles} \rightsquigarrow Z(S^1 \times S^1 \times S^2) = \varepsilon \left( \begin{array}{c} \text{blue line with boxes } \boxed{c}, \boxed{\Lambda}, \boxed{c}, \boxed{\Lambda} \\ \boxed{1} \end{array} \right) = \dots$$

# Properties

Let  $H$  be chromatic non-degenerate.

[Prop:  $Z(W \# W') = Z(W) \cdot Z(W')$

proof:  $\varepsilon(- \cdot -) = \varepsilon(-) \cdot \varepsilon(-)$

[Cor: If  $Z(S^2 \times S^2) \neq 0$ ,  $Z$  cannot detect exotic structures.

proof: Gompf proved  $W \stackrel{\text{homeo}}{\cong} W' \Rightarrow W \# (S^2 \times S^2) \stackrel{\text{diffeo}}{\cong} W' \# (S^2 \times S^2)$

{ Expectation: The TQFT is fully extended.

↳ Reuter's thm: if  $H$  is chromatic compact, then  $Z$  cannot detect exotic structures.

[Cor: If  $Z(\mathbb{C}P^2) \neq 0$ ,  $Z$  depends only on  $\chi(W)$ ,  $\pi_1(W)$ ,  $\sigma(W)$  and  $c_2(W) \in H_4(\pi, \mathbb{Z})$

proof: Kreck's classification up to  $\# \mathbb{C}P^2$

[Thm: If  $H$  is factorisable, then  $Z(W)$  depends only on  $\chi(W)$  and  $\sigma(W)$ .  
cancellation in the s.c. case



## Examples of Hopf algebras

Small quantum  $sl_2$  at  $q^{2R} = 1$ :

$$H = \mathbb{C}\langle E, F, K \mid E^R = F^R = K^R - 1 = 0, \begin{matrix} KE = q^2 EK \\ KF = q^{-2} FK \end{matrix}, EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \rangle$$

Thm:  $H$  is **chromatic compact**. It is **factorisable** iff  $R$  is odd.  
[  **$Z(\mathbb{C}P^2)$**   $\neq 0$  iff  $R$  is not a multiple of 4.  **$Z(S^2 \times S^2)$**   $\neq 0$ .

Positive characteristic:  $\text{char } k = p$ ,  $H = k[Z/pZ]$

Thm:  $H$  is **chromatic non-degenerate** but not **chromatic compact**.  
[  **$Z(\mathbb{C}P^2)$**   $= Z(S^2 \times S^2) = 1$

# Wish list

$H$  a finite Ribbon unimodular Hopf algebra s.t:

$\phi \in \mathcal{R} \xrightarrow{\mathcal{Z}(\pi)} \mathcal{R}_{\text{RT}(\pi)} \xrightarrow{\mathcal{Z}(\phi)} \mathcal{R}$

$\mathcal{R}$  (circle)  $\mathcal{K}$  (circle)  $\mathcal{Z}(\phi)$  renormalize

\* chromatic non-degenerate

$$\exists g \in H, (\lambda \otimes \text{Id}) R_{21} R_{12} \cdot g = \Lambda$$

\*  $\mathcal{Z}(S^2 \times S^2) = (\lambda \otimes \lambda) R_{21} R_{12} = 0$

\* NOT chromatic compact

$$\mathcal{E}(g) = 0 \Rightarrow H \text{ is non-semisimple}$$

\*  $\mathcal{Z}(\mathbb{C}P^2) = \lambda(\sigma) = 0$

Can  $\mathcal{Z}$  distinguish exotic structures?

Thank you