Anomalous TQFTs from the Cobordism Hypothesis

Higher Structures & Field Theory seminar

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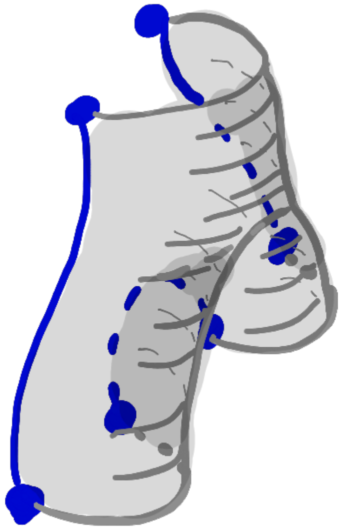
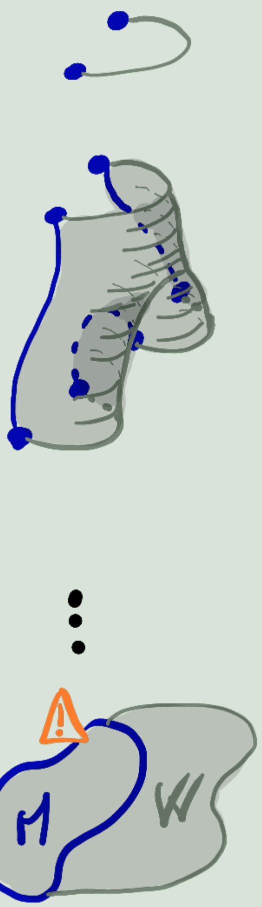


Motivations

(non-semisimple) WRT : $\mathrm{Bord}_3^{\text{filled}} \longrightarrow \Omega\mathcal{C}^{\text{odd opp}}$



$$\begin{tikzcd} \mathrm{Triv} \arrow[r, "\partial(\cdot)"] \arrow[rr, bend right, "\mathrm{WRT}"'] & Z(\cdot) \arrow[r, "Z(\supset)"] & \mathrm{Triv} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Triv} \arrow[r, "\partial(\cdot)"] \arrow[d, equal] & Z(\cdot) \arrow[r, "Z(\supset)"] \arrow[d, "Z(\cap)"] & \mathrm{Triv} \arrow[d, equal] \\ \mathrm{Triv} \arrow[r, "\partial(\cdot\,\cdot)"'] \arrow[ur, phantom, "\Nearrow \partial(\cap)"] & Z(\cdot\,\cdot) \arrow[r, "Z(\supset\supset)"'] \arrow[ur, phantom, "\Nearrow Z(\ )"] & \mathrm{Triv} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Triv} \arrow[r, "\partial(M)"] \arrow[rr, bend right, "\mathrm{WRT}"'] & Z(M) \arrow[r, "Z(W)"] & \mathrm{Triv} \end{tikzcd}$$

Claim: We can reconstruct (non-ss) WRT from $Z(\cdot)$ and $\partial(\cdot)$

Plan:
- I: Relative TQFTs
- II: Anomalous TQFTs
- III: Cobordism Hypothesis
- IV: WRT

Relative TQFTs

boundary conditions

Idea (Freed–Teleman):

Given $Z : \mathrm{Bord}_{n-1} \to \mathcal{C}$ once categorified

A boundary condition is a "transformation" $\partial : \mathrm{Triv} \Rightarrow Z$

Σ closed $\rightsquigarrow$ $\mathrm{Triv} \xrightarrow{\partial(\Sigma)} Z(\Sigma)$

$\Sigma \xrightarrow{M} \Sigma'$ $\rightsquigarrow$

$$\begin{tikzcd} \mathrm{Triv} \arrow[r, "\partial(\Sigma)"] \arrow[d, equal] \arrow[dr, phantom, "\partial(M)"] & Z(\Sigma) \arrow[d, "Z(M)"] \\ \mathrm{Triv} \arrow[r, "\partial(\Sigma')"'] & Z(\Sigma') \end{tikzcd}$$

$\vdots$ $\qquad\qquad$ $\vdots$

Definition (Johnson-Freyd–Scheimbauer):

An oplax ∂ condition is:

$$\begin{tikzcd} \partial : \mathrm{Bord}_{n-1} \arrow[r] \arrow[rrr, bend left, "\mathrm{Triv}"] \arrow[rrr, bend right, "Z"'] & \mathcal{C}^{\to} \arrow[rr, shift left, "s"] \arrow[rr, shift right, "t"'] & & \mathcal{C} \end{tikzcd}$$

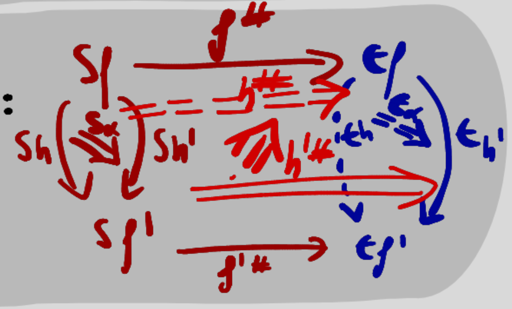
where $\mathcal{C}^{\to}$ has:

obj: $\quad S_f \xrightarrow{f^\#} t_f$

1-m:

$$\begin{tikzcd} S_f \arrow[r, "f^\#"] \arrow[d, "S_h"'] \arrow[dr, phantom, "\Nearrow h^\#"] & t_f \arrow[d, "t_h"] \\ S_{f'} \arrow[r, "f'^\#"'] & t_{f'} \end{tikzcd}$$

2-m:



Anomalous TQFTs

Take $Z : \mathrm{Bord}_n \to \mathcal{C}$, $\partial : \mathrm{Triv} \Rightarrow Z|_{\mathrm{Bord}_{n-1}}$

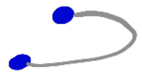
Def: $\mathcal{A}_{Z,\partial} : \mathrm{Bord}_{n-1}^{\text{filled}} := \mathrm{Bord}_n^{\to\emptyset} \longrightarrow \Omega\mathcal{C}^{\text{odd opp}}$

$$\begin{tikzcd} \mathrm{Bord}_{n-1}^{\text{filled}} \arrow[r, bend left, "\partial"] \arrow[r, bend right, "Z"'] & \mathcal{C}^{\to} \end{tikzcd}$$

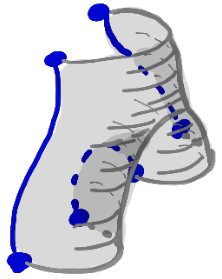
$Z : Z \Rightarrow \mathrm{Triv}$

$\partial : \mathrm{Triv} \Rightarrow Z$

$\mathcal{A} : \mathrm{Triv} \Rightarrow \mathrm{Triv}$

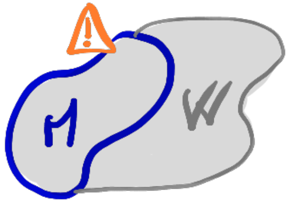


$$\begin{tikzcd} \mapsto \mathrm{Triv} \arrow[r, "\partial(\cdot\,\cdot)"] \arrow[rr, bend right, "\mathcal{A}"'] & Z(\cdot\,\cdot) \arrow[r, "Z(\supset)"] & \mathrm{Triv} \end{tikzcd}$$



$$\begin{tikzcd} \mapsto \mathrm{Triv} \arrow[r, "\partial(\cdot\,\cdot)"] \arrow[d, equal] & Z(\cdot\,\cdot) \arrow[r, "Z(\supset)"] \arrow[d, "Z(\cap\cap)"] & \mathrm{Triv} \arrow[d, equal] \\ \mathrm{Triv} \arrow[r, "\partial(\cdot\,\cdot\,\cdot\,\cdot)"'] \arrow[ur, phantom, "\Uparrow \partial(\cap\cap)"] & Z(\cdot\,\cdot\,\cdot\,\cdot) \arrow[r, "Z(\supset\supset)"'] \arrow[ur, phantom, "\Uparrow Z(\text{surface})"] & \mathrm{Triv} \end{tikzcd}$$

$\vdots$



$$\begin{tikzcd} \mapsto \mathrm{Triv} \arrow[r, "\partial(M)"] \arrow[rr, bend right, "\mathcal{A}"'] & Z(M) \arrow[r, "Z(W)"] & \mathrm{Triv} \end{tikzcd}$$

The ^{nc}"Cobordism Hypothesis"

Baez & Dolan
Lurie
Schommer-Pries

$$\mathrm{Bord}_n^{fr,nc} \longrightarrow \mathcal{C}$$

$$G^{\uparrow} \longmapsto X$$

$\bullet^+$ $\bullet^-$

... ↷ | ↺

... ~~↶~~ ... ↶

2 1 1 0

X X^*

... ev_X | $coev_X$ ~~ev_X^R~~ $coev_X^R$

... ...

$R_{co}(ev_X)$ | $R_u(ev_X)$ $R_{co}(coev_X)$ | $R_u(coev_X)$

All handle attachments

$$\begin{array}{ccc} & k & \\ n-k & & n-k \\ n+1-k \mid n-k & & k+1 \mid k \end{array}$$

dim n

dim $n+1$

$$\longmapsto \simeq R_{co}^{\,k} R_u^{\,n-k}(X)$$

$$0 \leq k < n$$

Conjecture (The Cobordism Hypothesis) $\mathcal{C}^{\otimes}$ s.m. (∞,n)-category

There are equivalences of groupoids

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \xrightarrow[\mathrm{ev}_{\mathrm{at}\,\ast}]{\sim} (\mathcal{C}^{fd})^{\sim}$$

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{or}, \mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{fd,\sim})^{SO(n)}$$

$X \in \mathcal{C}$ s.t either

(*) $\underbrace{R_{co}(L_u(\cdots R_u}_{\leq n}(x))\cdots)$ exist

(**) $R_{co}^{k} R_u^{n-k}(x)$ exist $0 \leq k \leq n$

Conjecture (Non-compact cobordism hypothesis)

There are equivalences of groupoids

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr,nc}, \mathcal{C}) \xrightarrow[\mathrm{ev}_{\mathrm{at}\,\ast}]{\sim} (\mathcal{C}^{nc\text{-}d})^{\sim}$$

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{or,nc}, \mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{nc\text{-}d,\sim})^{SO(n)}$$

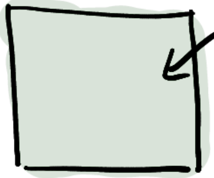
$X \in \mathcal{C}$ s.t

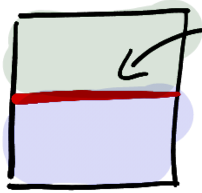
$R_{co}^{k} R_u^{n-k}(x)$ exist

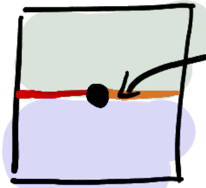
$0 \leq k < n$

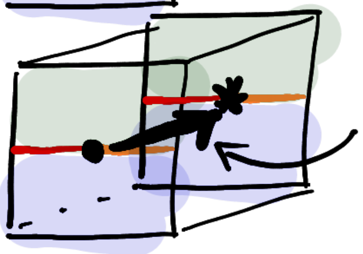
Choosing $\mathcal{C}$ for WRT theories

Def BrTens (or $\mathrm{Alg}_2(\mathrm{Pr}_R)$) has:

obj:  cocomplete braided tensor categories

1-m:  monoidal bimodule category

2-m:  bimodule category

3-m:  bimodule functor

4-m: bimodule natural transformation

$\mathcal{V}$ ribbon category $(\mathrm{Ind}(\mathcal{V}))$

Thm (Gwilliam–Scheimbauer) $\mathcal{V}$ is 2-dualizable

Thm (Ayala–Francis, Scheimbauer) The 2-TQFT associated with $\mathcal{V}$ is given by factorization homology. (In the semisimple case, by skein categories)

Thm (Brochier–Jordan–Snyder) $\mathcal{V}$ is 3-dualizable

Conjecture (Walker, Johnson-Freyd) In the semisimple case, the 3-TQFT associated with $\mathcal{V}$ is given by skein modules

Thm (Brochier–Jordan–Safronov–Snyder) If $\mathcal{V}$ is either fusion or modular, then $\mathcal{V}$ is 4-dualizable

Dualizability of unit inclusion

$\mathcal{V}$ (non-semisimple) modular

WRT obtained from $\mathcal{Z}$ and ∂

classified by $\mathcal{Z}(\bullet^+) \cong \mathcal{V}$
4-dualizable in BrTens

classified by $\partial(\bullet^+) = \mathcal{A}_\eta : \mathrm{Vect} \to \mathcal{V}$
unit inclusion

Thm (H.) $\mathcal{A}_\eta \in \mathrm{BrTens}^{\to}$ is (non-compact) 3-dualizable

$\xrightarrow{\text{nc-CH}}$ $\partial : \mathrm{Triv} \Rightarrow \mathcal{Z}|_{\mathrm{Bord}_3}$

$\mathcal{V}$ 4-dualizable :

$\mathcal{A}_\eta$ 3-dualizable $\overset{*}{\Longleftrightarrow}$ $\mathcal{V}$ fusion

Rebuilding WRT

$$\mathcal{A}_{Z,\partial} : \mathrm{Bord}_3^{\text{filled},nc} \longrightarrow \Omega\mathrm{BrTens} = \mathrm{Tens}$$

$$\mathcal{A}_{Z,\partial}(S^1) \simeq \mathcal{V} \;\Big|_{\mathcal{A}_y}^{\mathcal{A}_n} \circlearrowleft \mathcal{V} \rightsquigarrow \mathcal{V} \qquad \mathrm{Triv} \xrightarrow{\partial} Z\big|_{\mathrm{Bord}_{n-1}} \xrightarrow{Z} \mathrm{Triv}$$

Conjecture:

$$\begin{tikzcd}
\Omega\mathrm{Bord}_3^{\text{filled},nc} \arrow[rr, "\mathcal{A}_{Z,\partial}"] \arrow[d] & \arrow[d, phantom, "\circlearrowleft"] & \Omega\mathrm{Tens} = \mathrm{Pr} \\
\widetilde{\mathcal{C}ob}_{321}^{nc} \arrow[rr, "\mathrm{WRT}", "\mathrm{DGGPR}"'] & & \mathrm{Cat}_k \arrow[u, "\mathrm{Free}"']
\end{tikzcd}$$

Description of the 4-TQFT

$\mathcal{V}$ (non-semisimple) modular

Informal conjecture The 4-TQFT associated with $\mathcal{V}$ is given by:

dim : 0 1 2 | 3 | 4

$Z_{\mathcal{V}}$: (Admissible Free skein categories) // factorization homology | Admissible skein module functors | (3+1 CGHP's formulas for handle attachments)

∂ : $\emptyset \hookrightarrow \Sigma$, TRIV $\to \int_{\Sigma}^{Pr} \mathcal{V}$) $\emptyset$ skein object. | $\emptyset$ skein ⚠ DNC | $\emptyset_\varepsilon$

Conjecture check : (regular $\otimes$ $\mathcal{V}$ $\mapsto$

$\simeq \mathcal{V}$ ✓

$\emptyset \hookrightarrow \mathrm{SkCat}(\Sigma)$ $\mathrm{SkMod}(H) \to$ triv

$\mathrm{SkMod}(H, \emptyset)$ ✓

wrt surgery formula