Relative Topological Quantum Field Theories and the Cobordism Hypothesis

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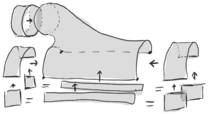
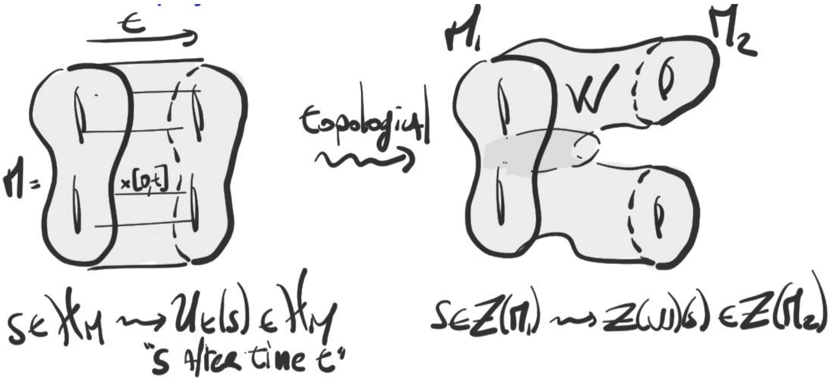


Topological Quantum Field Theories



Definition (Atiyah, Segal):

A **TQFT** is a symmetric monoidal functor $\mathcal{Z} : Cob_n \to \mathrm{Vect}$.
A **fully extended** (or local) **TQFT** is a symmetric monoidal n-functor

$$\mathcal{Z} : Bord_n \to n\mathrm{Vect}.$$

Examples

1. The trivial theory $Triv : \begin{cases} Bord_n & \to & \mathcal{C} \\ \text{any object} & \mapsto & \mathbf{1} \\ \text{any morphism} & \mapsto & Id_{\mathbf{1}} \end{cases}$.
2. Witten–Reshetikhin–Turaev theories:

   Input: semisimple modular tensor category $\mathcal{V}$.

   $$\begin{array}{ccc} \tilde{Cob}_3 & \to & Vect \\ \Sigma & \mapsto & k\langle \text{links in a handlebody} \rangle / \sim \\ M & \mapsto & \text{some 3-manifold invariant} \end{array}$$

3. Non-semisimple variants $\tilde{Cob}_3^{adm} \to Vect$ (partially defined).

Conclusion:

Constructing TQFTs is hard!

Results and conjectures

Theorem (Brochier–Jordan–Safronov–Snyder):

Using the Cobordism Hypothesis, one gets a (framed) 4-TQFT $\mathcal{Z}_{\mathcal{V}} : Bord_4^{fr} \to 4Vect$ for any modular tensor category $\mathcal{V}$.

Theorem (H.):

Using the Cobordism Hypothesis, one gets a (framed) 3-TQFT relative to $\mathcal{Z}_{\mathcal{V}}$, $\mathcal{R}_{\mathcal{V}} : Bord_3^{fr} \to 4Vect^{\to}$ for semisimple modular tensor categories $\mathcal{V}$.
$+$ partially defined one in the non-semisimple case.

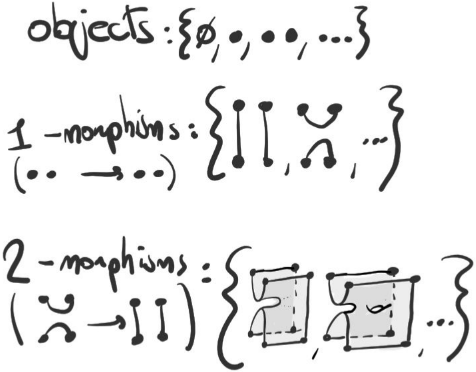
Conjecture (H–J–S, Freed–Teleman, Walker)

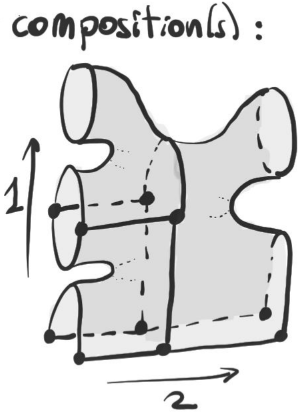
One can reconstruct the Witten–Reshetikhin–Turaev anomalous TQFTs $WRT_{\mathcal{V}} : \tilde{Cob}_3 \to Vect$ from the pair $(\mathcal{Z}_{\mathcal{V}}, \mathcal{R}_{\mathcal{V}})$.

Conjecture (H–J–S):

Also recovers the non-semisimple variants.

The source category *$Bord_n$*





Theorem (Lurie, Calaque–Scheimbauer):

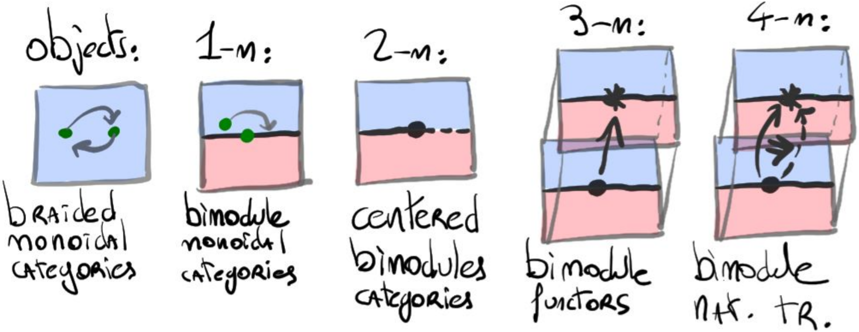
We can build this!

The target category nVect

Definitions of 2Vect:

Idea 1: $Pr =$ 2-category of presentable linear categories and cocontinuous functors.
Idea 2: $Alg_1(Vect) =$ 2-category of algebras and bimodules.

4Vect $\overset{today}{=}$ $Alg_2(Pr)$:



Theorem (Haugseng + Johnson-Freyd–Scheimbauer):

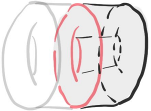
We can build this!

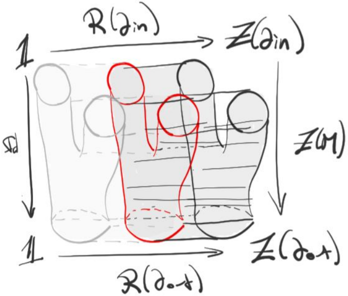
Relative TQFTs

Idea of definition (Freed-Teleman):

A **relative theory** to $\mathcal{Z}$ is a transformation $\mathcal{R} : Triv \to \mathcal{Z}$.

 $: \emptyset \to pt \quad \longmapsto \quad \mathcal{R}($  $) : \mathbf{1} \to \mathcal{Z}(pt)$

 $\longmapsto \quad \mathcal{R}($  $\times \Sigma) : Triv(\Sigma) \to \mathcal{Z}(\Sigma)$

$M : \partial_{in} \to \partial_{out} \overset{\mathcal{R}(\text{ } \times -)}{\longmapsto}$ 

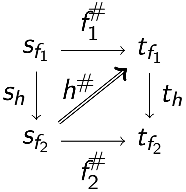
All $n-1$-dimensional!

Arrow category

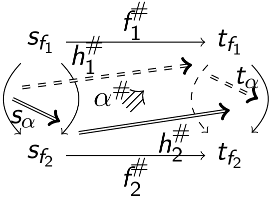
Definition (Johnson-Freyd–Scheimbauer):

$\mathcal{C}^{\to} = \begin{cases} \text{objects} & : \ (s_f \in \mathcal{C}, t_f \in \mathcal{C}, f^{\#} : s_f \to t_f) \\ \text{1-morphisms} & : \ h = (s_h, t_h, h^{\#}), \\ \text{2-morphisms} & : \\ n-1\text{-morphisms} & : \ \ldots \end{cases}$

1-morphisms: $h = (s_h, t_h, h^{\#})$,



2-morphisms:



$n-1$-morphisms: $\ldots$

- symmetric monoidal
- source and target functors $s, t : \mathcal{C}^{\to} \to \mathcal{C}$

Defintion (J–S):

Let $\mathcal{Z} : Bord_{n-1} \to \mathcal{C}$ be a 'categorified' TQFT.
A **TQFT relative to** $\mathcal{Z}$ is a s.m. functor $\mathcal{R} : Bord_{n-1} \to \mathcal{C}^{\to}$ such that $s \circ \mathcal{R} = Triv$ and $t \circ \mathcal{R} = \mathcal{Z}$.

$\mathcal{R}(\Sigma) : Triv(\Sigma) \to \mathcal{Z}(\Sigma)$ and

$$\begin{array}{ccc} Triv(\partial_{in}) & \xrightarrow{\mathcal{R}(\partial_{in})} & \mathcal{Z}(\partial_{in}) \\ {\scriptstyle Triv(M)}\downarrow & \overset{\mathcal{R}(M)}{\nearrow} & \downarrow{\scriptstyle \mathcal{Z}(M)} \\ Triv(\partial_{out}) & \xrightarrow[\mathcal{R}(\partial_{out})]{} & \mathcal{Z}(\partial_{out}) \end{array}.$$

Example (categorical number = 1):

$\mathcal{Z}_1, \mathcal{Z}_2 : \mathcal{B} \to \mathcal{C}$, $\mathcal{R} : \mathcal{B} \to \mathcal{C}^{\to}$, $s \circ \mathcal{R} = \mathcal{Z}_1$ and $t \circ \mathcal{R} = \mathcal{Z}_2$ means:

$\mathcal{R}(X \text{ object}) : \mathcal{Z}_1(X) \to \mathcal{Z}_2(X)$, and

$$\begin{array}{ccc} \mathcal{Z}_1(X) & \xrightarrow{\mathcal{R}(X)} & \mathcal{Z}_2(X) \\ {\scriptstyle \mathcal{Z}_1(f)}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mathcal{Z}_2(f)} \\ \mathcal{Z}_1(Y) & \xrightarrow[\mathcal{R}(Y)]{} & \mathcal{Z}_2(Y) \end{array}.$$

Anomalous TQFTs

Construction (Bulk+Relative=Anomalous):

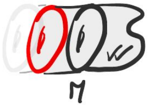
Bulk n-TQFT $\mathcal{Z} : Bord_n \to \mathcal{C} \rightsquigarrow \mathcal{Z}^{\to \mathbf{1}} : \begin{cases} Bord_n^{\to \emptyset} & \to & \mathcal{C}^{\to \mathbf{1}} \\ M \xrightarrow{W} \emptyset & \mapsto & \mathcal{Z}(M) \xrightarrow{\mathcal{Z}(W)} \mathbf{1} \end{cases}$

$+ \mathcal{R} : Bord_{n-1} \to \mathcal{C}^{\mathbf{1} \to}$ relative to $\mathcal{Z}$

$$\rightsquigarrow \mathcal{A} : \quad Bord_n^{\to\emptyset} \begin{array}{l} \xrightarrow{\mathcal{Z}^{\to \mathbf{1}}} \mathcal{C}^{\to \mathbf{1}} \\ \xrightarrow[s]{} Bord_{n-1} \xrightarrow{\mathcal{R}} \mathcal{C}^{\mathbf{1}\to} \end{array} \xrightarrow{\text{compose}} End_{\mathcal{C}}(\mathbf{1})$$

In our example (conjectural WRT):

$\mathcal{A}(M \xrightarrow{W} \emptyset) : k \xrightarrow{\mathcal{R}(M)} \mathcal{Z}(M) \xrightarrow{\mathcal{Z}(W)} k$ is a scalar.



$\mathcal{A}(\Sigma \xrightarrow{H} \emptyset) : Vect \xrightarrow{\mathcal{R}(\Sigma)} \mathcal{Z}(\Sigma) \xrightarrow{\mathcal{Z}(H)} Vect$ is vector space.

How to construct TQFTs easily

Cobordism Hypothesis (Baez–Dolan, Lurie, Grady–Pavlov, Ayala–Francis, Schommer-Pries):

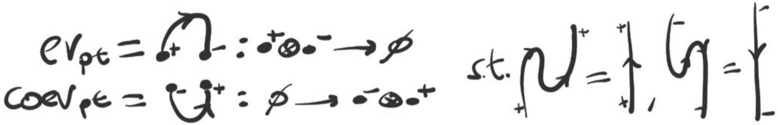
TQFTs are classified by their value at the point, which has to be a fully dualizable object:

$$ev_{pt} : Fun^{\otimes}(Bord_n^{fr}, \mathcal{C}) \;\; \xrightarrow{\sim} \;\; (\mathcal{C}^{fd})^{\sim}$$

is an equivalence of ∞-groupoids.

Idea (n=1):

The point $\bullet^+$ has a dual in $Bord_n$, $\bullet^-$:



And s.m. functors preserve dualizability data

Rigidity

Definition:

A **dual** of $X \in \mathcal{C}^{\otimes}$ is $(X^*, ev_X : X \otimes X^* \to \mathbf{1}_{\mathcal{C}}, coev_X : \mathbf{1} \to X^* \otimes X)$ s.t.

$\cap\cup = (ev_X \otimes Id_X) \circ (Id_X \otimes coev_X) \simeq Id_X = |$ and $\cup\cap \simeq |$.

Proposition:

Duals are unique up to essentially unique isomorphism.

Definition:

An **adjoint** of a k-morphism $F : X \to Y$ is
$(F^R : Y \to X, Rco(F) : F \circ F^R \Rightarrow Id_Y, Ru(f) : Id_X \Rightarrow F^R \circ F)$ s.t. $\cap\cup \simeq |$ and $\cup\cap \simeq |$.

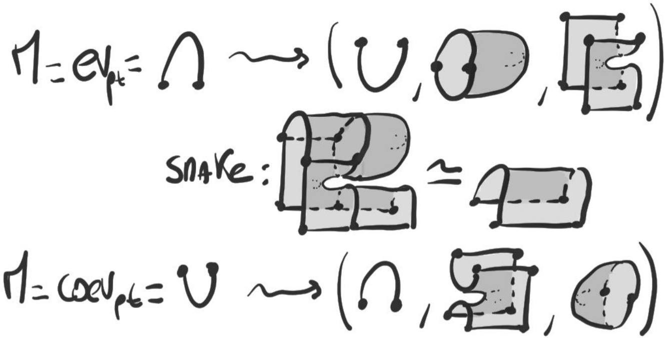
Proposition:

Adjoints are unique up to essentially unique isomorphism.

Higher dualizability

Observation:

For $0 \leq k < n$, every k-morphism $M : \partial_{in} \to \partial_{out}$ in $Bord_n$ has adjoint $(\overline{M}, M \times I/_{\partial_{in} \times I}, M \times I/_{\partial_{out} \times I})$.



Definition:

A s.m. category $\mathcal{C}$ **has duals** if for $0 \leq k < n$, all k-morphisms have adjoints.
An object $X \in \mathcal{C}$ is **fully dualizable** if X lies in a sub-n-category with duals.

Proof of the Cobordism Hypothesis

Injectivity ($\mathcal{Z}$ is determined by its value on the point):

Higher dualizability data of the point generates $Bord_n$
+ s.m. functors preserve higher dualizability data:

$[pt \mapsto X] \quad \Rightarrow \quad [\cap \mapsto ev_X \text{ and } \cup \mapsto coev_X]$

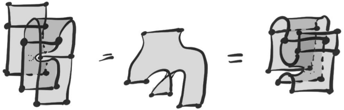
$\Rightarrow$ [ $\mapsto Ru(coev_X)$ and  $\mapsto Rco(coev_X)$]

And so on...

Surjectivity (a fully dualizable object induces a TQFT):

Set $\cap \mapsto ev_X$, $\cup \mapsto coev_X$,  $\mapsto Ru(coev_X)$ and so on...

Check that every relation in $Bord_n^{fr}$ holds in general for any fully dualizable object.
$\cap$ and $\cup$ biadjoints $\Rightarrow$ $coev_X$ and ev_X biadjoints.

 $\Rightarrow Rco(coev_X) \sim\sim Ru(ev_X)$.

Recap

Theorem (B–J–S–S):

$\mathcal{V}$ modular tensor category $\Rightarrow$ $\mathcal{V}$ is fully dualizable in $Alg_2(Pr)$.

Theorem (H.):

Unit inclusion $\eta : Vect \to \mathcal{V} \quad \rightsquigarrow \mathcal{A}_\eta \in Alg_2(Pr)^{\to}$
$\mathcal{A}_\eta$ fully dualizable $\Leftrightarrow$ $\mathcal{V} \simeq$ Free(rigid braided monoidal category).
Non-semisimple case: 2-dualizable and partially 3-dualizable.

Conclusion:

$$\begin{array}{lll} \mathcal{V} & \overset{C-H}{\rightsquigarrow} & \mathcal{Z}_{\mathcal{V}} : Bord_4^{fr} \to Alg_2(Pr) \\ & & \qquad\qquad + \\ \eta & \overset{C-H}{\rightsquigarrow} & \mathcal{R}_{\mathcal{V}} : Bord_3^{fr} \to Alg_2(Pr)^{\to} \\ & & \qquad\qquad = \\ & & \mathcal{A}_{\mathcal{V}} : Bord_4^{fr, \to\emptyset} \to Alg_1(Pr) \quad \overset{?}{\simeq} WRT_{\mathcal{V}} \end{array}$$

Non-semisimple case: $\mathcal{R}_{\mathcal{V}}$ partially defined, $\mathcal{A}_{\mathcal{V}} \overset{?}{\simeq}$ non-semisimple variants

Any questions?

Thank you!