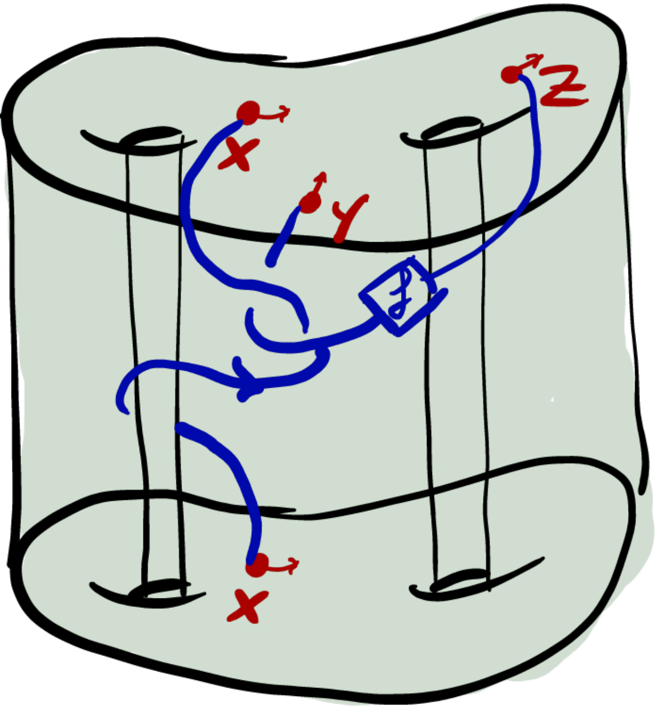
Non-semisimple fully extended skein TQFTs

GCS Doc & PostDoc Seminar



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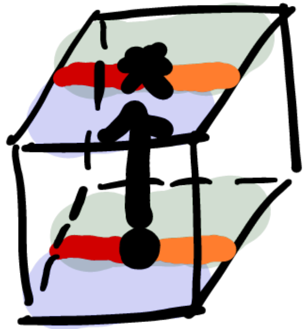
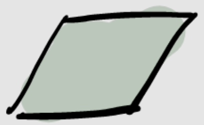
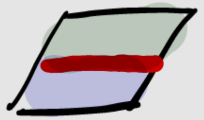


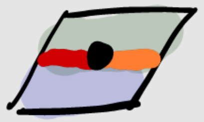
Context

$\mathbb{E}_2(\mathrm{Pr})$

Def BrTens (or $\mathrm{Alg}_2(\mathrm{Pr})$) has:

obj =  braided tensor cocomplete categories

1-m =  monoidal bimodule categories

2-m =  central bimodule categories

3-m =  cocontinuous bimodule functors

4-m = bimodule natural transformations

Thm (Brochier-Jordan-Safronov-Snyder)

$\mathcal{V}$ modular tensor category
↳ Lukas' setting

Then $\widehat{\mathrm{Proj}(\mathcal{V})}$ is 4-dualizable
↳ Luuk's setting

and invertible

Question What is the associated 4-TQFT

$Z : \mathrm{Bord}_4 \to \mathrm{BrTens}$?

What is an orientation structure?

I. Description of the 4-TQFT $\mathcal{Z}$

II. (2+1)-TQFTs at the boundary

Dimensions 0, 1 and 2:

Thm (Scheimbauer, Ayala-Francis)

Factorization Homology gives

$$\int_{-} \widehat{\mathrm{Proj}(\mathcal{V})} : \mathrm{Bord}_2 \longrightarrow \mathrm{BrTens}$$

↳ uniqueness in CH ⇒ Agree with $Z|_{\mathrm{Bord}_2}$

Thm (Cooke, semisimple / CAR $\mathcal{V} = \mathrm{Proj}(\mathcal{V})$) – in Progress (j.w. Brown)

Factorization Homology agrees with skein categories

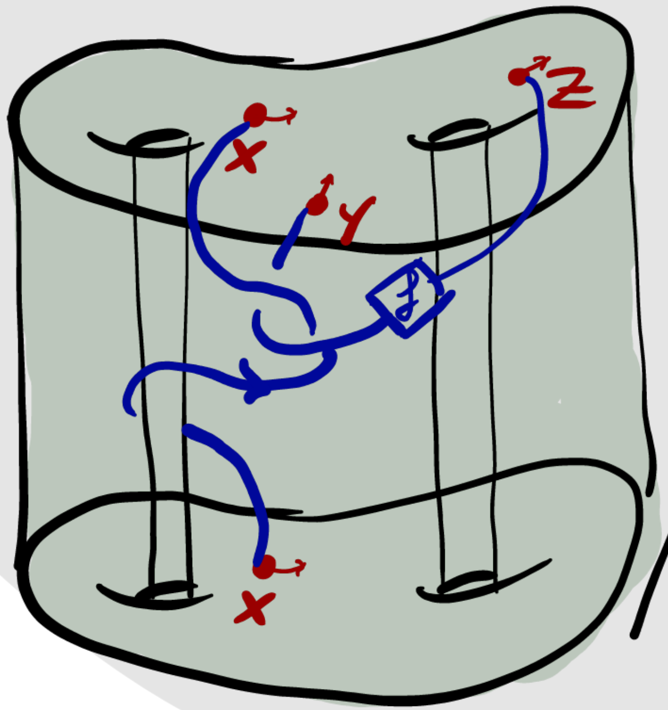
$$\int_{-} \widehat{\mathrm{Proj}(\mathcal{V})} \simeq \widehat{\mathrm{SkCat}^{\mathrm{adm}}_{\mathcal{V}}}(-)$$

Def: Admissible Skein categories $\mathrm{SkCat}^{\mathrm{adm}}_{\mathcal{V}}(\Sigma)$

obj:

at least one projective

mor:



/ skein Relations

Dimensions 2 and 3:

Thm (Walker, Tham, semisimple case) – in progress (follows Jordan–Safronov)

the assignment $\Sigma \mapsto \widehat{\mathrm{SkCat}}^{\mathrm{Adm}}_{\mathcal{V}}(\Sigma)$

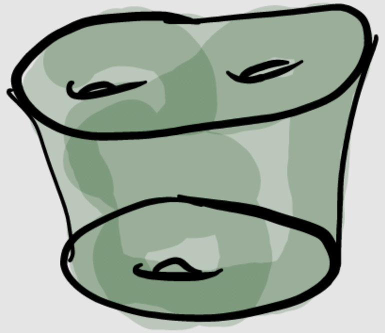
extends to categorified (2+1)-TQFT

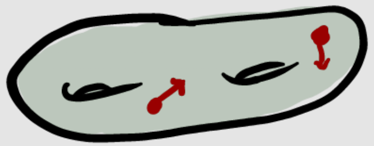
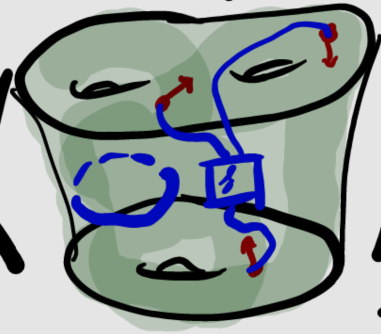
$$\mathrm{Sk}^{\mathrm{Adm}}_{\mathcal{V}} : \mathrm{Cob}_{2+1} \to h_1(\mathrm{Pr})$$

→ expectations: extends down and agrees with $Z|_{\mathrm{Bord}_3}$

Def: Admissible skein module functor

$$M:\Sigma \to \Sigma' \mapsto \mathrm{Sk}^{\mathrm{adm}}_{\mathcal{V}}(M) : \mathrm{SkCat}^{\mathrm{adm}}_{\mathcal{V}}(\Sigma) \otimes \mathrm{SkCat}^{\mathrm{adm}}_{\mathcal{V}}(\Sigma')^{op} \to \mathrm{Vect}$$



 $\otimes$  $\mapsto$ $\langle$  $\rangle$ / admissible skein relations

Dimensions 3 and 4:

Thm (Costantino–Geer–H.–Patureau-Mirand)

The assignment $M \mapsto SK_\nu^{adn}(M)$ extends to a $(3+1)$-TQFT
closed

$$SK_\nu^{adn} : \mathcal{C}ob_{3+1} \longrightarrow \mathcal{V}ect$$

↳ should extend down and agree with $\mathcal{Z}$

Idea: The 4-handle $S^3 \xrightarrow{B^4} \emptyset$ is mapped to the modified trace.

Other handles should satisfy handle cancelations
↳ defines them

~ Walker–Reutter

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Conjecture For $\mathcal{V}$ ~~modular~~ ribbon, finite + ..., the assignment

$$SK_{\mathcal{V}}^{adm} : \mathrm{Bord}_4^{or} \longrightarrow \widehat{\mathrm{BrTens}}$$

$$\bullet \longmapsto \mathrm{Proj}(\mathcal{V})$$

$$\Gamma \longmapsto \widehat{SKCat}_{\mathcal{V}}^{adm}(\Gamma \times I)$$

$$\Sigma \longmapsto \widehat{SKCat}_{\mathcal{V}}^{adm}(\Sigma)$$

$$M \longmapsto SK_{\mathcal{V}}^{adm}(M)$$

$W \longmapsto$ handle attachment formulas of CGHP

defines a fully extended 4-TQFT

I. Description of the 4-TQFT $\mathcal{Z}$

II. (2+1)-TQFTs at the boundary

3-manifold invariant

Idea: $M \xrightarrow{W} \emptyset$

$$\rightsquigarrow \quad \mathrm{Sk}_{\nu}^{\mathrm{adm}}(M) \qquad \mathrm{Sk}_{\nu}^{\mathrm{adm}}(W) \longrightarrow \mathbb{C}$$

depends mildly on W for ν modular

Thm (CGHP) Given $M \xrightarrow{W} \emptyset$ made of 2 and 4 handles

$$B(M, T \in \mathrm{Sk}_{\nu}^{\mathrm{adm}}(M)) := \frac{\mathrm{Sk}_{\nu}^{\mathrm{adm}}(S^4)^{\frac{3\sigma(W) - 2\chi(W) + |\pi_0(M)|}{2}}}{\mathrm{Sk}_{\nu}^{\mathrm{adm}}(\mathbb{CP}^2)^{\sigma(W)}} \, \mathrm{Sk}_{\nu}^{\mathrm{adm}}(W)(T)$$

does not depend on W.

Thm (CGHP) ν semisimple: $B(M, T = \emptyset) = \mathrm{WRT}(M)$

ν non-semisimple: $B(M, T) = \mathrm{DGGPR}(M, T)$

(up to a global factor $\sqrt{\mathrm{Sk}_{\nu}^{\mathrm{adm}}(S^4)}^{\,|\pi_0(M)|}$)

De Renzi Gainutdinov Geer Patureau Runkel

↳ expectations:

extends down

Boundary condition ∂

Def (Johnson-Freyd – Scheimbauer):
A boundary condition to
$$Z : \mathrm{Bord}_4 \to \mathcal{C}$$
is s.m functor
$$\partial : \mathrm{Bord}_3 \to \mathcal{C}^{\to} \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{C}$$
s.t
$$s(\partial) = \mathrm{Triv}$$
$$t(\partial) = Z|_{\mathrm{Bord}_3}$$

Thm (H.) The unit inclusion
$$\eta : \mathrm{Vect} \to \mathcal{V}$$
seen as an object of $\mathrm{BrTens}^{\to}$ is:

* 3-dualizable if $\mathcal{V}$ is semisimple
* 2-dualizable and almost 3 if $\mathcal{V}$ is nonsemisimple

enough to induce non-compact
$$\partial : \mathrm{Bord}_3^{nc} \to \mathrm{BrTens}^{\to}$$
every 3-cobordism have $\neq \emptyset$ outgoing ∂.

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2+1 theories

Def: $$\begin{array}{c}\mathrm{Bord}_3^{\text{filled},\,nc} \\ \rotatebox{90}{$\subseteq$} \\ \mathrm{Bord}_4^{\to\emptyset}\end{array} \quad \text{has} \quad \begin{cases} \text{obj: } p \xrightarrow{c} \emptyset \\ \text{1-m: } \gamma \xrightarrow{S} \emptyset \\ \text{2-m: } \Sigma \xrightarrow{H} \emptyset \\ \text{3-m: } M \xrightarrow{W} \emptyset \end{cases}$$

have outgoing boundary

$$\begin{tikzcd} p \arrow[r, "c"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "S \nearrow"] & \emptyset \arrow[d, equal] \\ p' \arrow[r, "c'"'] & \emptyset \end{tikzcd}$$

Def: $$A : \mathrm{Bord}_3^{\text{filled},} \longrightarrow \Omega\,\mathrm{BrTens}$$

$$X \xrightarrow{Y} \emptyset \longmapsto \mathbb{1} \xrightarrow{\partial(X)} Z(X) \xrightarrow{Z(Y)} \mathbb{1}$$

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Conjecture $\mathcal{V}$ (non-semisimple) modular:

$$
\begin{tikzcd}
\Omega \mathrm{Bord}_3^{\text{filled},} \arrow[rr, "\mathcal{A}"] \arrow[d] \arrow[drr, phantom, "\circlearrowright"] & & \Omega\Omega \mathrm{BrTens} \simeq \mathrm{Pr} \\
\widetilde{\mathrm{Cob}}_{3,2,1}^{nc} \arrow[r, "\mathrm{WRT}", "\mathrm{DGGPR}"'] & \mathrm{Cat} \arrow[ur, "\widehat{(-)}"'] &
\end{tikzcd}
$$

Thank you!