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Une approche aux invariants quantiques non-semisimples via l'algèbre supérieure

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Contents

Résumé

Dans cette thèse, nous étudions les théories des champs quantiques topologiques (TQFT) construites à partir d'une catégorie enrubannée. Nous sommes particulièrement intéressés par le cas non-semisimple. Notre angle d'attaque est de faire communiquer la topologie de basse dimension avec l'algèbre supérieure. Dans un sens, les constructions explicites basées sur des écheveaux guident l'algèbre supérieure vers les exemples qu'on sait intéressants. Dans l'autre, l'hypothèse du cobordisme prédit de nouvelles constructions.

Nous construisons des TQFTs de dimension 4 à partir de catégories non-semisimples et finies qui vérifient des conditions de non-dégénérescence. Cette contruction est un travail en collaboration avec Costantino, Geer et Patureau-Mirand. À l'inverse de la plupart des constructions non-semisimples précédentes, notre TQFT est bien définie sur tous les 4-cobordismes. Cette propriété était en fait prévisible par l'hypothèse du cobordisme. Notre construction est très explicite et nous étudions quelques exemples. Sous des hypothèses de non-dégénérescence supplémentaire, nous définissons des invariants de 3-variétés fermées décorées, qui sont calculés par notre TQFT sur une 4-variété bordante. Nous prouvons que ces invariants retrouvent les invariants de Lyubashenko renormalisés. Ils fournissent l'ingrédient de base des 3-TQFTs de DGGPR, qui sont des généralisations non-semisimples des TQFTs de Witten–Reshetikhin–Turaev.

Nous défendons l'idée que ce point de vue est très fructueux pour étudier ces théories non-semisimples à la WRT, et qu'il permet de les voir comme des TQFTs pleinement étendues. Quand la catégorie enrubannée $\mathscr{V}$ est modulaire, la (3+1)-TQFT que nous définissons plus haut est inversible. Il avait déjà été montré par Brochier, Jordan, Snyder et Safronov que la catégorie $\mathscr{V}$ est inversible vue comme un objet d'une 4-catégorie de catégories tressées. Nous nous attendons naturellement à ce que la TQFT pleinement étendue $\mathscr{Z}$ associée à $\mathscr{V}$ par l'hypothèse du cobordisme retrouve celle que nous avons décrite plus haut. De plus, on devrait pouvoir retrouver les TQFTs de DGGPR en appliquant les mêmes idées que plus haut. Plus précisément, nous nous attendons à ce qu'il existe une condition de bord pleinement étendue à $\mathscr{Z}$ qui, composée avec la théorie $\mathscr{Z}$ évaluée sur une variété bordante, retrouve DGGPR. Nous montrons que l'inclusion de l'unité dans $\mathscr{V}$, dont on s'attend à ce qu'elle soit associée à cette condition de bord, est, en effet, suffisamment dualisable. Nous montrons en fait qu'elle est quasiment, mais pas entièrement, 3-dualisable. Nous décrivons une version dite non-compacte de l'hypothèse du cobordisme, et définissons la notion associée d'objet dualisable non-compact. Ces objets donnent sous l'hypothèse du cobordisme des TQFTs partiellement définies, que nous appelons non-compactes. Cette dualisabilité partielle explique précisément pourquoi les TQFTs de DGGPR ne sont pas définies sur tous les 3-cobordismes. Nous conjecturons que l'hypothèse du cobordisme, appliquée à l'inclusion de l'unité et à $\mathscr{V}$, retrouve, par une procédure que nous détaillons, les TQFTs de DGGPR.

Sur les surfaces, la 4-TQFT $\mathscr{Z}$ est décrite par l'homologie de factorisation, qui est elle-même décrite par des catégories de modules sur les algèbres d'écheveaux internes de Brochier, Ben-Zvi et Jordan. Nous donnons une correspondance précise entre ces algèbres et les algèbres d'écheveaux à états de Lê, montrant en particulier qu'elles en sont une généralisation raisonnable. Notre preuve est explicite et montre directement que les algèbres d'écheveaux à états vérifient la propriété universelle qui définit les algèbres d'écheveaux internes. De plus, nous montrons des propriétés de recollement pour n'importe quelle catégorie enrubannée, un résultat que n'est pas connu pour d'autres généralisations des algèbres d'écheveaux à états.

Abstract

In this manuscript, we study Topological Quantum Field Theories built from a ribbon tensor category. We are particularly interested in the non-semisimple case. The main angle of this work is to make low-dimensional topology and higher algebra communicate. In one direction, explicit constructions from skein theory guide the higher algebra towards interesting examples. In the other, the cobordism hypothesis predicts new constructions.

We construct 4-dimensional TQFTs from non-semisimple finite tensor categories satisfying some non-degeneracy conditions. This construction is joint work with Costantino, Geer and Patureau-Mirand. Unlike most other non-semisimple constructions, this TQFT is defined on every 4-cobordism. This feature was actually predictable from the cobordism hypothesis. Our construction is very explicit and we study some examples. Under some extra non-degeneracy conditions, we also provide an invariant of decorated 3-manifolds which is computed by our TQFT on a bounding 4-manifold. We relate this invariant to the renormalized Lyubashenko's invariants. These invariants provide the building block of DGGPR 3-dimensional TQFTs, which are non-semisimple variants of the well-known Witten–Reshetikhin–Turaev TQFTs.

We argue that this point of view is very fruitful to understand these non-semisimple WRT theories and enables one to understand them as fully extended TQFTs. In the case where the ribbon category $\mathscr{V}$ is modular, the (3+1)-TQFT described above is invertible. It is also shown by Brochier, Jordan, Snyder and Safronov that the category $\mathscr{V}$ is invertible when thought of as an object of a 4-category of braided tensor categories. It is natural to expect that the TQFT $\mathscr{Z}$ associated to $\mathscr{V}$ by the cobordism hypothesis coincides with the one described above. Moreover, one should be able to recover DGGPR theories in a similar way, in a fully extended setting. More precisely, it is expected that there exists a fully extended boundary condition to $\mathscr{Z}$ which, when composed with $\mathscr{Z}$ on a bounding manifold, recovers DGGPR. We show that the unit inclusion, expected to be associated to this boundary condition under the cobordism hypothesis, is indeed sufficiently dualizable. Actually, we show that it is almost, but not entirely, 3-dualizable. We describe a so-called non-compact version of the cobordism hypothesis, and introduce the associated notion of non-compact dualizable object. Such objects give a partially defined, which we call non-compact, TQFT under the cobordism hypothesis. This explains precisely why the DGGPR theories are not defined on every 3-cobordim. We conjecture that the cobordism hypothesis applied on the unit inclusion and the modular category recovers, through a construction we describe, the non-semisimple WRT theories.

On surfaces, the fully extended 4-TQFT is known to give factorization homology, which is described as modules over the so-called internal skein algebras by Brochier, Ben-Zvi and Jordan. We relate these internal skein algebras to Lê's stated skein algebras and study some of their properties. We give an explicit proof, and show that stated skein algebras do satisfy the universal property defining internal skein algebras. In particular, we argue that internal skein algebras are a very reasonable generalization of stated skein algebras. Moreover, we show gluing properties of internal skein algebras in any ribbon category, a result which is not known for other generalizations of stated skein algebras.

Introduction

Quantum topology is a branch of mathematics inspired by modern quantum field theory. No physics will be discussed in this manuscript, but many tools are either motivated or inspired by physics.

Our main object of study will be Atiyah–Segal's notion of an n-dimensional Topological Quantum Field Theory (abbreviated n-TQFT). This notion encodes and formally defines mathematically the physical notion of a Quantum Field Theory in a very particular case where the theory is so-called topological, and does not depend on any metric structure. It contains two kinds of data. The first are vector spaces, called state spaces, associated to $(n-1)$-dimensional manifolds M. We think of them as the physical states on the space M. The second is linear map between these state spaces associated to an n-dimensional manifold whose boundary is decomposed into an incoming and an outgoing space. We call such an n-manifold a cobordism. We think of them as space-times, and of the linear map assigned to them as time evolution operators. This data should satisfy coherence relations for gluing and disjoint union. Atiyah defines in [Ati88], following Segal's Conformal Field Theories [Seg88], an n-TQFT to be a symmetric monoidal functor

$$\mathcal{Z} : \mathbf{Cob}_n^{\sqcup} \to \mathrm{Vect}_{\Bbbk}^{\otimes}$$

where $\mathbf{Cob}_n$ is the category of closed oriented smooth $(n-1)$-manifolds with n cobordisms between them, and $\mathrm{Vect}_{\Bbbk}$ is the category of vector spaces over some field $\Bbbk$ with linear maps between them.

Since Atiyah's pioneering paper, the construction of Topological Quantum Field Theories (TQFTs) of smooth compact oriented manifolds has flourished. Witten predicted in [Wit89] the existence of a 3-TQFT associated with Chern–Simons gauge theories. It was constructed mathematically a few years later by Reshetikhin and Turaev [RT91, Tur94]. They developed a theory which we will refer to as skein theory. This is a strong example of an event where ideas from physics gave birth to new insights in mathematics. These TQFTs give rise to computable invariants of 3-manifolds and representations of mapping class groups of surfaces that are very interesting from a purely mathematical perspective.

A *skein*, named after the English word for a piece of yarn, is a graph embedded in a 3-manifold. We will furthermore demand that the graph is oriented, framed (it comes equipped with a normal vector coming out of it at every point, or equivalently that the graph is actually made of ribbons) and colored. By colored we mean that every edge should be given a label from a fixed set of labels, and every vertex where edges meet should be given a label from a label set that depends on the labels of the adjacent edges. These set of labels actually arrange in a category $\mathcal{C}$: its objects are the labels for edges, and its morphisms give the labels for vertices. This category should moreover be endowed with the structure of a *ribbon category*. The historical example is when $\mathcal{C}$ is the category of representations of a simple algebraic group G, which is the gauge group of the Chern–Simons theory. In this context, a loop Γ in a 3-manifold M coloured by a G-representation

V corresponds to a function on the moduli space of flat G-principal bundles on M. It is the function obtained by taking monodromy along Γ and trace in V. Skeins are considered up to isotopy and *skein relations*: two skeins are equivalent when they correspond to the same function.

Witten–Reshetikhin–Turaev (abbreviated WRT) 3-TQFTs take as input some finite semisimple ribbon category satisfying some non-degeneracy condition, which they call *modular categories*. Their TQFTs were generalized to non-semisimple inputs in, for example, [KL01, BCGP16, DGG+22]. They also call their input categories modular, and we will adopt the convention to include both semisimple and non-semisimple categories under the word modular. These non-semisimple TQFTs have new properties, often proving more powerful than their semisimple analogs. For example, non-semisimple TQFTs have been shown to distinguish diffeomorphism types of homotopically equivalent lens spaces which were not distinguished by semisimple TQFTs.

Crane and Yetter defined in [CY93] a 4-TQFT from the same kind of data. It was noticed by Roberts [Rob95] that Crane–Yetter TQFTs could be described using skein theory, making the relation between the two theories more obvious, but not very clear yet. Something we haven't mentioned yet is that WRT theories are not well-defined on the usual category of cobordisms, but need some additional structure. The insight discovered by Walker [Walb] is that this data corresponds to the value of the Crane–Yetter theory on a bounding manifold. For example, given a closed 3-manifold M, one has to choose a bounding 4-manifold W. It can be seen as a cobordism $W : M \to \emptyset$ and via the 4-TQFT gives a linear map $\mathcal{Z}(W) : \mathcal{Z}(M) \to \Bbbk$. Evaluating it on a preferred element of $\mathcal{Z}(M)$ produces a scalar which coincides with the WRT 3-manifold invariant up to a renormalization scalar. We say that the Crane–Yetter theory describes the *anomaly* of WRT. This story is expected to extend in lower-dimensional manifolds. This is the point of view we will adopt and try to develop in this manuscript.

It is legitimate to ask whether we will forever keep on discovering new interesting TQFTs, or whether we will eventually have found them all. This raises the question of classifying TQFTs: finding them all, understanding their properties and their similarities. This is achieved in low dimensions, 0, 1 and 2, but becomes very hard in higher dimension. However, for a particular class of TQFTs, called *fully extended*, or fully local, there is a very strong classification result, the *Cobordism Hypothesis*. In these TQFTs cobordisms can be cut into small pieces and glued back together, and therefore the TQFT is completely determined by its local behavior. Mathematically, we encode this behavior by demanding that the TQFT give values to manifolds of dimensions lower that $(n-1)$ too and can be cut and paste along them. It was conjectured by Baez and Dolan in [BD95] that such fully extended TQFTs are classified by their value at the point, and what kind of value at the point come from a TQFT. Namely, the value at the point has to be a *fully-dualizable* object in the target higher category. An object X of a symmetric monoidal higher category $\mathcal{C}$ is said 1-dualizable if it has a dual in the usual sense. In particular there is an evaluation and a coevaluation 1-morphisms "witnessing" the duality. It is called 2-dualizable if these evaluation and coevaluation 1-morphisms have all adjoints. In particular they have unit and counit 2-morphisms witnessing the adjunctions and we can ask for them to have adjoints and so on. The cobordism hypothesis states that if X is n-dualizable, then there exists a unique-up-to-isomorphism fully extended framed n-TQFT $\mathcal{Z}_X$ which assigns the value X to the point. Making it an oriented n-TQFT corresponds to equipping X with an *$SO(n)$-homotopy-fixed-point structure*. Note that the terminology n-TQFT here is a bit loose and in general we will ask that $\mathcal{C}$ is a reasonable target "nVect" so that in particular restricting to closed $(n-1)$-manifolds and n-cobordisms we recover the usual notion of n-TQFT with values in Vect.

Despite giving such a satisfactory answer, the cobordism hypothesis has its drawbacks. It requires the language and techniques of higher category theory, which is a field still under development. Writing the statement of the cobordism hypothesis is difficult, as well as giving explicit examples. Given one of the objects that classify TQFTs, it is very hard to rebuild the associated TQFT. The proof is, of course, harder. A sketch of proof is given by Hopkins and Lurie in [Lur09b], and there are many proofs in progress or in review, but to the best of our knowledge this result is still a widely believed conjecture at this day.

Examples of n-dualizable objects in higher categories nVect have been studied in [DSS20] and [BJS21,BJSS21] for $n=3$ and $n=4$. However they seem to recover only the Turaev–Viro and Crane–Yetter state sum TQFTs. It follows from [DSS20] that Witten–Reshetikhin–Turaev TQFTs cannot be fully extended with values in the expected higher category 3Vect. The insight of Walker [Walb], see also [Fre, FT14], is that WRT theories have an anomaly, described by Crane–Yetter theories, and that they should be considered as boundary conditions to these. Therefore, WRT theories should be fully extended, but not with values in the usual 3Vect, rather in an arrow category of 4Vect. They should have trivial source and target corresponding to Crane–Yetter. The appropriate notion of arrow categories and "twisted" TQFTs is defined in [JS17].

Goals

This manuscript is motivated by the quest to bridge topological and higher-categorical constructions of TQFTs. In the first approach one explicitly defines an n-manifold invariant and works their way to a TQFT, adding structure or extra conditions as necessary. This is the approach behind WRT TQFTs and their non-semisimple variants, as well as Crane-Yetter TQFTs. The second approach classifies "vanilla" TQFTs (i.e. fully extended and without the extra structures/conditions of the examples above) using the Cobordism Hypothesis. To bridge the two approaches, we must answer the questions:

Question : *Can the Cobordism Hypothesis recover the interesting, hand-built examples we know?*

Having answered this question, one can ask furthermore:

Question : *Can these examples be generalized?*

We claim the answer to both questions is yes, though the answer to the first one is still conjectural.
In different words, we ask whether Crane–Yetter and WRT theories fall into that class of fully extended TQFTs that are classified by the cobordism hypothesis. Since the cobordism hypothesis classifies all such TQFTs, it should then be possible to decide whether we have found all similar TQFTs. The answer is not straightforward, especially in the case of WRT theories for which will need to use Walker's insight of seeing WRT theories as boundary conditions to Crane–Yetter.

The main object of study of this manuscript is a fully extended 4-dimensional TQFT $\mathcal{Z}_{\mathscr{V}}$ associated with a possibly non-semisimple ribbon category $\mathscr{V}$. However, *this fully extended 4-TQFT is yet to be defined.* It should be described by skein theory, should be an analogue of Crane–Yetter in the semisimple case and should recover WRT-type

TQFTs at its boundary in the modular case. There certainly should be constraints on $\mathscr{V}$, starting with finiteness, but they are not fully understood. In some sense, the goal of this manuscript is to present skein theory as a fully extended 4-dimensional TQFT. There are two ways to approach it.

The "top down" approach is to start from the top, i.e. construct a non-extended (3+1)-TQFT, and try to extend it down. In the semisimple case, this (3+1)-TQFT is already known: it is the Crane–Yetter TQFT. It is rather clear how one would extend this TQFT to lower-dimensional manifolds using skein categories, and is described informally in [Walb]. However, it is difficult to give formal statements in this context, as higher category theory is sometimes not well-suited to such hands-on approaches. Nevertheless, this understanding of what should happen in every dimension provides great guidance. It also provides many motivations and applications.

The "bottom up" approach is to find the value on the point and use the cobordism hypothesis. The cobordism hypothesis needs an n-dualizable object to produce a *framed* fully extended TQFT, and an $SO(n)$-homotopy-fixed-point structure to produce a fully extended TQFT. The name bottom-up refers to the fact that it is easier to show that an object is 2-dualizable rather than 3 or 4-dualizable, so historically these TQFTs are first defined in low dimensions, and then extended up. It is shown in [BJS21, BJSS21] that if $\mathscr{V}$ is either fusion (rigid finite semisimple with simple unit) or modular, then it is indeed 4-dualizable in an appropriate target category **BrTens**. Therefore $\mathcal{Z}_{\mathscr{V}}$ is defined, in those cases, as a framed TQFT. However, little is known about orientation structures. Moreover, the existence alone is not fully satisfying and one would like to identify the values of this TQFT and show they agree with the expected ones.

Though none of these approaches has been fully carried out, there are some partial results and many of the "shadows" of $\mathcal{Z}_{\mathscr{V}}$ are well-defined and understood. We will focus on three of these in this manuscript.

The first is the $(3+1)$-part, i.e. the first step in the top-down approach. It generalizes Crane–Yetter TQFTs to non-semisimple cases. It does not involve higher category theory and we construct it using only skein theory in Chapter 3. We check that it does coincide with Crane–Yetter TQFTs if $\mathscr{V}$ is semisimple. We extend it to non-semisimple cases under some finiteness and non-degeneracy conditions on $\mathscr{V}$. We believe these conditions are sufficient for the existence of the fully extended $\mathcal{Z}_{\mathscr{V}}$, and may provide good guidance.

The second is boundary conditions to $\mathcal{Z}_{\mathscr{V}}$. In the case where $\mathscr{V}$ is modular, the TQFT $\mathcal{Z}_{\mathscr{V}}$ is invertible, and one should be able to extract an anomalous 3-TQFT from a boundary condition to $\mathcal{Z}_{\mathscr{V}}$. These anomalous TQFTs are expected to recover WRT theories and their non-semisimple variants. We construct the anomalous 3-TQFT in the framed case using the cobordism hypothesis in Chapter 4. We give conjectures that it can be oriented, and how. We also give conjectures that it does recover WRT-type TQFTs.

The last is the 0-1-2-part of $\mathcal{Z}_{\mathscr{V}}$, i.e. the first step of the bottom-up approach. It has been described as factorization homology [Sch14a], skein categories [Coo23], and modules over internal skein algebras [BBJ18a, GJS23]. We compare these last internal skein algebras with stated skein algebras in Chapter 5.

Context

Low-Dimensional Topology: an overview on known constructions of TQFTs

In the 1980's Jones introduced in [Jon85] an extremely powerful polynomial invariants of links. It was very mysterious for a long time, and is still not well understood today. For example, we do not know whether there exists a knot with same Jones polynomial as the unknot!

Dimension 3

A milestone in quantum topology in dimension 3 is Witten's understanding that the Jones polynomial is related to a physical (and topological) Quantum Field Theory: the Chern–Simons theory with gauge group $SU(2)$. In particular, he predicts that (for any choice of "level" k, which for us corresponds to evaluating the variable at a k-th root of unity) the Jones polynomial gives rise to an invariant of closed 3-manifolds. These 3-manifold invariants were constructed by Reshetikhin and Turaev [RT91] using skein theory, and extended to 3-TQFTs in [BHMV95, Tur94]. They were first defined for categories of representations of quantum groups associated with the gauge group of the Chern–Simons theory, and were later extended to arbitrary semisimple modular categories.

These TQFTs are not plain TQFTs in Atiyah's sense: they are anomalous. They are defined on a category $\widetilde{\mathbf{Cob}}_3$ of cobordisms equipped with extra structure, which is usually described as an integer for 3-cobordisms and a Lagrangian in the first homology group of surfaces.

WRT TQFTs are known to extend once, and give values to the circle. To make sense of this statement, one first has to describe a bicategory $\widetilde{\mathbf{Cob}}_{321}$ of cobordisms of dimension 1-2-3 with extra structure, whose category of endomorphisms of the empty 1-manifold is $\widetilde{\mathbf{Cob}}_3$. Then one has to specify a target bicategory $2\mathrm{Vect}_{\Bbbk}$, whose category of endomorphisms of the unit object is $\mathrm{Vect}_{\Bbbk}$. Finally one has to give a symmetric monoidal functor $\mathrm{WRT} : \widetilde{\mathbf{Cob}}_{321} \to 2\mathrm{Vect}_{\Bbbk}$ whose restriction to endomorphisms of the unit coincides with the usual WRT theory. This is done, for example, in [De 17, Theorem 1.1.1] with $2\mathrm{Vect} := \widehat{\mathrm{Cat}}_{\Bbbk}$ the bicategory of Cauchy-complete $\Bbbk$-linear categories.

There have been many generalizations of WRT theories, and we will be interested in those with non-semisimple input. The first instances of these generalizations are due to Lyubashenko [Lyu95] and Hennings [Hen96]. They were extended to TQFTs in [KL01], though in a "connected" setting not fitted for our presentation. The first, and to our knowledge most powerful, example of non-semisimple analogs of WRT theories that fit in a more usual framework of TQFTs is due to Blanchet, Costantino, Geer and Patureau-Mirand [BCGP16]. It is however not perfectly standard either: now one needs a lot of extra structure on cobordisms, namely a cohomology class and an embedded skein that have to satisfy an admissibility condition.

In this manuscript, we will be mostly interested in the construction of [DGG$^+$22] which morally corresponds to the case of cohomology class 0 in [BCGP16]. They take as input a possibly non-semisimple modular tensor category, and produce a 3-TQFT for cobordisms equipped with a skein that satisfies an admissibility condition. If we restrict to the empty skein, which we will, they produce a TQFT defined on cobordisms with the same extra structure as WRT theories, but the admissibility condition means that it is not defined on every 3-cobordism. It is only defined for 3-cobordisms that have non-empty incoming boundary in every connected component. We call such TQFTs *non-compact*. Similarly to WRT theories, BCGP as well as DGGPR theories extend to the circle [De 17, De 21].

These non-semisimple TQFTs are very interesting from a topological point of view. They give sensitive 3-manifold invariants that distinguish homotopically equivalent lens spaces, and very interesting (projective) representations of Mapping Class Groups of surfaces. Because of the dependence on a cohomology class, BCGP theories naturally only give rise to a representation of the Torelli subgroup of the mapping class group, which are believed to be faithful. DGGPR theories (as well as BCGP theories for the null cohomology class) give rise to projective representations of the whole mapping class group for which no kernel is known for non-trivial choices of modular category.

Dimension 4

Less research has been done in dimension 4. A notable exception is due to Crane and Yetter [CY93] who defined invariants of closed smooth compact oriented 4-manifolds associated to any semisimple modular category. These invariants determine (3+1)-TQFTs, called Crane–Yetter TQFTs. However, the underlying 4-manifold invariants depends only on the Euler characteristic of the manifold and its signature (this was first observed for simply-connected 4-manifolds in [CKY97], and follows in general from work of Schommer-Pries that we recall below).

This construction was generalized to certain non-modular semisimple categories by Crane, Kauffman and Yetter [CKY97] (then the underlying 4-manifold invariants depends only on the Euler characteristic of the manifold and its signature for simply-connected 4-manifolds). In [BB18], Bärentz and Barrett generalize the CKY construction to pairs of finite semisimple categories $\mathcal{C} \to \mathcal{D}$ (the CKY construction corresponds to the inclusion of $\mathcal{C}$ in its Drinfeld center). The underlying 4-manifold invariants of these TQFTs, at least in a broad list of cases, are conjecturally related to the Euler characteristic, the signature and the fundamental group, see [BB18, Conjecture 8.1]. This implies that the invariant cannot distinguish exotic pairs of 4-manifolds, i.e. pairs of homeomorphic but non diffeomorphic manifolds. A different kind of quantum topology construction in dimension 4 has recently been defined by Beliakova and De Renzi in [BD23, BD]. They consider connected 4-manifolds which admit a handle decomposition without 3 and 4-handles, in particular with non-empty and connected boundary, and considered up to a suitable 2-equivalence relations.

Let us discuss what is known about the topological content of these TQFTs.

There is a general obstruction which is that, because of their gluing properties, the 4-manifold invariants obtained from TQFTs are usually multiplicative under connected sum, see [Reu23] for a discussion. It is shown by Gompf [Gom84], extending a previous result of Wall, that two compact orientable 4-manifolds (possibly with boundary) which are homeomorphic, become diffeomorphic after some finite sequence of connected sums with $S^2 \times S^2$. Therefore, a $(3+1)$-TQFT which is multiplicative under connected sum and assigns a non-zero value to $S^2 \times S^2$ cannot distinguish exotic pairs. Building on this idea, Reutter shows in [Reu23] that a 4-TQFT which is "semisimple" (for which $Z(S^3)$ and $Z(S^2 \times S^1)$ are semisimple Frobenius algebras, with structure induced by cobordisms) cannot detect exotic pairs. He shows moreover that once-extended 4-TQFTs are semisimple.

A similar restriction is that if one stabilizes a closed simply connected 4-manifold by a sequence of connected sums with $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$ then it can make it diffeomorphic to a connected sum of the form $\#_m\mathbb{CP}^2\#_n\overline{\mathbb{CP}^2}$ for some $m, n \in \mathbb{N}$. It follows, see [BB18, Lemma 3.12], that if a 4-TQFT has 1-dimensional state space on S^3 and invertible values on $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$, its underlying 4-manifold invariant depends only on the Euler characteristic and signature for simply-connected 4-manifolds.

For modular categories, the Crane–Yetter theories are known to be invertible. Remember that TQFTs form a monoid by taking tensor product, and a TQFT is called invertible if it is in this monoid (equivalently, if it assigns 1-dimensional vector spaces to 3-manifolds and invertible maps to 4-cobordisms). In this case the underlying invariant of 4-dimensional manifolds only depends on the Euler characteristic and the signature. It is shown in [Sch14b, Theorem 7.6] that invertible (3+1)-TQFTs are classified by two scalars, one of which corresponds to the signature σ and one to the half sum $\frac{\chi+\sigma}{2}$.

Higher Algebra: classifying fully extended TQFTs

Stating the Cobordism Hypothesis

The Baez–Dolan cobordism hypothesis [BD95] suggests that the correct notion of fully extended TQFTs will be much easier to study than their non-extended analogs, and be classified by their value at the point. This notion however is not immediate to pin down: a fully extended n-TQFT should be a symmetric monoidal (∞, n)-functor

$$\mathcal{Z} : \mathbf{Bord}_n \to n\mathrm{Vect}$$

from the (∞, n)-category of cobordisms in dimension less than or equal to n into a preferred target (∞, n)-category.

There are multiple well-developed notions of (∞, n)-categories and the one usually used in this context is Barwick's complete n-fold Segal spaces [Bar05, Lur09b, CS19]. Many notions we will be interested in are not fully developed in the context of (∞, n)-categories, and it is often sufficient to understand them for $(\infty, 1)$-categories. There is a well-developed theory of $(\infty, 1)$-categories called ∞-categories or quasi-categories [Lur09a, Lur]. Model categories also often provide a useful framework [JS17, Appendix A]. The (∞, n)-category of bordisms $\mathbf{Bord}_n$ is constructed in [Lur09b, CS19]. Note that it really is an (∞, n)-category, with $(n+1)$-morphisms corresponding to diffeomorphisms of n-cobordisms, and higher morphisms to homotopies.

The cobordism hypothesis is stated in [Lur09b, Theorems 2.4.6 and 2.4.26]. Note that Lurie only claims to give a sketch of proof, and a full proof is still under development. The first version, the framed cobordism hypothesis, gives an equivalence between *framed* fully extended TQFTs with values in a chosen nVect with *fully-dualizable* objects of nVect. An object is called m-dualizable if it lies in a sub-(∞, n)-category where every object has duals and every k-morphism has adjoints for $k < m$. It is called fully dualizable if it is n-dualizable. The second version, the oriented cobordism hypothesis, gives an equivalence between oriented fully extended TQFTs with values in nVect with fully-dualizable objects of nVect equipped with an $SO(n)$-homotopy-fixed-point structure.

The cobordism hypothesis provides a new angle to study and construct TQFTs. One "simply" has to find a fully dualizable object in a higher category, and it induces a (framed) fully extended TQFT. If one can show we have found all fully dualizable objects, then we have found all fully extended TQFTs.

The target category: Even Higher Morita categories

There is still one choice to be made, which is that of the target (∞, n)-category nVect. As the name suggest, a reasonable choice for $n = 1$ is Vect. To simplify let us talk about characteristic 0 below.

For $n = 2$, it is standard to consider a bicategory of linear categories. There are multiple choices here, as one could consider linear, Cauchy-complete, finitely cocomplete or cocomplete categories. We will mostly be interested in the bicategory $\mathbf{Pr}$ of cocomplete presentable linear categories studied in [BCJ15], and in its full subcategory Bimod of categories with enough compact-projectives (called tiny objects there). Note that there are completion procedures relating these categories, in particular the free cocompletion Free : $\mathrm{Cat}_{\mathbb{k}} \to \mathbf{Pr}$ from linear or Cauchy-complete categories to cocomplete categories, and the Ind-completion Ind : $\mathrm{Rex}_{\mathbb{k}} \to \mathbf{Pr}$ from finitely cocomplete to cocomplete categories. We will use them implicitly in this introduction. One could also consider the Morita bicategory of algebras, bimodules and bimodule morphisms. This bicategory is related to the bicategory of cocomplete categories by the Eilenberg–Watts theorem.

For $n = 3$, one would expect some 3-category of linear bicategories. This gets difficult to study, and an easier choice is to consider the subcategory of bicategories with only one object, which correspond to monoidal categories. TQFTs with this target are called 1-affine in [Joh21]. A standard choice for 3Vect is the 3-category **Tens** whose objects are (some kind of) monoidal categories, 1-morphisms are bimodule categories and 2 and 3 morphisms are functors and natural transformations, see [DSS20, BJS21].

For $n = 4$ one would expect the 4-category of linear 3-categories, but as above we will restrict to 3-categories with only one object and one 1-morphism. They correspond to braided categories. A standard choice for 4Vect is the 4-category **BrTens** whose objects are (some kind of) braided monoidal categories, 1-morphisms are monoidal-bimodule categories, 2-morphisms are bimodule categories and 3 and 4 morphisms are functors and natural transformations, see [BJS21, Joh21]. TQFTs with this target are called 2-affine in [Joh21].

The two examples above involving bimodule categories are usually called higher Morita categories. Giving the list of 0–4 morphisms is unfortunately not enough to define an $(\infty, 4)$-category, but there is a general construction of higher Morita categories. We will be interested in the *even higher Morita category* $\mathbf{Alg}_2(\mathbf{Pr})$ of $\mathbb{E}_2$-algebras in the bicategory $\mathbf{Pr}$ of cocomplete presentable categories.

Higher Morita categories were first defined in [Hau17] and [Sch14a]. Haugseng's construction takes as input a symmetric monoidal $(\infty, 1)$-category $\mathcal{S}$ and produce an $(\infty, n+1)$-category of $\mathbb{E}_n$-algebras and successive bimodules in $\mathcal{S}$. The construction is rather combinatorial and uses some variants of ∞-operads. Scheimbauer's construction is more geometric and uses factorization algebras. This is a very handy feature and makes explicit descriptions of TQFTs in this target easier, see [Sch14a]. However, it produces pointed bimodules, and as we will see this is going to be problematic.

This is not quite what we described above yet: for $\mathbb{E}_2$-algebras in categories it will only produce an $(\infty, 3)$-category, and we want an $(\infty, 4)$-category. Somehow higher Morita categories don't take into account the (non-invertible) 2-morphisms between categories. We need to go "even higher".

Even higher Morita categories are defined in [JS17]. There are two constructions, that extend Haugseng's and Scheimbauer's higher Morita categories. They take as input a symmetric monoidal (∞, k)-category $\mathcal{S}$ and produce an $(\infty, n+k)$-category $\mathbf{Alg}_n(\mathcal{S})$. Its 0 up to n morphisms are $\mathbb{E}_n$-algebras and successive bimodules in $\mathcal{S}$, and its $n+1$ up to

$n+k$ morphisms are higher morphisms in $\mathcal{S}$. For $\mathcal{S} = \mathbf{Pr}$ the bicategory of cocomplete presentable categories and cocontinuous functors, their construction applies and we can consider the even higher Morita 4-categories $\mathbf{Alg}_1(\mathbf{Pr})$ and $\mathbf{Alg}_2(\mathbf{Pr})$ as a models for 3Vect and 4Vect.

Dualizability in nVect

Dualizability in 2Vect is studied in [BCJ15]. Every object of Bimod is 1-dualizable, and finite semisimple categories are fully dualizable. They are all the 1- and 2- dualizable objects for all choices of 2Vect mentioned above [BDSV, Appendix A], except for $\mathbf{Pr}$ for which the answer is not known. It is however conjectured in [BCJ15, Remark 3.6] that there are no other examples in $\mathbf{Pr}$.

Dualizability in higher Morita categories is studied in [GS] for Scheimbauer's pointed model. They show that every object is n-dualizable in $\mathbf{Alg}_n^{pointed}(\mathcal{S})$. However, the pointing prevents any higher dualizability, and the only $(n+1)$-dualizable object is the unit. For this reason, we consider even higher Morita categories built from Haugseng's model below.

Dualizability in 3Vect is studied in [DSS20] for 3Vect a 3-category of finite rigid tensor categories. They show (in characteristic 0) that finite rigid tensor categories are 2-dualizable and that the fully dualizable objects are exactly the rigid finite semisimple categories (though it is enough to study fusion [FT21, Appendix D]). We will be interested in a larger choice for 3Vect, namely $\mathbf{Alg}_1(\mathbf{Pr})$. Dualizability in $\mathbf{Tens} = \mathbf{Alg}_1(\mathbf{Pr})$ is studied in [BJS21]. They show that cp-rigid categories (with enough compact-projectives and whose compact-projectives have duals) are 2-dualizable and that fusion categories are 3-dualizable. They make no claim that this describes all the fully dualizable objects, and indeed there are some non-semisimple invariants simply because semisimplicity is not a Morita invariant notion, see [BJS21, Remark 1.13].

Dualizability in 4Vect is studied in [BJS21, BJSS21]. They show that cp-rigid braided tensor categories are 3-dualizable and that braided fusion categories are fully dualizable in $\mathbf{BrTens} = \mathbf{Alg}_2(\mathbf{Pr})$ in [BJS21]. They show that possibly non-semisimple modular tensor categories are fully dualizable, and actually invertible, in [BJSS21, Theorem 1.1]. This is the first example of a non-semisimple fully dualizable object, which is rather surprising as it was expected that semisimplicity would be a necessary condition.

This last dualizability result raises a question:

Question : *What is the associated non-semisimple 4-TQFT?*

We claim that it corresponds to the (3+1)-TQFT we define in Chapter 3.

From Algebra to Topology: building TQFTs from the cobordism hypothesis

Using the cobordism hypothesis, every dualizability result mentioned above should have a topological counterpart, and induce a Topological Quantum Field Theory.

Note that partial dualizability results are still interesting: a k-dualizable object in nVect for $k < n$ will induce a symmetric monoidal functor $\mathbf{Bord}_k^{fr} \to n$Vect. However we would not want to call this a k-TQFT, because it doesn't have values in kVect. In particular, closed k-manifolds will not be assigned scalars, but something that looks like $(n-k-1)$-categories (with the convention that 0-categories are vector spaces and (-1)-categories are scalars). We will adopt the point of view that they are wishing to be

n-TQFTs, but don't have all the dualizability required and are only defined on $\mathbf{Bord}_k^{fr}$. There are many names for this kind of TQFTs. For example if $k = 3$ and $n = 4$, such a functor $\mathbf{Bord}_3^{fr} \to 4\mathrm{Vect}$ may be called a categorified 3-TQFT in [BJS21], a 3-TQFT with moduli level 1 in [FV15] or (though this is usually in a non-extended setting) a $(3+\varepsilon)$-TQFT.

Let us note too that every TQFT we were interested in in low-dimensional topology is oriented, and we are so far producing framed TQFTs. The oriented cobordism demands an $SO(n)$-homotopy-fixed-point structure, which we will often abbreviate $SO(n)$-structure. Very little is known about them.

Dimension 3: Turaev–Viro TQFTs

Fully extended 3-TQFTs obtained from fully dualizable objects of [DSS20] are expected to be related to Turaev–Viro state sum theories. We talk about tensor categories below, the general case where the unit is not simple is similar, see [BDSV].

Orientation structures in this context are studied in [DSS20, Sch14c]. They conjecture that pivotal categories are $SO(2)$-homotopy-fixed-points and that spherical categories are $SO(3)$-homotopy-fixed-points.

For the oriented 2-dualizable objects, so possibly non-semisimple pivotal tensor categories, it is natural to expect that the categorified 2-TQFT will assign to surfaces the admissible skein modules of [CGP] (for the ideal of projectives). Note that in the spherical case these were extended to non-compact (2+1)-TQFTs in [CGPVb] (so, not defined on all 3-cobordisms, but still much better that only defined on $\mathbf{Bord}_2$).

For the oriented 3-dualizable objects, so spherical fusion categories, the associated 3-TQFTs are expected to coincide with Turaev–Viro–Barrett–Westbury TQFTs [TV92, BW96]. These are already known to be extended TQFTs by [BK].

Dimension 4: Crane–Yetter TQFTs

Fully extended 4-TQFTs obtained from fully dualizable objects of [BJS21] are expected to be related to Crane–Yetter state sum theories.

Remember that every braided tensor category is 2-dualizable in an even higher Morita category. It follows from the construction of both framed and oriented TQFTs of [Sch14a] that an $SO(2)$-structure on an $\mathbb{E}_2$-algebra is a structure of $\mathbb{E}_2^{or}$-algebra. Note that the notations for $\mathbb{E}_n$-algebra are often confusing, and what is usually called a framed $\mathbb{E}_2$-algebra is what gives results on oriented manifolds. We denote $\mathbb{E}_2$ the operad associated with embeddings of framed disk, and $\mathbb{E}_2^{or}$ the operad associated with embeddings of oriented disks. For braided tensor categories, this corresponds to the data of a balancing.

Moreover, Scheimbauer computes explicitly the twice-categorified 2-TQFTs associated with a braided tensor category: it is given by [AFT17]'s factorizarion homology. This factorization homology has been computed by [Coo23] by skein categories when the input braided tensor category is the free cocompletion of a ribbon category. We will often call this case the semisimple case, though at this point this name is a little bit misleading. Note that these results apply for twice-categorified TQFTs with values in Scheimbauer's pointed even higher Morita category. One expects that Scheimbauer's pointed version should sit inside Haugseng's unpointed version, and that the description above holds in Haugseng's model too.

For 3-dualizable cp-rigid braided tensor categories, it is expected that a ribbon structure on the category of dualizable objects induces an $SO(3)$-structure. In the semisimple case, it is expected following work of Walker [Walb] that the induced categorified 3-TQFT corresponds to skein modules. A 3-cobordism between two surfaces induces a cocontinuous functor between their free cocompletions by considering every skein module in that 3-manifold with boundary conditions prescribed by the objects of the skein categories, see [Walb, GJS23, Tha21]. It is natural to expect that in the non-semisimple case the analogous definition with admissible skein modules of [CGP] will give the correct answer.

For 4-dualizable objects of [BJS21], i.e. fusion categories, little is known about orientation structures. However, following the work of Walker, one can conjecture that they correspond to a choice of trace on the ribbon category, which we do in Chapter 4. The associated fully extended 4-TQFTs are expected to coincide with Crane–Yetter TQFTs [CY93, CKY97]. They were already expected to be extended TQFTs in [Walb, Tha21].

For new 4-dualizable objects of [BJSS21], i.e. non-semisimple modular tensor categories, no low-dimensional topology construction preexisted the result. We propose one in Chapter 3.

Witten–Reshetikhin–Turaev TQFTs

It seems that we have found the TQFTs associated with every dualizable object mentioned, but a key player of low-dimensional topology is still missing.

Question : *Can we recover Witten–Reshetikhin–Turaev theories and its non-semisimple variants from the cobordism hypothesis?*

The naive answer appears to be no. Work of Douglas, Schommer-Pries and Snyder [DSS20] show that WRT theories extend to the point if and only if they are of Turaev–Viro type. Of course the answer depends on the choice of the target category 3Vect, and this result holds for finite rigid tensor categories. One approach that we will not pursue it to build an entirely new target category, see [Hen17, FT21]. Instead, we will use 4Vect.

The complete answer appears to be yes, but it is more complicated. It was never fully spelled out in written form, but has been communicated in talks and informal notes in the semisimple case [Fre, Walb]. In the non-semisimple case, this is expectations of Jordan and Safronov.

An obstacle to obtaining the Witten–Reshetikhin–Turaev theories from the cobordism hypothesis is that these theories are defined on a category $\widetilde{\mathbf{Cob}}_3$ of cobordisms equipped with some extra structure. It is well understood that this extra structure is the shadow of a bounding manifold: the integer is the signature of a bounding 4-manifold, and the Lagrangian is the kernel of the inclusion in a bounding 3-manifold. It was understood by Walker [Walb] that the anomaly of WRT theories is described by Crane–Yetter 4-TQFTs. In other words, this “shadow” of a bounding manifold is the value of Crane–Yetter on that bounding manifold. We only need to remember the signature of a 4-manifold because Crane–Yetter only depends on the signature (note that for this to be true one needs to pick the trace very well) and similarly in lower dimensions.

Freed and Teleman suggest that the WRT theory should be thought of as a boundary theory for the Crane–Yetter theory. An adequate description of relative field theory was given by Freed and Teleman [FT14] and formalized by Johnson-Freyd and Scheimbauer [JS17]. More precisely, consider an (∞, n)-category $\mathcal{C}$ and an n-TQFT $\mathcal{Z} : \mathbf{Bord}_n \to \mathcal{C}$. They construct an (∞, n)-category $\mathcal{C}^{\to}$ of so-called "oplax" arrows in $\mathcal{C}$ which has source and a target functors $s, t : \mathcal{C}^{\to} \to \mathcal{C}$. A relative theory to $\mathcal{Z}$, or a boundary condition to $\mathcal{Z}$, or an $(n-1)$-TQFT with anomaly $\mathcal{Z}$, or again an oplax-$\mathcal{Z}$-twisted $(n-1)$-TQFT is a symmetric monoidal functor

$$\mathcal{R} : \mathbf{Bord}_{n-1} \to \mathcal{C}^{\to}$$

whose target coincides with $\mathcal{Z}$ restricted to $\mathbf{Bord}_{n-1}$ and whose source coincides with the trivial TQFT.

Walker's insight is that composing the arrow $\mathcal{R}(-) : \mathbb{1} \to \mathcal{Z}(-)$ with the value of $\mathcal{Z}$ on a bounding manifold $\mathcal{Z}(-) \to \mathbb{1}$ recovers WRT theories for $\mathcal{Z}$ the Crane–Yetter theory and $\mathcal{R}$ the inclusion of the unit. This also answers [DSS20]'s objection: WRT does not give a value on the point simply because the point cannot be equipped with a bounding 1-manifold.

Another obstacle to obtaining non-semisimple DGGPR theories from the cobordism hypothesis is that these TQFTs are non-compact. They are defined on a restricted class of 3-cobordism that have non-empty incoming boundary in every connected component. We need to use a non-compact version of the Cobordism Hypothesis to work with this restricted category of cobordisms. This non-compact version appears as an intermediate step in the sketch of proof of the Cobordism Hypothesis proposed by Hopkins and Lurie [Lur09b]. Note that there is independent work in progress of Reutter–Walker in this direction.

Summing up, one should be able to recover the WRT theories (resp. their non-semisimple variants) from a 4-TQFT and a (resp. non-compact) boundary theory for this 4-TQFT, both of which are fully extended and obtained from the Cobordism Hypothesis. The 4-TQFT should be induced by the modular tensor category $\mathscr{V}$ and the boundary theory by the inclusion of the unit in $\mathscr{V}$, see the last slide of [Fre]. Then the WRT theory is obtained by composing the boundary condition with the 4-TQFT on a bounding manifold.

Note that this construction using a bounding manifold is what we need to answer the question above, namely compare with the existing constructions. It is also necessary to obtain a scalar invariant out of a 3-manifold. However, in some sense, it is not very fundamental. The "right" Topological Quantum Field Theory is the part which we called the boundary theory above. It does not assign a scalar to a 3-manifold, but an element in a one-dimensional vector space, the state space of the invertible Crane–Yetter TQFT, which is just as sensible from a physical point of view.

Results

Most new results are in Chapters 3, 4 and 5. Let us however mention that Theorem 1.5.10 and the detailed proof of Theorem 2.3.29 are new.

Chapter 3: Non-semisimple skein (3+1)-TQFTs

All results presented in this chapter are based on work in collaboration with Francesco Costantino, Nathan Geer and Bertrand Patureau-Mirand. They appeared in [CGHP].

We construct (3+1)-TQFTs that take as input a possibly non-semisimple finite ribbon tensor category $\mathcal{C}$ satisfying some non-degeneracy condition. Its state space on a 3-manifold M is the $\mathrm{Proj}(\mathcal{C})$-admissible skein module $\mathscr{S}_{\mathcal{C}}(M)$ of M is defined in [CGP]. We prescribe the TQFT on handle attachments, in a spirit very close to WRT, [DGG+22] and [CGPVb] theories.

In the case where $\mathcal{C}$ is either modular or fusion, in characteristic 0 in the latter case, the non-degeneracy conditions are satisfied and we expect to recover the (3+1)-part of the fully extended 4-TQFT associated with $\mathcal{C}$ seen as a fully dualizable object of **BrTens**. We show that we recover Crane–Yetter–Kauffman theories in the semisimple case. In general, we still expect the theory to be fully extended. Our conditions for existence may serve as a guideline for conditions of full dualizability. Actually, in characteristic 0 it is expected in [BJSS21] that a finite tensor category with semisimple Müger center is fully dualizable, and this indeed seems to be related to our non-degeneracy conditions. We also expect that our construction is an explicit realization of the work of Kevin Walker and David Reutter announced in [Wala].

The main non-degeneracy conditions we ask is for $\mathcal{C}$ to be unimodular and *chromatic-non-degenerate*. This is only enough to produce a non-compact TQFT, and for the full result we must ask $\mathcal{C}$ to be moreover *chromatic compact*. We work over an algebraically closed field $\Bbbk$ of any characteristic.

Theorem A (Theorem 3.4.4): *Let $\mathcal{C}$ be a finite unimodular ribbon tensor category. If $\mathcal{C}$ is chromatic non-degenerate, then the* $\operatorname{Proj}(\mathcal{C})$*-admissible skein modules extend to a non-compact* $(3+1)$*-TQFT*

$$\mathscr{S}_{\mathcal{C}} : \boldsymbol{Cob}^{nc}_{3+1} \to \mathrm{Vect}_{\Bbbk}$$

by explicit handle attachment formulas.

If $\mathcal{C}$ is moreover chromatic compact, then this non-compact theory extends to a (3+1)-TQFT

$$\mathscr{S}_{\mathcal{C}} : \boldsymbol{Cob}_{3+1} \to \mathrm{Vect}_{\Bbbk}$$

We use the work of Juhász [Juh18] to build our TQFT by giving the state spaces and explicit formulas for every handle attachment. Let us describe the main ingredients in these formulas.

The 4-handle will be given by the *modified trace* studied in [GPT09, GKP13, GKP11, GKP22], which is the usual key ingredient in non-semisimple skein theory. It is shown to exist and to be unique up scalar in a unimodular ribbon tensor category in [GKP22, Corollary 5.6]. It is shown to be equivalent to a linear form on the admissible skein mopdule of S^3 in [CGP, Theorem 3.1]. Our TQFT will depend on a choice of modified trace, but simply by a term depending only on the Euler characteristic, see Proposition 3.4.7.

The 3-handle will be given by the *cutting morphism* first introduced in [CGPT20] and already used in [CGPVb]. It is associated with the copairing of the modified trace and its existence uses the non-degeneracy of the modified trace provided by [GKP22].

The 2-handle will be given by the *chromatic morphism* studied in [CGPVa]. It plays the role of the Kirby color, and in the abelian case is another way to phrase the integral in the coend used in [DGG+22]. It is shown to exist in a finite unimodular ribbon tensor category, and actually in suitably finite spherical categories, in [CGPVa]. This chromatic morphism is not claimed to be unique in any way, but our construction does not depend on a choice, see Proposition 3.2.1.

The 1-handle will be given by the *gluing morphism*. This is a new notion which we introduce. The category $\mathcal{C}$ is called *chromatic non-degenerate* when a gluing morphism exists. It can indeed be rephrased by asking that a certain morphism involving the chromatic morphism and the double braiding is non-zero, see Proposition 3.1.8. Again, our construction will not depend on the choice of gluing morphism, see Proposition 3.4.1.

The 0-handle will be given by the *global dimension* of $\mathcal{C}$. It is only the choice of a scalar because the state space of the 3-sphere is one-dimensional. However, for a coherent choice to exist, it will impose conditions on the gluing morphism, namely that it is invertible, see Lemma 3.2.6. We call this condition chromatic compact because it is a strengthening of the above that yields to fully-defined TQFTs instead of non-compact ones.

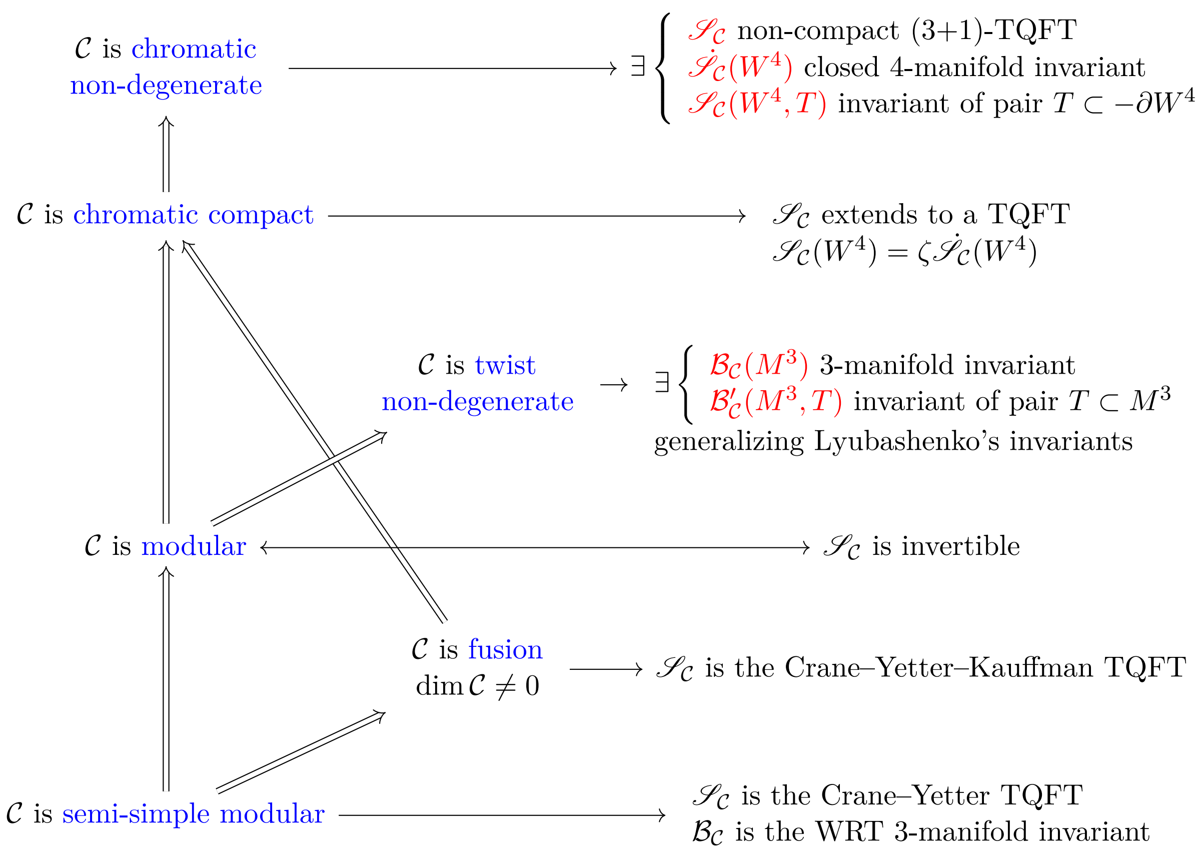


Figure 1: This figure represents different properties on a ribbon chromatic category $\mathcal{C}$ and their relationships and corresponding 3-manifold invariants and TQFTs. A category at the tail of a double arrow implies the property at the head of the arrow. For example, chromatic compact implies chromatic non-degenerate. A category at the tail of a single arrow implies the existence of the invariant at the head of the arrow. For example, a chromatic non-degenerate category gives rise to a non-compact $(3+1)$-TQFT $\mathscr{S}_{\mathcal{C}}$.

A particularity of skein TQFTs is that vectors in the TQFT space are represented by $\mathcal{C}$-colored links or ribbon graphs in 3-manifolds. Hence the abstract TQFT comes with an invariant of pair (W, T) where T is a $\mathcal{C}$-colored links or ribbon graph in the boundary of W.

We show in Theorem 3.3.2 that if $\mathcal{C}$ is *twist non-degenerate*, now without the chromatic non-degenerate assumption, we can define an invariant of 3-manifolds equipped with an admissible skein using the chromatic morphism on a surgery link. We show in Theorem 3.4.9 that if the category is both twist non-degenerate and chromatic non-degenerate, this 3-manifold invariant is indeed given by renormalizing the contribution of W in the invariant of pairs (W, T) for W a 4-manifold bounding M made of 2-handles. We show that it also coincides with [DGG$^+$22]'s renormalized Lyubashenko's 3-manifold invariant.

These invariants might be much stronger than the TQFT itself. Indeed the invertible CY TQFTs are known to only depend on two complex numbers [Sch14b, Theorem 7.6], but given the empty skein they recover the WRT invariant of the boundary of a 4-manifold. This extra data corresponds to a boundary condition for the TQFT. We will conjecture in Chapter 4 that such a boundary condition (in a fully extended setting) on the invertible TQFT associated with a non-semisimple modular category will recover the non-semisimple (2+1)-TQFT from [DGG$^+$22]. Assuming the expectations above, Theorem 3.4.9 can be seen as a partial confirmation of this conjecture.

We also expect our construction to be related to work of Beliakova and De Renzi [BD23, BD], though a complete comparison appears to need the fully extended version of our TQFTs. They only give values to 4-dimensional 2-handlebodies, so it is natural that their construction is defined for more general input (in particular, they don't need our chromatic non-degenerate condition, which we need precisely to define the missing handles). Our construction would give the extension to every 4-manifolds, when it exists.

One advantage of our construction is that it is elementary. The techniques are based on algebraic data, e.g. modified traces and chromatic morphisms, which are easy to formulate with low-level technology using monoidal categories. Their properties are easily represented graphically and most of our proofs reduce to diagrammatic ones. We can easily study the TQFTs we produce: we characterize their invertibility, study their behavior under connected sum and provide some examples.

Figure 1 displays all the different constructions and what condition they impose on the category $\mathcal{C}$.

Chapter 4: Anomalous theories

The results of this chapter appeared in [Haï].

This chapter is motivated by the quest to obtain non-semisimple WRT theories from the cobordim hypothesis. We give the first step towards executing the program described above.

Remember that Walker, Freed and Teleman predict that the inclusion of the unit η in a semisimple modular tensor category $\mathcal{V}$ will induce a 3-dimensional boundary condition to the 4-TQFT associated with $\mathcal{V}$, and that together they recover WRT TQFTs. In the non-semisimple case, Jordan and Safronov expect that the same story will apply, except that the boundary condition will only be partially defined, which we call non-compact.

We first show the relevant dualizability of the unit inclusion, and then, conjecturally, reconstruct WRT-type TQFTs from the induced relative theory.

Dualizability results

We recall Lurie's sketch of proof of the non-compact cobordism hypothesis and introduce the corresponding notion of non-compact-n-dualizable object. We use the framework of [JS17] to prove that the unit inclusion in a semisimple (resp. non-semisimple) modular tensor category is 3-dualizable (resp. non-compact-3-dualizable) and therefore induces a (resp. non-compact) relative 3-TQFT under the Cobordism Hypothesis.

Our main theorem is the following.

Theorem B (Theorems 4.2.12, 4.2.15 and 4.2.14): *Let $\mathcal{V} \in$ **BrTens** be a braided tensor category, and $\mathcal{A}^\flat_\eta$ the object of [JS17]'s oplax arrow category* $\mathbf{BrTens}^{\to}$ *induced by the inclusion of the unit. Then:*

1. *$\mathcal{A}^\flat_\eta$ is always 2-dualizable.*

If $\mathcal{V}$ has enough compact-projectives, then:

2. *$\mathcal{A}^\flat_\eta$ is non-compact-3-dualizable if and only if $\mathcal{V}$ is cp-rigid.*
3. *$\mathcal{A}^\flat_\eta$ is 3-dualizable if and only if $\mathcal{V}$ is the free cocompletion of a small rigid braided monoidal category if and only if $\mathcal{V}$ is cp-rigid with compact-projective unit.*

We give explicitly the dualizability data of $\mathcal{A}^\flat_\eta$. For the 1-dualizability data we obtain mates of the bimodule categories $\mathcal{M}_\eta$ and $\mathcal{M}_T$ induced by the unit and the tensor product of $\mathcal{V}$. For the 2-dualizability data we obtain mates of the functors η, T, and $T_{bal} : \mathcal{V} \underset{\mathcal{V}\boxtimes\mathcal{V}}{\boxtimes} \mathcal{V} \to \mathcal{V}$ which is induced by T. For the 3-dualizability data, we obtain mates of the unit and counits of the right adjunctions of these functors.

Our arguments work any braided monoidal functor $F : \mathcal{V} \to \mathcal{W}$, though we loose that they are criteria, see Theorem 4.2.22. The object $\mathcal{A}^\flat_F \in \mathbf{BrTens}^{\to}$ induced by F is always 2-dualizable. It is non-compact-3-dualizable as soon as $\mathcal{V}$ and $\mathcal{W}$ are cp-rigid. In this case, it is 3-dualizable if and only if F preserves compact-projectives.

On the examples of interest, the main theorem gives:

Corollary C: *Let $\mathscr{V}$ be a modular tensor category in the sense of [DGG⁺22], $\mathcal{V} :=$* Ind$(\mathscr{V})$ *its Ind-completion, and $\mathcal{A}^\flat_\eta$ induced by the unit inclusion in $\mathcal{V}$. Then:*

1. *If $\mathscr{V}$ is semisimple, $\mathcal{A}^\flat_\eta$ is 3-dualizable.*
2. *If $\mathscr{V}$ is non-semisimple, $\mathcal{A}^\flat_\eta$ is not 3-dualizable, but is non-compact-3-dualizable.*

Freed and Teleman study the dualizability of the unit inclusion in the 3-category $\mathbf{Alg}_1(\mathrm{Rex}_{\mathbb{C}})$ in [FT21, Theorem B]. They show that $\mathscr{V} \in \mathbf{Alg}_1(\mathrm{Rex}_{\mathbb{C}})$ is finite semisimple if and only if $\mathcal{M}_\eta$ is 2-dualizable, i.e. lies in a subcategory with duals (the forward implication is [DSS20]).

We show the analogous statement one categorical number higher in Theorem 4.2.16. Suppose $\mathcal{V} \in$ **BrTens** has enough compact-projectives. Then $\mathcal{A}_\eta$ is 3-dualizable if and only if $\mathcal{V}$ is finite semisimple.

Using this result and to-appear results of Will Stewart, we can show that the free cocompletion of a non-semisimple rigid braided monoidal category is not 4-dualizable. One should really take Ind-completions.

Reconstructing WRT-type TQFTs

We explain how one can obtain a theory defined on filled cobordisms from a relative theory together with its bulk theory. We give conjectures on how these theories can be oriented in the case of possibly non-semisimple modular tensor categories. We state the conjectures that these recover the WRT and [DGG+22] theories.

By Corollary C, we obtain under the Cobordism Hypothesis a framed, possibly non-compact, 3-TQFT $\mathcal{R}_{\mathcal{V}} : \mathbf{Bord}_3^{fr} \to \mathbf{BrTens}^{\to}$ relative to the 4-TQFT $\mathcal{Z}_{\mathcal{V}}$ associated with $\mathcal{V}$.

We give conjectures that these theories can be oriented.

Conjecture A (Conjectures 4.3.11 and 4.3.13): *Let $\mathscr{V}$ be a modular tensor category and $\mathcal{V} = \mathrm{Ind}(\mathscr{V})$, which is invertible and in particular 5-dualizable by [BJSS21]. Then,*

1. *The ribbon structure of $\mathscr{V}$ induces an $SO(3)$-homotopy-fixed-point structure on $\mathcal{V}$.*
2. *The ribbon structure of η induces an $SO(3)$-homotopy-fixed-point structure on $\mathcal{A}_\eta^\flat$.*
3. *A choice of modified trace $\mathtt{t}$ on $\mathscr{V}$ induces an $SO(4)$-homotopy-fixed-point structure on $\mathcal{V}$.*
4. *Exactly two modified traces induce an $SO(5)$-homotopy-fixed-point structure on $\mathcal{V}$, namely $\pm\mathscr{D}^{-1}\mathtt{t}$ for $\mathscr{D}$ a square root of the global dimension of $\mathscr{V}$.*

Let us include here a conjecture that reflects the expectations from the previous chapter, which we can state now that we have conjectured orientation structures on $\mathcal{V}$.

Conjecture B: *Choose $\mathtt{t}$ a modified trace on $\mathscr{V}$ and let $\mathcal{Z}_{\mathcal{V}}$ be the associated oriented 4-TQFT. Then one has a natural isomorphism*

$$\mathscr{S}_{\mathscr{V}} \simeq h_1\Omega^3\mathcal{Z}_{\mathcal{V}}$$

between the (3+1)-TQFT defined in Chapter 3 and the (3+1)-part of $\mathcal{Z}_{\mathcal{V}}$.

We construct the "anomalous" theory $\mathcal{A}_{\mathcal{V}} : \mathbf{Bord}_3^{filled} \to \mathbf{Tens}$ associated with $\mathcal{R}_{\mathcal{V}}$ and $\mathcal{Z}_{\mathcal{V}}$. It is defined on a 3-category of cobordisms equipped with a filling, i.e. a bounding higher manifold, which degenerates to the more usual $\widetilde{\mathbf{Cob}}$. The anomalous theory is non-compact when $\mathcal{R}_{\mathcal{V}}$ is.

We can now state the main conjectures.

Conjecture C (Conjecture 4.3.15): *Let $\mathscr{V}$ be a semisimple modular tensor category with a chosen square root of its global dimension. The anomalous theory $\mathcal{A}_{\mathcal{V}}$ induced by the associated oriented 4-TQFT $\mathcal{Z}_{\mathcal{V}}$ and oriented oplax-$\mathcal{Z}_{\mathcal{V}}$-twisted 3-TQFT $\mathcal{R}_{\mathcal{V}}$ recovers the once-extended Witten–Reshetikhin–Turaev theory as its 321-part.*

Conjecture D (Conjecture 4.3.16): *Let $\mathscr{V}$ be a non-semisimple modular tensor category with a chosen modified trace $\mathtt{t}$ and square root of its global dimension. The anomalous theory $\mathcal{A}_{\mathcal{V}}$ induced by the associated oriented 4-TQFT $\mathcal{Z}_{\mathcal{V}}$ and oriented non-compact oplax-$\mathcal{Z}_{\mathcal{V}}$-twisted 3-TQFT $\mathcal{R}_{\mathcal{V}}$ recovers the once-extended DGGPR theory for cobordisms with trivial decoration as its 321-part.*

We show that the values on the circle coincide by computing $\mathcal{R}_\mathcal{V}$ explicitly and using the factorization homology description of $\mathcal{Z}_\mathcal{V}$. For higher-dimensions, this second description stops (remember that surfaces are filled) and we need more values of $\mathcal{Z}_\mathcal{V}$.

An interesting consequence of these conjectures is that since this is the 321-part of $\mathcal{A}_\mathcal{V}$, these theories actually extend down. Not to the point, as $\mathcal{A}_\mathcal{V}$ is not defined on the point which is not bordant, but to the pair of points S^0.

Note that we use both the oplax-twisted 3-TQFT and the 4-TQFT in this construction. Therefore, not every case of 3-oplax-dualizability in Theorem B induces such a theory. We also need $\mathcal{V}$ to be 4-dualizable. The assumption that $\mathcal{Z}_\mathcal{V}$ is invertible however can be dropped. The anomalous theory would then strongly depend on the filling, and give interesting invariants of 4-manifolds with boundary.

Chapter 5: Stated versus internal skein algebras

The results of this chapter appeared in [Haï22].

The goal of this chapter is to draw a comprehensive link between stated skein algebras of [Lê18, CL22] and internal skein algebras of [BBJ18a, GJS23], including their structures and properties. The interest of such a link is to benefit from the nice features of both sides. Stated skein algebras are defined very explicitly, and have numerous applications. Internal skein algebras are much more theoretical, they are defined for all ribbon categories of coefficients, and their basic (in particular, excision) properties derive formally.

The main result of this chapter is the following:

Theorem D (Theorem 5.2.4 and Proposition 5.3.6): *Let $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$–comod^{fin} and Σ a compact oriented surface with a right boundary edge. Then one has an isomorphism of $\mathcal{O}_{q^2}(SL_2)$-comodule-algebras*

$$A_\Sigma \simeq (\mathscr{S}(\Sigma))^{bop}$$

between [BBJ18a, GJS23] internal skein algebras and braided opposites of stated skein algebras.

This result was already known as folklore, though the braided opposite was missing. It is stated in a slightly weaker form (only as algebras in $\mathrm{Vect}_\Bbbk$) in [LY22, Theorem 4.4], [LS, Theorem 9.1] and [GJS23, Remark 2.21]. Both algebras are also known to be isomorphic to the Alekseev–Grosse–Schomerus moduli algebras. In [Fai, Theorem 5.3] and [Kora] the authors provide an $\mathcal{O}_{q^2}(SL_2)$-comodule algebra isomorphism between these AGS algebras and stated skein algebras. In [BBJ18a, Section 7] the authors provide an isomorphism between AGS algebras and internal skein algebras. It is not explicitly given and might be where the braided opposite hides, i.e. it might be that they deform the representation variety in the direction of the opposite Poisson structure. We introduce the relevant notion of left against right internal skein algebras to resolve this issue.

Our proofs are mostly based on skein theory and we do not require the reader to be familiar with the formalism of higher categories. We give explicitly the natural isomorphism exhibiting the stated skein algebra as the left internal skein algebra, and we show that right internal skein algebras are braided opposites of left internal skein algebras.

Using the well-developed properties of internal skein algebras, we get:

Corollary E: *There is an equivalence of categories*

$$SK_{\mathcal{V}}(\Sigma) \simeq mod_{\mathcal{O}_{q^2}(SL_2)\text{–}comod} - \mathscr{S}(\Sigma)$$

between the free cocompletion (or presheaf category) of the skein category and the category of right modules over the stated skein algebra, internally to $\mathcal{O}_{q^2}(SL_2)$*-comodules.*

We also extend the definition of internal skein algebras to multiple boundary edges. We show that they still agree with stated skein algebras with appropriate braided opposites inserted in Theorem 5.3.28. In the proof, we use a half-twist on the category $\mathcal{O}_{q^2}(SL_2)$*–comod*^{fin}.

It follows that stated skein algebras generalize to any ribbon category $\mathcal{C}$, because internal skein algebras do. Note though they will be algebra-objects in the free cocompletion of $\mathcal{C}$ which is maybe unusual. It becomes more familiar if $\mathcal{C}$ is the category of comodules over a semisimple coribbon Hopf algebra. Then they will be H-comodule algebras.

We prove in Theorem 5.3.32 that multi-edges internal skein algebras always satisfy excision properties similar to those on stated skein algebras. To fully relate the excision properties, we need to use the half-twist on $\mathcal{O}_{q^2}(SL_2)$*–comod*^{fin} introduced above. We argue in particular that there was a non-trivial choice which corresponds to this half-twist made in the definition of stated skein algebras, actually in the way of expressing their cutting properties, see Remark 5.3.34.

Chapter 1

Low-dimensional Topology

In this first preliminary chapter we recall the basics of skein theory and the constructions of skein-theoretic TQFTs.

In Section 1.1, we introduce the central example of skein theory: the Kauffman bracket skein relations. In Section 1.2, we recall Reshetikhin–Turaev theory of graphical calculus in ribbon categories [RT91, Tur94]. In Section 1.3, we recall the definition of a TQFT and Juhasz's presentation of the cobordism category by generators and relations [Juh18]. These sections can safely be skipped by the expert.

In Section 1.4, we recall the skein-theoretic construction of Crane–Yetter and Witten–Reshetikhin–Turaev TQFTs associated with semisimple modular tensor categories. We take a slightly unusual point of view and our exposition serves as an illustration of the construction of Chapter 3 and of the idea of boundary conditions of Chapter 4.

In Section 1.5, we recall the tools from non-semisimple skein theory and [DGG$^+$22] TQFTs. Again we emphase the idea that they are boundary conditions to the (3+1)-TQFTs defined in Chapter 3. To this end, we prove Theorem 1.5.10 which relates DGGPR state spaces to admissible skein modules, and which is new.

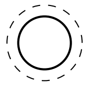
1.1 Kauffman Bracket Skein Theory

Kauffman introduced in 1987 [Kau87] a state model for the Jones polynomial. It is an invariant of unoriented framed links that one can compute from a link diagram by iteratively reducing this diagram using the Kauffman bracket skein relations

These relations are local, they change a small part of the link, and, on top of allowing one to compute the Jones polynomial easily, they are very straightforward to generalize for links not living in S^3.

1.1.1 Kauffman Bracket Skein Modules and Algebras

The Kauffman bracket skein modules, introduced in [Prz91, Tur88] have been studied extensively and are still an active research subject [GJS23, Kin, DKS]. Their integral version are even richer, but we will mainly focus on the skein module over a field in this manuscript.

Definition 1.1.1: Let M be an oriented 3-manifold. Let $Link(M)$ denote the $\mathbb{Z}[q, q^{-1}]$-module freely generated by isotopy classes of unoriented framed links in M, i.e. smooth closed sub-1-manifolds equipped with a trivialization of their normal bundle. For every oriented embedding $\phi : \mathbb{D}^3 \hookrightarrow M$ and framed link L in M intersecting the image of ϕ exactly in the form  or  (with blackboard framing), we introduce the relations

where the left hand side represents $L \in Link(M)$ and the right hand side another linear combination of links in M, which are as depicted inside the image of ϕ, and coincide with L outside. The *Kauffman Bracket Skein Module* $Sk(M)$ of M is the quotient of $Links(M)$ by these relations.
When $M = S^3$, using these relations every link reduces to a scalar times the empty link, and this scalar is uniquely defined. Hence, $Sk(S^3)$ is isomorphic to $\mathbb{Z}[q, q^{-1}]$, generated by the empty link. The Kauffman bracket link invariant is the quotient map $Links(S^3) \to Sk(S^3) \simeq \mathbb{Z}[q, q^{-1}]$.
When $M = \Sigma \times (0, 1)$ is a thickened surface, its skein module has a natural algebra structure given by superposition $\Sigma \times (0,1) \sqcup \Sigma \times (0,1) \hookrightarrow \Sigma \times (0,1)$ induced on the height coordinate by $(0,1) \sqcup (0,1) \simeq (\frac{1}{2}, 1) \sqcup (0, \frac{1}{2}) \hookrightarrow (0,1)$. We denote this algebra $Sk(\Sigma)$, the *skein algebra* of the surface Σ.
When M is a manifold with boundary, its skein module is naturally a module over the skein algebra of its boundary. ♢

Proposition 1.1.2 (Corollary 4.1 in [SW07]): *The $\mathbb{Z}[q, q^{-1}]$-module $Sk(\Sigma)$ is free with basis the set of isotopy classes of simple unoriented framed links (without double points nor trivial circles) in Σ.*

Contrary to skein modules of thickened surfaces that are not finitely generated, Witten predicted that the skein module of a closed 3-manifold seen over $\mathbb{Q}(q)$ is finite dimensional. This was recently proven in [GJS23]:

Theorem 1.1.3 (Gunningham–Jordan–Safronov): *Let M be an oriented closed 3-manifold. The $\mathbb{Q}(q)$-vector-space $Sk(M, \mathbb{Q}(q)) := Sk(M) \underset{\mathbb{Z}[q,q^{-1}]}{\otimes} \mathbb{Q}(q)$ is finite-dimensional.*

However, computing the exact dimension of skein modules is still a difficult problem. Recent developments in this direction can be found in [Kin, DKS].
A similar behavior is expected for 3-manifolds with boundary. The following is sometimes called the Detcherry conjecture.

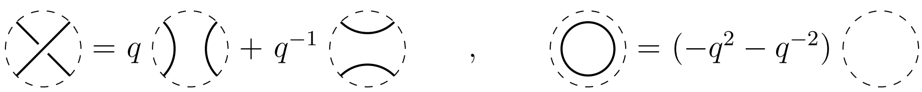
Conjecture 1.1.4: *Let M be an oriented compact 3-manifold with boundary. Then $Sk(M, \mathbb{Q}(q))$ is finitely generated over the skein algebra of its boundary.*

1.1.2 Stated skein algebras

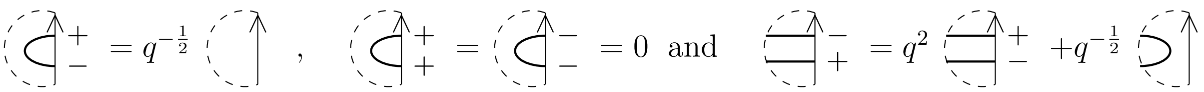
Algebra presentations for skein algebras are not known except for a few examples, despite the computation of a basis as modules. Their representation theory is a difficult and active subject, see [Korb] for a survey. Stated skein algebras generalise skein algebras for tangles on a marked surface with boundary, see [BW11], [Mul] and [Lê18]. These tangles can be cut and stated skein algebras satisfy nice excision properties, which makes computations much simpler in stated skein algebras. In particular, for a non-closed connected surface, one can chunk it into triangles, and embed the skein algebra into a quantum torus [BW11]. This is very fruitful to obtain representations of skein algebras. We will recall here the approach of [CL22]. Stated skein algebras can be defined over integral rings of coefficients, but we will work over the field $\Bbbk$ being either $\mathbb{Q}(q^{\frac{1}{2}})$ or $\mathbb{C}$ with $q^{\frac{1}{2}} \in \mathbb{C}^\times$ generic, as our proofs in Chapter 5 only hold in this context.

Definition 1.1.5: A marked surface is a compact oriented surface with boundary $\overline{\mathfrak{S}}$ with a finite set $P \subseteq \partial\overline{\mathfrak{S}}$ of boundary points, called marked points. We write $\mathfrak{S} = \overline{\mathfrak{S}} \smallsetminus P$ and call this the marked surface. We write $\partial_P\overline{\mathfrak{S}}$ the boundary components of $\overline{\mathfrak{S}}$ that contains a point of P and $\partial\mathfrak{S} := \partial_P\overline{\mathfrak{S}} \smallsetminus P$. Namely we only consider boundary components of $\mathfrak{S}$ that contain a marked point, which we remove in $\mathfrak{S}$, so all components of $\partial\mathfrak{S}$ are intervals. The circular boundary components are discarded and give punctures in $\mathfrak{S}$. The boundary structure of stated skein algebras will depend on a way to cut out a bigon out of a boundary edge, and that of internal skein algebras on a way to insert one from a boundary edge. To avoid choices, we suppose that marked surfaces come equipped with a thickening of their boundary edges inside the surface.
A stated tangle α on $\mathfrak{S}$ is an unoriented, framed, compact, properly embedded 1-submanifold of $\mathfrak{S} \times (0,1)$ whose boundary $\partial\alpha \subseteq \partial\mathfrak{S} \times (0,1)$ has positively vertical framing and comes equipped with a state $st : \partial\alpha \to \{+,-\}$. We call height the $(0,1)$-coordinate of a point, and require that all boundary points of α lying over a same component of $\partial\mathfrak{S}$ have distinct heights. An isotopy of stated tangles is an isotopy with values in stated tangles, in particular preserving the height order over a same boundary component. A stated tangle in $\mathfrak{S} \times (0,1)$ can always be represented as a diagram with blackboard framing in $\mathfrak{S}$, with its under/over crossing information, such that the height order of boundary points corresponds to a given orientation on the boundary edges, see [BW11, Section 3.5]. $\diamondsuit$

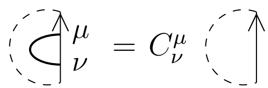
Definition 1.1.6 (Section 2.5 in [CL22]): The stated skein algebra $\mathscr{S}(\mathfrak{S})$ of a marked surface $\mathfrak{S}$ is the $\Bbbk$-vector space generated by isotopy classes of stated tangles on $\mathfrak{S}$ modulo usual skein relations:



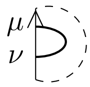
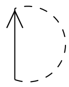
and the boundary skein relations:

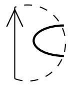
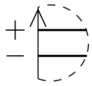
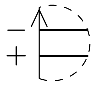


where the arrows on the boundary edges represent the relative height order of the two points.
It is an algebra with product given by vertical superposition and unit the empty link.

We denote by C^μ_ν the coefficient such that  , namely $C^+_+ = C^-_- = 0$,

$C_-^+ = q^{-\frac{1}{2}}$ and one can compute $C_+^- = -q^{-\frac{5}{2}}$, see [CL22, Lemma 2.3 (13)]. We also write $C(\nu) := C_\nu^{-\nu}$. ♢

Proposition 1.1.7 (Lemmas 2.3 and 2.4 in [Lê18]): *These relations express equivalently with the boundary at the left, namely:*  $= {}^{\mu}_{\nu}\text{Ↄ}$ *, where* ${}^{+}_{+}\text{Ↄ} = {}^{-}_{-}\text{Ↄ} = 0$*,*

${}^{+}_{-}\text{Ↄ} = -q^{\frac{5}{2}}$ *and* ${}^{-}_{+}\text{Ↄ} = q^{\frac{1}{2}}$*, and*  $= q^{-\frac{1}{2}}$  $-q^{-\frac{5}{2}}$ .

Remark 1.1.8: It is easy to check that $\mathscr{S}(\mathfrak{S} \sqcup \mathfrak{S}') \simeq \mathscr{S}(\mathfrak{S}) \otimes_k \mathscr{S}(\mathfrak{S}')$ since all relations happen in a connected disk. ♢

Definition 1.1.9 (see Section 3.1 in [Lê18] for details): Let $\mathfrak{S}$ be a marked surface and c an ideal arc on $\mathfrak{S}$, joining two marked points in $\overline{\mathfrak{S}}$. Denote by $Cut_c(\mathfrak{S})$ the marked surface obtained by cutting $\mathfrak{S}$ along c.
Given a stated tangle α on $\mathfrak{S}$, one can cut it along c and get a tangle $Cut_c(\alpha)$ on $Cut_c(\mathfrak{S})$. This tangles has new boundary points, two lifts per points of $\alpha \cap c$, and we may give any state to these points. The obtained stated tangle is called a lift of α if the two lifts of a point of $\alpha \cap c$ have same state. ♢

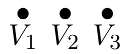
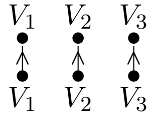
This definition is not perfectly innocent and it seems that one could have chosen different state-matching patterns for lifts, see Remark 5.3.34.

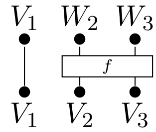
Theorem 1.1.10 (Theorem 3.1 in [Lê18]): *Let* $\mathfrak{S}$ *be a marked surface and* c *an ideal arc. The map* $\rho_c : \mathscr{S}(\mathfrak{S}) \to \mathscr{S}(Cut_c(\mathfrak{S}))$, $\alpha \mapsto \sum_{lifts} \tilde{\alpha}$ *is well-defined (it only depends on the isotopy class of* α*) and is an injective algebra morphism. Moreover, the splitting morphisms* ρ_c *and* $\rho_{c'}$ *associated to two disjoint ideal arcs* c *and* c' *commute.*

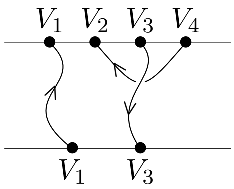
1.2 Reshetikhin–Turaev skein theory

Following Witten's description of Chern–Simons theory as a Topological Quantum Field Theory [Wit89], in the modern point of view on skein theory every strand of a tangle should be colored by a representation of the gauge group of the Chern–Simons theory. More generally, one colors by an object of some linear "ribbon" category. This theory was developed mathematically by Reshetikhin and Turaev [RT91].

1.2.1 Graphical calculus

In the following, we fix some monoidal category $\mathcal{C}$ and develop a graphical calculus for $\mathcal{C}$. We will add conditions and structure on $\mathcal{C}$ for this calculus to be well-defined.
We want to represent objects by points on a line, and tensor product of objects by juxtaposition of points. We think of the drawing  as representing $V_1 \otimes V_2 \otimes V_3$.
Any drawings linking two sets of points (read from bottom to top) represents a morphism between the associated objects. In particular, a bunch of straight lines 

represents the identity. There may be inserted coupons 

, colored by corresponding morphisms, here $f : V_2 \otimes V_3 \to W_2 \otimes W_3$. One may also twist and braid the strands representing every object: 

Composition is represented by vertical superposition.

Graphical calculus is very handy to represent various compositions of morphisms between tensor product of objects, but it not clear that it is well-defined, i.e. that there is always one and only one way to interpret a drawing as a morphism. Reshetikhin–Turaev theory is precisely about this point. A good survey on graphical calculus can be found in [Sel11].

The golden standard of a category admitting a graphical calculus is the category of drawings themselves.

Definition 1.2.1: A *framed tangle* α in a 3-manifold with boundary M is an oriented framed compact properly embedded 1-sub-manifold of M. Remember that a framing is a trivialisation of the normal bundle of the submanifold α. Its boundary $\partial\alpha \subseteq \partial M$ is a finite set of points. Given a finite set of oriented framed points in ∂M, one can consider the set of tangles that have this fixed boundary and whose framing matches the framing of the given boundary points. For the orientation, one also needs to specify whether the boundary is incoming or outgoing, in which case one want the orientation of the boundary points with inward or with outward normal to match the orientation of the tangle. ◊

Note that when we work up to isotopy, framed tangles are equivalent to ribbon tangles, by slightly thickening them in the direction of the framing.

Definition 1.2.2: The category Tan^{fr} of framed tangles in $\mathbb{R}^2 \times [0,1]$ has objects finite sets of oriented framed points of the form $[n] = \{1, \dots, n\} \times \{0\} \subseteq \mathbb{R}^2$ with upward vertical framing (in the second coordinate). An object of Tan^{fr} is described by a finite sequence $\vec{\eta}$ of $+$'s and $-$'s, which we identify with the set of oriented framed points in $\mathbb{R}^2$ it represents.
Morphisms from $\vec{\eta}$ to $\vec{\mu}$ are given by isotopy classes of framed tangles in $\mathbb{R}^2 \times [0,1]$ with boundary the union of $\vec{\eta} \subseteq \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \{0\}$ and $\vec{\mu} \subseteq \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \{1\}$. This imposes that a strand coming out of a positively oriented point should be oriented upward, and of a negatively oriented downward. In the definition above, we orient $\mathbb{R}^2 \times [0,1]$ by saying that $\mathbb{R}^2 \times \{0\}$ has inward normal and $\mathbb{R}^2 \times \{1\}$ has outward normal.
Composition is given by vertical juxtaposition, and contraction $[0,1] \sqcup [0,1] \xrightarrow{[\frac{1}{2},1] \sqcup [0,\frac{1}{2}]} [0,1]$, see Figure 1.1. Identities are given by n straight lines $[n] \times [0,1]$. ◊

The category Tan^{fr} is monoidal with tensor product given by horizontal juxtaposition, see Figure 1.2. On objects, we get concatenation $\vec{\eta} \otimes \vec{\mu} = \vec{\eta\mu}$. The unit object is the empty set.

Definition 1.2.3: A *braided category* is a monoidal category $\mathcal{C}^{\otimes}$ equipped with a natural isomorphism $c : \otimes \to \otimes^{op}$ called the braiding. Its (U,V)-component is denoted $c_{U,V} =$

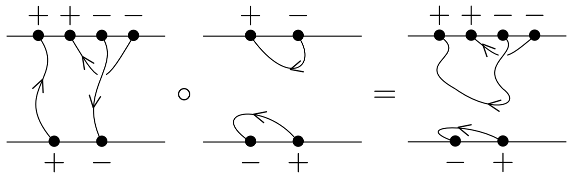


Figure 1.1: Example of composition in Tan^{fr}

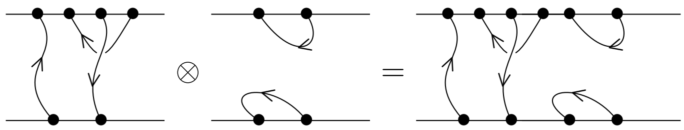
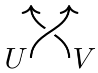
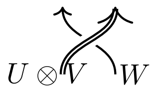


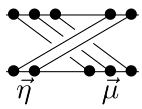
Figure 1.2: Example of tensor product in Tan^{fr}



. The braiding should satisfy:

$c_{U\otimes V,W} =$  $=$  $= (c_{U,W} \otimes Id_V) \circ (Id_U \otimes c_{V,W})$.

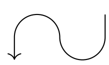
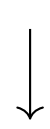
It is called symmetric if $c_{X,Y} \circ c_{Y,X} = Id_{X\otimes Y}$ for all objects $X, Y \in \mathcal{C}$. ♢

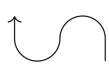
The category Tan^{fr} is braided with $c_{\vec{\eta},\vec{\mu}} =$  with blackboard framing (coming out of the page) and appropriate orientation. It is natural by sliding tangles across the intersection.

Definition 1.2.4: A *rigid category* is a monoidal category $\mathcal{C}^{\otimes}$ with both left and right duals for every object.
A left dual for an object $V \in \mathcal{C}$ is an object V^* together with two morphisms

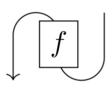
$ev =$  $: V^* \otimes V \to k$ and $coev =$  $: k \to V \otimes V^*$

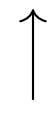
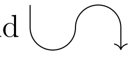
satisfying the so-called snake identities:

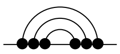
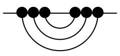
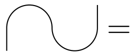
$(ev \otimes Id_{V^*}) \circ (Id_{V^*} \otimes coev) =$  $=$  $= Id_{V^*}$ and

$(Id_V \otimes ev) \circ (coev \otimes Id_V) =$  $=$  $= Id_V$.

These identities imply (by pre or post-composing with ev and $coev$) that $- \otimes V$ is left adjoint to $- \otimes V^*$, and that $V^* \otimes -$ is right adjoint to $V \otimes -$.
The left dual is unique up to canonical isomorphism. It is functorial (contravariant) with

$f^* :=$  .

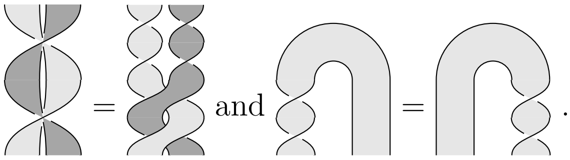
There is a similar notion of right dual ${}^*V$ with $ev =$  $: V \otimes {}^*V \to k$ and $coev =$  $: k \to {}^*V \otimes V$ satisfying  $=$  and  $=$  . The right dual of the left dual of V is canonically identified with V. ♢

The category Tan^{fr} is rigid with both left and right duals given by taking the opposite signs, and mirroring the sequence, $(\overrightarrow{\eta})^* = \overleftarrow{-\eta}$. Evaluation is  and coevaluation is  with blackboard framing and appropriate orientations. The identities  and  hold as we take isotopy classes of tangles.

Note that in a braided category $\mathcal{C}^{\otimes}$ there is always a naive way to obtain right duals from left duals by setting:

In Tan^{fr} however, this naive ways doesn't quite match the drawings we gave above, and it differs by a twist in the framing.

Definition 1.2.5: A twist on a braided category $\mathcal{C}^{\otimes}$ is a natural isomorphism $\theta : Id_{\mathcal{C}} \Rightarrow Id_{\mathcal{C}}$ compatible with the braiding: $\theta_{V\otimes W} = \theta_V \otimes \theta_W \circ c_{W,V} \circ c_{V,W}$. A braided category endowed with a twist is called balanced.
A *ribbon category* is a rigid balanced category $\mathcal{C}$ whose twist is compatible with duality: $\theta_{V^*} = (\theta_V)^*$. ♢

Remark 1.2.6: The category Tan^{fr} with twist  is ribbon. The compatibility conditions are  ♢

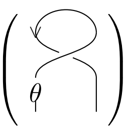
We showed that the category of drawings satisfies certain properties, namely it is ribbon. The main theorem of this section is that being ribbon is exactly what is needed for graphical calculus to make sense.

Theorem 1.2.7 (Theorem 2.5 in [Tur94]): *Let $\mathcal{C}$ be a ribbon category and V an object of $\mathcal{C}$. There exists a unique monoidal functor, called the Reshetikhin–Turaev functor,*

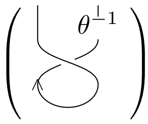
$$\mathrm{RT}_V : Tan^{fr} \to \mathcal{C}$$

such that on objects $\mathrm{RT}_V(+) = V$ *and* $\mathrm{RT}_V(-) = V^*$ *and on morphisms* $\mathrm{RT}_V($$) = c_{V,V}$, $\mathrm{RT}_V($$) = ev$, $\mathrm{RT}_V($$) = coev$ *and* $\mathrm{RT}_V($$) = \theta_V$.

Remark 1.2.8: Here, the right duality morphisms are mapped to

$RT_V($$) = RT_V$  $= ev_V \circ c_{V,V^*} \circ (\theta_V \otimes Id_{V^*})$

and

$RT_V($$) = RT_V$  $= (Id_{V^*} \otimes \theta_V^{-1}) \circ c^{-1}_{V,V^*} \circ coev_V$.

Remark 1.2.9: A framed link in $\mathbb{R}^3$ can be seen as an endomorphism of the empty sequence of points in Tan^{fr}. Under the functor above, it maps to an endomoprhism of the monoidal unit $\mathbf{1}_\mathcal{C}$ of $\mathcal{C}$. We therefore get an invariant of framed links with values in $End_\mathcal{C}(\mathbf{1}_\mathcal{C})$. ◇

Definition 1.2.10: Let U be the 0-framed unknot and $V \in \mathcal{C}$. We call $\mathrm{qdim}(V) := \mathrm{RT}_V(U) \in \Bbbk$ the *quantum dimension* of V. ◇

The theorem above can be thought of as giving some graphical calculus, as we have a way to interpret any tangle as a morphism in $\mathcal{C}$. But it is a bit restrictive, as we will only see the object V appearing, and braidings of V with itself and so on. We want to generalise a little this graphical calculus. The object V here is called the color, and one can extend this constructions to tangles whose strands are colored with different colors.

Definition 1.2.11: The category $Tan^{fr}_\mathcal{C}$ of $\mathcal{C}$-colored tangles has objects finite sets $\bar{X}$ of colored points, namely framed oriented points of the form $[n]$ endowed with a color which is an object of $\mathcal{V}$. Objects are described by finite sequences $(\vec{V},\vec{\eta})$ of pairs (V_i,η_i), $V_i \in \mathcal{C}, \eta_i \in \pm$. It has morphisms from $\bar{X}$ to $\bar{Y}$ isotopy classes of colored tangles from $\bar{X}$ to $\bar{Y}$, i.e. framed oriented tangles with each strand colored by an object of $\mathcal{C}$, and such that the two extremity points of a (non-circular) strand have same color as the strand. ◇

Finally, one can add morphisms of $\mathcal{V}$ that are not compositions and tensor products of identities, braidings, duality morphisms and twists, namely which are not necessarily represented by a tangle. Concretely, one adds coupons which are colored by morphisms of $\mathcal{V}$.

Definition 1.2.12: A *$\mathcal{C}$-colored ribbon grap*h between colored points is a colored framed tangle with coupons. A coupon is an embedding of a little square $[0,1]^2$ in $\mathbb{R}^2 \times (0,1)$, and the endpoints of a framed strand may be glued to either a point $(X,\pm)$, respecting the framing, or to a part of the top or bottom edge of a coupon, with horizontal framing. One marks the end of a framed strand glued to a coupon with $+$ if it is going upward, and $-$ if it is going downward in the coupon. Each framed strand is colored by an object of $\mathcal{C}$ and each coupon with framed strands coming from the bottom face $((X_1,\pm),\dots,(X_n,\pm))$, in this order, and from the top face $((Y_1,\pm),\dots,(Y_m,\pm))$ is colored by a morphism $f : X_1^\pm \otimes \dots \otimes X_n^\pm \to Y_1^\pm \otimes \dots \otimes Y_m^\pm$, where $X^+ = X$ and $X^- = X^*$. See Figure 1.3. ◇

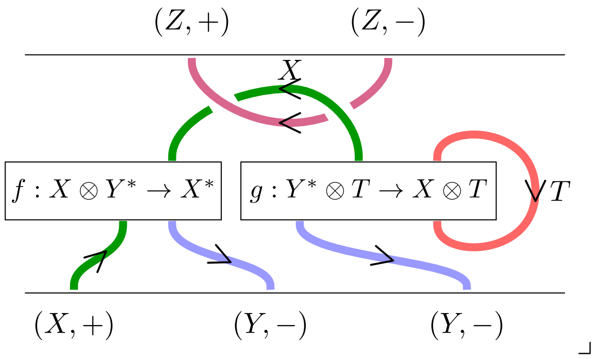


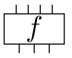
Figure 1.3: A $\mathcal{C}$-colored ribbon graph

Definition 1.2.13: The category $\mathrm{Rib}_\mathcal{C}$ of $\mathcal{C}$-colored ribbon graphs has objects finite sets of colored points and morphisms the $\Bbbk$-vector space generated by isotopy classes of $\mathcal{C}$-

colored ribbon graphs. It is monoidal by juxtaposition, and rigid, braided and ribbon by the usual morphisms (without use of coupons). ♢

Theorem 1.2.14 (Theorem 2.5 in [Tur94]): *Let $\mathcal{C}$ be a $\Bbbk$-linear ribbon category, there is an essentially unique $\Bbbk$-linear monoidal functor*

$$\mathrm{RT}_{\mathcal{C}} : \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C}$$

such that $\mathrm{RT}((X, \pm)) = X^{\pm}$, RT() $= c_{X,Y}$, RT() $= ev_X$, RT() $=$ $coev_X$, RT() $= \theta_X$ *and* RT() $= f$.

Definition 1.2.15: Let $U^{\square}$ be the 0-framed unknot with a single coupon. For $V \in \mathcal{C}$ and $f : V \to V$, denote U_V^f the closed ribbon graph $U^{\bullet}$ with strand colored by V and coupon by f. We call *trace* of f the scalar $\mathrm{tr}(f) := \mathrm{RT}_{\mathcal{C}}(U_V^f)$. ♢

1.2.2 Skein categories

The skein theory for tangles in $\mathbb{R}^2 \times [0, 1]$ can be extended to any 3-manifold. It generalizes the ideas of skein and stated skein algebras and modules that were introduced in the case of the Kauffman bracket skein theory. Skein categories were first introduced by Kevin Walker, though informally and the only written reference is the incomplete draft [Walb]. They were first defined in [Joh21], where they were studied as being (conjecturally) the values on surfaces of a fully extended skein 4-TQFT. This is the point of view we will adopt later in this manuscript. They were also studied in [Coo23] and shown to coincide with Factorization Homology of [AFT17], itself known to form a fully extended 2-TQFT [Sch14a]. This result was independently proven in [KT22] in the semisimple case.

Definition 1.2.16: Let M be a compact oriented 3-manifold, possibly with boundary, and whose boundary is oriented (each connected component has the choice of having orientation induced by the inward or outward normal). A $\mathcal{C}$-colored ribbon graph T in M is a framed tangle with coupons embedded in M such that the endpoints of every framed strand is either at the top or bottom face of a coupon, or at the boundary of M. The strands and coupons are colored by objects and morphisms as before. The boundary data $T \cap \partial M$ of T is a finite set X of framed colored oriented points in ∂M. Given such a set X, we denote by $\mathrm{Rib}_{\mathcal{C}}(M, X)$ the vector space freely generated by isotopy classes of $\mathcal{C}$-colored ribbon graphs in M with boundary data X. ♢

Definition 1.2.17: The skein relation on $\mathrm{Rib}_{\mathcal{C}}(M, X)$ is the equivalence relation generated by:
a linear combination of $\mathcal{C}$-colored ribbon graphs $\sum_i \lambda_i T_i$ is equivalent to 0 if there exists a little cube $\phi : \mathbb{R}^2 \times [0, 1] \hookrightarrow M$ such that all of the T_i's coincide outside its image, intersect $\phi(\mathbb{R}^2 \times [0, 1])$ on either the top or the bottom face, transversely, at a finite set of points of the form $[n]$ and give the zero morphism in $\mathcal{C}$ after evaluation under $\mathrm{RT}_{\mathcal{C}}$ on this little cube. More precisely, every $\phi^{-1}(T_i \cap \operatorname{im} \phi)$ gives a $\mathcal{C}$-colored ribbon graph in $\mathbb{R}^2 \times [0, 1]$, namely a morphism in $\mathrm{Rib}_{\mathcal{C}}$. As the T_i's coincide outside the cube, they all have same source and target. The skein relation holds if $\sum_i \lambda_i \, \mathrm{RT}_{\mathcal{C}}(\phi^{-1}(T_i \cap \operatorname{im} \phi)) = 0$.
Put differently, let T be a ribbon graph and ϕ a little cube of M such that $G := T \cap \operatorname{im} \phi$ is a (possibly complicated) ribbon graph. Then $\mathrm{RT}_{\mathcal{C}}(G)$ is a morphism in $\mathcal{C}$, and we allow ourselves to replace G with a single coupon colored by this morphism.

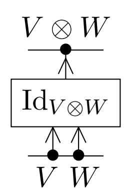
The *skein module* $Sk_{\mathcal{C}}(M, X)$ *of* M *with boundary data* X is the quotient of $\mathrm{Rib}_{\mathcal{C}}(M, X)$ by the skein relations. ♢

Definition 1.2.18: Let Σ be a compact oriented surface possibly with boundary. The $\Bbbk$-linear category $\mathrm{Rib}_{\mathcal{C}}(\Sigma)$ has objects finite sets of framed colored oriented points in Σ. Orient $\Sigma \times [0, 1]$ by saying that $\Sigma \times \{0\}$ has inward normal and $\Sigma \times \{1\}$ has outward normal. Morphisms from X to Y is the vector space $\mathrm{Rib}_{\mathcal{C}}(\Sigma \times [0, 1], X \times \{0\} \cup Y \times \{1\})$. Composition is given by gluing along the $[0, 1]$-coordinate. Identity is given by the ribbon graph $X \times [0, 1]$.
The *skein category* $Sk_{\mathcal{C}}(\Sigma)$ has same objects as $\mathrm{Rib}_{\mathcal{C}}(\Sigma)$ and morphisms from X to Y the quotient of $\mathrm{Hom}_{\mathrm{Rib}_{\mathcal{C}}(\Sigma)}(X, Y)$ by skein relations. Namely $\mathrm{Hom}_{Sk_{\mathcal{C}}(\Sigma)}(X, Y) = Sk_{\mathcal{C}}(\Sigma \times [0, 1], X \times \{0\} \cup Y \times \{1\})$. ♢

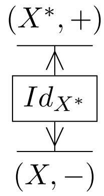
Remark 1.2.19: There are many equivalent ways of defining skein modules and skein categories. One can require the strands to be actual ribbons, and the endpoints to be small intervals as is done in [Joh21, Tur94] or small embedded disks as in [GJS23]. We preferred to only remember the "direction of thickening" and work with framed tangles and framed points. As we work up to isotopy, one can make a ribbon arbitrarily thin, and the resulting vector spaces $\mathrm{Rib}_{\mathcal{C}}(M, X)$ and $Sk_{\mathcal{C}}(M, X)$ are isomorphic for every choice. The analogous skein categories are equivalent. Similarly, we chose to work with coupons, but one could shrink the coupons arbitrarily small, and work with embedded framed graphs (with the vertices colored by morphisms). One still needs to remember how the strands came into the coupon, and it is sufficient to remember the cyclic ordering of the strands, as is done in [KT22]. Again, the resulting skein theories are equivalent. ♢

Remark 1.2.20: In the definitions of $\mathrm{Rib}_{\mathcal{C}}(\Sigma)$ and $Sk_{\mathcal{C}}(\Sigma)$ in the special case where $\Sigma = \mathbb{R}^2$, we allow any finite set of framed points as objects. However, in our initial discussion on tangles in $\mathbb{R}^2 \times [0, 1]$, and in the definition of $\mathrm{Rib}_{\mathcal{C}}$, we did not allow points to be anywhere in $\mathbb{R}^2$, but only to be of the form $[n]$. This simplifies the construction of the Reshetikhin–Turaev functor. Clearly, $\mathrm{Rib}_{\mathcal{C}}$ forms a full subcategory of $\mathrm{Rib}_{\mathcal{C}}(\mathbb{R}^2)$, and its quotient by the skein relation is a full subcategory of $Sk_{\mathcal{C}}(\mathbb{R}^2)$. These categories are actually equivalent, with equivalence given by the inclusion, but it is not trivial to choose a quasi-inverse. One has to choose an isomorphism from each object of $Sk_{\mathcal{C}}(\mathbb{R}^2)$ to one of the form $[n]$. This can be done for example by giving lexicographical order on $\mathbb{R}^2$ and pushing all points in good position while preserving this order, then turning the framing clockwise until it is vertical.
Once this choice has been made (we make the one above), we can see that the Reshetikhin–Turaev evaluation gives a monoidal functor $\mathrm{RT}_{\mathcal{C}} : Sk_{\mathcal{C}}(\mathbb{R}^2) \to \mathcal{C}$. It induces an equivalence there. Its quasi-inverse is given by the inclusion of $\mathcal{C}$ in $Sk_{\mathcal{C}}(\mathbb{R}^2)$ as objects with a single point, and morphisms a single straight line, possibly with a coupon in the middle. ♢

Remark 1.2.21: Note that this inclusion is monoidal but not strictly monoidal. The isomorphism from the two framed points [2] respectively colored by V and W to the framed point [1] colored by $V \otimes W$ is the three-legged graph with a single coupon $\mathrm{Id}_{V \otimes W}$ as shown at the right.



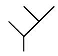
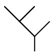
Remark 1.2.22: In any skein category, the identity coupon $\mathrm{Id}_{X^*} : X^* \to X^*$ with entry a downward oriented X-colored ribbon and output an upward oriented X^*-colored ribbon depicted at the right gives an identification $(X, -) \simeq (X^*, +)$.



1.2.3 Quantum groups

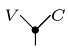
Quantum groups, or Hopf algebras, provide very interesting and well-studied examples of ribbon categories. We will see that the algebraic structures match naturally. Coalgebras give abelian categories, bialgebras give monoidal categories, Hopf algebras give rigid categories, coquasitriangular Hopf algebras give rigid braided categories and finally coribbon Hopf algebras give ribbon categories. There is a sort of converse, given a fibre functor, see [EGNO15]. We discuss categories of comodules here, but there is an analogous story with categories of modules, see [Maj95]. Let $\Bbbk$ be a field.

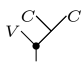
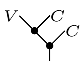
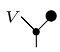
Definition 1.2.23: A coalgebra is a $\Bbbk$-vector space C equipped with:
a coproduct $\Delta =$  $: C \to C \otimes C$ which we denote $\Delta(x) = x_{(1)} \otimes x_{(2)}$ with implicit summation and
a counit $\varepsilon =$  $: C \to \Bbbk$, satisfying:

$(Id_C \otimes \Delta) \circ \Delta =$  $=$  $= (\Delta \otimes Id_C) \circ \Delta$ (coassociativity)

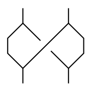
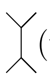
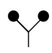
$(Id_C \otimes \varepsilon) \circ \Delta =$  $= | = Id_C =$  $= (\varepsilon \otimes Id_C) \circ \Delta$ (counit)

A right C-comodule is a $\Bbbk$-vector space V equipped with a coaction

$\Delta_V =$  $: V \to V \otimes C$

such that  $=$  (coassociativity) and  $= |^V = Id_V$ (counit).
We denote C–*comod* the category of right C-comodules, with morphisms the linear maps $f : V \to W$ preserving the coaction, namely $\Delta_W \circ f = (f \otimes Id_C) \circ \Delta_V$. We denote C–*comod*^{fin} the full subcategory spanned by finite dimensional comodules. ♢

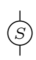
Definition 1.2.24: A bialgebra is an algebra ($A, m =$ $, \mathbf{1} =$ ) equipped with a coalgebra structure (A, Δ, ε) such that Δ and ε are algebra morphisms, namely:

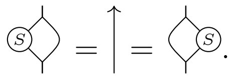
 $=$  (the crossing represents the flip of tensors), and  $=$ 

The category A–*comod* of comodules over a bialgebra A is monoidal with $\otimes = \otimes_\Bbbk$ with coaction on tensor product given by:

$$V \otimes_\Bbbk W \overset{\Delta_V \otimes \Delta_W}{\longrightarrow} (V \otimes A) \otimes (W \otimes A) \overset{flip}{\longrightarrow} V \otimes W \otimes A \otimes A \overset{Id_V \otimes Id_W \otimes m}{\longrightarrow} V \otimes W \otimes A$$

Definition 1.2.25: A Hopf algebra is a bialgebra H equipped with an antipode

$S =$  $: H \to H$

such that . We moreover assume that S is invertible. It is slightly unusual but is true in general as soon as H is finite dimensional or (co)-quasi-triangular.
The category H–*comod*^{fin} is rigid. Given a comodule V, its dual has underlying vector-space V^*. The H-coaction on a form $f : V \to k$ is the element $\Delta f \in V^* \otimes H \simeq Hom(V, H)$ described, for $v \in V$, by $\Delta f(v) = f(v_{(1)}) \otimes Sh_{(2)}$ for the left dual, and by $\Delta f(v) = f(v_{(1)}) \otimes S^{-1}h_{(2)}$ for the right dual. ♢

Definition 1.2.26: A co-quasi-triangular Hopf algebra H is a Hopf algebra equipped with a co-R-matrix, or R-form, $R : H \otimes H \to \Bbbk$ which is invertible by convolution, i.e. there exists $R^{-1} : H \otimes H \to \Bbbk$ such that

$$\forall a, b \in H,\ R(a_{(1)} \otimes b_{(1)})R^{-1}(a_{(2)} \otimes b_{(2)}) = \varepsilon(a)\varepsilon(b)$$

satisfying

$$b_{(1)}.a_{(1)}.R(a_{(2)} \otimes b_{(2)}) = R(a_{(1)} \otimes b_{(1)}).a_{(2)}.b_{(2)}\ ,$$

$$R(ab \otimes c) = R(a \otimes c_{(1)}).R(b \otimes c_{(2)}) \text{ and}$$

$$R(a \otimes bc) = R(a_{(1)} \otimes c).R(a_{(2)} \otimes b).$$

The category H–*comod* is braided with braiding $c_{V,W} : V \otimes W \to W \otimes V$ given by $c_{V,W}(v \otimes w) = w_{(1)} \otimes v_{(1)}.R(v_{(2)} \otimes w_{(2)})$. ♢

Definition 1.2.27: A coribbon Hopf algebra is a co-quasi-triangular Hopf algebra H equipped with a coribbon functional, i.e. a map $\theta : H \to \Bbbk$ such that :
(1) θ is invertible by convolution: there exists $\theta^{-1} : H \to \Bbbk$ such that $\theta(a_{(1)})\theta^{-1}(a_{(2)}) = \theta^{-1}(a_{(1)})\theta(a_{(2)}) = \varepsilon(a)$,
(2) θ is central: $\theta(a_{(1)})a_{(2)} = a_{(1)}\theta(a_{(2)})$,
(3) compatibility with product: $\theta(ab) = R(b_{(1)} \otimes a_{(1)})\theta(b_{(2)})\theta(a_{(2)})R(a_{(3)} \otimes b_{(3)})$ and
(4) compatibility with antipode: $\theta \circ S = \theta$.
The category H–*comod* of comodules over a coribbon Hopf algebra H is balanced with twist on a H-comodule V given by $\theta_V : V \xrightarrow{\Delta_V} V \otimes H \xrightarrow{Id_V \otimes \theta} V$. The category H–*comod*^{fin} is ribbon. ♢

Remark 1.2.28: In the literature the coribbon functional is often defined to be θ^{-1} in our definition. The compatibility condition are hence deformed. ♢

1.2.4 The coribbon Hopf algebra $\mathcal{O}_{q^2}(SL_2)$

On can recover the Kauffman bracket link invariant, and its skein relations, from a particular example of a coribbon Hopf algebra associated with SL_2.

Definition 1.2.29: Let $\Bbbk = \mathbb{C}(q^{\frac{1}{2}})$. The coribbon Hopf algebra $\mathcal{O}_{q^2}(SL_2)$ is the $\Bbbk$-algebra with: (one should read the matrix equations component-wise, and the tensor product of matrices should be computed as a usual matrix product with tensor products of coefficients instead of products)

generators	:	a, b, c, d
relations	:	$ca = q^2ac, \quad db = q^2bd, \quad ba = q^2ab, \quad dc = q^2cd,$ $bc = cb, \quad ad - q^{-2}bc = 1 \quad \text{and} \quad da - q^2cb = 1$
coproduct	:	$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
counit	:	$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
antipode	:	$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^2b \\ -q^{-2}c & a \end{pmatrix}$
co-R-matrix	:	$R\begin{pmatrix} a\otimes a & a\otimes b & b\otimes a & b\otimes b \\ a\otimes c & a\otimes d & b\otimes c & b\otimes d \\ c\otimes a & c\otimes b & d\otimes a & d\otimes b \\ c\otimes c & c\otimes d & d\otimes c & d\otimes d \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1} & q-q^{-3} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$
coribbon functional	:	$\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -q^3 & 0 \\ 0 & -q^3 \end{pmatrix}$ ◇

Remark 1.2.30: Note that we use here an unusual coribbon functional. It is studied in [Tin] where the author proves that it gives precisely the Kauffman-bracket relation under the Reshetikhin–Turaev functor, whereas the usual coribbon functional gives relations which differ by a sign, and give the Jones polynomial after writhe renormalisation. ◇

Definition 1.2.31: The quantum plane $\mathbb{A}^2_{q^2}$ is the free $\Bbbk$-algebra generated by x and y modulo the relation $yx = q^2xy$. It is an $\mathcal{O}_{q^2}(SL_2)$-comodule algebra with

$$\Delta\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

on generators.
Its subspace of homogeneous polynomials of degree n forms a sub-comodule which we denote V_n.
In particular the generators span the comodule V_1 which we abbreviate V and call the standard co-representation of $\mathcal{O}_{q^2}(SL_2)$. It is more conventional to call its basis $v_+ = x$ and $v_- = y$. This comodule is self dual. The left dual V^* of V has basis v_+^* and v_-^*, and one has an isomorphism of $\mathcal{O}_{q^2}(SL_2)$-comodules $\varphi : \begin{cases} V & \to & V^* \\ v_+ & \mapsto & -q^{\frac{5}{2}}v_-^* \\ v_- & \mapsto & q^{\frac{1}{2}}v_+^* \end{cases}$. ◇

Proposition 1.2.32 (see VII.7.1 and VII.5.1 in [Kas95]): *For $n \geq m \in \mathbb{N}$, the comodule $V_n \otimes V_m$ splits as the direct sum*

$$V_n \otimes V_m \simeq V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m} \ .$$

Proposition 1.2.33: *The $\mathcal{O}_{q^2}(SL_2)$-comodules V_n, $n \in \mathbb{N}$, are all the simple $\mathcal{O}_{q^2}(SL_2)$-comodules. Moreover, the categories $\mathcal{O}_{q^2}(SL_2)$–comod and $\mathcal{O}_{q^2}(SL_2)$–comod^{fin} are semi-simple, namely any $\mathcal{O}_{q^2}(SL_2)$-comodule splits as a direct sum of V_n's.*

Proof : The statement for finite comodules is classical, see [KS97, Section 4.2.1]. The extension to all comodules simply comes from the fact that any element of a comodule lives in a finite dimensional subcomodule [Swe69, Theorem 2.1.3 b)] and thus in a direct sum of simples. Finally, a module is semi-simple if and only if it is the (non-direct) sum of its simple submodules, see [AF92, Theorem 9.6]. □

Using the Proposition 1.2.32 we can show that the comodule $V^{\otimes n}$ has one simple component isomorphic to V_n . Hence the standard co-representation V generates all comodules V_n and hence all comodules under tensor product, direct summand and direct sum. These results are sometimes better known in the $\mathcal{U}_{q^2}(SL_2)$-modules point of view.

Definition 1.2.34: The Hopf algebra $\mathcal{U}_{q^2}(SL_2)$ is the $\Bbbk$-algebra with:

generators	:	E, F, K
relations	:	$KE = q^4 EK, \quad KF = q^{-4} FK \quad \text{and} \quad EF - FE = \dfrac{K - K^{-1}}{q^2 - q^{-2}}$
coproduct	:	$\Delta(K) = K \otimes K, \quad \Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1$
counit	:	$\varepsilon(K) = 1$ and $\varepsilon(E) = \varepsilon(F) = 0$
antipode	:	$S(K) = K^{-1}, \quad S(E) = -EK^{-1}$ and $S(F) = -KF$

♢

Definition 1.2.35: The left $\mathcal{U}_{q^2}(SL_2)$-module $V_{\pm,n}$ is the vector space of dimension $n+1$ on which $\mathcal{U}_{q^2}(SL_2)$ acts as $K = \pm \begin{pmatrix} q^{2n} & 0 & \cdots & 0 \\ 0 & q^{2(n-2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q^{-2n} \end{pmatrix}$, $E = \begin{pmatrix} 0 & [n]_{q^2} & & 0 \\ & \ddots & \ddots & \\ \vdots & & \ddots & [1]_{q^2} \\ 0 & \cdots & & 0 \end{pmatrix}$

and $F = \pm \begin{pmatrix} 0 & & \cdots & 0 \\ [1]_{q^2} & \ddots & & \vdots \\ & \ddots & \ddots & \\ 0 & & [n]_{q^2} & 0 \end{pmatrix}$, where $[n]_{q^2} = \dfrac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}$.

The modules of the form $V_{+,n}$ are called of type 1, and the others are discarded here. The module $V_{+,1}$ is denoted V and called the standard representation of $\mathcal{U}_{q^2}(SL_2)$. ♢

Definition 1.2.36: An $\mathcal{U}_{q^2}(SL_2)$-module W is called *locally finite* if for each $w \in W$, its orbit $\mathcal{U}_{q^2}(SL_2) \cdot w$ forms a finite dimensional vector space. In other words, if W is the union of its finite dimensional submodules. ♢

Proposition 1.2.37 (Theorem VII.2.2 in [Kas95]): *The $\mathcal{U}_{q^2}(SL_2)$-modules $V_{\pm,n}$, $n \in \mathbb{N}$ are all the locally finite simple $\mathcal{U}_{q^2}(SL_2)$-modules. Moreover, the categories of finite and locally finite left $\mathcal{U}_{q^2}(SL_2)$-modules are semi-simple.*

Their full subcategories generated by direct sums of simple modules of type 1 are still semi-simple, and closed under tensor product. We denote $\mathcal{U}_{q^2}(SL_2)$–mod^{lf} and $\mathcal{U}_{q^2}(SL_2)$–mod^{fin} will represent these full subcategories of respectively locally finite and finite modules of type 1.

Definition 1.2.38: We can define a dual paring, namely a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : \mathcal{U}_{q^2}(SL_2) \otimes \mathcal{O}_{q^2}(SL_2) \to \Bbbk$ satisfying

$$\begin{aligned} &\langle x, y.y' \rangle = \langle x_{(1)}, y \rangle \langle x_{(2)}, y' \rangle \ , \\ &\langle x.x', y \rangle = \langle x, y_{(1)} \rangle \langle x', y_{(2)} \rangle \ , \\ &\langle x, 1 \rangle = \varepsilon(x) \ , \\ &\langle 1, y \rangle = \varepsilon(y) \text{ and} \\ &\langle S(x), y \rangle = \langle x, S(y) \rangle \ . \end{aligned}$$

It is given on generators by $\left\langle K, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix}$, $\left\langle E, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\left\langle F, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
A right $\mathcal{O}_{q^2}(SL_2)$-comodule structure on some vector space W induces a left $\mathcal{U}_{q^2}(SL_2)$-module structure by $x \cdot w = w_{(1)}.\langle x, w_{(2)} \rangle$, $x \in \mathcal{U}_{q^2}(SL_2)$, $w \in W$. ♢

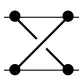
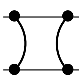
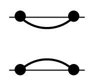
Proposition 1.2.39: *This correspondence induces equivalences of categories $\mathcal{O}_{q^2}(SL_2)$–comod^{fin} $\simeq$ $\mathcal{U}_{q^2}(SL_2)$–mod^{fin} and $\mathcal{O}_{q^2}(SL_2)$–comod $\simeq$ $\mathcal{U}_{q^2}(SL_2)$–mod^{lf} between right $\mathcal{O}_{q^2}(SL_2)$-comodules and type 1 locally finite left $\mathcal{U}_{q^2}(SL_2)$-modules. The simple comodules V_n are mapped on the simple modules $V_{+,n}$.*

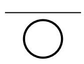
Proof : The equivalence between locally finite modules on a Hopf algebra and comodules on its restricted dual is given in [Abe80, (3.3) p.126] and the link between $\mathcal{O}_{q^2}(SL_2)$ and the restricted dual $\mathcal{U}_{q^2}(SL_2)^\circ$ of $\mathcal{U}_{q^2}(SL_2)$ is given in [Tak02, Theorem 7.9]. Namely, $\mathcal{O}_{q^2}(SL_2)$ is a sub-Hopf-algebra of $\mathcal{U}_{q^2}(SL_2)^\circ$. The standard co-representation is mapped on the standard representation and the simple comodules V_n on the simple modules $V_{+,n}$, see [Kas95, VII.5.1]. The correspondence preserves direct sums (it is the identity on vector spaces) hence maps direct sums of V_n's to direct sums of $V_{+,n}$'s. □

Note that this correspondence also preserves the monoidal (and actually ribbon) structure.

We are now ready to study the skein theory associated with $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$–*comod*^{fin}. In this case, we have seen in Definition 1.2.31 that the standard corepresentation V is isomorphic to its dual V^* in $\mathcal{V}$ by φ. Thus one gets an identification $(V, +) \simeq (V, -)$ by the coupon φ in the skein category of any surface. Sliding this coupon along a strand changes the orientation of the strand. Consequently, one can switch signs of points and orientations of strands. In this example, we can stop mentioning orientations and talk about unoriented framed tangles. The Reshetikin–Turaev functor is still well-defined on unoriented framed tangles, see [Tin, Theorem 4.2]. An unoriented tangle gives a ribbon graph by choosing an arbitrary orientation, coloring every strand by V and replacing V^*'s imposed on the boundary points by V's using φ or φ^{-1}. For example, for the unoriented cap $\cap$ one can orient it either as pointing to the left or to the right, so one has to check that $RT($$) \circ (\varphi \otimes Id_V) = RT($$) \circ (Id_V \otimes \varphi)$. Note that this would not hold with the "standard" coribbon element in $\mathcal{O}_{q^2}(SL_2)$, and our choice is important.

Proposition 1.2.40: *Let* $\mathrm{RT}_V : Tan^{fr,unoriented} \to \mathcal{O}_{q^2}(SL_2)$–*comod*^{fin} *be the functor with* $\mathrm{RT}_V(+) = V$ *the standard co-representation given in Theorem 1.2.7 and [Tin, Theorem 4.2]. Then,*

RT_V  $= q.\mathrm{RT}_V$  $+q^{-1}.\mathrm{RT}_V$  *and*

RT_V  $= (-q^2 - q^{-2}).\mathrm{RT}_V$.

In particular, the invariant of framed links obtained from RT_V *is the Kauffman bracket.*

Proposition 1.2.41 ([MM92, ALZ15] or Theorem 3.3.4 in [CFS95]): *The algebra* $\mathrm{Hom}_{\mathcal{V}}(V^{\otimes n}, V^{\otimes n})$ *is isomorphic to the Temperley-Lieb algebra* TL_n*, as defined in [Tur94, Section XII.3]. The full subcategory of* $\mathcal{V}$ *of objects of the form* $V^{\otimes n}$, $n \in \mathbb{N}$*, is equivalent to the category Temperley–Lieb* TL*, defined in [Tur94, Section XII.2].*

In the following, we will call this full subcategory TL.

Theorem 1.2.42: *Let* $n.V$ *and* $m.V$ *be two objects of* $Sk_{\mathcal{V}}(\Sigma)$ *given respectively by* n *and* m *points colored by* V*. Any morphism in* $\mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(n.V, m.V)$ *can be represented by a linear combination of unoriented tangles. Moreover, two linear combinations of unoriented tangles represent the same morphism in* $\mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(n.V, m.V)$ *if and only if one can get from one to another by a sequence of isotopies and Kauffman bracket skein relations.*
Said differently, the category $Sk_{TL}(\Sigma)$ *is a full subcategory of* $Sk_{\mathcal{V}}(\Sigma)$*.*

Proof : In the case of $\Sigma = \mathbb{R}^2$, one has $Sk_{\mathcal{V}}(\mathbb{R}^2) \simeq \mathcal{V}$ and $n.V \simeq V^{\otimes n}$, and the result follows from the proposition above.
On a general surface Σ, on each connected component one can pull every coupon of a given ribbon graph inside a disk on Σ, so there are only strands outside this disk. The color W of such a strand embeds in some $V^{\otimes n}$ and adding the coupons $W \hookrightarrow V^{\otimes n} \twoheadrightarrow W$ and sliding the second across the strand, one can have only strands colored by $V^{\otimes n}$ outside the disk. Finally the part on the disk can be represented by a tangle by the result on $\mathbb{R}^2$.
For the second part, we want to say that all relations happen in $\mathbb{R}^2$, where we know that the only relations are isotopies and Kauffman bracket skein relations. Skein relations happen in $\mathbb{R}^2 \times [0, 1]$ by definition, and isotopies (which, in our smooth setting are ambient isotopies) can be decomposed into a composition of "local" isotopies, each supported over a disk [EK71, Corollary 1.3]. □

1.3 Topological Quantum Field Theories

A Quantum Field Theory in physics is a model that should associate to a space M (which we think of as our universe) a vector space $\mathcal{Z}(M)$ (actually a Hilbert space) of physical states on M, and to a space-time W linking two spaces M and M', a time-evolution operator $\mathcal{Z}(W) : \mathcal{Z}(M) \to \mathcal{Z}(M')$. The precise notion of a Quantum Field Theory is not figured out mathematically yet, but we have a definition when we assume that the theory is topological, i.e. $\mathcal{Z}(W)$ depends on W only up to diffeomorphism.

Definition 1.3.1: Let M and M' be closed oriented n-manifolds. A *cobordism* from M to M' is a compact oriented $(n+1)$-manifold W together with a decomposition of its boundary into two unions of connected components $\partial^+ W, \partial^- W$, an orientation reversing diffeomorphism $\partial^- W \to M$ and an orientation preserving diffeomorphism $\partial^+ W \to M'$.
A diffeomorphism between two cobordism W, W' from M to M' is a diffeomoprhism $\varphi : W \to W'$ preserving the identifications of the boundary with M and M'. ♢

Definition 1.3.2 ([Ati88]): An $(n+1)$-*TQFT* is a symmetric monoidal functor

$$\mathcal{Z} : \mathbf{Cob}_{n+1}^{\sqcup} \to \mathrm{Vect}_{\Bbbk}^{\otimes}.$$

where $\mathbf{Cob}_{n+1}$ is the category whose objects are closed oriented n-manifolds $M, M', \dots$ and whose morphisms $M \to M'$ are diffeomorphism classes of cobordisms. The identity cobordism is the cylinder $M \times [0,1]$ and composition is given by gluing, which admits a unique smooth structure up to diffeomorphism. It is symmetric monoidal with disjoint union. $\mathrm{Vect}_{\Bbbk}$ is the category of vector spaces over a fixed field $\Bbbk$ and linear maps. It is symmetric monoidal with tensor product. ♢

Remark 1.3.3: The index $n+1$ in $\mathbf{Cob}_{n+1}$ is subject to some disagreements in the literature. In particular our notation disagrees with the one of [Juh18] which we will refer to extensively in the rest of this section. We make the consistent choice of always referring to the highest dimension present. We will also sometimes say 4-TQFT for (3+1)-TQFT, and such. ♢

Example 1.3.4: Fix $\lambda \in \Bbbk^\times$. The Euler characteristic TQFT assigns the vector space $\Bbbk$ to every n-manifold M and the linear map which is multiplication by the scalar $\lambda^{\chi(W)-\chi(M)}$ to a cobordism $W : M \to M'$. ♢

In the non-semisimple case, we will sometimes need a notion of partially-defined TQFT.

Definition 1.3.5: The *category of non-compact cobordisms* $\mathbf{Cob}^{\mathsf{nc}}_{n+1}$ is the subcategory of $\mathbf{Cob}_{n+1}$ with the same objects and with morphisms the cobordisms that have non-empty incoming boundary in every connected component (said differently, $\pi_0(\partial^- W) \to \pi_0(W)$ is surjective). It is still symmetric monoidal with disjoint union.
A *non-compact* $(n+1)$*-TQFT* is a symmetric monoidal functor $\mathcal{Z} : \mathbf{Cob}^{\mathsf{nc}}_{n+1} \to \mathrm{Vect}_{\Bbbk}$. ♢

1.3.1 Diffeomorphisms and handle attachments as cobordisms

There are two particular kinds of cobordisms that, we will see, generate every cobordism: mapping cylinders of diffeomorphisms and handle attachments.

Definition 1.3.6: Let $f : M \to M'$ be an orientation-preserving diffeomorphism. The *mapping cylinder of* f is the cobordism $C_f = M \times [0,1]$ with boundary decomposition $\partial^- C_f = M \times \{0\} \overset{Id_M}{\simeq} M$ and $\partial^+ C_f = M \times \{1\} \overset{f}{\simeq} M'$.
The gluing of two such mapping cylinders is diffeomorphic to the mapping cylinder of the composition of the diffeomorphism. Hence this construction gives a symmetric monoidal functor $\mathbf{Man}_n \to \mathbf{Cob}_{n+1}$, where $\mathbf{Man}_n$ is the category of oriented closed n-manifolds and diffeomorphisms. This functor is constant on isotopy classes. ♢

Definition 1.3.7 (Definition 1.2 of [Juh18]): Let M be a closed oriented n-manifold and $0 \leq k \leq n+1$. A framed $(k-1)$-sphere in M is an orientation preserving embedding $\mathbb{S} : S^{k-1} \times D^{n+1-k} \hookrightarrow M$. The *k-handle attachment* along $\mathbb{S}$ is the cobordism

$$W(\mathbb{S}) = M \times [0,1] \cup_{\mathbb{S}} D^k \times D^{n-k}$$

with $\partial^- W(\mathbb{S}) = M \times \{0\}$ and $\partial^+ W(\mathbb{S})$ is the complement in the boundary, denoted $M(\mathbb{S})$. It can be endowed with a smooth structure unique up to diffeomorphism (see for instance [Mil65, Theorem 3.13]). We call $M(\mathbb{S})$ the surgery of M along $\mathbb{S}$. It is obtained by removing the image of $\mathbb{S}$ in M and gluing $D^k \times S^{n-k}$ at the created $S^k \times S^{n-k}$ boundary.♢

1.3.2 Presentation of the $(n+1)$-cobordism category

A generator and relation presentation of the $(n+1)$-cobordism category is given in [Juh18]. The (3+1)-TQFT constructions given in Chapter 3 will use this work.

Let $\mathcal{G}$ be the directed graph with vertices closed oriented n-manifolds and with edges an edge $e_d : M \to M'$ for every diffeomorphism $d : M \to M'$ and an edge $e_{M,\mathbb{S}} : M \to M(\mathbb{S})$ for every framed $(k-1)$-sphere $\mathbb{S}$ in M.
Let $\mathcal{G}^{\mathsf{nc}}$ be the subgraph of $\mathcal{G}$ obtained by removing the edges $e_{M,\mathbb{S}}$ where $\mathbb{S}$ is a -1-sphere (so removing 0-handles). Let $\mathcal{F}(\mathcal{G})$ (resp. $\mathcal{F}(\mathcal{G}^{\mathsf{nc}})$) be the free category generated by $\mathcal{G}$ (resp. $\mathcal{G}^{\mathsf{nc}}$). It is symmetric monoidal with disjoint union.

In [Juh18, Definition 1.4] Juhász considers a set of relations $\mathcal{R}$ in $\mathcal{F}(\mathcal{G})$ which we recall now. If w and w' are words consisting of composing arrows, then we write $w \sim w'$ if $(w, w') \in \mathcal{R}$.

(R1) For composable diffeomorphisms d and d' we have the relation $e_{d \circ d'} \sim e_d \circ e_{d'}$. For $d : M \to M$ a diffeomorphism isotopic to the identity we have $e_d \sim \mathrm{Id}_M$, where Id_M is the empty word.

(R2) If $d : M \to M'$ is an orientation preserving diffeomorphism between 3-manifolds and $\mathbb{S}$ a framed sphere in M then let $\mathbb{S}' = d \circ \mathbb{S}$ be the framed sphere in M'; then let $d^{\mathbb{S}} : M(\mathbb{S}) \to M'(\mathbb{S}')$ be the induced diffeomorphism. Then the commutativity of the following diagram defines a relation:

$$\begin{array}{ccc} M & \xrightarrow{e_{M,\mathbb{S}}} & M(\mathbb{S}) \\ {\scriptstyle e_d}\downarrow & & \downarrow{\scriptstyle e_{d^{\mathbb{S}}}} \\ M' & \xrightarrow{e_{M',\mathbb{S}'}} & M'(\mathbb{S}'). \end{array}$$

(R3) If $\mathbb{S}, \mathbb{S}'$ are disjoint framed sphere in an oriented 3-manifold M then $M(\mathbb{S})(\mathbb{S}') = M(\mathbb{S}')(\mathbb{S})$ and we denote this 3-manifold by $M(\mathbb{S}, \mathbb{S}')$. The commutativity of the following diagram defines a relation:

$$\begin{array}{ccc} M & \xrightarrow{e_{M,\mathbb{S}}} & M(\mathbb{S}) \\ {\scriptstyle e_{M,\mathbb{S}'}}\downarrow & & \downarrow{\scriptstyle e_{M(\mathbb{S}),\mathbb{S}'}} \\ M(\mathbb{S}') & \xrightarrow{e_{M(\mathbb{S}'),\mathbb{S}}} & M(\mathbb{S}, \mathbb{S}'). \end{array}$$

(R4) Let $\mathbb{S}$ be a framed k-sphere in M and $\mathbb{S}'$ a framed k'-sphere in $M(\mathbb{S})$. If the attaching sphere $\mathbb{S}'(S^{k'} \times \{0\}) \subset M(\mathbb{S})$ intersects the belt sphere $\{0\} \times S^{-k+2} \subset M(\mathbb{S})$ once transversally, then there is a diffeomorphism (well defined up to isotopy) $\phi : M \to M(\mathbb{S}, \mathbb{S}')$ (defined in Definition 2.17 of [Juh18]) and the following is a relation:

$$e_{M(\mathbb{S}),\mathbb{S}'} \circ e_{M,\mathbb{S}} \sim e_\phi.$$

(R5) If $\mathbb{S} : S^k \times D^{3-k} \hookrightarrow M$ is a framed k-sphere then there is a relation $e_{M,\mathbb{S}} \sim e_{M,\bar{\mathbb{S}}}$ where $\bar{\mathbb{S}}$ is the framed k-sphere given by $\bar{\mathbb{S}}(x, y) = \mathbb{S}(r_{k+1}(x), r_{3-k}(y))$ with $x \in S^k \subset \mathbb{R}^{k+1}, y \in D^{3-k} \subset \mathbb{R}^{3-k}$ and $r_m(x_1, x_2, \ldots, x_m) = (-x_1, x_2, \ldots, x_m)$.

Let $\mathcal{R}^{\mathsf{nc}}$ be the subset of $\mathcal{R}$ consisting of relations where all involved edges are in $\mathcal{G}^{\mathsf{nc}}$.
Recall that given a category $\mathcal{C}$ and a set of relations $\sim$ on its morphisms, the quotient category $\mathcal{C}/\sim$ has the same objects as $\mathcal{C}$ and equivalence classes of morphisms of $\mathcal{C}$ as morphisms.

Theorem 1.3.8 (Theorem 1.7 of [Juh18]): *The functor $c : \mathcal{F}(\mathcal{G}) \to \boldsymbol{Cob}_{n+1}$ which is the identity on objects and maps an edge e_d to the mapping cylinder C_d and an edge $e_{M,\mathbb{S}}$ to the handle attachment $W(\mathbb{S})$ induces an isomorphism of symmetric monoidal categories*

$$\mathcal{F}(\mathcal{G})/\mathcal{R} \to \boldsymbol{Cob}_{n+1}\,.$$

Theorem 1.3.9: *The restriction of the same functor $c : \mathcal{F}(\mathcal{G}^{\mathsf{nc}}) \to \boldsymbol{Cob}^{\mathsf{nc}}_{n+1}$ induces an isomorphism of symmetric monoidal categories*

$$\mathcal{F}(\mathcal{G}^{\mathsf{nc}})/\mathcal{R}^{\mathsf{nc}} \to \boldsymbol{Cob}^{\mathsf{nc}}_{n+1}.$$

Proof : This follows from Juhász's argument using parameterized Cerf decomposition. The fact that 0–1-handle cancellations can be avoided so that one only needs to consider the equivalence relation generated by $\mathcal{R}^{\mathsf{nc}}$ is stated in [Juh18, Theorem 2.24]. For an argument directly based on the statements of Juhász's theorems one can easily translate in dimension 4 the one made in dimension 3 of [CGPVb, Corollary 4.3]. □

1.4 Semisimple skein TQFTs

We recall here the constructions of skein-theoretic TQFTs coming from semisimple ribbon categories [RT91, Tur94, CY93, CKY97]. We use a slightly unusual point of view to emphasize the relation between Crane–Yetter and Witten–Reshetikhin–Turaev TQFTs associated with the same ribbon category. Most proofs are special cases of the non-semisimple cases that will be studied in greater detail in Chapter 3, and we do not rewrite them. Note that, even though we refer to our work for the proofs, the semisimple constructions preexisted.

1.4.1 Algebraic data

Let $\Bbbk$ be an algebraically closed field and let $\mathcal{C}$ be a ribbon fusion $\Bbbk$-linear category in the sense of [EGNO15]: a finite semisimple abelian ribbon category which is $\Bbbk$-linear, whose tensor product is bilinear, and such that $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk.\,\mathrm{Id}_{\mathbb{1}}$. We denote by $O(\mathcal{C})$ the set of isomorphism classes of simple objects of $\mathcal{C}$. By assumption, it is finite, and every object of $\mathcal{C}$ is isomorphic to a finite direct sum of simple objects.

Definition 1.4.1: Let $\mathcal{C}$ be a ribbon fusion category and $S_1, \dots, S_n$ be representatives of $O(\mathcal{C})$. We set $G := \oplus_i S_i$ and call it the generator. We define

$$\mathsf{c} = \oplus_i \,\mathrm{qdim}(S_i)\,\mathrm{Id}_{S_i} : G \to G$$

and call it the Kirby color. We call global dimension of $\mathcal{C}$ the scalar

$$d(\mathcal{C}) := \sum_{1\leq i\leq n} \mathrm{qdim}(S_i)^2 = \mathrm{tr}(\mathsf{c}).$$

We call $\mathcal{C}$ *chromatic-non-degenerate* if $d(\mathcal{C}) \neq 0$. It is automatically true if $\Bbbk$ is of characteristic 0 by [EGNO15, Theorem 7.21.12].

We call *Gauss sums* of $\mathcal{C}$ the scalars $\Delta_\pm := \mathrm{tr}(\theta_G^\pm \circ \mathsf{c})$. We call $\mathcal{C}$ *twist-non-degenerate* if they are both non-zero.
We call $\mathcal{C}$ *modular* if the matrix $S = (\mathrm{tr}(c^{-1}_{S_j,S_i} \circ c_{S_i,S_j}))_{0\leq i,j\leq n}$ is invertible. This implies chromatic-non-degenerate by [EGNO15, Proposition 8.14.2] and twist-non-degenerate by [EGNO15, Proposition 8.15.4].
We call an object $X \in \mathcal{C}$ *transparent* if for every $Y \in \mathcal{C}$ we have $c^{-1}_{Y,X} \circ c_{X,Y} = \mathrm{Id}_{X\otimes Y}$. We call *Müger center* $\mathrm{Mü}(\mathcal{C})$ the full subcategory of transparent objects of $\mathcal{C}$. By [EGNO15, Proposition 8.20.12] $\mathcal{C}$ is modular if and only if it has trivial Müger center, namely every transparent object is isomorphic to a direct sum of the unit $\mathbb{1}$, so $\mathrm{Mü}(\mathcal{C}) \simeq \mathrm{Vect}_{\Bbbk}^{fd}$. ♢

Definition 1.4.2: Let X, Y be objects of $\mathcal{C}$. We have a non-degenerate pairing

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,X) & \to & \Bbbk \\ (f\ ,\ g) & \mapsto & \mathrm{tr}(f\circ g) \end{array}$$

We denote $\Omega_{X,Y} \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,X)$ the associated copairing, namely

$$\Omega_{X,Y} = \sum_j \phi_j \otimes \phi^j$$

for some pair of dual basis $(\phi_j)_j$ of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $(\phi^j)_j$ of $\mathrm{Hom}_{\mathcal{C}}(Y,X)$. We abbreviate $\Omega_X := \Omega_{X,\mathbb{1}}$.

1.4.2 Crane–Yetter (3+1)-TQFTs

Let $\mathcal{C}$ be a chromatic-non-degenerate ribbon fusion category. We define a (3+1)-TQFT $CY_{\mathcal{C}} : \mathbf{Cob}_{3+1} \to \mathrm{Vect}_{\Bbbk}$ by using Theorem 1.3.8. We do not give the historical state sum definition of Crane and Yetter [CY93] (extended to possibly non-modular categories later in [CKY97]), but a skein-theoretic one. Both approaches match using Robert's chain-mail construction, see [Tha21] for an account in this context.
We start by defining the state spaces.

Definition 1.4.3: Let M be a closed oriented 3-manifold. The *$\mathcal{C}$-skein module $Sk_{\mathcal{C}}(M)$* of M is the vector space generated by isotopy classes of $\mathcal{C}$-colored framed ribbon graphs in M modulo skein relations, namely the skein module with empty boundary conditions from Definition 1.2.17. ♢

Skeins, as embedded ribbon graphs, transport naturally under a diffeomoprhism $M \to M'$. The relations being local, one can easily check that $Sk_{\mathcal{C}}(M_1 \sqcup M_2) \simeq Sk_{\mathcal{C}}(M_1) \otimes Sk_{\mathcal{C}}(M_2)$. This construction gives a symmetric monoidal functor $Sk_{\mathcal{C}} : \mathbf{Man}_3 \to \mathrm{Vect}_{\Bbbk}$. We now simply have to describe maps associated to handle attachments.

Definition 1.4.4 (The 0-handle): Let $\mathbb{S} : \emptyset \hookrightarrow M$ be a framed -1-sphere. The 0-handle is the cobordism $W(\mathbb{S}) = D^4 \sqcup M \times [0,1] : M \to S^3 \sqcup M$. We set

$$CY_{\mathcal{C}}(W(\mathbb{S})) : \begin{array}{ccccc} Sk_{\mathcal{C}}(M) & \to & Sk_{\mathcal{C}}(S^3) & \otimes & Sk_{\mathcal{C}}(M) \\ T & \mapsto & d(\mathcal{C}).\emptyset & \otimes & T \end{array}$$

where $\emptyset$ denotes the empty skein. ♢

Definition 1.4.5 (The 1-handle): Let $\mathbb{S} : S^0 \times D^3 \hookrightarrow M$ be a framed 0-sphere. The 1-handle is a cobordism $W(\mathbb{S}) : M \to M(\mathbb{S})$ where $M(\mathbb{S})$ is obtained from M by removing the two balls $\mathbb{S}$ and joining them by a $[0,1] \times S^2$. For $T \in Sk_{\mathcal{C}}(M)$, one can isotope T so that it does not intersect $\mathbb{S}$. This gives a skein in $Sk_{\mathcal{C}}(M \smallsetminus \mathbb{S})$. As $M \smallsetminus \mathbb{S}$ is a submanifold of $M(\mathbb{S})$, it transports to a skein $T' \in Sk_{\mathcal{C}}(M(\mathbb{S}))$. See Figure 1.4. We set

$$CY_{\mathcal{C}}(W(\mathbb{S}))(T) = \frac{1}{d(\mathcal{C})} T'.$$

This process is well-defined because two skeins avoiding $\mathbb{S}$ that are isotopic in M are also isotopic by an isotopy in $M \smallsetminus \mathbb{S}$. $\diamondsuit$

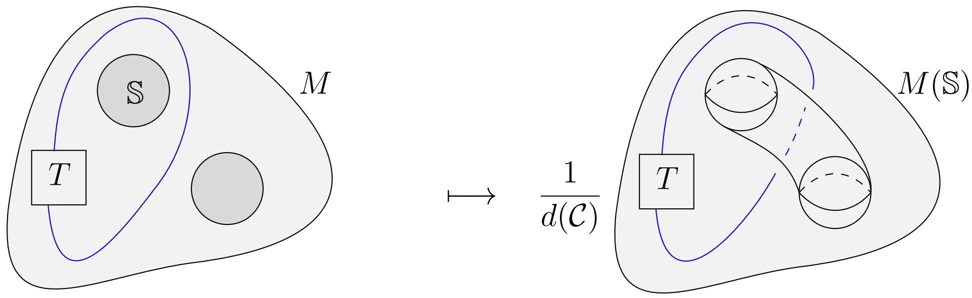


Figure 1.4: The 1-handle

Definition 1.4.6 (The 2-handle): Let $\mathbb{S} : S^1 \times D^2 \hookrightarrow M$ be a framed 1-sphere. The 2-handle is a cobordism $W(\mathbb{S}) : M \to M(\mathbb{S})$ where $M(\mathbb{S})$ is obtained by doing surgery on M along $\mathbb{S}$, namely removing $\mathbb{S}$ and gluing back another solid torus but switching meridian and longitude. We call $l = \mathbb{S}(S^1 \times 1) \subseteq \partial(M \smallsetminus \mathbb{S})$ the longitude and $m = \mathbb{S}(1 \times S^1) \subseteq \partial(M \smallsetminus \mathbb{S})$ the meridian. The meridian (in red in Figure 1.5) bounds a disk in M, but may not bound one in $M(\mathbb{S})$, and the opposite is true for the longitude (in dark purple).
Again a skein T can be pushed to a skein T' disjoint from $\mathbb{S}$, and is mapped to itself union a copy m_G^{c} of the meridian colored by the generator G and with a single coupon colored by the chromatic morphism c, see Figure 1.5. So

$$CY_{\mathcal{C}}(W(\mathbb{S}))(T) = T' \cup m_G^{\mathsf{c}} \ .$$

This process is well-defined by Proposition 3.4.2, in the special case of a semisimple category. $\diamondsuit$

Definition 1.4.7 (The 3-handle): Let $\mathbb{S} : S^2 \times I \hookrightarrow M$ be a framed 2-sphere. The 3-handle is a cobordism $W(\mathbb{S}) : M \to M(\mathbb{S})$ where $M(\mathbb{S})$ is obtained by cutting $\mathbb{S}$ at the middle sphere $S^2 \times \{\frac{1}{2}\}$, and filling both sides with a 3-ball.
This time in general a skein cannot pushed to be disjoint from $\mathbb{S}$, as it may go through it. It can however be isotoped to be transverse to $S^2 \times \{\frac{1}{2}\}$, and (fusing tensor products) to have only one strand going through it.
A skein $T \subset M$ with only one transverse intersection with $S^2 \times \{\frac{1}{2}\}$ at a strand colored by an object X is mapped to a linear combination of skeins obtained from T by cutting the strand X and adding two coupons on either sides. These coupons are colored by Ω_X. Writing $\Omega_X = \sum_j \phi_j \otimes \phi^j$ as in Definition 1.4.2, we set

$$CY_{\mathcal{C}}(W(\mathbb{S}))(T) = \sum_j T_j^{cut}$$

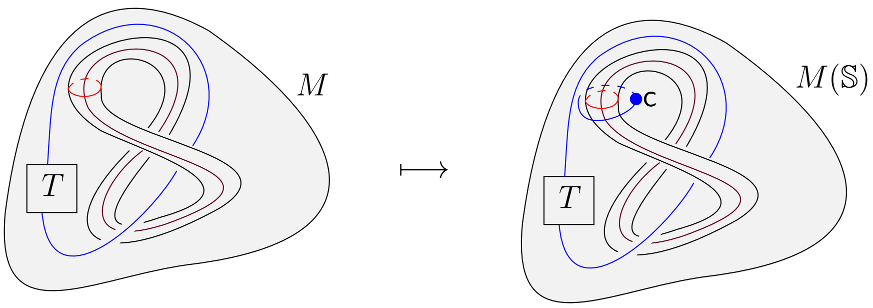


Figure 1.5: The 2-handle

where T_j^{cut} is the skein T, cut at $T \cap S^2 \times \{\frac{1}{2}\}$, filled with two coupons (so now living in $M(\mathbb{S})$) couloured by ϕ_j and ϕ^j, see Figure 1.6.
This process is well-defined by Proposition 3.4.3, in the special case of a semisimple category. ◊

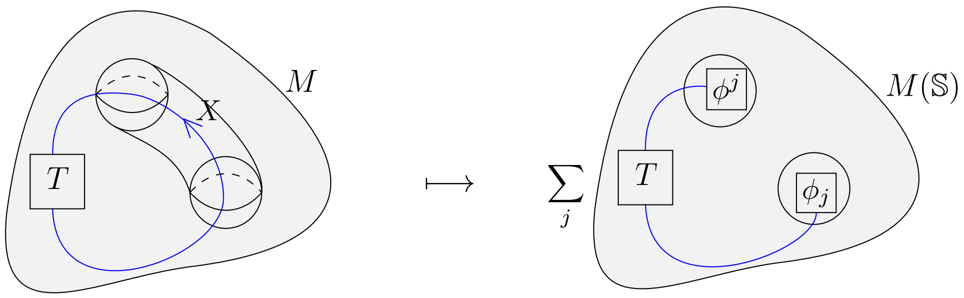


Figure 1.6: The 3-handle

Definition 1.4.8 (The 4-handle): Let $\mathbb{S} : S^3 \hookrightarrow M$ be a framed 3-sphere. It has to be a connected component of M diffeomorphic to S^3. So up to diffeomorphism we can consider that $M = S^3 \sqcup M'$. The 4-handle is the cobordism $W(\mathbb{S}) = D^4 \sqcup M' \times [0,1] : S^3 \sqcup M' \to M'$. We set

$$CY_{\mathcal{C}}(W(\mathbb{S})) : \begin{array}{ccccc} Sk_{\mathcal{C}}(S^3) & \otimes & Sk_{\mathcal{C}}(M') & \to & Sk_{\mathcal{C}}(M') \\ T & \otimes & T' & \mapsto & \mathrm{RT}_{\mathcal{C}}(T).T' \end{array}$$

where $\mathrm{RT}_{\mathcal{C}}$ is the Reshetikhin–Turaev evaluation. ◊

Theorem 1.4.9 (special case of Theorem 3.4.4): *There exists a unique TQFT, called the Crane–Yetter TQFT associated with $\mathcal{C}$,*

$$CY_{\mathcal{C}} : \textbf{\textit{Cob}}_{3+1} \to \mathrm{Vect}_{\Bbbk}$$

which coincides with $Sk_{\mathcal{C}}$ on $\textbf{\textit{Man}}_3$ and with the definitions of $CY_{\mathcal{C}}(W(\mathbb{S}))$ given above on handle attachments.

The fact it does coincide with the usual definition of the Crane–Yetter TQFT using state sums is shown in Theorem 3.5.1 using Robert's chain-mail construction as detailed in [Tha21].

Remark 1.4.10: It is strongly expected [Walb, Joh21, BJS21, Tha21] that the Crane–Yetter theory extends down and assigns values to surfaces (and actually 0 and 1-manifolds) and is functorial with respect to 3-cobordisms and 4-cobordisms with corner. A more precise notion of extended TQFT will be introduced in Chapter 2. The expected values of the Crane–Yetter theory on surfaces are skein categories described in Section 1.2.2. 3-dimensional cobordisms induce some kind of bimodule between skein categories via the skein module functor construction [Walb, GJS23, Tha21].

1.4.3 Witten–Reshetikhin–Turaev (2+1)-TQFTs

In this section we pick a ribbon fusion category $\mathcal{C}$ which is modular. In this case, it is known that the Crane–Yetter TQFT described above is very simple: every state space is one-dimensional, and the scalar invariants associated with 4-manifolds depend only on the Euler characteristic χ and the signature σ, see Theorem 3.4.8 and [CKY97, Proposition 6.2]. However, there is still an interesting theory living "at its boundary". Again, this point of view is not the historical definition of [RT91, Tur94], but was developed later by Kevin Walker to understand the anomaly of Witten–Reshetikhin–Turaev theories.

Remark 1.4.11: In every state space $Sk_{\mathcal{C}}(M)$ of the Crane–Yetter TQFT, there is a canonical vector given by the empty skein, denoted $\emptyset$. ◇

Definition 1.4.12: Let M be an oriented closed 3-manifold and W a bounding 4-manifold. We define a scalar invariant of pairs (M, W) as:

$$WRT_{\mathcal{C}}(M, W) := CY_{\mathcal{C}}(W)(\emptyset)$$

To obtain a (2+1)-TQFT we expect an invariant of closed 3-manifold, not depending on the data of a bounding 4-manifold. It is possible to renormalize the contribution of the 4-manifold W because the Crane–Yetter TQFT is very simple for a modular category $\mathcal{C}$.

Theorem 1.4.13 (special case of Theorems 3.3.2 and 3.3.3): *Let M be an oriented connected closed 3-manifold. It is obtained from S^3 as surgery on some link L. Denote $\dot{W} : M \to S^3$ the 4-cobordism obtained by composing the 2-handles described by the link L. Denote $W : M \to \emptyset$ the 4-cobordism obtained by composing $\dot{W}$ with the 4-handle $D^4 : S^3 \to \emptyset$. Choose $\mathscr{D}$ a square root of $d(\mathcal{C})$. Then*

$$\begin{aligned} WRT_{\mathcal{C}}(M) &:= \frac{\mathscr{D}^{-1}}{\Delta_+^{\frac{\chi(W)+\sigma(W)-1}{2}} \Delta_-^{\frac{\chi(W)-\sigma(W)-1}{2}}} CY_{\mathcal{C}}(W)(\emptyset) \\ &= \mathscr{D}^{-\chi(W)} \left(\frac{\Delta_+}{\mathscr{D}}\right)^{-\sigma(W)} CY_{\mathcal{C}}(W)(\emptyset) \end{aligned}$$

is a well-defined invariant of 3-manifolds, and does not depend on the choice of L and hence of W. (Note that $\frac{\chi(W)+\sigma(W)-1}{2}$ and $\frac{\chi(W)-\sigma(W)-1}{2}$ are integers)

Remark 1.4.14: We have added a factor $\mathscr{D}^{-1-b_1(M)}$ to the definition of Theorem 3.3.2 so that our invariant coincides with the usual definition of [Tur94, Theorem 2.2.2]. The first expression emphasizes that the choice of $\mathscr{D}$ is unnecessary to define the 3-manifold invariant (and is not made in Theorem 3.3.2).

The second expression emphasizes that the factor in front of $CY_{\mathcal{C}}(W)(\emptyset)$ is here to renormalize the contribution of the choice of W. Since in this case the Crane–Yetter 4-manifold invariant depends only on the Euler characteristic χ and the signature σ, we only need these two factors $\lambda^\chi\mu^\sigma$. Indeed for any two choices W and W' of bounding manifolds for M, the maps $CY_{\mathcal{C}}(W)$ and $CY_{\mathcal{C}}(W')$ differ by a scalar $\lambda^{\chi(W)-\chi(W')}\mu^{\sigma(W)-\sigma(W')}$, so that $\lambda^{-\chi(W)}\mu^{-\sigma(W)}CY_{\mathcal{C}}(W)$ is independent of the choice of W.

Note that there are no closed 4-manifolds with signature 0 and Euler characteristic 1 (they always have same parity) so we cannot find the appropriate coefficients λ and μ simply by evaluating the Crane–Yetter TQFT on some 4-manifolds and one needs to make a choice of square root. For example, taking $W' = W \sqcup S^4$ above we obtain that $\lambda^2 = CY_{\mathcal{C}}(S^4) = d(\mathcal{C})$ and taking $W' = W \sqcup \mathbb{C}P^2$ we obtain that $\lambda^3\mu = CY_{\mathcal{C}}(\mathbb{C}P^2) = d(\mathcal{C})\Delta_+$, but we cannot get any other condition. These condition will determine an index 2 lattice for the values of χ and σ, and we have the choice of a square root for $\lambda^2 = d(\mathcal{C})$.◇

The WRT TQFT is then usually obtained from this 3-manifold invariant through a process called the universal construction, see [Tur94, BHMV95]. We can describe them alternatively using the expected extension of the Crane–Yetter theory to dimension 2 by skein categories.

Remark 1.4.15: In every skein category $Sk_{\mathcal{C}}(\Sigma)$ there is a canonical object given by the empty set of points. ◇

Definition 1.4.16: Let Σ be a closed oriented surface and M a bounding 3-manifold. We define the state space as the relative skein modules defined in Definition 1.2.17

$$WRT_{\mathcal{C}}(\Sigma, M) := Sk_{\mathcal{C}}(M, \emptyset)$$

Again, these vector spaces depend only mildly on the bounding manifold M, and any two choices give isomorphic state spaces. However, one needs to track these isomorphisms, and there is no purely canonical way of defining $WRT_{\mathcal{C}}(\Sigma)$ as there was for the scalar invariant above. The WRT TQFT will in fact not be a TQFT in the usual sense, but be *anomalous*. It is sometimes said that the Crane–Yetter TQFT is the anomaly of the WRT TQFT. We will not recall in detail how to fix this anomaly. The following theorem follows from [Tur94]. For a more modern treatment closer to the point of view we have used here see [De 17, Chapter 1].

Theorem 1.4.17: *There exists a symmetric monoidal functor*

$$WRT_{\mathcal{C}} : \widetilde{\boldsymbol{Cob}}_{2+1} \to \mathrm{Vect}_{\Bbbk}$$

where $\widetilde{\boldsymbol{Cob}}_{2+1}$ is the category of surfaces equipped with a Lagrangian in their first homology group, and 3-cobordisms equipped with an integer. Composition is given by usual composition and adding the integers plus the Maslov index of the three featured Lagrangians.
It satisfies that for a closed 3-manifold M equipped with the integer 0, the scalar $WRT_{\mathcal{C}}(M, 0)$ coincides with $WRT_{\mathcal{C}}(M)$ defined above, and for a closed surface Σ equipped with a Lagrangian L, the vector space $WRT_{\mathcal{C}}(\Sigma, L)$ is isomorphic to $WRT_{\mathcal{C}}(\Sigma, M)$ for any choice of bounding 3-manifold M.

Remark 1.4.18: From our point of view, we can interpret the extra data in the definition of $\widetilde{\mathbf{Cob}}_{2+1}$ as coming from bounding manifolds, though we only remember the data needed to renormalize the contribution of the bounding manifold. This way, Lagrangians $L \subseteq H_1(\Sigma)$ correspond to the choice of a bounding 3-manifold M with $L = \ker(H_1(\Sigma) \to H_1(M))$. The integer that 3-cobordisms are equipped with corresponds to the data of the signature of a bounding 4-manifold. The reason we have to keep track of this data is that two 4-manifold with corners with signature 0, glued on a corner surface, may give a 4-manifold of non-zero signature. The Maslov index computes precisely this defect. On the other hand, the Euler characteristic glues very nicely, and actually the whole Crane–Yetter TQFT can be twisted to not depend on the Euler characteristic, see Proposition 3.4.7. ◇

Remark 1.4.19: The observation that WRT can be obtained from Crane–Yetter in this way is due to Kevin Walker, though the full construction is not formalized yet. Chapter 4 is an attempt towards a formalization. ◇

1.5 Non-semisimple skein theory

When the category $\mathcal{C}$ is no longer supposed to be semisimple, a lot of the techniques above fall apart. One can check that if $\mathcal{C}$ is ribbon tensor and *not* semisimple, then the unit $\mathbb{1}$ is not projective [EGNO15, Corollary 4.2.13]. Every projective object has 0 quantum dimension [GKP11, Corollary 4.4.2 with $J = \mathbb{1}$]. The Reshetikhin–Turaev invariant of any closed $\mathcal{C}$-colored ribbon graph containing some projective color is zero [GKP11, Theorem 1.4.1 (2) with $J = \mathbb{1}$] and the trace pairing is degenerate. There is however a way of obtaining information from these graphs with projectives, which we present here.

1.5.1 Non-degenerate modified traces

We use the notation and terminology of [CGP]. Let $\mathcal{C}$ be a ribbon tensor category. Recall that in [EGNO15] a *tensor category* is defined as a locally finite $\Bbbk$-linear abelian rigid monoidal category with $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$.

Definition 1.5.1: Let Proj be the ideal of projective objects of $\mathcal{C}$. A *non-degenerate m-trace on* Proj is family of linear maps $\mathsf{t} = \{\mathsf{t}_P : \mathrm{End}_{\mathcal{C}}(P) \to \Bbbk\}_{P \in \mathrm{Proj}}$ satisfying the following properties:

1. **Cyclicity:** For all $U, V \in$ Proj and morphisms $V \xrightarrow{f} U$, $U \xrightarrow{g} V$, we have $\mathsf{t}_V(gf) = \mathsf{t}_U(fg)$.
2. **Right partial trace:** If $U \in$ Proj and $W \in \mathcal{C}$, then for any $f \in \mathrm{End}_{\mathcal{C}}(U \otimes W)$,
$$\mathsf{t}_{U \otimes W}(f) = \mathsf{t}_U\Big((\mathrm{Id}_U \otimes \overrightarrow{\mathrm{ev}}_W)(f \otimes \mathrm{Id}_{W^*})(\mathrm{Id}_U \otimes \overleftarrow{\mathrm{coev}}_W)\Big). \tag{1.1}$$
3. **non-degeneracy**: For any $P \in$ Proj, the pairing $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1}) \to \Bbbk$ given by $(x, y) \mapsto \mathsf{t}_P(x \circ y)$ is non-degenerate. ◇

Note since $\mathcal{C}$ is ribbon the m-trace also satisfies the left partial trace property similar to Equation (1.1).

Definition 1.5.2: Let $\mathrm{Rib}_{\mathcal{C}}^{adm}$ be the subset of all closed $\mathcal{C}$-colored ribbon graphs in S^3 obtained as the braid closure of a (1,1)-ribbon graph T_P whose open edge is colored with an object $P \in \mathrm{Proj}$. Equivalently it is the set of $\mathcal{C}$-colored ribbon graphs in S^3 with at least one edge colored by a projective object. The elements of $\mathrm{Rib}_{\mathcal{C}}^{adm}$ are called *admissible ribbon graphs* in S^3. ♢

Recall that $\mathrm{RT}_{\mathcal{C}}$ is the Reshetikhin-Turaev functor from the category $\mathrm{Rib}_{\mathcal{C}}$ of $\mathcal{C}$-colored ribbon graphs in $\mathbb{R}^2 \times [0,1]$ to $\mathcal{C}$.

Theorem 1.5.3 (see [GPT09, GP18]): *The assignment*

$$F' : \mathrm{Rib}_{\mathcal{C}}^{adm} \to \Bbbk \text{ given by } F'(L) = \mathsf{t}_P(\mathrm{RT}_{\mathcal{C}}(T_P))$$

is an isotopy invariant of $\mathcal{C}$-colored ribbon graphs L in S^3.

Definition 1.5.4: A category is *unimodular* if its unit object has a self dual projective cover. ♢

Theorem 1.5.5 (Corollary 5.6 in [GKP22]): *If $\mathcal{T}$ is unimodular and pivotal then it has a unique (up to scalar) non-degenerate m-trace on* Proj.

1.5.2 Admissible skein modules

The invariant of ribbon graphs defined above is only defined for graphs that have at least one projective color: the elements of $\mathrm{Rib}_{\mathcal{C}}^{adm}$. We can define the appropriate notion of skein relations and skein modules for these admissible graphs.

Definition 1.5.6: Let M be a compact oriented 3-manifold. An *admissible ribbon graph* in M is a $\mathcal{C}$-colored ribbon graph in M where each connected component of M contains at least one edge colored with a projective object.
A *projective skein relation* is a skein relation as in Definition 1.2.17, namely a linear combination of admissible ribbon graphs $\sum_i \lambda_i T_i$ is equivalent to 0 if they agree outside a cube ϕ and evaluate to 0 inside the cube, but with the extra condition that each T_i must have some projective-colored strand not entirely inside the cube ϕ.
The quotient of $\mathrm{Rib}_{\mathcal{C}}^{adm}(M)$ by projective skein relations is called the *admissible skein module $\mathscr{S}_{\mathcal{C}}(M)$ of M*[1]. ♢

Theorem 1.5.7: *If $\mathcal{C}$ is finite, for any oriented compact 3-manifold M the vector space $\mathscr{S}_{\mathcal{C}}(M)$ is finite dimensional.*

Proof : In the case of surfaces this theorem is [CGP, Theorem 5.10]. Let M be a connected 3-manifold, decomposed into one 0-handle, g index 1 handles, and some index 2 and 3 handles. Then $\mathscr{S}_{\mathcal{C}}(M)$ is generated by admissible skeins in the genus g handlebody H_g formed by the handles of index 0 and 1. Then we conclude by observing that $H_g = \Sigma \times [-1,1]$ for some orientable surface Σ, therefore $\dim \mathscr{S}_{\mathcal{C}}(H_g) = \dim \mathscr{S}_{\mathcal{C}}(\Sigma) < \infty$.

[1] Since we only work with the ideal of projective objects, in this manuscript, we denote by $\mathscr{S}_{\mathcal{C}}$ the skein module $\mathcal{S}_{\mathrm{Proj}}$ of [CGP].

Finally for non connected manifolds the skein modules are tensor products of those of the connected components. □

By [CGP, Proposition 2.2] the assignment $M \to \mathscr{S}_{\mathcal{C}}(M)$ extends to a functor

$$\mathscr{S}_{\mathcal{C}} : \mathrm{Emb}_n \to \mathrm{Vect}_{\Bbbk}$$

where Emb_n is the category whose objects are oriented n-dimensional manifolds and morphisms are isotopy classes of orientation preserving proper embeddings.

1.5.3 DGGPR (2+1)-TQFTs

A (2+1)-TQFT was built in [DGG+22] using some of the techniques introduced above. They are expected to be related to the non-semisimple TQFTs of [BCGP16], and they share similar features. These TQFTs are anomalous in the same way WRT TQFTs are. They are not defined on the usual category of cobordism but on a category of cobordisms with extra structure which corresponds to the data needed to renormalize the contribution of a bounding manifold. In the semisimple case this contribution was noticed to correspond to the Crane–Yetter TQFT by Kevin Walker, but the analogous (3+1)-TQFT for non-semisimple categories was only introduced later in [CGHP] (and is the subject of Chapter 3). The [DGG+22] and [BCGP16] TQFTs are also non-compact. This means that they are not defined on the whole category of cobordisms, but only for some class of admissible 3-cobordisms.

Definition 1.5.8: The category $\widetilde{\mathbf{Cob}}_{2+1}^{nc}$ is the subcategory of $\widetilde{\mathbf{Cob}}_{2+1}$ containing every object and only those 3-cobordisms that have incoming boundary in every connected component. Said differently, $M : \Sigma \to \Sigma'$ in $\widetilde{\mathbf{Cob}}_{2+1}$ belongs to $\widetilde{\mathbf{Cob}}_{2+1}^{nc}$ if and only if $\pi_0(\Sigma) \to \pi_0(M)$ is surjective. ♢

Theorem 1.5.9 ([DGG+22]): *Let $\mathcal{C}$ be a finite ribbon tensor category which is modular in the sense that it's Müger center is trivial. Then there exists a symmetric monoidal functor*

$$\mathrm{DGGPR}_{\mathcal{C}} : \widetilde{\boldsymbol{Cob}}_{2+1}^{nc} \to \mathrm{Vect}_{\Bbbk}$$

We can identify part of their construction as in Theorem 1.4.17:

Theorem 1.5.10: *The state space $\mathrm{DGGPR}_{\mathcal{C}}(\Sigma)$ on a closed surface Σ is isomorphic to the admissible skein module $\mathscr{S}_{\mathcal{C}}(H)$ where H is a bounding handlebody.*

Note that a surjection $\pi : \mathscr{S}_{\mathcal{C}}(H) \twoheadrightarrow \mathrm{DGGPR}_{\mathcal{C}}(\Sigma)$ was already constructed in [DGG+22, Proposition 4.11] (in our case bichrome graphs can be turned blue by [DGG+22, Lemma 4.5]). We give another isomorphism below, but our proof in particular implies that π is an isomorphism.
We will first need a lemma which gives a more explicit description of $\mathscr{S}_{\mathcal{C}}(H)$.

Lemma 1.5.11: *Let $\mathcal{C}$ be a ribbon tensor category and H a genus-g handlebody. Then there is a vector space isomorphism*

$$\mathscr{S}_{\mathcal{C}}(H) \simeq \Big(\bigoplus_{(P_i)_i \in \mathrm{Proj}^g} \mathrm{Hom}_{\mathcal{C}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1}) \Big) / \langle (f, \mathrm{Id}) \sim (\mathrm{Id}, f^*), f : P_i \to P_i' \rangle$$

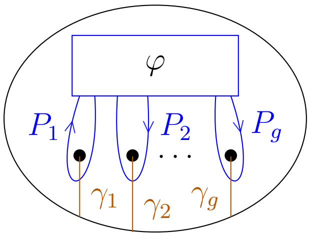
where for $f : P_i \to P'_i$ and $\psi : P_1 \otimes P_1^* \otimes \cdots \otimes P'_i \otimes P_i^* \otimes \cdots \otimes P_g \otimes P_g^* \to \mathbb{1}$, the relation $(f, \mathrm{Id}) \sim (\mathrm{Id}, f^*)$ denotes the usual "coend" relation

$$\psi \circ (\mathrm{Id} \otimes \cdots \otimes f \otimes \mathrm{Id}_{P_i^*} \otimes \cdots \otimes \mathrm{Id}) \sim \psi \circ (\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}_{P_i} \otimes f^* \otimes \cdots \otimes \mathrm{Id}) \ .$$

Proof : Denote by E_g the vector space on the right hand side.

Observe that H is a thickening of a g-punctured disk D_g, hence $\mathscr{S}_\mathcal{C}(H) \simeq \mathscr{S}_\mathcal{C}(D_g)$ as defined in [CGP]. Choose disjoint arcs $\gamma_1, \ldots, \gamma_g$ joining each puncture to the boundary. Cutting along these arcs gives a disk $D = D_g \smallsetminus (\cup_i \gamma_i)$.

There is a map $E_g \to \mathscr{S}_\mathcal{C}(D_g)$ which maps a morphism φ to a unique coupon in D colored by φ with g pairs of strand coming out of it, each going around one of the punctures, so intersecting γ_i once, and colored by P_i, as shown below.



It is well-defined because the relations $(f, \mathrm{Id}) \sim (\mathrm{Id}, f^*)$ are satisfied by isotopy and skein relations in $\mathscr{S}_\mathcal{C}(D_g)$. We now construct its inverse.

An admissible ribbon graph in D_g is said to be in good position if it intersects each γ_i transversely and along at least one projective edge.

From an admissible ribbon graph T in D_g in good position we can obtain an element of E_g as follows. Let P_i denote the tensor product of the colors of the edges crossing γ_i. It is projective by our assumption. Then, cutting T along the γ_i's, we obtain a ribbon graph T' in D which can be evaluated to a single coupon φ. The morphism φ has source the tensor product of the boundary point of T' and target $\mathbb{1}$, namely

$$\varphi : P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^* \to \mathbb{1} \ .$$

We need to put every admissible ribbon graph in good position and study the relations between them.

Consider the vector space $\mathrm{Rib}_\mathcal{C}^{adm}(D_g)$ freely generated by admissible ribbon graphs in D_g which intersect the arcs γ_i transversely. It is enough to consider these because generically every ribbon graph intersects the arcs transversely. The admissible skein module it its quotient by isotopy and admissible skein relations.

Moreover, generically every isotopy can be decomposed into a serie of isotopies of the form: 1) isotopies in D, 2) a coupon crossing a γ_i and 3) a cup or a cap crossing a γ_i. Finally, up to isotopy, every admissible skein relation can be supposed to happen in D.

In other words, $\mathscr{S}_\mathcal{C}(D_g)$ is the quotient of $\mathrm{Rib}_\mathcal{C}^{adm}(D_g)$ by the equivalence relation generated by the three isotopy moves described above and admissible skein relations in D.

Now let us produce an admissible ribbon graph in good position from an admissible ribbon graph T in $\mathrm{Rib}_\mathcal{C}^{adm}(D_g)$. Find a projective strand of T in D and, doing a snake move and adding a coupon, produce a strand colored by $Q := P \otimes P^*$ coming out of the P-strand, and ending with a coupon $coev_P : Q \to \mathbb{1}$. One could equivalently use the trick of [DGG+22, Figure 8] to produce a $P_\mathbb{1}$ strand ending with a coupon ε. Then drag this strand to the boundary of D_g crossing above every other edge (using braiding coupons) and go around the whole boundary, intersecting every γ_i. The resulting ribbon graph is in good position and can therefore be interpreted as an element of E_g.

We need to show that this process is well-defined on $\mathrm{Rib}_{\mathcal{C}}^{adm}(D_g)$. Suppose you have chosen a different projective strand colored by P', or indeed the same strand at a different place, and that you joined the boundary using a different path. Then one can do both operations, and have two projective strands going through every γ_i. Now one can undo either of these operations, isotoping the Q or Q' strand back in its initial position, and the ribbon graph will stay in good position throughout because there is always the other projective. This isotopy is obtained as a sequence of relations in E_g whenever the coevaluation coupon crosses a γ_i, and the map $\mathrm{Rib}_{\mathcal{C}}^{adm}(D_g) \to E_g$ is well-defined.

Now we can check it descends to $\mathscr{S}_{\mathcal{C}}(D_g)$. Admissible skein relations and isotopies in D do not affect φ. Coupons, cups and caps crossing a γ_i are relations in E_g, up to doing the operation above to ensure that there is always projective intersecting γ_i.

It is clear from the definition that these two maps are inverses to each other, and we get the isomorphism. ☐

Proof (of Theorem 1.5.10): The description above can now more easily be related to the coend used in [DGG+22]. Remember that the coend $\mathscr{L}$ is defined as the colimit

$$\mathscr{L} = \int^{X\in\mathcal{C}} X \otimes X^* = \Big(\oplus_{X\in\mathcal{C}} X \otimes X^* \Big) / \langle (f, \mathrm{Id}) \sim (\mathrm{Id}, f^*), f : X \to Y \rangle$$

We only consider projectives in our case, but this does not change this colimit, and $\mathscr{L} \simeq \int^{P\in\mathrm{Proj}} P \otimes P^*$ by [KL01, Proposition 5.1.7]. Note that by [KL01, Corollary 5.1.8], the infinite nature of this colimit is unnecessary, and we could allow only $P = G$ the projective generator. We will still denote it $\int^{P\in\mathrm{Proj}}$, but it will be useful to remember that everything is finite.
It is shown in [DGG+22, Proposition 4.17 and Lemma 4.1 at $V = \mathbb{1}$] that

$$\mathrm{DGGPR}_{\mathcal{C}}(\Sigma) \simeq \Big(\mathrm{Hom}_{\mathcal{C}}(\mathscr{L}^{\otimes g}, \mathbb{1})\Big)^* .$$

Using the definition of the colimit, the vector space $\mathrm{Hom}_{\mathcal{C}}(\mathscr{L}^{\otimes g}, \mathbb{1})$ is obtained as a limit: the subspace of the product $\Pi\, \mathrm{Hom}_{\mathcal{C}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1})$ of the collection that satisfy the $(f, \mathrm{Id}) \sim (\mathrm{Id}, f^*)$ relations. The dual of this limit is then (using the fact everything is finite) the colimit

$$\mathrm{DGGPR}_{\mathcal{C}}(\Sigma) \simeq \Big(\bigoplus_{(P_i)_i\in\mathrm{Proj}^g} \mathrm{Hom}_{\mathcal{C}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1})^* \Big) / \langle (f, \mathrm{Id}) \sim (\mathrm{Id}, f^*), f : P_i \to P_i' \rangle$$

This is almost the same as the formula we gave for $\mathscr{S}_{\mathcal{C}}(H)$, though there are duals. We have an isomorphism $\mathrm{Hom}_{\mathcal{C}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1})^* \simeq \mathrm{Hom}_{\mathcal{C}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1})$ given by the modified trace paring, and noticing that by design $P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*$ is self-dual up to isomorphism. These isomorphisms preserve the $(f, \mathrm{Id}) \sim (\mathrm{Id}, f^*)$ relations, and induce an isomorphism on the quotient. Hence the result. ☐

The TQFT of [DGG+22] is actually defined on a bigger category of cobordisms where cobordisms have embedded colored graphs. In this case, they give invariants of closed 3-manifolds M equipped with an admissible ribbon graph $T \subseteq M$. We will see in Theorem 3.3.3 that this invariant coincides, up to a factor $\mathscr{D}^{-1-b_1(M)}$, with the one given in Theorem 3.3.2 for the same $\mathcal{C}$.

Chapter 2

Higher Algebra

In this second preliminary chapter we will review the higher-algebraic tools that we intend to use to describe and classify skein-theoretic Topological Quantum Field Theories. We will recall the general notion of fully extended TQFT and their classification by the cobordism hypothesis, mostly following [Lur09b].

This uses a fair amount of higher algebra and we will recall as much as is feasible. The reader unfamiliar with higher category theory may still find interest in the study of presentable cocomplete categories in Section 2.3.1 and in the explicit descriptions of objects and higher morphisms of **BrTens** in Section 2.3.4.

In Section 2.1, we define complete n-fold Segal space, our model for higher categories, in detail. We give some standard constructions there, in particular the notion of dualizability in Section 2.1.6. However, we will only give a quick overview of the definition of higher functors, which relies on model category theory.

In Section 2.2, we recall the broad lines of the construction of the symmetric monoidal complete n-fold Segal space $\mathbf{Bord}_n$. We state the cobordism hypothesis, and its oriented version.

In Section 2.3, we recall the definition of even higher Morita categories $\mathbf{Alg}_2(\mathbf{Pr})$ by [Hau17, JS17]. We only sketch Haugseng's construction of higher Morita categories which relies on Lurie's ∞-categories and the straightening-unstraightening equivalence. Note that Haugseng identifies his construction quite well, which we recall. We give in detail Johnson-Freyd–Scheimbauer's extension to "even higher" Morita categories. We recall the explicit description of $\mathbf{Alg}_2(\mathbf{Pr})$ under the name **BrTens** from [BJS21]. We prove in Theorem 2.3.29 that this description does recover the underlying bicategories of $\mathbf{Alg}_2(\mathbf{Pr})$. Finally we recall the known dualizability results in **BrTens**.

2.1 Higher categories

Higher category theory is a recent and still under development field. Its starting point is the study of structures that have coherences *up to homotopy*, or more generally whose coherences represent *some extra data* that should itself be subject to higher coherences. The initial example is the space of based loop ΩX on a topological space X. It has a product given by concatenation of loops, which is associative up to homotopy, namely there is a path in ΩX joining any two way of parenthesizing a product of loops. Moreover these associativity-paths are themselves coherent: on a product of many loops, there are multiple ways to link two parenthesizings, but all resulting associativity-paths are homotopic. We say that ΩX is an A_∞-algebra in the category of topological spaces.

Now this implicitly uses some extra structure on the category of topological spaces: homotopies. We want to make sense of an equally true statement: the complex of singular chains $Sing(\Omega X)$ on ΩX forms an A_∞-algebra in the category of chain complexes. But what is a homotopy in the category of chain complexes? We do know how to make sense of this (it is a degree 1 map of chain complexes), but these are just examples and we want to develop a general theory. What is a category *with homotopies*? How do we say that a functor (e.g. *Sing* above) preserves homotopies? One approach that has been more pursued in the context of algebraic topology described above is the notion of model categories, where one makes sense abstractly of homotopies and weak equivalences. An approach which will be more fruitful in our context is to consider that homotopies are extra data of morphisms between morphisms. In a higher category, there are objects, morphisms between these objects, morphisms between theses morphisms and so on.
A first model of higher category is that of a topological category. A topological category $\mathcal{C}$ is a category enriched in topological spaces, namely every Hom space $\mathrm{Hom}_\mathcal{C}(X,Y)$ between two objects X and Y is equipped with the structure of a topological space, and composition maps $\mathrm{Hom}_\mathcal{C}(X,Y)\times\mathrm{Hom}_\mathcal{C}(Y,Z) \to \mathrm{Hom}_\mathcal{C}(X,Z)$ are continuous. Homotopies (morphisms between morphisms) are then by definition paths in these spaces, and so on. This notion, though equivalent (in the sense of model categories) to the following, is not ideal to work with. A better-accepted definition of higher category has been developed by Lurie in [Lur09a]. It is based on simplicial complexes, hence is more combinatorial by nature and reflects well the idea of having a combinatorial data of objects, 1-morphisms between objects, 2-morphisms between 1-morphisms and so on.
Lurie's notion of ∞-category, though important in what follows, does not encapsulate every example we will be interested in. It is clear in the topological setting: every 2-morphism (by definition, homotopy) is invertible up to higher homotopy (as a homotopy can be read both ways). We will be very interested in the case where there are non-invertible higher morphisms.
We say that a model for higher categories is a model for (∞,n)-categories if every k-morphism for $k > n$ is invertible. We will say it is a model for (m,n)-categories if moreover every k-morphism for $k > m$ is an identity. In this terminology, topological categories, and Lurie's notion of an ∞-category, are models for $(\infty,1)$-categories. There is a combinatorial model for (2,2)-categories: bicategories, which we will use to describe dualizability. However, for greater m and n, we quickly run short of combinatorial models. We will describe below Barwick's notion of complete n-fold Segal spaces [Bar05], which is a model for (∞,n)-category, and will be our definition of (∞,n)-category in this manuscript. We will mostly follow [Lur09b], [Sch14a] and [CS19].

2.1.1 Segal spaces

A Segal space is a model for an $(\infty,1)$-category, developed in [Rez01]. As for most constructions of models for higher categories, the definition of a Segal space starts with some usual construction for strict categories, and tries to weaken some conditions and add some structure to get a category which is only associative up to homotopy.

Definition 2.1.1: Let $\mathcal{C}$ be a category. Its *nerve* $N(\mathcal{C})$ is the simplicial set with $N(\mathcal{C})_0 = Ob(\mathcal{C})$ and $N(\mathcal{C})_k$ is the set of tuples of k composable arrows in $\mathcal{C}$. The face maps are given by either forgetting the first or last arrow, or composing two of them. The degeneracy maps are given by adding an identity. ♢

Example 2.1.2: An element of $N(\mathcal{C})_3$: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$. ◊

Proposition 2.1.3: *The simplicial set $N(\mathcal{C})$ satisfies the strict Segal condition: the canonical map, called the Segal map,*

$$N(\mathcal{C})_n \to N(\mathcal{C})_1 \times_{N(\mathcal{C})_0} \cdots \times_{N(\mathcal{C})_0} N(\mathcal{C})_1 \ ,$$

given by including every $(k, k+1)$-edge, $0 \leq k \leq n-1$, in Δ_n, is a bijection.
Moreover, any simplicial space $X_\bullet$ satisfying the strict Segal condition is isomorphic to some $N(\mathcal{C})$.

Proof : The first statement is simply the definition of $N(\mathcal{C})_n$ as tuples of n composable arrows.
For the second, we construct $\mathcal{C}$ as follow:
Its objects are given by X_0, and its morphisms by X_1. The source and target of a morphism is known from the two face maps $X_1 \rightrightarrows X_0$, and the identities by the degeneracy map $X_0 \to X_1$. The composition law is read in X_2. A pair of composable arrows is exactly an element of $X_1 \times_{X_0} X_1 \simeq X_2$. Then two of the face maps $X_2 \rightrightarrows X_1$ are the given two morphisms, the third one defines their composition. Unity of identities is given by degenerate elements of X_2, and associativity is read in X_3. □

In an $(\infty, 1)$ category, we want more structure than in a category, higher morphisms, but in a controlled manner. Every Hom set should now be an $(\infty, 0)$-category, namely a space. We denote Space $\subseteq$ sSet the full subcategory of Kan complexes [Kan57, Definition 1.1] inside the category of simplicial sets. To avoid confusion with the other types of simplicial structure, we will often consider them as topological spaces. The set of 0-simplices of a space X is called its set of points.

Definition 2.1.4 (Definition 1.4 in [CS19]): A *Segal space* is a simplicial space $X_\bullet$: $\Delta^{op} \to$ Space satisfying the Segal condition: the canonical map

$$X_n \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1,$$

is a weak equivalence. ◊

Remark 2.1.5: Note the homotopy fiber product above. By definition, the inclusion of the $(k, k+1)$-edges in Δ^n induces a map $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$. There is always a canonical map from the usual fiber product to the homotopy fiber product (a homotopy fiber product $X \times^h_Y Z$ can be described as the space of pairs of objects of X and Z together with a path between their images in Y, the usual fiber product includes as the space of trivial paths). The map displayed above is the composition $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1 \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$. Note that the second map between the usual and homotopy fiber products would always be a weak equivalence under the Reedy fibrant hypothesis in [Rez01]. However, following [Lur09b] and [CS19], we dropped this hypothesis. ◊

The idea is the same as above, X_0 is the space of objects of the category, and X_1 the space of morphisms. Source, target, identities are given by structure maps. The composition law is given by X_2 and uses an inverse to the weak equivalence above, so it is only well-defined up to equivalence. Unity and associativity are no longer strict, but are controlled through all of the X_n's. All the higher constraints are encoded there.

Definition 2.1.6: A *morphism of Segal spaces* is a morphism of simplicial spaces. One can check that on the strict version a functor is indeed the data of a morphism of simplicial sets between the nerves. ♢

Definition 2.1.7: Let X be a Segal space and x, y two point of X_0, their *Hom space* is

$$\mathrm{Hom}_X(x,y) := \{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\}.$$

Note that we allow paths in X_0 as morphisms by this homotopy fiber product.
One can extract a usual category from a Segal space X: its *homotopy category* $h_1(X)$. Its set of objects is the set of points of X_0, and is has morphisms from x to y the set

$$\mathrm{Hom}_{h_1(X)}(x,y) := \pi_0(\mathrm{Hom}_X(x,y)).$$

A morphism $f \in X_1$ is said to be *invertible* if it is in $h_1(X)$. We denote X_1^{inv} the subspace of invertible morphisms. ♢

Remark 2.1.8: In the definition above, we see that whether two objects $x, y \in X_0$ are isomorphic cannot be read entirely inside X_0: they may be isomorphic by an isomorphism in X_1 which is not realized as a path in X_0. This raises a problem. In a Segal space, there are two ways to encode an isomorphism: as a path in X_0, or as an invertible element of X_1. Therefore, there might be two Segal paces that have non-equivalent spaces of objects and of morphisms, that actually represent the "same" higher category. The appropriate notion of equivalence is the following. ♢

Definition 2.1.9: A morphism of Segal spaces $f : X \to Y$ is called a *Dwyer–Kan equivalence* if it induces an equivalence of categories $h_1(X) \to h_1(Y)$ and weak equivalences between Hom spaces $\mathrm{Hom}_X(x,y) \to \mathrm{Hom}_Y(f(x), f(y))$ for every $x, y \in X_0$. ♢

Remark 2.1.10: It is unpleasant that the homotopy types of the X_n's are ill-defined up to Dwyer–Kan equivalences, and in every construction on Segal spaces one would have to be very careful about whether the construction is invariant under such equivalences. A good solution is to decide how isomorphisms should be described, instead of leaving the choice between paths in X_0 and elements of X_1^{inv}. In the following definition, our solution is to ask that every isomorphism comes from a path in X_0. ♢

Definition 2.1.11 (Section 6 of [Rez01]): A Segal space $X_\bullet$ is called *complete* if the degeneracy map (i.e. the identities) $X_0 \to X_1^{inv}$ is a weak equivalence. ♢

In other words this demands that every invertible morphism is homotopic to an identity in a coherent way, which is the case for paths. Informally, X_1^{inv} should look like the space of free paths in X_0, which is indeed homotopy equivalent to X_0.

Theorem 2.1.12 (Section 14 in [Rez01]): *For every Segal space X, one can construct a complete Segal space $\widehat{X}$ together with a Dwyer–Kan equivalence $X \to \widehat{X}$ which is universal among maps to a complete Segal space.*

Remark 2.1.13: The completion $\widehat{X}$ is constructed very explicitly in [Rez01, Section 14]. The space $\widehat{X}_0$ is taken to have 0-cells (which we call points) the same points as X_0, but 1-cells of $\widehat{X}_0$ consist of 1-cells in the space X_1^{inv} of invertible 1-morphisms. They include in particular both 1-cells in X_0 (using the identities), and points of X_1^{inv} (seen as constant paths). k-cells are taken to be k-cells in the space of k-tuples of invertible 1-morphisms. To describe the spaces $\widehat{X}_n$, one has to make sense of "invertible morphism of n-simplices" and this is encoded as morphism out of a product $\Delta_n \times E(1)$ where $E(1)$ is the nerve of the walking isomorphism. In general, the set $\widehat{X}_n^k$ of k-cells in $\widehat{X}_n$ is defined to be

$$\widehat{X}_n^k := \mathrm{Hom}_{\mathrm{sSet}}(\Delta_n \times E(k), X_\bullet^k)$$

where $E(k)$ is the nerve of the walking k-tuple of isomorphisms. We use exponents for the spacial simplicial structure to avoid confusion with the (soon to be n-uple) categorical simplicial structure. ◇

Remark 2.1.14: The theory of model categories describes how to deal with a notion of equivalences in a category, as we did above. It suggests to replace every object by an equivalent "nice" objects (called fibrant). In our case, there is a model structure on $\mathrm{Fun}(\Delta^{op}, \mathrm{sSet})$ whose fibrant objects are the complete Segal spaces. The completion procedure we described above is called a fibrant replacement in this context. See [CS19, Section 1.4] and [JS17, Appendix A] for a more detailed account on how complete Segal spaces appear from model category theory. ◇

2.1.2 n-fold Segal spaces

We now have a tool to create a recursive model for (∞, n)-categories. We first define n-uple Segal spaces, then correct this definition to n-fold Segal spaces.

Tentative definition: A Segal object in an ∞-category $\mathcal{C}$ with finite limits is a simplicial $\mathcal{C}$-object $X : \Delta^{op} \to \mathcal{C}$ satisfying the Segal condition: the canonical map $X_n \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$ is a weak equivalence.
An *n-uple Segal space* is a Segal object in the ∞-category of $(n-1)$-uple Segal spaces.◇

There are two problem with this definition. The first is that we did not describe higher morphisms between Segal spaces, but this can be done using model category theory. The second is that this definition allows morphisms between morphisms that have different sources and targets. Let us see this in more detail, and try to correct it.

Example 2.1.15 (2-uple Segal space): We want a functor $X : (\Delta^{op})^{\times 2} \to \mathrm{Space}$, which is indeed equivalent to a functor $\Delta^{op} \to \mathrm{Fun}(\Delta^{op}, \mathrm{Space})$. It should satisfy first that for every first index k, the simplicial space $X_{k,\bullet}$ is a Segal space, and second that the $X_{\bullet,\bullet}$ satisfy the Segal condition in the first variable. This second condition has to be checked levelwise, and is equivalent to asking that for every index l, the simplicial space $X_{\bullet,l}$ is a Segal space.
Now, as one can expect, $X_{0,0}$ is the space of objects, but there are two spaces $X_{1,0}$ and $X_{0,1}$ of "vertical" and "horizontal" morphisms. An element of $X_{1,1}$ is now a 2-morphism, but its "boundary" is a square: 2 vertical morphisms with possibly different sources and targets, and two horizontal morphisms linking these sources and targets, see Figure 2.1. This encodes the notion of a "double category", with these two directions of morphisms and squares between them, but it is not exactly what we want for (∞, n)-categories. So we ask there to be no horizontal morphisms but identities, which should solve the problem.◇

Definition 2.1.16: An *n-fold Segal space* is an n-uple simplicial topological space $X_{\bullet,\ldots,\bullet} : (\Delta^{op})^{\times n} \to \mathrm{Space}$ satisfying:

$$\forall k_1, \ldots, \overset{\vee}{k_i}, \ldots, k_n, \qquad X_{k_1,\ldots,\bullet,\ldots,k_n} \text{ is a Segal space} \tag{2.1}$$

$$\begin{array}{ccc} A & \xrightarrow{v \in X_{0,1}} & A' \\ f \in X_{1,0} \downarrow & h \in X_{1,1} & \downarrow g \in X_{1,0} \\ B & \xrightarrow[w \in X_{0,1}]{} & B' \end{array}$$

Figure 2.1: A 2-morphism in a 2-uple Segal space

$$\forall k_1, \dots, k_{i-1}, \quad X_{k_1,\dots,k_{i-1},0,\bullet,\dots,\bullet} \text{ is essentially constant} \tag{2.2}$$

where essentially constant means level-wise weakly equivalent to a constant $(n-i)$-uple simplicial object.
A map of n-fold Segal spaces is a map of n-uple simplicial objects. We denote by SeSp_n the category of n-fold Segal spaces.
An element of $X_{0,\dots,0}$ is called an object of X, and an element of $X_{1,\dots,1,0,\dots,0}$, with k 1's, a k-morphism of X. The face maps give two source and target maps from k-morphisms to $(k-1)$-morphisms. The degeneracy map gives a map in the other direction, and we denote the image of a $(k-1)$-morphism x by Id_x. ♢

The higher indices encode composition laws and coherence conditions, e.g. associativity. By the Segal condition, the space $X_{1,2,0,\dots}$ is described by pairs of composable 2-morphisms, and the third face map gives "horizontal" composition for 2-morphisms, see Figure 2.2.

$$\begin{array}{ccccc} A & = & A & = & A \\ f_1 \downarrow & \overset{h}{\Rightarrow} & f_2 \downarrow & \overset{h'}{\Rightarrow} & \downarrow f_3 \\ B & = & B & = & B \end{array}$$

Figure 2.2: Horizontal composition of 2-morphisms

The space $X_{2,0,\dots,0}$ is described by pairs of composable arrows, but is has a third face map: their composition. The space $X_{2,1,0,\dots}$ is described by pairs of 2-morphisms whose sources and targets are composable, and the third face map gives their "vertical" composition, see Figure 2.3. We sometimes call it composition of 2-morphisms in the direction of 1-morphisms.

$$\begin{array}{ccc} A & = & A \\ f \mid & \overset{h_1}{\Rightarrow} & \mid f' \\ B & = & B \\ g \downarrow & \overset{h_2}{\Rightarrow} & \downarrow g' \\ C & = & C \end{array}$$

Figure 2.3: Vertical composition of 2-morphisms

Definition 2.1.17: Let X be an n-fold Segal space and $1 \leq j \leq k \leq n$. Choose a quasi-inverse of the Segal map

$$(f_{0,1}, f_{1,2}) : X_{1,\dots,1,2,1\dots,1,0\dots,0} \to X_{1,\dots,1,1,1\dots,1,0\dots,0} \underset{X_{1,\dots,1,0,1\dots,1,0\dots,0}}{\times^h} X_{1,\dots,1,1,1\dots,1,0\dots,0}$$

where there are k 1's in the right hand terms, and the 2 is in position j in the left hand-side.

Let $f, g \in X_{1,\dots,1,0\dots,0}$ be k-morphisms in X with compatible sources and targets (if $j = k$, we ask $t(f) = s(g)$) so that (f, g) is in the left hand-side above. We call *composition of f and g in the direction of j-morphisms* the k-morphism $g \circ_j f$ obtained as the image of (f, g) under the map

$$X_{1,\dots,1,1,1\dots,1,0\dots,0} \underset{X_{1,\dots,1,0,1\dots,1,0\dots,0}}{\times^h} X_{1,\dots,1,1,1\dots,1,0\dots,0} \overset{(f_{0,1},f_{1,2})^{-1}}{\longrightarrow} X_{1,\dots,1,2,1\dots,1,0\dots,0} \overset{f_{0,2}}{\longrightarrow} X_{1,\dots,1,2,1\dots,1,0\dots,0}$$

where $f_{0,2}$ is the face map obtained by the inclusion $\begin{matrix} \{0<1\} & \to & \{0<1<2\} \\ 0,1 & \mapsto & 0,2 \end{matrix}$.

Note that the composition is not well-defined on the nose, as it depends on the choice of a quasi-inverse, but it is well-defined up to coherent isomorphism. ♢

Definition 2.1.18: An *n-by-m-fold Segal space* is an $(n+m)$-uple simplicial space X satisfying Segal conditions and partial essentially constancy conditions:

$$\text{for fixed } k_1, \dots, k_n,\ X_{k_1,\dots,k_n,\bullet,\dots,\bullet} \text{ is an } m\text{-fold Segal space}$$

$$\text{for fixed } k_{n+1}, \dots, k_{n+m},\ X_{\bullet,\dots,\bullet,k_{n+1},\dots,k_{n+m}} \text{ is an } n\text{-fold Segal space}$$

In particular a 1-by-1-fold Segal space is a 2-uple Segal space as described in the example above. ♢

Definition 2.1.19 (Definition 1.5.3 of [Sch14a]): Let $\mathcal{C}$ be an n-fold Segal space, and $x, y \in \mathcal{C}_{0,\dots,0}$ be two objects of $\mathcal{C}$. The $(n-1)$*-fold Segal space* $\mathrm{Hom}_{\mathcal{C}}(x, y)$ *of morphisms* from x to y is defined as:

$$\mathrm{Hom}_{\mathcal{C}}(x,y)_{\bullet,\cdots,\bullet} = \{x\} \times^h_{\mathcal{C}_{0,\bullet,\cdots,\bullet}} \mathcal{C}_{1,\bullet,\cdots,\bullet} \times^h_{\mathcal{C}_{0,\bullet,\cdots,\bullet}} \{y\}.$$

Inductively, we can define an $(n-k-1)$-fold Segal space $\mathrm{Hom}_{\mathcal{C}}(x, y)$ between k-morphisms $x, y \in \mathcal{C}_{1,\dots,1,0\dots,0}$ (with k 1's) with $s(x) = s(y)$ and $t(x) = t(y)$ where s and t are the two face maps $\mathcal{C}_{1,\dots,1,1,0\dots,0} \to \mathcal{C}_{1,\dots,1,0,0\dots,0}$. We set $\mathrm{Hom}_{\mathcal{C}}(x, y) := \mathrm{Hom}_{\mathrm{Hom}_{\mathcal{C}}(s(x),t(x))}(x, y)$. Note that for $\mathrm{Hom}_{\mathcal{C}}(s(x), t(x))$ to be defined, we need $s(s(x)) = s(t(x))$ and $t(s(x)) = t(t(x))$, which is given by essential constancy of $\mathcal{C}_{1,\dots,1,0,\bullet,0\dots,0}$.
If $\mathcal{C}$ has a distinguished object x, we denote $\Omega\mathcal{C} := \mathrm{Hom}_{\mathcal{C}}(x, x)$ the loop $(n-1)$-fold Segal space. It itself has a distinguished objects Id_x, and this construction can be iterated. In general if $\mathcal{C}$ is monoidal the distinguished object x will be the monoidal unit. ♢

Definition 2.1.20 (Definition 2.12 in [CS19]): Let $\mathcal{C}$ be an n-fold Segal space. We define its *homotopy 1-category* $h_1(\mathcal{C})$ inductively. We denote $h_0(\mathcal{C})$ the set of isomorphism classes of objects of $h_1(\mathcal{C})$. The case $n = 1$ is Definition 2.1.7. For general n we set the objects of $h_1(\mathcal{C})$ to be the underlying set of $\mathcal{C}_{0,\dots,0}$, the objects of $\mathcal{C}$. Morphisms between objects x and y are isomorphism classes of morphisms from x to y in $h_1(\mathrm{Hom}_{\mathcal{C}}(x, y))$, i.e:

$$\mathrm{Hom}_{h_1(\mathcal{C})}(x, y) = h_0(\mathrm{Hom}_{\mathcal{C}}(x, y))$$

where $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is the $(n-1)$-fold Segal space defined above, and $h_0(\mathrm{Hom}_{\mathcal{C}}(x, y))$ is defined by induction. Composition is given by Definition 2.1.17. ♢

Dwyer–Kan equivalences between n-fold Segal spaces can be defined inductively:

Definition 2.1.21: A map of n-fold Segal space $f : \mathcal{C} \to \mathcal{D}$ is a *Dwyer–Kan equivalences* if it induces an equivalence of categories $h_1(f) : h_1(\mathcal{C}) \to h_1(\mathcal{D})$ and Dwyer–Kan equivalences of $(n-1)$-fold Segal spaces between Hom $(n-1)$-fold Segal space $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(f(x), f(y))$ for every objects $x,\ y$ of $\mathcal{C}$. ♢

Definition 2.1.22: An n-fold Segal space $X_{\bullet,\cdots,\bullet}$ is said *complete* if for every $k_1,\ldots,k_{i-1}$, the Segal space $X_{k_1,\ldots,k_{i-1},\bullet,0,\ldots,0}$ is complete. ♢

Applying iteratively Rezk's completion, one can construct the completion $\widehat{X}$ of an n-fold Segal space X with a Dwyer–Kan equivalence $X \tilde{\to} \widehat{X}$ which is universal among maps to a complete n-fold Segal space [CS19, Definition 2.14].

Remark 2.1.23: Again, there is a model structure on $\mathrm{Fun}((\Delta^{\times n})^{op}, \mathrm{sSet})$ whose fibrant objects are complete n-fold Segal spaces. They behave well with Dwyer–Kan equivalences in the sense that a Dwyer–Kan equivalence between two fibrant objects is a level-wise weak equivalence. In particular, the levels of a complete n-fold Segal space are well-defined up to weak equivalences. The completion procedure is again fibrant replacement. ♢

Definition 2.1.24: An (∞,n)-*category*, which we will often abbreviate as n-category, is a complete n-fold Segal space. An n-functor is a map of n-fold Segal spaces. We denote CSS_n the category of complete n-fold Segal spaces. ♢

Remark 2.1.25: The notion of n-functors $F : \mathcal{C} \to \mathcal{D}$ is both incomplete and imperfect. It is incomplete because we only have a set $\mathrm{Hom}(\mathcal{C},\mathcal{D})$ but, because the category $\mathrm{Fun}((\Delta^{op})^{\times n}, \mathrm{sSet})$ is sSet-enriched, these are only the points of a simplicial set of morphisms $\mathrm{maps}(\mathcal{C},\mathcal{D})$. It is imperfect because it may give non-equivalent outputs on equivalent inputs (and the mapping spaces just mentioned may fail to be spaces). The right notion of derived mapping space $\mathrm{maps}^h(\mathcal{C},\mathcal{D})$ is defined in [JS17, Definition 2.11] using model category theory. ♢

2.1.3 Standard constructions

We recall the notion of derived mapping spaces, internal Homs, nerves, truncation, and extension.

We will need some model category theory, which we do not recall in detail. See [Toë14, §2.1] for a survey and [Hov99] for details. We will recall just enough to make Remark 2.1.25 explicit. The following is very well explained in [JS17, Appendix A]. We use their projective model structure.

There are two model category structures on $\mathrm{Fun}((\Delta^{op})^{\times n}, \mathrm{sSet})$ whose fibrant objects are respectively complete n-fold Segal spaces and complete n-uple Segal spaces. They have the same cofibrant objects.

The notion of derived mapping spaces $\mathrm{maps}^h(\mathcal{C},\mathcal{D})$ is defined by taking usual mapping spaces from a cofibrant replacement $\widetilde{\mathcal{C}} \tilde{\to} \mathcal{C}$ to a fibrant replacement $\mathcal{D} \tilde{\to} \widehat{\mathcal{D}}$, i.e:

$$\mathrm{maps}^h(\mathcal{C},\mathcal{D}) := \mathrm{maps}(\widetilde{\mathcal{C}},\widehat{\mathcal{D}}) \ .$$

In particular we have a map $\mathrm{maps}(\mathcal{C},\mathcal{D}) \to \mathrm{maps}^h(\mathcal{C},\mathcal{D})$.

As cofibrant objects agree in the two model structures, the derived mapping space between two complete n-fold Segal spaces agrees with the one computed in complete n-uple Segal spaces. To avoid confusion, we will nevertheless write maps^h_{uple} in the latter case.

To compute derived mapping spaces we have to find a cofibrant replacement for the source $\mathcal{C}$. This is not easy, but we know one family of examples: given an object $[k_1] \times \cdots \times [k_n]$ of $\Delta^{\times n}$, the representable presheaf $\mathrm{Hom}_{\Delta^{\times n}}(-,[k_1]\times\cdots\times[k_n])$ seen as a discrete simplicial set is always cofibrant. This is exploited in [JS17, Remarks 3.2 and 3.4 and Example 5.13] to compute some examples of derived mapping spaces.

There is yet another notion, using the fact that the model category in question is monoidal. Given two n-categories $\mathcal{C}$ and $\mathcal{D}$, their internal Hom n-category is the n-category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ defined by the universal property

$$\mathrm{maps}^h(\mathcal{B}, \mathrm{Fun}(\mathcal{C},\mathcal{D})) \simeq \mathrm{maps}^h(\mathcal{B}\times\mathcal{C},\mathcal{D})$$

following [JS17, Definition 5.14] and [Lur09b, Notation 2.4.1].

Definition 2.1.26: A strict n-category is defined inductively as a category enriched in strict $(n-1)$-category. A strict n-functor is a functor of enriched categories. We denote Cat_n^{str} the category of strict n-categories. ♢

Unsurprisingly, they can bee seen as special cases of n-categories as defined above.
Let $k_1,\dots,k_n \geq 0$. There is a strict n-category $\Theta^{k_1\dots,k_n}$ freely generated by $k_1\times\cdots\times k_n$ n-morphisms on a $k_1\times\cdots\times k_n$ grid, see [JS17, Definition 5.1]. Let us recall the precise definition for $n=2$.

Definition 2.1.27 (Definition 5.1 of [JS17] for $i=2$): Let $k,l\geq 0$, the strict bicategory $\Theta^{k,l}$ has $k+1$ objects $\{v_0,\dots,v_k\}$, $l+1$ 1-morphisms $\{e_0^{a,b},\dots,e_l^{a,b}\}$ from v_a to any v_b for $b>a$ (otherwise it has none or only the identity $e_0^{a,a}$) and a unique 2-morphism $b_{i,j}^{a,b}$ from $e_i^{a,b}$ to $e_j^{a,b}$ if $i\leq j$ (and none otherwise).
Said differently, it is the strict 2-category freely generated by the morphisms displayed in Figure 2.4. ♢

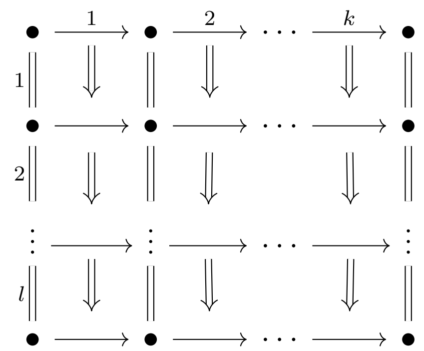


Figure 2.4: The 2-category $\Theta^{k,l}$ is a grid of k-by-l squares

These categories can be used to implement a higher (strict) nerve construction. The following mirrors the definition of [Ara14] in the context of n-quasi-categories.

Definition 2.1.28: Let S be a strict n-category. Its *strict nerve* $N^{str}(S)$ is the (discrete) n-fold Segal space with levels the sets

$$N^{str}(S)_{k_1,\dots,k_n} := \mathrm{Hom}_{\mathrm{Cat}_n^{str}}(\Theta^{k_1,\dots,k_n}, S)$$

with simplicial structure induced by the maps between the $\Theta^{k_1,\dots,k_n}$'s as in [JS17, Section 5]. The strict Segal condition is satisfied, and coincides with the Segal condition with homotopy fiber products because every space is discrete. The strict constancy condition is verified, because it is on the $\Theta^{k_1,\dots,k_n}$'s. ♢

We will usually still write S when we want to see it as an n-fold Segal space, even though we mean $N^{str}(S)$. Note that $N^{str}(S)$ is almost never complete, and we will often have to complete it. One exception is the nerves of the $\Theta^{k,l}$s, which are complete because the $\Theta^{k,l}$s are gaunt (have no non-trivial isomorphisms).

Definition 2.1.29 (Remark 5.4 in [JS17]): The *underlying n-fold Segal space $U_{fold}\mathcal{C}$* of a complete n-uple Segal space $\mathcal{C}$ has levels

$$(U_{fold}\mathcal{C})_{k_1,\dots,k_n} := \mathrm{maps}^h_{uple}(\Theta^{k_1,\dots,k_n}, \mathcal{C})$$

with simplicial structure induced by the maps between the $\Theta^{k_1,\dots,k_n}$'s. ♢

Definition 2.1.30 (Sections 2.4.1 and 2.4.2 in [CS19]): One can obtain an $(n-1)$-fold Segal space from an n-fold Segal space by setting the last coordinate to 0. This process is called truncation. Let $\mathcal{C}$ be an n-fold Segal space. For $m \leq n$ its m-truncation $\tau_m\mathcal{C}$ is the m-fold Segal space with

$$(\tau_m\mathcal{C})_{k_1,\dots,k_m} := (\mathcal{C})_{k_1,\dots,k_m,0,\dots,0}.$$

This process does not give equivalent results from (Dwyer–Kan) equivalent inputs if the n-fold Segal space $\mathcal{C}$ is not supposed to be complete. To avoid this issue, one has to always complete $\mathcal{C}$ before taking its truncation.
In the other direction, one can extend an n-fold Segal space to an $(n+1)$-fold Segal space with only invertible $(n+1)$-morphisms. This process is called extension. We set

$$(\varepsilon\mathcal{C})_{k_1,\dots,k_n,k_{n+1}} := (\mathcal{C})_{k_1,\dots,k_n} \ .$$

It sends complete n-fold Segal spaces to complete $(n+1)$-fold Segal space by [CS19, Lemma 2.16]. It is left adjoint to τ_n. ♢

Remark 2.1.31: If one sees two n-categories as $(n+1)$-categories, their $(n+1)$-category of functors is actually an n-category and is the usual one. More precisely, given $\mathcal{C}, \mathcal{D}$ two n-categories,

$$\mathrm{Fun}(\varepsilon\mathcal{C}, \varepsilon\mathcal{D}) \simeq \varepsilon\,\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \ .$$

More generally, if a functor $F : \mathcal{M} \to \mathcal{N}$ between two model categories has a left adjoint G that preserves finite products, then for any two objects $\mathcal{C}$ and $\mathcal{D}$, one has

$$\mathrm{Fun}(\mathcal{C}, F\mathcal{D}) \simeq F\,\mathrm{Fun}(G\mathcal{C}, \mathcal{D})$$

Indeed, using the defining universal property of internal Hom's,

$$\begin{aligned} \mathrm{maps}^h(\mathcal{B}, \mathrm{Fun}(\mathcal{C}, F\mathcal{D})) &\simeq \mathrm{maps}^h(\mathcal{B} \times \mathcal{C}, F\mathcal{D}) \simeq \mathrm{maps}^h(G(\mathcal{B} \times \mathcal{C}), \mathcal{D}) \\ &\simeq \mathrm{maps}^h(G\mathcal{B} \times G\mathcal{C}, \mathcal{D}) \simeq \mathrm{maps}^h(G\mathcal{B}, \mathrm{Fun}(G\mathcal{C}, \mathcal{D})) \simeq \mathrm{maps}^h(\mathcal{B}, F\,\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \end{aligned}$$

If moreover F is fully faithful, or if the counit $GF \Rightarrow \mathrm{Id}$ is an equivalence, then F preserves inner Hom's, i.e. for any two objects $\mathcal{C}$ and $\mathcal{D}$, one has

$$\mathrm{Fun}(F\mathcal{C}, F\mathcal{D}) \simeq F(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$$

The example above is $F = \varepsilon : \mathrm{CSS}_n \to \mathrm{CSS}_{n+1}$ whose left adjoint is the "inverting" η from [CS19, Section 2.4.3]. Here η preserves finite products as geometric realization does, and the counit $\eta\varepsilon \Rightarrow \mathrm{Id}$ is an equivalence.
We will also consider $F = \tau_n : \mathrm{CSS}_{n+1} \to \mathrm{CSS}_n$ whose left adjoint is ε. Now ε does preserve finite products, but the counit is not an equivalence (this time the unit is). ♢

2.1.4 Symmetric monoidal structures

We recall the notion of symmetric monoidal n-category following [Sch14a, Section 1.6] and [CS19, Section 3].

Definition 2.1.32: *Segal's category* Γ has objects the pointed finite sets

$$\langle m\rangle = \{*, 1, \dots, m\}\ ,\quad m \geq 0,$$

and morphisms from $\langle m\rangle$ to $\langle n\rangle$ maps of sets preserving the pointing. For every $m \geq 0$ and every $1 \leq j \leq m$ we call Segal morphism the map $\gamma_j^m \langle m\rangle \to \langle 1\rangle$ which maps j to 1 and every other element to $*$. ♢

Definition 2.1.33: A *symmetric monoidal n-fold Segal space* X is a functor

$$X : \Gamma \to \mathrm{SeSp}_n$$

from Segal's category Γ to the category of n-fold Segal spaces such that for every $m \geq 0$, the Segal morphisms induce an equivalence $X(\prod_j \gamma_j^m) : X(\langle m\rangle) \to \prod_j X(\langle 1\rangle)$. By abuse, we sometimes call $X(\langle 1\rangle)$ the symmetric monoidal n-fold Segal space, and X is the symmetric monoidal structure.
A *symmetric monoidal n-category* is a symmetric monoidal n-fold Segal space with values in complete n-fold Segal spaces.
A *symmetric monoidal n-functor* is a map of Γ-objects. ♢

Remark 2.1.34: The notion above is surprisingly strict, one would expect a much laxer one where symmetry is verified only up to coherent isomorphisms. Why this notion suffices is explained in [CS19, Section 3.1]. However symmetric monoidal n-functors, just like n-functors before, may behave badly. As before, one should consider derived mapping spaces in the model category of [JS17, Example A.11]. See also [CS19, Definition 3.6]. We will denote $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ the symmetric monoidal n-category defined as the internal Hom in this model category as in [Lur09b, Variant 2.4.3]. ♢

Remark 2.1.35: The completion of a symmetric monoidal n-fold Segal space yields to a symmetric monoidal n-category, see [Sch14a, Lemma 1.6.6]. ♢

Remark 2.1.36: There is another way of describing monoidal n-fold Segal space, as categories of endomorphisms of an object in an $(n+1)$-fold Segal space. One expects (or defines [CS19, Section 3.2]) that a monoidal n-fold Segal space $\mathcal{C}$ is the same data as an $(n+1)$-fold Segal space $B\mathcal{C}$ with only one object, such that $\Omega B\mathcal{C} \simeq \mathcal{C}$. A 2-monoidal (read braided monoidal) n-fold Segal space $\mathcal{C}$ is the same data as an $(n+2)$-fold Segal space $B^2\mathcal{C}$ with only one object and one 1-morphism such that $\Omega^2 B^2\mathcal{C} \simeq \mathcal{C}$, and so on. This formalism is described in [CS19, Section 3.2.2].
One could now define a symmetric monoidal n-category to be an ∞-monoidal n-category, i.e. an infinite tower of $(n+k)$-categories $B^k\mathcal{C}$ such that $\Omega B^{k+1}\mathcal{C} \simeq B^k\mathcal{C}$ and $B^0\mathcal{C} = \mathcal{C}$. It is expected, similarly to [Lur, Corollary 5.1.1.5], that this notion is equivalent to that of Γ-object given above, and it is shown in [CS19, Section 3.3] that a Γ-object induces such an infinite tower. In particular if an n-fold Segal space $\mathcal{C}$ is symmetric monoidal (in the sense of Γ-objects), there exists a delooping $(n+1)$-fold Segal space $B\mathcal{C}$ with essentially one object such that $\mathcal{C} \simeq \Omega B\mathcal{C}$. We will use this delooping $(n+1)$-fold Segal space to define dualizability later. ♢

2.1.5 Underlying bicategories

In the description above we saw that the definition of an n-category is not as nice and combinatorial as the definition of a bicategory where one simply gives a list of objects, 1-morphism and 2-morphisms, some composition rules and identities, and checks some coherences between them. However, we will see that an n-category induces a "chain" of bicategories at every level. The n-category is not clearly determined by it, but dualizability in the n-category, a notion we will extensively study in this manuscript, is.

Definition 2.1.37 (Section 5 of [Rom], sketched in Def. 1.4.1 in [Sch14a]): Let $\mathcal{C}$ be an n-fold Segal space. Its *homotopy bicategory* $h_2(\mathcal{C})$ has objects points of $\mathcal{C}_{0,\dots,0}$ the objects of $\mathcal{C}$. The category of morphisms between objects x and y is the homotopy category of the $(n-1)$-fold Segal space of morphisms from x to y:

$$\mathrm{Hom}_{h_2(\mathcal{C})}(x,y) := h_1\Big(\mathrm{Hom}_{\mathcal{C}}(x,y)\Big)$$

Compositions are given by Definition 2.1.17. ◇

Remark 2.1.38: If $\mathcal{C}$ is complete, then taking isomorphism classes of objects can be seen at the level of the space $\mathcal{C}_{0,\dots,0}$, and does not involve $\mathcal{C}_{1,0,\dots,0}$. Namely, $h_0(\mathcal{C}) = \pi_0(\mathcal{C}_{0,\dots,0})$. Similarly, $h_1(\mathcal{C})$ depends only on $\mathcal{C}_{\bullet,0,\dots,0}$ and $h_2(\mathcal{C})$ on $\mathcal{C}_{\bullet,\bullet,0,\dots,0}$. If $\mathcal{C}$ is not complete, we need all levels to determine whether a given morphism is an isomorphism (it must have a weak inverse, such that the composition is isomorphic to the identity, but this "isomorphic" needs even higher morphisms to make sense, and so on). From this remark we see that it would be difficult to define the notion of invertibility in an (∞,∞)-category. ◇

Remark 2.1.39: The definition above is pleasant because it is very explicit and natural. However, it is difficult to show that this construction by hand indeed gives a bicategory, with expected associativity and unit conditions. This was done very recently in [Rom]. Note that the notion of homotopy bicategory could already be defined, see for example [Cam20], if one is happy to use equivalent models of higher categories, in this case 2-quasi-categories. Its properties there are well-understood, and they construct its right adjoint, a nerve construction, which will be used in Remark 2.3.18. ◇

Remark 2.1.40: Given an n-fold Segal space $\mathcal{C}$ we can extract a chain of bicategories:

- the bicategory $h_2(\mathcal{C})$
- a bicategory $h_2(\mathrm{Hom}_{\mathcal{C}}(x,y))$ for every pair of objects $x, y \in \mathcal{C}_{0,\dots,0}$
- a bicategory $h_2(\mathrm{Hom}_{\mathcal{C}}(f,g))$ for every pair of morphisms $f, g : x \to y$
- $\cdots$
- a bicategory $h_2(\mathrm{Hom}_{\mathcal{C}}(\alpha,\beta))$ for every pair of k-morphisms α, β with same source and target.

We will see that this "naïve" data is sufficient to describe dualizability in $\mathcal{C}$. ◇

Proposition 2.1.41: *A Dwyer–Kan equivalence $f : \mathcal{C} \to \mathcal{D}$ between two n-fold Segal spaces induces equivalences of bicategories*

$$h_2(f) : h_2(\mathcal{C}) \to h_2(\mathcal{D})$$

and
$$h_2(\mathrm{Hom}_{\mathcal{C}}(\alpha,\beta)) \to h_2(\mathrm{Hom}_{\mathcal{D}}(f(\alpha),f(\beta)))$$
for every pair of k-morphisms α,β with same source and target in $\mathcal{C}$.

Proof : By definition f induces an equivalences $h_1(\mathcal{C}) \to h_1(\mathcal{D})$, so $h_2(f)$ is essentially surjective. Given any pair of objects $x,y \in \mathcal{C}_{0,\dots,0}$, it induces a Dwyer–Kan equivalence

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(f(x),f(y)),$$

hence an equivalence

$$h_1(\mathrm{Hom}_{\mathcal{C}}(x,y)) \to h_1(\mathrm{Hom}_{\mathcal{D}}(f(x),f(y))).$$

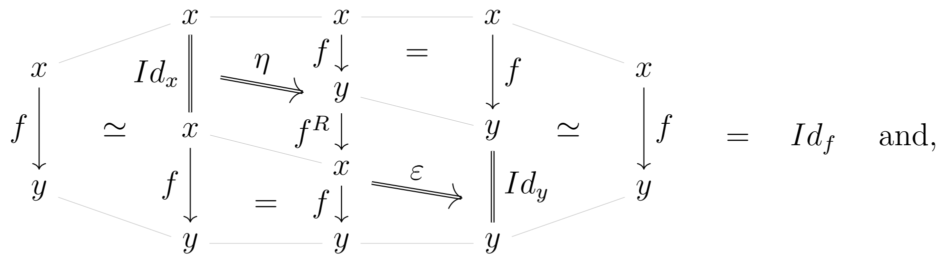
So $h_2(f)$ is fully faithful, and an equivalence of bicategories.
By definition a Dwyer–Kan equivalence induces a Dwyer–Kan equivalence of $(n-k-1)$-fold Segal spaces $\mathrm{Hom}_{\mathcal{C}}(\alpha,\beta) \to \mathrm{Hom}_{\mathcal{D}}(f(\alpha),f(\beta))$, hence an equivalence of bicategories between their homotopy bicategories by the paragraph above. □

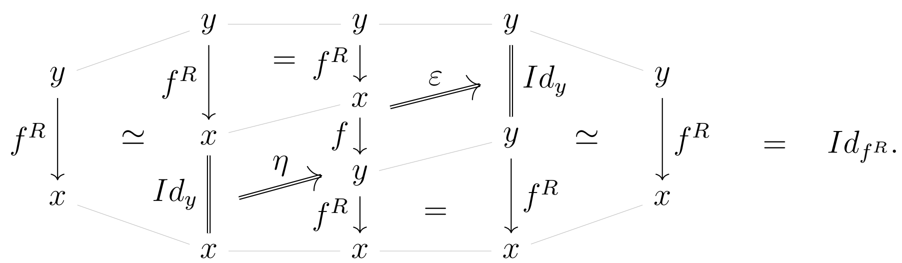
In particular, the naïve data described above of an n-fold Segal space X is equivalent to the one of its completion $\widehat{X}$.

2.1.6 Dualizability in higher categories

We recall the notion of higher dualizability and of fully dualizable objects in a higher category.

Definition 2.1.42: Let $\mathcal{C}$ be a bicategory, and $f : x \to y$ a 1-morphism in $\mathcal{C}$. A *right adjoint* for f is a morphism $f^R : y \to x$ together with two 2-morphisms $\varepsilon : f \circ f^R \Rightarrow Id_y$ called the counit and $\eta : Id_x \Rightarrow f^R \circ f$ called the unit, satisfying the so-called snake identities:





We say that f has a right adjoint f^R and that f^R has a left adjoint f.
This definition extends to higher categories. Let $\mathcal{C}$ be an n-category, $2 \le k \le n$ and $f : x \to y$ a k-morphism between two $k-1$-morphisms $x,y : a \to b$ in $\mathcal{C}$. A right adjoint for f is a right adjoint for f seen as a 1-morphism in the bicategory $h_2(\mathrm{Hom}_{\mathcal{C}}(a,b))$. If $k=1$ then we demand a right adjoint of f in $h_2(\mathcal{C})$. If $k=0$ and $\mathcal{C}$ is a monoidal category

then we demand a right adjoint of f in $h_2(B\mathcal{C})$, where $B\mathcal{C}$ is the delooping $(n+1)$-fold Segal space with essentially one object with endomorphisms $\mathcal{C}$ and composition the monoidal structure of $\mathcal{C}$ from Remark 2.1.36.

Definition 2.1.43: Let $\mathcal{C}$ be a symmetric monoidal n-category. It is said to *have duals up to level* m if every k-morphism of $\mathcal{C}$, $0 \leq k < m$, has both a left and a right adjoint. It is said to *have duals* if it has duals up to level n.
An object $X \in \mathcal{C}$ is called *m-dualizable* if it lies in a sub-n-category with duals up to level m. It is called *fully dualizable* if it is n-dualizable.
We denote $\mathcal{C}^{fd}$ the maximal sub-n-category of $\mathcal{C}$ that has duals. Its objects constitute of the fully dualizable objects of $\mathcal{C}$. ♢

Proposition 2.1.44: *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between n-categories and f a k-morphism in $\mathcal{C}$ which has a right adjoint. Then $F(f)$ has a right adjoint in $\mathcal{D}$.*
In particular if F is a symmetric monoidal functor between symmetric monoidal n-categories, it maps fully dualizable objects to fully dualizable objects.

Proof : The image under F of the adjoint, unit and counit of f in $\mathcal{C}$ form adjoint, unit and counit for $F(f)$ in $\mathcal{D}$. □

One may also define a notion of m-dualizability for k-morphisms.

Definition 2.1.45: A k-morphism f of $\mathcal{C}$ is called *m-dualizable* if it lies in a sub-n-category with duals up to level $m+k$. It is called fully dualizable if it is $(n-k)$-dualizable.♢

However, we will see later that this definition is very strong (in particular, it demands dualizability of the source and target of f), and that one may relax it.

2.2 Fully extended TQFTs and the Cobordism Hypothesis

A fully extended TQFT is a symmetric monoidal functor from the n-category $\mathbf{Bord}_n$ of bordisms of dimension up to n to some target n-category $\mathcal{C}$. The target is left to be determined, but in general it will be of linear nature and satisfy $h_1(\Omega^{n-1}\mathcal{C}) \simeq \mathrm{Vect}_{\Bbbk}$. Fully extended TQFTs are classified by the cobordism hypothesis in terms of fully dualizable objects in $\mathcal{C}$. The goal of this section is to give a precise meaning to these sentences.

2.2.1 The n-category of bordisms

The construction of the n-category of cobordism is done in [Lur09b] and precised in [CS19]. We recall the broad ideas here.
We want to define an n-fold Segal space, so an n-uple simplicial space $\mathbf{Bord}_n : (\Delta^{op})^n \to$ Space. For indices $k_1, \ldots, k_n$, $(\mathbf{Bord}_n)_{k_1,\ldots,k_n}$ should be a space of n-manifolds M that are obtained as compositions of k_1 bordisms in the first direction, k_2 in the second and so on. We can represent this by asking that the n-manifold M lives over a "grid" in $(0,1)^n$ defined by $k_1 - 1$ hyperplanes orthogonal to the first direction, $k_2 - 1$ orthogonal to the second and so on. The part of M which lives over one of the hyperplanes should behave nicely (to allow to describe M as a gluing there), while the part over the "cubes" is free and is where the cobordism happens, see Figure 2.6.

An n-fold Segal space of such grids is described in [CS19, Section 4].

Definition 2.2.1: The simplicial space $\mathrm{Int}_\bullet$ is defined as follows. Its underlying simplicial set of points $int_\bullet : \Delta^{op} \to Set$ is defined with

$$int_k = \left\{ (a_0, b_0, \ldots, a_n, b_n) \in (0,1)^{2n},\ \forall i\ a_i < b_i,\ \begin{array}{c} 0 = a_0 \leq a_1 \leq \cdots \leq a_n, \\ b_0 \leq b_1 \leq \cdots \leq b_n = 1 \end{array} \right\}.$$

We think of them as increasing sequences of $k+1$ intervals $I_k = [a_k, b_k]$. We will typically represent them as disjoint small intervals over which the composition happen, but [CS19] makes the point that they should be allowed to overlap. The simplicial structure is given by forgetting or doubling one of intervals, and if necessary restriction and rescaling $(0,1)$. Every level int_k is the set of points of a space Int_k whose higher simplices are defined by considering only smooth deformations of $2n$-uples of points, see [CS19, Section 4.2].
The *n-uple simplicial space of n-grids $Grid_n$* is $(\mathrm{Int}_\bullet)^{\times n}$. A $(k_1, \ldots, k_n)$-grid is a point of $(Grid_n)_{k_1,\ldots,k_n}$. ♢

The grids there are thickened to ensure that the gluings are collar gluings, and might overlap. There are two thickened hyperplanes “at infinity” describing source and target, see Figure 2.5. The Segal condition (2.1) is satisfied because we asked that M behave

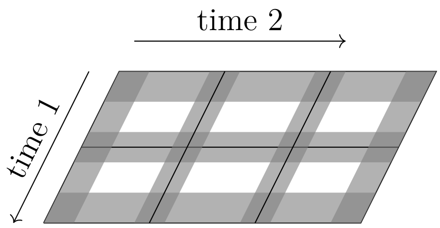


Figure 2.5: A (2,3)-grid. Here $k_1 = 2$ and $k_2 = 3$.

nicely over an hyperplane and is obtained as a gluing. To ensure the essential constancy requirement (2.2), we need to ensure that M is particularly simple over the k_i-hyperplanes for small i's, see Figure 2.6.

Definition 2.2.2 (Definition 5.1 of [CS19]): Let $N \geq 1$. The n-uple simplical space $P\,\mathbf{Bord}_n^N$ is defined as follows. The space $(P\,\mathbf{Bord}_n^N)_{k_1,\ldots,k_n}$ has points oriented n-manifolds $M \subseteq \mathbb{R}^N \times (0,1)^n$ such that the projection $M \to (0,1)^n$ is proper, together with a $(k_1, \ldots, k_n)$-grid $((I_0^i, \ldots, I_{k_i}^i))_{1 \leq i \leq n}$. They must satisfy that the projection over the i'th coordinate $p_i : M \to (0,1)$ has no critical point over any of the thickened k_j-hyperplanes, so on $p_j^{-1}(I_k^j),\ 0 \leq k \leq k_j$ for $j \leq i$. Note that this means that there are much more conditions on the $p_j^{-1}(I_k^j)$ for small j's (all the p_i's, $i \geq j$, must be submersive). See Figure 2.6.
The higher simplices are again given by considering smooth deformations in the Whitney C^∞-topology on embeddings of M in $\mathbb{R}^N \times (0,1)^n$, see [CS19, Section 5.2.1].
The face maps are given by forgetting a thickened hyperplane, or restricting $(0,1)^n$ (and therefore M above it), see [CS19, Definition 5.15]: ♢

Proposition 2.2.3 (Proposition 5.19 in [CS19]): *The n-uple simplicial spaces $P\,\mathbf{Bord}_n^N$ are n-fold Segal spaces.*

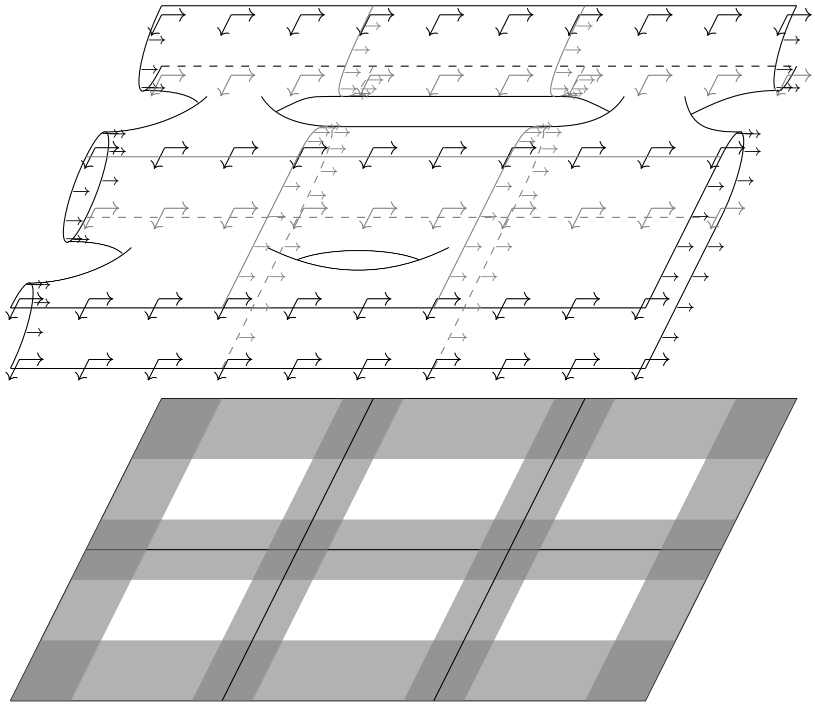


Figure 2.6: A manifold living over a (2,3)-grid. Over the k_1-hyperplanes, it is trivial. Over the k_2-hyperplanes, it is trivial in one direction, and maybe not in the other.

Definition 2.2.4 (Definitions 5.22 and 5.24 in [CS19]): The n-fold Segal space $P\,\mathbf{Bord}_n$ is the homotopy colimit over N of the n-fold Segal spaces $P\,\mathbf{Bord}_n^N$. In particular, its levels are obtained as $(P\,\mathbf{Bord}_n)_{k_1,\dots,k_n} := \mathrm{hocolim}_N((P\,\mathbf{Bord}_n^N)_{k_1,\dots,k_n})$.
It may not be complete for large n [Lur09b, Warning 2.2.8] and the n-category $\mathbf{Bord}_n$ is defined to be its completion. $\diamondsuit$

Remark 2.2.5: One may give a similar definition for framed manifolds denoted $\mathbf{Bord}_n^{fr}$, or indeed for manifolds equipped with any kind of tangential structure [CS19, Section 9]. $\diamondsuit$

The symmetric monoidal structure on $\mathbf{Bord}_n$ is given by disjoint union [CS19, Section 7.1]. Remember that in Definition 2.1.33 of symmetric monoidal structures, we do not explicitly build the disjoint union of two cobordisms, we instead give the space of cobordisms that happen to be disjoint unions, and we have to check a Segal property.

Definition 2.2.6: Let $m, N \geq 0$. The space $(P\,\mathbf{Bord}_n^N\langle m\rangle)_{(k_1,\dots,k_n)}$ has points m-uples of oriented manifolds $(M_1,\dots,M_m)$ each living in $\mathbb{R}^N \times (0,1)^n$ together with a $(k_1,\dots,k_n)$-grid $\bar{I}$ such that each $(M,\bar{I})$ is a point of $(P\,\mathbf{Bord}_n^N)_{(k_1,\dots,k_n)}$ and all the M_i's are disjoint in $\mathbb{R}^N \times (0,1)^n$. The higher simplices and simplicial structures are given similarly.
The space $(P\,\mathbf{Bord}_n\langle m\rangle)_{(k_1,\dots,k_n)}$ is the homotopy colimits of the spaces $(P\,\mathbf{Bord}_n^N\langle m\rangle)_{(k_1,\dots,k_n)}$. They arrange into an n-fold Segal space $P\,\mathbf{Bord}_n\langle m\rangle$.
The n-fold simplicial spaces $P\,\mathbf{Bord}_n$ is symmetric monoidal with

$$\begin{array}{ccc} \Gamma & \to & \mathrm{SeSp}_n \\ \langle m\rangle & \mapsto & P\,\mathbf{Bord}_n\langle m\rangle \end{array}$$

Functoriality and the fact that Segal morphisms induce an equivalence are shown in [CS19, Proposition 7.2]. The n-category $\mathbf{Bord}_n$ is then symmetric monoidal by [Sch14a, Lemma 1.6.6]. $\diamondsuit$

Definition 2.2.7: A *fully extended n-dimensional Topological Quantum Field Theory* (or fully extended n-TQFT) with values in a symmetric monoidal n-category $\mathcal{C}$ is a symmetric monoidal n-functor

$$\mathcal{Z} : \mathbf{Bord}_n \to \mathcal{C}$$

◇

Remark 2.2.8: By universality of the completion, and as $\mathcal{C}$ is assumed to be complete, one could equivalently consider $P\,\mathbf{Bord}_n$ above. ◇

Proposition 2.2.9 (Proposition 2.5.1 in [Sch14a]): *There is an equivalence of symmetric monoidal categories*

$$h_1(\Omega^{n-1}\,\mathbf{Bord}_n) \simeq \boldsymbol{Cob}_n$$

Therefore, a fully extended TQFT $\mathcal{Z}$ induces in particular an ordinary TQFT

$$h_1(\Omega^{n-1}\mathcal{Z}) : \boldsymbol{Cob}_n \to h_1(\Omega^{n-1}\mathcal{C}).$$

2.2.2 The cobordism hypothesis

The cobordism hypothesis describes fully extended Topological Quantum Field Theories with values in a higher category $\mathcal{C}$ in terms of fully dualizable objects of $\mathcal{C}$. It was formulated in [BD95]. A sketch of proof was given in [Lur09b], a more formal version is work in progress of Schommer-Pries. An independent proof of a more general result is proposed in [GP]. Another independent proof using factorization homology is work in progress, see [AF].

Conjecture 2.2.10 (The Cobordism Hypothesis, Theorem 2.4.6 in [Lur09b]): *Let $\mathcal{C}$ be a symmetric monoidal n-category. Fully extended framed n-TQFTs* $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{fr}, \mathcal{C})$ *form an n-category which happens to be an ∞-groupoid, therefore a space.*
Evaluation at the point induces a weak equivalence of spaces

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{fr}, \mathcal{C}) \simeq (\mathcal{C}^{fd})^{\sim}$$

between framed fully extended n-TQFTs with values in $\mathcal{C}$ and the underlying ∞-groupoid of the subcategory of fully dualizable objects of $\mathcal{C}$, $(\mathcal{C}^{fd})^{\sim} := (\mathcal{C}^{fd})_{0,\dots,0}$.

For $X \in \mathcal{C}$ a fully dualizable object, we denote $\mathcal{Z}_X$ (a choice of representant of) the associated fully extended framed n-TQFT.

Remark 2.2.11: Often enough, we will be interested in the case where $\mathcal{C}$ is actually an m-category for $m > n$. Then $\mathrm{Fun}(\mathbf{Bord}_n^{fr}, \mathcal{C})$ is defined by seeing $\mathbf{Bord}_n^{fr}$ as an m-category using the extension from Definition 2.1.30. By Remark 2.1.31, one has an equivalence

$$\mathrm{Fun}(\varepsilon\,\mathbf{Bord}_n^{fr}, \mathcal{C}) \simeq \varepsilon\,\mathrm{Fun}(\mathbf{Bord}_n^{fr}, \tau_n\mathcal{C})$$

and the cobordism hypothesis as stated above applies. ◇

Remark 2.2.12: The group $O(n)$ acts on the n-category $\mathbf{Bord}_n^{fr}$ by changing the framing. Therefore it acts on $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{fr}, \mathcal{C})$ and by the Cobordism Hypothesis on $(\mathcal{C}^{fd})^{\sim}$. Oriented theories are fixed points under the $SO(n)$-action. The oriented cobordism hypothesis claims a converse. ◇

Conjecture 2.2.13 (The oriented CH, Theorem 2.4.26 in [Lur09b]): *Let $\mathcal{C}$ be a symmetric monoidal n-category. Fully extended n-TQFTs $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n, \mathcal{C})$ form an n-category which happens to be an ∞-groupoid, therefore a space.*
Evaluation at the point induces a weak equivalence of spaces

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n, \mathcal{C}) \simeq ((\mathcal{C}^{fd})^{\sim})^{hSO(n)}$$

between fully extended n-TQFTs with values in $\mathcal{C}$ and homotopy fixed points under the $SO(n)$-action on $(\mathcal{C}^{fd})^{\sim}$.

Remark 2.2.14: Note that the oriented cobordism hypothesis cannot be stated without first stating the framed version. Indeed the $SO(n)$-action on $(\mathcal{C}^{fd})^{\sim}$ is very non-trivial to define and is given by the framed cobordism hypothesis. It is explained in [Lur09b, Warning 2.4.13] that it does not extend to an $SO(n)$-action on the n-category $\mathcal{C}^{fd}$: it is only defined on the underlying ∞-groupoid. It is therefore unsurprisingly very difficult to exhibit an $SO(n)$-homotopy fixed fully dualizable object in a given n-category $\mathcal{C}$. ◇

2.3 Even Higher Morita Category of Braided Tensor Categories

We will work in the even higher Morita category $\mathbf{Alg}_2(\mathbf{Pr})$ of E_2-algebras in a bicategory of cocomplete categories which we study in Section 2.3.1. The higher Morita $(n+1)$-category of E_n-algebras in an ∞-category $\mathcal{S}$ was introduced in [Hau17] using a combinatorial/operadic description which we briefly recall in Section 2.3.2. A pointed version was introduced in [Sch14a] using very geometric means, namely factorization algebras. This geometric description allows for a good description of dualizability but the pointing prevents any higher dualizability, see [GS]. Even higher Morita categories are defined in [JS17, Section 8], for pointed and unpointed versions, which we recall in Section 2.3.3. They form an $(n+k)$-category $\mathbf{Alg}_n(\mathcal{S})$ for $\mathcal{S}$ a symmetric monoidal k-category. We will always use the strong version of even higher Morita categories of [JS17], and never mention lax and oplax versions. All of these construction assume that the monoidal structure of $\mathcal{S}$ behaves well with colimits, more precisely that $\mathcal{S}$ is $\otimes$-sifted cocomplete in [Sch14a], that it has good relative tensor products in [Hau17], and that $\mathcal{S}^{\boxed{\text{strong}}}$ is $\otimes$-sifted cocomplete in [JS17, Definition 8.3]. It is shown in [JS17, Example 8.11] that $\mathbf{Pr}$ satisfies all of these conditions. We consider the unpointed even higher Morita 4-category $\mathbf{Alg}_2(\mathbf{Pr})$. This formal description of a 4-category is technical, and for studying dualizability we only need the “naïve” data of all the bicategories of Remark 2.1.40. This data is described more explicitly in [BJS21], under the name **BrTens**, which we recall in Section 2.3.4.

2.3.1 Cocomplete categories

We begin by recalling some properties of the bicategory **Pr**. Let $\Bbbk$ be a field of characteristic zero.

Definition 2.3.1: The bicategory $\mathrm{Cat}_\Bbbk$ has objects small $\Bbbk$-linear categories, 1-morphisms $\Bbbk$-linear functors and 2-morphisms natural transformations. It is symmetric monoidal with tensor product described by $Ob(\mathscr{C}\otimes\mathscr{D}) := Ob(\mathscr{C})\times Ob(\mathscr{D})$ and $\mathrm{Hom}_{\mathscr{C}\otimes\mathscr{D}}((c,d),(c',d')) := \mathrm{Hom}_{\mathscr{C}}(c,c')\otimes_\Bbbk \mathrm{Hom}_{\mathscr{D}}(d,d')$. It is characterized by:

$$\mathrm{Hom}_{\mathrm{Cat}_\Bbbk}(\mathscr{A}\otimes\mathscr{B},\mathscr{C}) \simeq \mathrm{Hom}_{CAT_\Bbbk}(\mathscr{A},\mathrm{Hom}_{\mathrm{Cat}_\Bbbk}(\mathscr{B},\mathscr{C})).$$

where we write $CAT_\Bbbk$ when the categories involved are not necessarily small. The unit is the one object category whose endomorphisms are $\Bbbk$.

The bicategory **Pr** has objects cocomplete locally presentable $\Bbbk$-linear categories [AR94, Defintion 1.17] of [BCJ15, Definition 2.1]. These are cocomplete (cocomplete means have all colimits) categories that are generated under λ-filtered colimits by a small subcategory of λ-compact objects for some cardinal λ. It has 1-morphisms $\Bbbk$-linear cocontinuous functors between them (cocontinuous means preserve colimits), and 2-morphisms natural transformations. It is symmetric monoidal with the Kelly tensor product $\boxtimes$ for cocomplete categories, which is characterized by:

$$\mathrm{Hom}_{\mathbf{Pr}}(\mathcal{A}\boxtimes\mathcal{B},\mathcal{C}) \simeq \mathrm{Hom}_{\mathbf{Pr}}(\mathcal{A},\mathrm{Hom}_{\mathbf{Pr}}(\mathcal{B},\mathcal{C})) \simeq \mathit{Cocont}(\mathcal{A},\mathcal{B};\mathcal{C}),$$

see [Kel05, Section 6.5] and [Ram17, Theorem 2.45]. The unit is $\mathrm{Vect}_\Bbbk$. ♢

Definition 2.3.2: The *free cocompletion* of a small $\Bbbk$-linear category $\mathscr{C}$ is a cocomplete category $\mathrm{Free}(\mathscr{C})\in\mathbf{Pr}$ together with a functor $i:\mathscr{C}\to\mathrm{Free}(\mathscr{C})$ which is initial among functors to a cocomplete category. Namely, $i^*:\mathrm{Hom}_{\mathbf{Pr}}(\mathrm{Free}(\mathscr{C}),\mathcal{D})\to\mathrm{Hom}_{CAT_\Bbbk}(\mathscr{C},\mathcal{D})$ is an equivalence of categories for any $\mathcal{D}\in\mathbf{Pr}$. The free cocompletion $\mathrm{Free}(\mathscr{C})$ is unique up to essentially unique equivalence. A standard choice for the free cocompletion is the presheaf category $\mathrm{Hom}_{Cat_\Bbbk}(\mathscr{C}^{op},\mathrm{Vect}_\Bbbk)$, in which $\mathscr{C}$ embeds by the Yoneda embedding, see [Dug, Section 2.2]. We will often forget mentions of i and consider $\mathscr{C}$ as a subcategory of its free cocompletion.
With this description it is easy to see that free cocompletions arrange into a functor

$$\mathrm{Free} = \mathrm{Hom}_{\mathrm{Cat}_\Bbbk}((-)^{op},\mathrm{Vect}_\Bbbk) : \mathrm{Cat}_\Bbbk \to \mathbf{Pr}\,.$$

We denote $\mathrm{Bimod}_\Bbbk$ its essential image. Note that one can always take the tensor product of a presheaf P with a vector space V by taking pointwise tensor product. Moreover, V is a coproduct of copies of $\Bbbk$ and the presheaf $V\otimes P$ is a coproduct of copies of P. ♢

We now aim to show that the free cocompletion is characterized by the fact that it is cocomplete, contains $\mathscr{C}$ as a full subcategory, every object is a colimit of objects of $\mathscr{C}$, and objects of $\mathscr{C}$ are compact projective.

Lemma 2.3.3 (co-Yoneda): *Let $P\in\mathrm{Free}(\mathscr{C})$ be a presheaf. Then P is a colimit of objects of $\mathscr{C}$. This colimit is explicitly obtained as the coend*

$$P \simeq \int^{C\in\mathscr{C}} \mathrm{Hom}_{\mathrm{Free}(\mathscr{C})}(C,P)\otimes C \quad .$$

Definition 2.3.4: An object $C \in \mathcal{C}$ is called *compact-projective* (which we abbreviate cp) if the functor $\mathrm{Hom}_{\mathcal{C}}(C,-) : \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ is cocontinuous. The category $\mathcal{C}$ is said to *have enough compact projectives* if its full subcategory $\mathcal{C}^{cp}$ of compact projective objects generates $\mathcal{C}$ under colimits. A monoidal category $\mathcal{C}$ is called *cp-rigid* if it has enough cp and all its cp objects are left and right dualizable. ♢

Lemma 2.3.5: *Any object $C \in \mathscr{C}$ is compact-projective in* $\mathrm{Free}(\mathscr{C})$.

Proof : Colimits are computed pointwise in a presheaf category. □

The last two lemmas show that $\mathrm{Free}(\mathscr{C})$ has enough cp. The following is shown in [BCJ15, Proposition 2.2].

Proposition 2.3.6: *Let $\mathcal{C} \in \mathbf{Pr}$. The following are equivalent:*

1. *$\mathcal{C}$ has enough cp,*
2. *the canonical functor $\mathrm{Free}(\mathcal{C}^{cp}) \to \mathcal{C}$ induced by the universal property of the free cocompletion along the inclusion $\mathcal{C}^{cp} \subseteq \mathcal{C}$ is an equivalence, and*
3. *$\mathcal{C}$ lies in* $\mathrm{Bimod}_{\Bbbk}$

Proof : For the non-trivial implication $1 \Rightarrow 2$, suppose $\mathcal{C}$ has enough cp. The functor $F : \mathrm{Free}(\mathcal{C}^{cp}) \to \mathcal{C}$ is characterized by $F|_{\mathcal{C}^{cp}} = Id_{\mathcal{C}^{cp}}$ and F is cocontinuous. It is essentially surjective because $\mathcal{C}$ is generated by colimits of cp objects. For $C, D \in \mathrm{Free}(\mathcal{C}^{cp})$, write $C = \mathrm{colim}_i\, c_i$ and $D = \mathrm{colim}_j\, d_j$ with $c_i, d_j \in \mathcal{C}^{cp}$. Then:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(F(C), F(D)) = \quad & \overset{F\ cocont}{\simeq} && \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_i\, F(c_i), \mathrm{colim}_j\, F(d_j)) \\
& \overset{c_i\ cp}{\simeq} && \mathrm{colim}_i \lim_j \mathrm{Hom}_{\mathcal{C}}(F(c_i), F(d_j)) \\
& \overset{F|_{\mathcal{C}^{cp}} = Id_{\mathcal{C}^{cp}}}{\simeq} && \mathrm{colim}_i \lim_j \mathrm{Hom}_{\mathcal{C}^{cp}}(c_i, d_j) \\
& \overset{cp}{\simeq} && \mathrm{Hom}_{\mathrm{Free}(\mathcal{C}^{cp})}(C, D)
\end{aligned}$$

and F is fully faithful. □

Proposition 2.3.7: *Let $\mathcal{C}, \mathcal{D} \in$ $\mathrm{Bimod}_{\Bbbk}$, and write $\mathcal{C} \simeq$ $\mathrm{Free}(\mathscr{C})$ and $\mathcal{D} \simeq$ $\mathrm{Free}(\mathscr{D})$. There is an equivalence of categories*

$$\mathrm{Hom}_{\mathbf{Pr}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\Bbbk}}(\mathscr{C} \otimes \mathscr{D}^{op}, \mathrm{Vect}_{\Bbbk})$$

mapping a cocontinuous functors $F : \mathcal{C} \to \mathcal{D}$ to the "bimodule"

$$\mathcal{F}_F : \begin{array}{ccc} \mathscr{C} \otimes \mathscr{D}^{op} & \to & \mathrm{Vect}_{\Bbbk} \\ (C, D) & \mapsto & \mathrm{Hom}_{\mathcal{D}}(D, F(C)) \end{array} .$$

Proof : It follows from:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Pr}}(\mathcal{C}, \mathcal{D}) & \overset{i^*}{\simeq} \mathrm{Hom}_{CAT_{\Bbbk}}(\mathscr{C}, \mathcal{D}) \\
& \simeq \mathrm{Hom}_{CAT_{\Bbbk}}(\mathscr{C}, \mathrm{Hom}_{\mathrm{Cat}_{\Bbbk}}(\mathscr{D}^{op}, Vect_{\Bbbk})) \\
& \simeq \mathrm{Hom}_{\mathrm{Cat}_{\Bbbk}}(\mathscr{C} \otimes \mathscr{D}^{op}, \mathrm{Vect}_{\Bbbk})
\end{aligned}$$

using universality of free cocompletions and the characterization of tensor product in $\mathrm{Cat}_{\Bbbk}$. □

For $f : \mathscr{C} \to \mathscr{D}$ a $\Bbbk$-linear functor, we denote $\mathcal{F}_f : \begin{array}{ccc} \mathscr{C} \otimes \mathscr{D}^{op} & \to & \mathrm{Vect}_{\Bbbk} \\ (C, D) & \mapsto & \mathrm{Hom}_{\mathscr{D}}(D, f(C)) \end{array}$ the bimodule associated with $\mathrm{Free}(f)$. They form a subset of all morphisms from $\mathcal{C}$ to $\mathcal{D}$ that will be characterized in Proposition 2.3.8. In topos theory they get called "essential geometric morphisms".

One can describe composition of cocontinuous functors at the level of bimodules. Given $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$, we want to describe $\mathcal{F}_{G\circ F}(A, C) = \mathrm{Hom}_{\mathcal{C}}(C, G \circ F(A))$ for $A \in \mathscr{A}$ and $C \in \mathscr{C}$. The problem is that $F(A)$ in general is not in $\mathscr{B}$, but is obtained as a colimit of such by the co-Yoneda lemma: $F(A) \simeq \int^{B\in\mathscr{B}} \mathrm{Hom}_{\mathcal{B}}(B, F(A)) \otimes B$. Then by cocontinuity of G we obtain that $G(F(A)) = \int^{B\in\mathscr{B}} \mathrm{Hom}_{\mathcal{B}}(B, F(A)) \otimes G(B)$ and because C is compact projective, that $\mathrm{Hom}_{\mathcal{C}}(C, G \circ F(A)) = \int^{B\in\mathscr{B}} \mathrm{Hom}_{\mathscr{B}}(B, F(A)) \otimes \mathrm{Hom}_{\mathscr{C}}(C, G(B))$. This adds up to the formula $\mathcal{F}_{G\circ F}(-,-) = \int^{B\in\mathscr{B}} \mathcal{F}_F(-, B) \otimes \mathcal{F}_G(B, -)$, which we denote $\mathcal{F}_F \underset{\mathscr{B}}{\boxtimes} \mathcal{F}_G$.

Proposition 2.3.8: *Let $F : \mathcal{C} \to \mathcal{D}$ be a 1-morphism in* **Pr** *between two categories that have enough compact projectives. Let $\mathscr{C} = \mathcal{C}^{cp}$ and $\mathscr{D} = \mathcal{D}^{cp}$. Then the following are equivalent:*

- *F has a right adjoint in* **Pr**
- *The right adjoint of F in $CAT_{\Bbbk}$ is cocontinuous*
- *F preserves compact projective objects*
- *F is the cocontinuous extension of a functor $f : \mathscr{C} \to \mathscr{D}$*

When these hold, the right adjoint of F is given by the bimodule

$$\overline{\mathcal{F}}_F : \begin{array}{ccc} (\mathscr{C})^{op} \otimes \mathscr{D} & \to & Vect_k \\ (C, D) & \mapsto & \mathrm{Hom}_{\mathcal{D}}(F(C), D) \end{array} \tag{2.3}$$

with unit induced by F, and counit induced by composition in $\mathcal{D}$.

Proof : The first two and last two points are immediately equivalent.
If F^R is cocontinuous then for $C \in \mathscr{C}$ and $D = \mathrm{colim}_i D_i$ obtained as a colimit,

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{D}}(F(C), D) & \simeq & \mathrm{Hom}_{\mathcal{C}}(C, F^R(D)) \\ & \overset{F^R \text{ cocont}}{\simeq} & \mathrm{Hom}_{\mathcal{C}}(C, \mathrm{colim}_i F^R(D_i)) \\ & \overset{C \text{ cp}}{\simeq} & \mathrm{colim}_i \mathrm{Hom}_{\mathcal{C}}(C, F^R(D_i)) \simeq \mathrm{colim}_i \mathrm{Hom}_{\mathcal{D}}(F(C), D_i) \end{array}$$

and $F(C)$ is compact projective, so F preserves cp.
The other direction is a classical construction, see [BDSV, Lemma 2.10], which we recall

here. The unit goes: $\eta : Id_{\mathscr{C}} \Rightarrow \overline{\mathcal{F}}_f \circ \mathcal{F}_f = \mathcal{F}_f \underset{\mathscr{D}}{\boxtimes} \overline{\mathcal{F}}_f$ where

$$\mathcal{F}_f \underset{\mathscr{D}}{\boxtimes} \overline{\mathcal{F}}_f : \begin{cases} \mathscr{C}^{op} \otimes \mathscr{C} & \to & \mathrm{Vect}_{\Bbbk} \\ (C, C') & \mapsto & \int^{D \in \mathscr{D}} \mathrm{Hom}_{\mathscr{D}}(f(C), D) \otimes \mathrm{Hom}_{\mathscr{D}}(D, f(C')) \simeq \mathrm{Hom}_{\mathscr{D}}(f(C), f(C')) \end{cases}$$

and is given by

$$\eta_{C,C'} : \begin{cases} Id_{\mathscr{C}}(C, C') = \mathrm{Hom}_{\mathscr{C}}(C, C') & \to & \mathcal{F}_f \underset{\mathscr{D}}{\boxtimes} \overline{\mathcal{F}}_f(C, C') = \mathrm{Hom}_{\mathscr{D}}(f(C), f(C')) \\ g & \mapsto & f(g) \end{cases}.$$

The counit goes $\varepsilon : \mathcal{F}_f \circ \overline{\mathcal{F}}_f = \overline{\mathcal{F}}_f \underset{\mathscr{C}}{\boxtimes} \mathcal{F}_f \Rightarrow Id_{\mathscr{D}}$, where

$$\overline{\mathcal{F}}_f \underset{\mathscr{C}}{\boxtimes} \mathcal{F}_f : \begin{cases} \mathscr{D}^{op} \otimes \mathscr{D} & \to & Vect_k \\ (D, D') & \mapsto & \int^{C \in \mathscr{C}} \mathrm{Hom}_{\mathscr{D}}(D, f(C)) \otimes \mathrm{Hom}_{\mathscr{D}}(f(C), D') \end{cases}$$

and is given by

$$\varepsilon_{D,D'} : \begin{cases} \int^{C \in \mathscr{C}} \mathrm{Hom}_{\mathscr{D}}(D, f(C)) \otimes \mathrm{Hom}_{\mathscr{D}}(f(C), D') & \to & \mathrm{Hom}_{\mathcal{D}}(D, D') \\ g \otimes h & \mapsto & h \circ g \end{cases}.$$

which is well-defined on the coend because the composition in $\mathscr{D}$ is associative. □

In general, given a cocontinuous functor $F : \mathcal{C} \to \mathcal{D}$, the bimodule $\overline{\mathcal{F}}_F$ of (2.3) makes sense, and is associated with a cocontinuous functor $\mathcal{D} \to \mathcal{C}$. It coincides with the right adjoint of F on the compact projectives, but it is cocontinuous and may not agree with F^R on all of $\mathcal{D}$ (it does when F^R is cocontinuous). We call it the "renormalization" of F^R and denote it $\widetilde{F^R}$. It is used when F^R is not cocontinuous.

Proposition 2.3.9: *The free cocompletion functor* Free : $\mathrm{Cat}_{\Bbbk} \to \mathbf{Pr}$ *is symmetric monoidal.*

Proof : For $\mathscr{A}, \mathscr{B} \in \mathrm{Cat}_{\Bbbk}$ and $\mathcal{C} \in \mathbf{Pr}$, we have:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Pr}}(\mathrm{Free}(\mathscr{A}) \boxtimes \mathrm{Free}(\mathscr{B}), \mathcal{C}) &\simeq \mathrm{Hom}_{\mathbf{Pr}}(\mathrm{Free}(\mathscr{A}), \mathrm{Hom}_{\mathbf{Pr}}(\mathrm{Free}(\mathscr{B}), \mathcal{C})) \\ &\simeq \mathrm{Hom}_{CAT_{\Bbbk}}(\mathscr{A}, \mathrm{Hom}_{CAT_{\Bbbk}}(\mathscr{B}, \mathcal{C})) \\ &\simeq \mathrm{Hom}_{CAT_{\Bbbk}}(\mathscr{A} \otimes \mathscr{B}, \mathcal{C}) \\ &\simeq \mathrm{Hom}_{\mathbf{Pr}}(\mathrm{Free}(\mathscr{A} \otimes \mathscr{B}), \mathcal{C}) \end{aligned}$$

and $\mathrm{Free}(\mathscr{A}) \boxtimes \mathrm{Free}(\mathscr{B})$ satisfies the universal property for $\mathrm{Free}(\mathscr{A} \otimes \mathscr{B})$. □

2.3.2 The higher Morita category $\mathfrak{alg}_n(\mathcal{S})$

We recall the construction of higher Morita categories of E_n-algebras in an ∞-category $\mathcal{S}$ introduced in [Hau17], see also [Hau].
Note that they actually describe an n-uple Segal object in ∞-categories, from which one can extract an $(n+1)$-uple Segal space, take its underlying $(n+1)$-fold Segal space, and complete to an $(n+1)$-category. Here ∞-category is understood in the sense developed in [Lur09a]. We do not recall this notion, but it is well-known [JT07] to be (Quillen) equivalent to the notion of complete Segal space we introduced in Section 2.1.1. Moreover, we will not consider the full construction of [Hau17] but truncate these ∞-categories

to their underlying groupoids. We describe the n-fold Segal space $\mathfrak{alg}_n(\mathcal{S})$ which is a truncation of the $(n+1)$-fold Segal space $U_{Seg}\mathfrak{i}\mathfrak{ALG}_n(\mathcal{S})$ defined in [Hau17, Definition 5.33]. This is the notion needed in [JS17] to define even higher Morita categories.

Let us recall the broad lines of the construction.
They introduce a notion of generalized Δ^n-∞-operad in [Hau17, Definition 5.8] as some particular class of functors from some ∞-category $\mathcal{M}$ into $(\Delta^n)^{op}$. The notion of a Δ^n-monoidal ∞-category $\mathcal{S}$ is a particular case [Hau17, Definition 5.1]. The relation with the more usual notion of n-times monoidal ∞-category is done by Lurie's straightening equivalence [Lur09a, Section 3.2], see [Hau17, Corollary A.31]. We will only be interested in the special cases of monoidal and symmetric monoidal (2,1)-categories. The ∞-category $Alg^n_{\mathcal{M}}(\mathcal{N})$ of $\mathcal{M}$-algebras in $\mathcal{N}$ is defined in [Hau17, Definition 5.12] for arbitrary Δ^n-∞-operads $\mathcal{M}$ and $\mathcal{N}$. Given integers $k_1, \dots, k_n$, the slice category $(\Delta/[k_1])^{op} \times \cdots \times (\Delta/[k_n])^{op}$ gives a generalized Δ^n-∞-operad with the forgetful functor [Hau17, Lemma 5.14].

Definition 2.3.10: Let $\mathcal{S}$ be a Δ^n-monoidal ∞-category with good relative tensor products. The level $(\mathfrak{ALG}_n(\mathcal{S}))_{k_1,\dots,k_n}$ of the n-uple simplicial ∞-category $\mathfrak{ALG}_n(\mathcal{S})$ is defined [Hau17, Definition 5.21] to be a sub-∞-category of $Alg^n_{(\Delta/[k_1])^{op}\times\cdots\times(\Delta/[k_n])^{op}}(\mathcal{S})$ of so-called composite bimodules. ♢

For every $0 \leq i \leq j \leq k$, the map $[1] \xrightarrow{i,j} [k]$ gives an object (i,j) of $\Delta/[k]$. Therefore an element M of $Alg^n_{(\Delta/[k_1])^{op}\times\cdots\times(\Delta/[k_n])^{op}}(\mathcal{S})$ associates an object $M(i_1, j_1, \dots, i_n, j_n)$ of $\mathcal{S}$ to a sequence $(0 \leq i_1 \leq j_1 \leq k_1, \dots, 0 \leq i_n \leq j_n \leq k_n)$. It is explained in [Hau17, Section 2.1 and 2.2] why these objects have algebra structures when $i_l = j_l$ and are bimodules over these algebras in general. It is explained in [Hau17, Corollary 4.20] that composite bimodules are those for which the $M(i_1, j_1, \dots, i_n, j_n)$ are obtained as relative tensor products of the $M(l_1, l_1+1, \dots, l_n, l_n+1)$'s for various l's. This property is what ensures the Segal condition.
The n-fold simplicial structure is given by Lurie's straightening equivalence and [Hau17, Theorem 5.31]. The Segal condition is shown in [Hau17, Corollary 5.25].

Definition 2.3.11: The n-fold Segal space $\mathfrak{alg}_n(\mathcal{S})$ is the underlying n-fold Segal space of the n-uple Segal space $(\mathfrak{ALG}_n(\mathcal{S}))^\sim$ whose levels $(\mathfrak{ALG}_n(\mathcal{S}))^\sim_{k_1,\dots,k_n}$ are the underlying groupoids of the ∞-categories $(\mathfrak{ALG}_n(\mathcal{S}))_{k_1,\dots,k_n}$.
The higher Morita n-category $\widehat{\mathfrak{alg}}_n(\mathcal{S})$ is its completion. ♢

The input data of a Δ^n-monoidal ∞-category is identified with the more usual notion of an $\mathbb{E}_n$-algebra in the symmetric monoidal ∞-category $\mathrm{Cat}_\infty^\times$ endowed with carthesian monoidal structure in [Lur, Proposition 2.4.2.5, Example 2.4.2.4 and Definition 2.1.2.13] and [Hau17, Corollary A.31].

The spaces of objects and morphisms of the higher Morita category $\mathfrak{alg}_n(\mathcal{S})$ are well-identified.

Theorem 2.3.12 (Section 5.2 and Corollary A.26 of [Hau17]): *Let $\mathcal{S}$ be a Δ^n-monoidal ∞-category, seen as an $\mathbb{E}_n$-monoidal ∞-category. There is an equivalence of ∞-categories*

$$(\mathfrak{ALG}_n(\mathcal{S}))_{0,\dots,0} := Alg^n_{(\Delta^n)^{op}}(\mathcal{S}) \simeq Alg_{/\mathbb{E}_n}(\mathcal{S})$$

between the ∞-category of objects of $\mathfrak{ALG}_n(\mathcal{S})$ and the ∞-category of $\mathbb{E}_n$-algebras in the $\mathbb{E}_n$-monoidal category $\mathcal{S}$ as defined in [Lur, Definition 2.1.3.1].
In particular, their underlying groupoids are equivalent.

Remark 2.3.13: We will only be interested in two special cases of $\mathbb{E}_n$-algebras in a $\mathbb{E}_n$-monoidal ∞-category $\mathcal{S}$. The first is when $n=2$ and $\mathcal{S}$ happens to be symmetric monoidal (which is a special case of $\mathbb{E}_2$-monoidal, for example using [Lur, Corollary 5.1.1.5]). It is shown in [Lur, Example 5.1.2.4] that if $\mathcal{S}$ is a subcategory of Cat_∞ consisting of (nerves of usual) categories, then an $\mathbb{E}_2$-algebra structure on an object in $\mathcal{S}$ consists exactly of a structure of braided category.
The second is when $n=1$. Then $\mathcal{S}$ is a homotopy-associative algebra object in Cat_∞ and an $\mathbb{E}_1$-algebra in $\mathcal{S}$ is a homotopy-associative algebra object in $\mathcal{S}$ in the usual sense, see [Lur, Example 5.1.0.7]. ◇

Definition 2.3.14 (Section 4.5 in [Hau17]): Let $\mathcal{S}$ be a Δ^1-monoidal ∞-category. We denote $\mathrm{Bimod}(\mathcal{S}) := Alg^1_{(\Delta/[1])^{op}}(\mathcal{S})$ the ∞-category of bimodules in $\mathcal{S}$. The inclusions $0,1:\Delta\hookrightarrow\Delta/[1]$ induce two functors $s,t:\mathrm{Bimod}(\mathcal{S})\to Ass(\mathcal{S}) := Alg^1_{\Delta^{op}}(\mathcal{S})$.
Let A,B be objects of $Ass(\mathcal{S})$. The ∞-category $\mathrm{Bimod}_{A,B}(\mathcal{S})$ is the fiber of $\mathrm{Bimod}(\mathcal{S})\to Ass(\mathcal{S})^{\times 2}$ at (A,B). ◇

It is shown in [Hau17, Corollary 5.45] that when $\mathcal{S}$ is Δ^n-monoidal with good relative tensor products and A,B are Δ^n-algebra objects, then $\mathrm{Bimod}_{A,B}(\mathcal{S})$ inherits a Δ^{n-1}-monoidal structure.

Proposition 2.3.15 (Propositions 4.53, 5.44 and Corollary A.77 in [Hau17]): *Denote $\otimes$ the monoidal structure on $\mathcal{S}$. The monoidal structure on $\mathrm{Bimod}_{A,B}(\mathcal{S})$ is given on two objects M,N by the relative tensor product*

$$M\underset{A,B}{\otimes}N := A\underset{A\otimes A}{\otimes}(M\otimes N)\underset{B\otimes B}{\otimes}B.$$

It is shown in [Hau17, Lemma 5.46] that $\mathrm{Bimod}_{A,B}(\mathcal{S})$ has good relative tensor product.

Theorem 2.3.16 (Theorem 5.49 in [Hau17]): *Let A,B be objects of $\mathfrak{alg}_n(\mathcal{S})$. Then there is a Dwyer–Kan equivalence of $(n-1)$-fold Segal spaces*

$$\mathrm{Hom}_{\mathfrak{alg}_n(\mathcal{S})}(A,B)\simeq\mathfrak{alg}_{n-1}(\mathrm{Bimod}_{A,B}(\mathcal{S})).$$

2.3.3 The even higher Morita 4-category $\mathbf{Alg}_2(\mathbf{Pr})$

We recall the extension of Haugseng's higher Morita categories into even higher Morita categories from [JS17] for $\mathcal{S}=\mathbf{Pr}$.
They define a 2-by-2-fold Segal space $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{\bullet,\bullet})_{\bullet,\bullet}$ and define $\mathbf{Alg}_2(\mathbf{Pr})$ to be the underlying 4-fold Segal space of its completion.
We are working in the particular case where $\mathcal{S}=\mathbf{Pr}$ is a strict bicategory. It can be seen as a particular case of 2-fold Segal space where every level is discrete by the discrete nerve construction.

Definition 2.3.17 (Definition 5.14 of [JS17]): Let $k,l\geq 0$. The symmetric monoidal (2,1)-category $\mathbf{Pr}^{\boxed{\text{strong}}}_{k,l}$ is the bicategory $\mathrm{Fun}(\Theta^{k,l},\mathbf{Pr})$ of bifunctors, strong natural transformations and invertible modifications from $\Theta^{k,l}$ to $\mathbf{Pr}$ (remember that the $\Theta^{k,l}$'s were defined above following [JS17, Definition 5.1]). It can be seen as a symmetric monoidal ∞-category via Duskin's nerve [Dus02] and one can apply Haugseng's construction on it. $\diamond$

Remark 2.3.18: The definition above is a slight abuse. In [JS17], the $(\infty, 2)$-category $\mathrm{Fun}(\Theta^{k,l}, \mathbf{Pr})$ is defined as an internal Hom in complete 2-fold Segal spaces, not in bicategories. One has to show that the naive way of computing internal Hom between bicategories described above does coincide with the one defined by seeing these bicategories as $(\infty, 2)$-categories. I am very thankful to Bertrand Toën for pointing out the following argument to me.

There exists a nerve functor $N : \mathrm{Bicat} \to \mathrm{CSS}_2$ which satisfies that for C, D two bicategories,

$$\mathrm{Fun}_{\mathrm{CSS}_2}(N(C), N(D)) \simeq N(\mathrm{Fun}_{\mathrm{Bicat}}(C, D))$$

Moreover, for $\mathcal{C}$ a strict 2-category, this nerve is weakly equivalent to the strict nerve, $N^{str}(\mathcal{C}) \simeq N(\mathcal{C})$.

The first statement is explained in [Cam20] when $(\infty, 2)$-categories are understood as 2-quasi-categories. They construct N in [Cam20, Corollary 5.11]. They construct its right adjoint, taking homotopy bicategories, in [Cam20, Theorem 6.29]. They show it preserves finite products in [Cam20, Lemma 6.27]. They show that N is fully faithful in [Cam20, Corollary 5.13]. Hence N satisfies the conditions from Remark 2.1.31 and preserves inner Hom's.

We can conclude in our context using well-known Quillen equivalences between different models for (∞, n)-categories, namely [Ara14, Theorem 8.4] relating n-quasi-categories to Rezk Θ_n-spaces and [BR20, Corollary 7.3] relating Rezk Θ_n-spaces to complete n-fold Segal spaces. Such Quillen equivalences in particular preserve internal Hom's.

The second statement is [Cam20, Theorem 10.10], again for N with values in 2-quasi-categories, and strict nerve understood in a sense very close to ours, by taking functors out of the strict bicategories $\Theta_2([n; \boldsymbol{m}])$ see [Ara14, Section 5.8]. These bicategories contain the $\Theta^{k,l}$'s as the $\Theta_2([k; (l \ldots, l)])$'s. Tracking down the equivalences above, we see that the equivalence of [BR20, Corollary 7.3] precisely restricts a presheaf on Θ_2 to the objects of the form $\Theta^{k,l}$ and [Ara14, Theorem 8.4] takes discrete spaces. Therefore Ara's strict nerve maps to ours, and Campbell's result applies. $\diamond$

Definition 2.3.19 (Section 8 in [JS17]): The 4-uple simplicial space

$$\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{\bullet,\bullet})_{\bullet,\bullet} : \begin{array}{ccc} (\Delta^{\times 4})^{op} & \to & \mathrm{Space} \\ (p,q,k,l) & \mapsto & \mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{k,l})_{p,q} \end{array}$$

is obtained by applying Haugseng's construction to various (2,1)-categories $\mathbf{Pr}^{\boxed{\text{strong}}}_{k,l}$. It's simplicial structure is given by Haugseng's construction for the first two coordinates, and the maps between the $\Theta^{k,l}$ bicategories together with functoriality of Haugseng's construction for the last two coordinates. $\diamond$

Theorem 2.3.20 (Theorem 8.5 in [JS17]): *The 4-uple simplicial space $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{\bullet,\bullet})_{\bullet,\bullet}$ is a 2-by-2-fold Segal space, and is complete in the last two coordinates.*
In particular, the 4-uple simplicial space $\widehat{\mathfrak{alg}}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{\bullet,\bullet})_{\bullet,\bullet}$ is a complete 2-by-2-fold Segal space.

We have a 2-fold simplicial diagram of 2-fold Segal spaces as in Figure 2.7. It is already complete in the simplicial directions displayed, and its completion is the component-wise completion in this diagram.

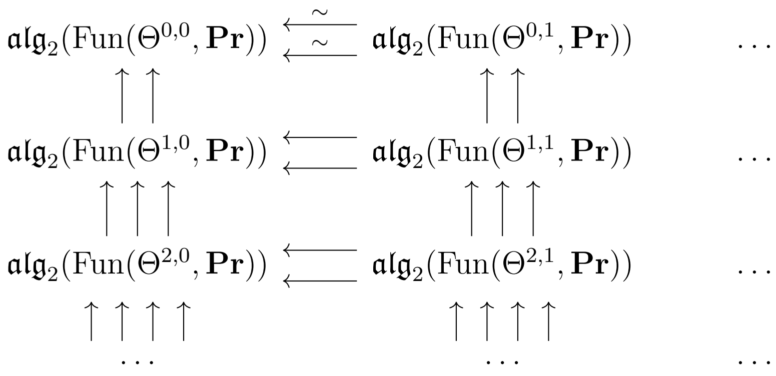


Figure 2.7: The 2-by-2-fold Segal space $\mathfrak{alg}_2(\mathbf{Pr}^{strong}_{\bullet,\bullet})_{\bullet,\bullet}$

Definition 2.3.21 (Definition 8.6 in [JS17]): The even higher Morita 4-category

$$\mathbf{Alg}_2(\mathbf{Pr}) := U_{fold}\widehat{\mathfrak{alg}}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{\bullet,\bullet})_{\bullet,\bullet}$$

is the underlying 4-fold Segal space of $\widehat{\mathfrak{alg}}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{\bullet,\bullet})_{\bullet,\bullet}$. ♢

Remember that by definition [JS17, Remark 5.4] taking underlying 4-fold Segal space means that

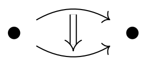
$$\mathbf{Alg}_2(\mathbf{Pr})_{p,q,k,l} := \mathrm{maps}^h(\Theta^{p,q,k,l}, \mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{\bullet,\bullet})_{\bullet,\bullet}),$$

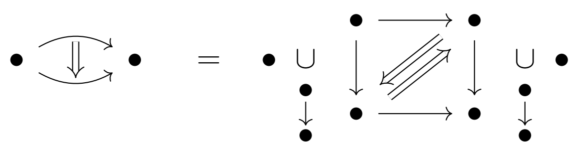
where $\Theta^{p,q,k,l}$ are defined in [JS17, Definition 5.1]. To compute the derived mapping spaces here we need to find a (projectively) cofibrant replacement of $\Theta^{p,q,k,l}$. The (nerves of the) $\Theta^{p,q,k,l}$'s are indeed not cofibrant (for $pq \neq 0$) but they are obtained as colimits of representable presheaves which are. Morphisms out of them are therefore obtained as homotopy limits.

We adopt [JS17] convention to not write the last chain of 0's in the indices, and we drop comas. For example, $\mathcal{C}_{11}$ denotes $\mathcal{C}_{1,1,0,0}$.

Proposition 2.3.22: *Let $\mathcal{C}$ be a complete 2-by-2-fold Segal space. Then we have equivalences*

$$\begin{array}{rcl} \mathrm{maps}^h(\Theta^{1,1,0,0}, \mathcal{C}) & \simeq & \mathcal{C}_{11} \\ \mathrm{maps}^h(\Theta^{1,1,1}, \mathcal{C}) & \simeq & \mathcal{C}_1 \underset{\mathcal{C}_{101}}{\times^h} \mathcal{C}_{111} \underset{\mathcal{C}_{101}}{\times^h} \mathcal{C}_1 \\ \mathrm{maps}^h(\Theta^{1,1,1,1}, \mathcal{C}) & \simeq & \mathcal{C}_1 \underset{\mathcal{C}_{1011}}{\times^h} \mathcal{C}_{1111} \underset{\mathcal{C}_{1011}}{\times^h} \mathcal{C}_1 \end{array}$$

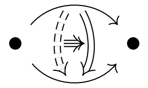
Proof : The first statement is [JS17, Remark 3.4]. We recall the proof to apply it to the other two cases. The bigon $\Theta^{1,1}$:  (or more formally, its strict nerve) is obtained as a quotient of the square by the two vertical edges

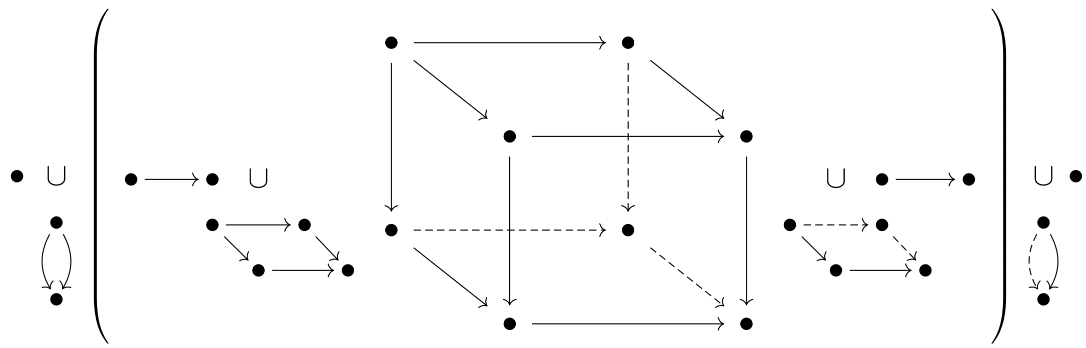
 (2.4)

Here the square is the presheaf represented by $[1]\times[1]$, the vertical edges are represented by $[0]\times[1]$, and the points by $[0]\times[0]$. As argued in [JS17, Remark 3.2], as the map from two vertical arrow to the square is an inclusion, a cofibrant replacement for $\Theta^{1,1}$ can be computed as the associated homotopy pushout. Finally, by the universal property of homotopy pushouts we can compute the derived mapping space into any complete 2-uple Segal space $\mathcal{C}$ as

$$\mathrm{maps}^h(\Theta^{1,1},\mathcal{C}) \simeq \mathcal{C}_0 \underset{\mathcal{C}_{01}}{\times^h} \mathcal{C}_{11} \underset{\mathcal{C}_{01}}{\times^h} \mathcal{C}_0$$

In particular, if $\mathcal{C}_{\bullet\bullet}$ is already essentially constant one has $\mathrm{maps}^h(\Theta^{1,1},\mathcal{C}) \simeq \mathcal{C}_{11}$ as claimed.

Similarly, $\Theta^{1,1,1} =$  is obtained as a quotient of the cube by collapsing its top and bottom face to an edge and collapsing its left and right face to a point. However, doing these operations successively, it is not true anymore that the left and right square faces inject into the partially collapsed cube. Indeed, only vertical bigons remain. For the arguments above to hold, we need injectivity and we have to write bigons instead. We represent only the 1-dimensional part, but everything below is filled.



(2.5)

Again, this is a homotopy pushout. We have:

$$\mathrm{maps}^h(\Theta^{1,1,1},\mathcal{C}) \simeq \mathrm{holim}\left(\begin{array}{ccccccc}
 & \mathcal{C}_0 & \xleftarrow{s} & \mathcal{C}_1 & \xrightarrow{t} & \mathcal{C}_0 & \\
 & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & \\
 & \mathcal{C}_{001} & \xleftarrow{s} & \mathcal{C}_{101} & \xrightarrow{t} & \mathcal{C}_{001} & \\
 & \wr\uparrow & & \uparrow & & \wr\uparrow & \\
\mathcal{C}_0 & \xrightarrow{\delta} \mathcal{C}_{011} & \xleftarrow{s} & \mathcal{C}_{111} & \xrightarrow{t} & \mathcal{C}_{011} \xleftarrow{\delta} & \mathcal{C}_0 \\
 & \wr\downarrow & & \downarrow & & \wr\downarrow & \\
 & \mathcal{C}_{001} & \xleftarrow{s} & \mathcal{C}_{101} & \xrightarrow{t} & \mathcal{C}_{001} & \\
 & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & \\
 & \mathcal{C}_0 & \xleftarrow{s} & \mathcal{C}_1 & \xrightarrow{t} & \mathcal{C}_0 &
\end{array}\right) \tag{2.6}$$

If one first computes the homotopy limits on columns, the resulting homotopy limit is exactly the one we get by the presentation of $\Theta^{1,1,1}$ as a homotopy pushout above. Indeed the homotopy limit of the central column corresponds to the big parenthesis in (2.5), and the columns on its side to the vertical bigons by the same argument as for $\Theta^{1,1}$.
By our 2-by-2-fold assumption, we have the displayed four equivalences. Now a cone on this diagram is equivalent to a cone on the central column, and therefore this homotopy

limit can also be computed as

$$\mathrm{maps}^h(\Theta^{1,1,1}, \mathcal{C}) \simeq \mathcal{C}_1 \underset{\mathcal{C}_{101}}{\times^h} \mathcal{C}_{111} \underset{\mathcal{C}_{101}}{\times^h} \mathcal{C}_1$$

as claimed.

Finally, $\Theta^{1,1,1,1}$ can be obtained as a quotient of the hypercube in the same way. We obtain a homotopy limit similar to (2.6), but now living on a 3-dimensional grid. Under our 2-by-2-fold assumption, the new direction is already essentially constant, and the homotopy limit reduces to that on the 2-dimensional grid, namely

$$\mathrm{maps}^h(\Theta^{1,1,1,1}, \mathcal{C}) \simeq \operatorname{holim} \left(\begin{array}{ccccccccc}
 & & \mathcal{C}_0 & \xleftarrow{s} & \mathcal{C}_1 & \xrightarrow{t} & \mathcal{C}_0 & & \\
 & \mathrel{//} & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & \mathrel{\backslash\backslash} & \\
 & & \mathcal{C}_{0011} & \xleftarrow{s} & \mathcal{C}_{1011} & \xrightarrow{t} & \mathcal{C}_{0011} & & \\
 & & \wr\uparrow & & \uparrow & & \wr\uparrow & & \\
\mathcal{C}_0 & \xrightarrow{\delta} & \mathcal{C}_{0111} & \xleftarrow{s} & \mathcal{C}_{1111} & \xrightarrow{t} & \mathcal{C}_{0111} & \xleftarrow{\delta} & \mathcal{C}_0 \\
 & & \wr\downarrow & & \downarrow & & \wr\downarrow & & \\
 & & \mathcal{C}_{0011} & \xleftarrow{s} & \mathcal{C}_{101} & \xrightarrow{t} & \mathcal{C}_{0011} & & \\
 & \mathrel{\backslash\backslash} & {\scriptstyle\delta}\uparrow & & {\scriptstyle\delta}\uparrow & & {\scriptstyle\delta}\uparrow & \mathrel{//} & \\
 & & \mathcal{C}_0 & \xleftarrow{s} & \mathcal{C}_1 & \xrightarrow{t} & \mathcal{C}_0 & &
\end{array} \right) \tag{2.7}$$

which for the same reason as above gives the claimed formula. □

2.3.4 BrTens and $\mathbf{Alg}_2(\mathbf{Pr})$

We recall the explicit description of the underlying bicategories of $\mathbf{Alg}_2(\mathbf{Pr})$ from [BJS21]. We use the name **BrTens** for their description, and include a comparison theorem with the original definition of $\mathbf{Alg}_2(\mathbf{Pr})$ that was left implicit.
We follow conventions from [BJS21], which means that we illustrate with drawings from factorization algebras though we use the formalism from [Hau17, JS17], and that the drawings are a 90 degree rotation from [GS]. So 1-morphisms read top-down and 2-morphisms read left-to-right. It will not appear in this chapter, but we follow the convention that a 3-manifold seen as a cobordism is read bottom-up. They all arrange in a nice picture, and the usual drawings are different 2 dimensional projections. We will mostly work in the (1,2)-plane in Figure 2.8.

Let us recall the description of **BrTens** from [BJS21].

Definition 2.3.23 (Section 2.4 in [BJS21]): An object $\mathcal{V}$ of **BrTens** is a locally presentable cocomplete $\Bbbk$-linear braided monoidal category. We call these *braided tensor categories* here, even though this name has many uses. Equivalently, it is an E_2-algebra in **Pr**. ♢

Remark 2.3.24: There are many equivalent descriptions of the same objects:

1. E_2-algebras in **Pr**,

2. E_1-algebras in E_1-algebras in **Pr**, used in [Hau17],

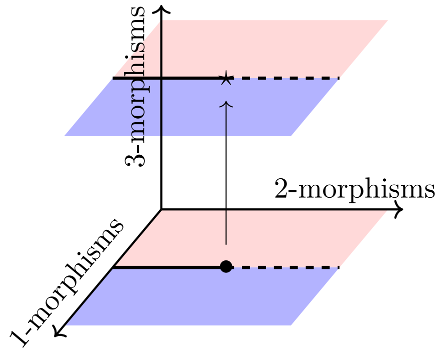


Figure 2.8: How to read morphism directions in **BrTens**

3. locally constant factorization algebras on $(0,1)^2$ with values in **Pr**, used in [Sch14a],
4. braided tensor categories.

The equivalence between 1. and 2. is Dunn's additivity [Lur, Section 5.1.2]. The equivalence between 1. and 3. is [Sch14a, Theorem 3.2.20] or [Lur, Theorem 5.4.5.9]. Now specifically for **Pr**, a sketch of the equivalence between 3. and 4. is given in [BJS21, Figure 1]. The equivalence between 2. and 4. is [Lur, Example 5.1.2.4].
Let us briefly recall one direction of this last equivalence that will be useful to keep in mind. For an E_1-algebra in E_1-algebras $\mathcal{V}$ one has both vertical and horizontal monoidal structures $\otimes_v$ and $\otimes_h$ on $\mathcal{V}$, with natural isomorphisms $\mu_{X,Y,Z,T} : (X \otimes_h Y) \otimes_v (Z \otimes_h T) \tilde{\to} (X \otimes_v Z) \otimes_h (Y \otimes_v T)$ satisfying higher compatibilities. However, a braided category has only one monoidal structure $\otimes$ and a braiding. One can obtain the description above by setting $\otimes_h = \otimes_v = \otimes$. On the drawings, we take the top elements and push them at the left to have everything on a line. The natural isomorphism μ is induced by the braiding as in Figure 2.9. ◇

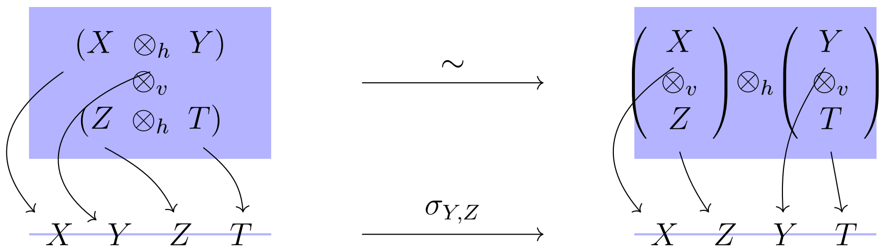
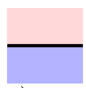


Figure 2.9: Braided categories are algebra objects in monoidal categories

Let us turn to 1-morphisms. In the factorization algebra picture , one can read that a 1-morphism between two braided tensor categories $\mathcal{V}$ (in red) and $\mathcal{W}$ (in blue) is a monoidal category $\mathcal{A} \in$ **Pr** (the horizontal line) with a top $\mathcal{V}$-action and a bottom $\mathcal{W}$-action that commute with respect to each other and that commute with the monoidal structure of $\mathcal{A}$ in a coherent way. Note that as $\mathcal{A}$ is monoidal, such an $\mathcal{V}$-action $\triangleright$ is determined by a monoidal functor $\mathcal{V} \to \mathcal{A}$ that maps V to $V \triangleright \mathbf{1}_{\mathcal{A}}$. See [BJS21, Figure 2].

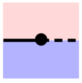
Definition 2.3.25 (Definition-Proposition 3.2 in [BJS21]): A 1-morphism between $\mathcal{V}$ and $\mathcal{W}$ in **BrTens** is a $\mathcal{V}$-$\mathcal{W}$-*central algebra* $\mathcal{A}$. Namely, it is an monoidal category

$\mathcal{A} \in \mathbf{Pr}$ equipped with a braided monoidal functor

$$(F_{\mathcal{A}}, \sigma^{\mathcal{A}}) : \mathcal{V} \boxtimes \mathcal{W}^{\sigma op} \to Z(\mathcal{A})$$

to the Drinfeld center of $\mathcal{A}$.
Remember that the Drinfeld center of $\mathcal{A}$ has objects pairs (y, β) where y is an object of $\mathcal{A}$ and $\beta : - \otimes y \overset{\sim}{\Rightarrow} y \otimes -$ is a natural isomorphism. Here $F_{\mathcal{A}}$ gives the object and $\sigma^{\mathcal{A}}$ gives the half braiding. We denote $V \triangleright A := F_{\mathcal{A}}(V) \otimes A$ and $A \triangleleft V := A \otimes F_{\mathcal{A}}(V)$. ♢

Composition of 1-morphism is relative tensor product over the corresponding braided tensor category, see [BJS21, Section 3.4]

Again in the factorization algebra picture , a 2-morphism $\mathcal{M}$ between two $\mathcal{V}$-$\mathcal{W}$-central algebras $\mathcal{A}$ and $\mathcal{B}$ is a $\mathcal{A}$-$\mathcal{B}$-bimodule category where $\mathcal{V}$ (resp. $\mathcal{W}$) acts similarly when acting through $\mathcal{A}$ or through $\mathcal{B}$.

Definition 2.3.26 (Definition 3.9 in [BJS21]): A 2-morphism between $\mathcal{A}$ and $\mathcal{B}$ in **BrTens** is a *$\mathcal{V}$-$\mathcal{W}$-centered $\mathcal{A}$-$\mathcal{B}$-bimodule category.* Namely, it is an $\mathcal{A}$-$\mathcal{B}$-bimodule category $\mathcal{M}$ equipped with natural isomorphisms

$$\eta_{v,m} : F_{\mathcal{A}}(v) \triangleright m \tilde{\to} m \triangleleft F_{\mathcal{B}}(v) \ , \quad v \in \mathcal{V} \, , \ m \in \mathcal{M} \ ,$$

satisfying coherences with tensor product in $\mathcal{V}$ and with the half braidings in $\mathcal{A}$ and $\mathcal{B}$.♢

Horizontal and vertical composition are again relative tensor product over the corresponding monoidal category.

Definition 2.3.27 (Section 3.6 in [BJS21]): A 3-morphism $F : \mathcal{M} \to \mathcal{N}$ is a functor of $\mathcal{A}$-$\mathcal{B}$-bimodules categories that preserves the $\mathcal{V}$-$\mathcal{W}$-centered structure.
A 4-morphism $\eta : F \Rightarrow G$ is a natural transformation of bimodule functors. ♢

Note that we have not defined a 4-category **BrTens** in this section, merely collections of 0–4 morphisms and ways to compose them, but this is enough to define a chain of bicategories analogous to Remark 2.1.40. The following is a slight abuse of notation to make our point clearer.

Definition 2.3.28: The bicategory $h_2(\mathbf{BrTens})$ has objects braided tensor categories, 1-morphisms central algebras, and 2-morphisms equivalence classes of centered bimodules.
Given braided tensor categories $\mathcal{V}, \mathcal{W}$, the bicategory $h_2(\mathrm{Hom}_{\mathbf{BrTens}}(\mathcal{V}, \mathcal{W}))$ has objects $\mathcal{V}$-$\mathcal{W}$-central algebras, 1-morphisms centered bimodules and 2-morphisms equivalence classes of centered bimodule functors.
Given $\mathcal{V}$-$\mathcal{W}$-central algebras $\mathcal{A}, \mathcal{B}$ the bicategory $h_2(\mathrm{Hom}_{\mathbf{BrTens}}(\mathcal{A}, \mathcal{B}))$ has objects $\mathcal{A}$-$\mathcal{B}$-centered bimodules, 1-morphisms centered bimodule functors and 2-morphisms bimodule natural transformations. ♢

The main theorem of this section is the following.

Theorem 2.3.29: *There is an equivalence of bicategories*

$$h_2(\mathbf{BrTens}) \simeq h_2(\mathbf{Alg}_2(\mathbf{Pr})) \ .$$

Given braided tensor categories $\mathcal{V}, \mathcal{W}$, seen as objects of $\mathbf{Alg}_2(\mathbf{Pr})$ by the equivalence above, one has:

$$h_2(\mathrm{Hom}_{\mathbf{BrTens}}(\mathcal{V}, \mathcal{W})) \simeq h_2(\mathrm{Hom}_{\mathbf{Alg}_2(\mathbf{Pr})}(\mathcal{V}, \mathcal{W})) \ .$$

Given $\mathcal{V}$-$\mathcal{W}$-central algebras $\mathcal{A}, \mathcal{B}$, seen as 1-morphisms of $\mathbf{Alg}_2(\mathbf{Pr})$ by the equivalence above, one has:

$$h_2(\mathrm{Hom}_{\mathbf{BrTens}}(\mathcal{A}, \mathcal{B})) \simeq h_2(\mathrm{Hom}_{\mathbf{Alg}_2(\mathbf{Pr})}(\mathcal{A}, \mathcal{B})) \ .$$

In particular the definition $\mathbf{BrTens} := \mathbf{Alg}_2(\mathbf{Pr})$ is consistent with the definitions above.

We will need a few lemmas before proving this theorem. We will begin by identifying the homotopy bicategories of the three essential 2-categories $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{0,0})$, $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{1,0})$ and $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\text{strong}}}_{1,1})$ that build morphisms in $\mathbf{Alg}_2(\mathbf{Pr})$. Then we will look at the procedure of taking underlying 4-fold Segal space.

The homotopy bicategory of $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\textbf{strong}}}_{0,0})$:

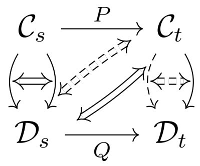
Here $\mathbf{Pr}^{\boxed{\text{strong}}}_{0,0} := \mathrm{Fun}(\Theta^{0,0}, \mathbf{Pr})$ is simply the underlying (2,1)-category of $\mathbf{Pr}$.

Lemma 2.3.30: *One has an equivalence of bicategories*

$$h_2(\mathbf{BrTens}) \simeq h_2(\mathfrak{alg}_2(\mathbf{Pr})) \ .$$

Proof : By Theorem 2.3.12, objects of $\mathfrak{alg}_2(\mathbf{Pr})$ are $\mathbb{E}_2$-algebras in $\mathbf{Pr}$. By Theorem 2.3.16, the Hom 1-fold Segal space between two objects $\mathcal{V}$ and $\mathcal{W}$ is $\mathfrak{alg}_1(\mathrm{Bimod}_{\mathcal{V},\mathcal{W}}(\mathbf{Pr}))$. The monoidal structure on the ∞-category $\mathrm{Bimod}_{\mathcal{V},\mathcal{W}}(\mathbf{Pr})$ is relative tensor product over $\mathcal{V}$ and $\mathcal{W}$ by Proposition 2.3.15 which agrees with [BJS21, Proposition 2.37].
Now, $\mathfrak{alg}_1(\mathrm{Bimod}_{\mathcal{V},\mathcal{W}}(\mathbf{Pr}))$ has objects algebra-objects $\mathcal{A}, \mathcal{B}$ in $\mathrm{Bimod}_{\mathcal{V},\mathcal{W}}(\mathbf{Pr})$, which is shown to be equivalent to the notion of $\mathcal{V}$-$\mathcal{W}$-central algebras in [BJS21, Definition-Proposition 3.2]. Isomorphism classes of morphisms from $\mathcal{A}$ to $\mathcal{B}$ are given by isomorphism classes of $\mathcal{A}$-$\mathcal{B}$-bimodules internal to $\mathrm{Bimod}_{\mathcal{V},\mathcal{W}}(\mathbf{Pr})$, which is shown to be equivalent to the notion of $\mathcal{V}$-$\mathcal{W}$-centered $\mathcal{A}$-$\mathcal{B}$-bimodules in [BJS21, Proposition 3.10]. Compositions of 1 and 2-morphisms are defined to be relative tensor products in [BJS21, Sections 3.4 and 3.5] as in [Hau17, Corollary 4.20]. □

The homotopy bicategory of $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\textbf{strong}}}_{1,0})$:

The (2,1)-category $\mathbf{Pr}^{\boxed{\text{strong}}}_{1,0}$ has objects arrows $\mathcal{C}_s \xrightarrow{P} \mathcal{C}_t$ in $\mathbf{Pr}$, 1-morphisms squares $\begin{array}{ccc} \mathcal{C}_s & \xrightarrow{P} & \mathcal{C}_t \\ {\scriptstyle R_s}\downarrow & \overset{\eta}{\swarrow}\!\!\sim & \downarrow{\scriptstyle R_t} \\ \mathcal{D}_s & \xrightarrow[Q]{} & \mathcal{D}_t \end{array}$ and 2-morphisms commuting thickened bigons . There are two symmetric monoidal functor $s, t : \mathbf{Pr}^{\boxed{\text{strong}}}_{1,0} \to \mathbf{Pr}$ that remembers only the part labelled "s" or "t", induced by the two functors $\Theta^{0,0} \to \Theta^{1,0}$.

Lemma 2.3.31: *Objects of $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\textit{strong}}}_{1,0})$ are equivalent to arrows*

$$\mathcal{V}_s \xrightarrow{F} \mathcal{V}_t$$

where $\mathcal{V}_s$ and $\mathcal{V}_t$ are braided tensor categories and F is a braided monoidal functor.

1-morphisms of $\mathfrak{alg}_2(\mathbf{Pr}^{\boxed{\textit{strong}}}_{1,0})$ from $\mathcal{V}_s \xrightarrow{F_1} \mathcal{V}_t$ to $\mathcal{W}_s \xrightarrow{F_2} \mathcal{W}_t$ are equivalent to arrows

$$\mathcal{A}_s \xrightarrow{G} \mathcal{A}_t$$

where $\mathcal{A}_s$ is a $\mathcal{V}_s$-$\mathcal{W}_s$-central algebra, $\mathcal{A}_t$ is a $\mathcal{V}_t$-$\mathcal{W}_t$-central algebra and G is a central monoidal functor.

2-morphisms in $h_2(\mathfrak{alg}_2(\mathbf{Pr}_{1,0}^{\boxed{strong}}))$ from $\mathcal{A}_s \xrightarrow{G_1} \mathcal{A}_t$ to $\mathcal{B}_s \xrightarrow{G_2} \mathcal{B}_t$ are equivalent to isomorphism classes of arrows

$$\mathcal{M}_s \xrightarrow{H} \mathcal{M}_t$$

where $\mathcal{M}_s$ is a $\mathcal{V}_s$-$\mathcal{W}_s$-centered $\mathcal{A}_s$-$\mathcal{B}_s$-bimodule, $\mathcal{M}_t$ is a $\mathcal{V}_t$-$\mathcal{W}_t$-centered $\mathcal{A}_t$-$\mathcal{B}_t$-bimodule and H is a centered bimodule functor.

The bicategory $h_2(\mathfrak{alg}_2(\mathbf{Pr}_{1,0}^{\boxed{strong}}))$ is equivalent to the bicategory of arrows as described above where every composition is given by relative tensor product.

Note that none of the functors described above will be composable, they always go from the "source" side to the "target" side. Composition of functors will be given by $\mathbf{Pr}_{2,0}^{\boxed{\text{strong}}}$.

Proof : By Theorem 2.3.12, objects of $\mathfrak{alg}_2(\mathbf{Pr}_{1,0}^{\boxed{\text{strong}}})$ are equivalent to $\mathbb{E}_2$-algebras $\mathcal{F}$ in $\mathbf{Pr}_{1,0}^{\boxed{\text{strong}}}$. Such an $\mathcal{F}$ is an arrow $\mathcal{V}_s \xrightarrow{F} \mathcal{V}_t$ endowed with an $\mathbb{E}_2$-monoidal structure.
The images $s(\mathcal{F}) = \mathcal{V}_s$ and $t(\mathcal{F}) = \mathcal{V}_t$ are therefore equipped with $\mathbb{E}_2$-monoidal structures, and are braided tensor categories. More generally, the source and target parts of each statement is given by Lemma 2.3.30.

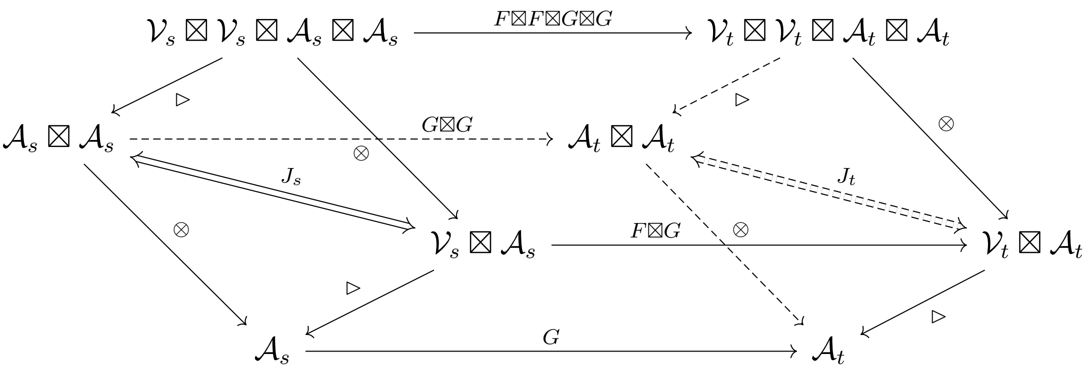
The product $\mathcal{F} \boxtimes \mathcal{F} \to \mathcal{F}$ is a square $\begin{array}{ccc} \mathcal{V}_s \boxtimes \mathcal{V}_s & \xrightarrow{F \boxtimes F} & \mathcal{V}_t \boxtimes \mathcal{V}_t \\ \downarrow{\scriptstyle \otimes_s} & \swarrow\!\!\!\nearrow & \downarrow{\scriptstyle \otimes_t} \\ \mathcal{V}_s & \xrightarrow{F} & \mathcal{V}_t \end{array}$ exhibiting F as a monoidal functor (compare with [JS17, Example 8.8]). Similarly the unit of $\mathcal{F}$ shows that F is unital. The braiding isomorphism $\otimes_{\mathcal{F}} \overset{\sim}{\Rightarrow} \otimes_{\mathcal{F}}^{op}$ coming from its $\mathbb{E}_2$-monoidal structure exhibits F as a braided monoidal functor.

By Theorem 2.3.16, a 1-morphism between two $\mathbb{E}_2$-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathfrak{alg}_2(\mathbf{Pr}_{1,0}^{\boxed{\text{strong}}})$ is an $\mathbb{E}_1$-algebra internal to $\mathcal{F}_1$-$\mathcal{F}_2$-bimodules. Such a $\mathcal{G}$ is an arrow $\mathcal{A}_s \xrightarrow{G} \mathcal{A}_t$ with compatible monoidal and bimodule structures. A bimodule structure

$$\mathcal{F}_1 \boxtimes \mathcal{G} \boxtimes \mathcal{F}_2 \xrightarrow{\triangleright_{\mathcal{F}_1} - \triangleleft_{\mathcal{F}_2}} \mathcal{G} \text{ , i.e. } \begin{array}{ccc} \mathcal{V}_s \boxtimes \mathcal{A}_s \boxtimes \mathcal{W}_s & \xrightarrow{F_1 \boxtimes G \boxtimes F_2} & \mathcal{V}_t \boxtimes \mathcal{A}_t \boxtimes \mathcal{W}_t \\ \downarrow{\scriptstyle act} & \swarrow\!\!\!\nearrow & \downarrow{\scriptstyle act} \\ \mathcal{A}_s & \xrightarrow{G} & \mathcal{A}_t \end{array} \tag{2.8}$$

endows $\mathcal{A}_s$ (resp. $\mathcal{A}_t$) with a $\mathcal{V}_s$-$\mathcal{W}_s$- (resp. $\mathcal{V}_t$-$\mathcal{W}_t$-) bimodule structure and exhibits G as a bimodule functor. A monoidal structure $\mathcal{G} \underset{\mathcal{F}}{\boxtimes} \mathcal{G} \to \mathcal{G}$ in $\mathcal{F}_1$-$\mathcal{F}_2$-bimodules is the data of a monoidal structure $\mathcal{G} \boxtimes \mathcal{G} \to \mathcal{G}$ in $\mathbf{Pr}_{1,0}^{\boxed{\text{strong}}}$ which is balanced. It endows $\mathcal{A}_s$ and $\mathcal{A}_t$ with a structure of monoidal categories, and exhibits G as a monoidal functor, as above. Compatibility between these two structures demands an isomorphism (we only write one "$\mathcal{F}$") $\begin{array}{ccc} \mathcal{F} \boxtimes \mathcal{G} \boxtimes \mathcal{G} & \xrightarrow{\otimes_{\mathcal{G}}} & \mathcal{F} \boxtimes \mathcal{G} \\ \downarrow{\scriptstyle \triangleright_{\mathcal{F}}} & \swarrow\!\!\!\nearrow & \downarrow{\scriptstyle \triangleright_{\mathcal{F}}} \\ \mathcal{G} \boxtimes \mathcal{G} & \xrightarrow{\otimes_{\mathcal{G}}} & \mathcal{G} \end{array}$ whereas a balancing is $\begin{array}{ccc} \mathcal{G} \boxtimes \mathcal{F} \boxtimes \mathcal{G} & \xrightarrow{\triangleleft_{\mathcal{F}}} & \mathcal{F} \boxtimes \mathcal{G} \\ \downarrow{\scriptstyle \triangleright_{\mathcal{F}}} & \swarrow\!\!\!\nearrow & \downarrow{\scriptstyle \otimes_{\mathcal{G}}} \\ \mathcal{G} \boxtimes \mathcal{G} & \xrightarrow{\otimes_{\mathcal{G}}} & \mathcal{G} \end{array}$. They are equivalent, using the unit of $\mathcal{F}$, to asking

$$\begin{array}{ccc} \mathcal{F} \boxtimes \mathcal{F} \boxtimes \mathcal{G} \boxtimes \mathcal{G} & \xrightarrow{\otimes_{\mathcal{F}} \boxtimes \otimes_{\mathcal{G}}} & \mathcal{F} \boxtimes \mathcal{G} \\ \downarrow{\triangleright_{\mathcal{F}} \boxtimes \triangleright_{\mathcal{F}}} & \Leftarrow & \downarrow{\triangleright_{\mathcal{F}}} \\ \mathcal{G} \boxtimes \mathcal{G} & \xrightarrow{\otimes_{\mathcal{G}}} & \mathcal{G} \end{array} \quad \text{, i.e.}$$



(2.9)

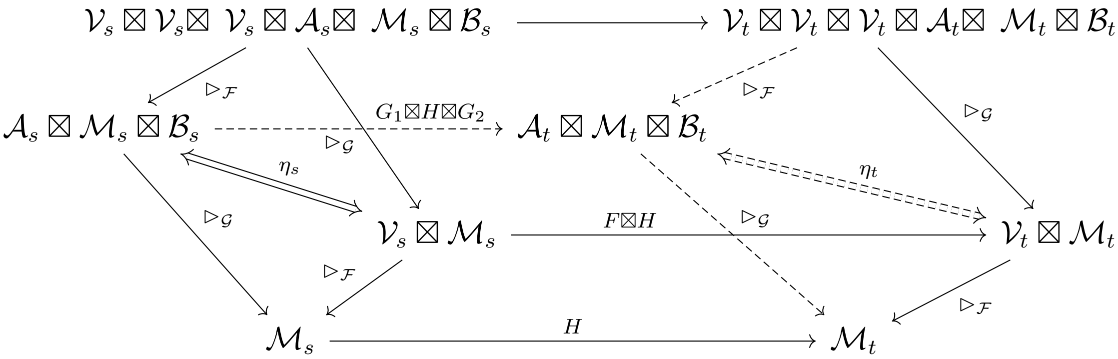
The natural isomorphisms J_s and J_t, using units of $\mathcal{V}$ and $\mathcal{A}$, exhibit $\mathcal{A}_s$ and $\mathcal{A}_t$ as central algebras. The missing natural isomorphisms are those described above exhibiting G as a monoidal and bimodule functor. Commutativity of the cube shows that G is a central monoidal functor.

A 2-morphism $\mathcal{H} : \mathcal{G}_1 \to \mathcal{G}_2$ is an arrow $\mathcal{M}_s \xrightarrow{H} \mathcal{M}_t$ with a bimodule structure

$$\mathcal{G}_1 \boxtimes \mathcal{H} \boxtimes \mathcal{G}_2 \xrightarrow{\triangleright_{\mathcal{G}}} \mathcal{H} \text{ , i.e.} \qquad \begin{array}{ccc} \mathcal{A}_s \boxtimes \mathcal{M}_s \boxtimes \mathcal{B}_s & \xrightarrow{G_1 \boxtimes H \boxtimes G_2} & \mathcal{A}_t \boxtimes \mathcal{M}_t \boxtimes \mathcal{B}_t \\ \downarrow{\triangleright} & \Leftarrow & \downarrow{\triangleright} \\ \mathcal{M}_s & \xrightarrow{H} & \mathcal{M}_t \end{array} \tag{2.10}$$

exhibiting H as a bimodule functor. Demanding that these are $\mathcal{F}_1$-$\mathcal{F}_2$-balanced and internal to $\mathcal{F}_1$-$\mathcal{F}_2$-bimodules demands a $\mathcal{F}_1$-$\mathcal{F}_2$-bimodule structure on $\mathcal{H}$ (which will have to agree with the ones obtained by acting through either $\mathcal{G}$'s by the following) and an isomorphism as above (we still write only one "$\mathcal{F}$"):

$$\begin{array}{ccc} \mathcal{F} \boxtimes \mathcal{F} \boxtimes \mathcal{F} \boxtimes \mathcal{G}_1 \boxtimes \mathcal{H} \boxtimes \mathcal{G}_2 & \xrightarrow{\otimes_{\mathcal{F}} \boxtimes \triangleright_{\mathcal{G}}} & \mathcal{F} \boxtimes \mathcal{H} \\ {\triangleright_{\mathcal{F}} \boxtimes \triangleright_{\mathcal{F}} \boxtimes \triangleright_{\mathcal{F}}}\downarrow & \Leftarrow & \downarrow{\triangleright_{\mathcal{F}}} \\ \mathcal{G}_1 \boxtimes \mathcal{H} \boxtimes \mathcal{G}_2 & \xrightarrow{\triangleright_{\mathcal{G}}} & \mathcal{H} \end{array} \quad \text{, i.e.}$$

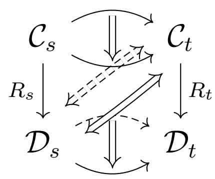


(2.11)

Here the natural isomorphisms η_s and η_t (using units of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{V}$) give the centered structure on $\mathcal{M}_s$ and $\mathcal{M}_t$, and commutativity of the cube demands that H is a centered bimodule functor.

Finally, compositions in every directions in $\mathfrak{alg}_2(\mathbf{Pr}_{1,0}^{\boxed{\text{strong}}})$ are defined to be relative tensor products. □

The homotopy bicategory of $\mathfrak{alg}_2(\mathbf{Pr}_{1,1}^{\boxed{\textbf{strong}}})$:

The $(2,1)$-category $\mathbf{Pr}_{1,1}^{\boxed{\text{strong}}}$ has objects bigons $\mathcal{C}_s \underset{P_d}{\overset{P_u}{\Downarrow \eta}} \mathcal{C}_t$, 1-morphisms commuting thickened bigons (in the opposite direction from before)  . A 2-morphism is only the data of a natural transformation in the source and the target side, commuting with rest of the diagram. There are now multiple notion of source and target. There are two inclusions $\Theta^{0,0} \to \Theta^{1,1}$ and we still denote $s, t : \mathbf{Pr}_{1,1}^{\boxed{\text{strong}}} \to \mathbf{Pr}$ the induced symmetric monoidal functors. There are also two inclusion $\Theta^{1,0} \to \Theta^{1,1}$ and we denote $u, d : \mathbf{Pr}_{1,1}^{\boxed{\text{strong}}} \to \mathbf{Pr}_{1,0}^{\boxed{\text{strong}}}$ the "up" side and the "down" side.

Lemma 2.3.32: *Objects of $\mathfrak{alg}_2(\mathbf{Pr}_{1,1}^{\boxed{\text{strong}}})$ are equivalent to bigons*

$$\mathcal{V}_s \underset{F_d}{\overset{F_u}{\Downarrow \nu}} \mathcal{V}_t$$

where $\mathcal{V}_s$ and $\mathcal{V}_t$ are braided tensor categories, F_u and F_d are braided monoidal functors and ν is a monoidal natural transformation.

1-morphisms of $\mathfrak{alg}_2(\mathbf{Pr}_{1,1}^{\boxed{\text{strong}}})$ are equivalent to bigons

$$\mathcal{A}_s \underset{G_d}{\overset{G_u}{\Downarrow \alpha}} \mathcal{A}_t$$

where $\mathcal{A}_s$ (resp. t) is a $\mathcal{V}_s$-$\mathcal{W}_s$-central algebra, G_u (resp. d) is a central monoidal functor and α is a monoidal natural transformation.

2-morphisms of $h_2(\mathfrak{alg}_2(\mathbf{Pr}_{1,1}^{\boxed{\text{strong}}}))$ are equivalent to isomorphism classes of bigons

$$\mathcal{M}_s \underset{H_d}{\overset{H_u}{\Downarrow \eta}} \mathcal{M}_t$$

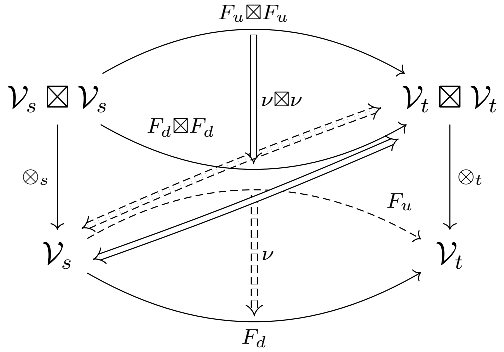
where $\mathcal{M}_s$ (resp. t) is a $\mathcal{V}_s$-$\mathcal{W}_s$-centered $\mathcal{A}_s$-$\mathcal{B}_s$-bimodule, H_u (resp. d) is a centered bimodule functor and η is a bimodule natural transformation.

The bicategory $h_2(\mathfrak{alg}_2(\mathbf{Pr}_{1,1}^{\boxed{\text{strong}}}))$ is equivalent to the bicategory of bigons as described above where every composition is given by relative tensor product.

Proof : The proof is very similar. The source and target parts of the statements are Lemma 2.3.30, and the up and down parts are Lemma 2.3.31. But except for objects, there is nothing else that the source, target, up and down parts, because there are no

3-morphisms in **Pr**. Therefore an algebra or module structure will be no extra structure, and non-trivial conditions coming only from the 1-morphisms.

A product on a bigon $\mathcal{V}_s \underset{F_d}{\overset{F_u}{\Downarrow \nu}} \mathcal{V}_t$ is given by



(2.12)

where the unlabelled natural isomorphisms are those exhibiting F_u and F_d as monoidal functors. Commutativity of this diagram, together with the analogous diagram for the unit, demands that ν is a monoidal natural transformation.
The diagram is the same for 1-morphism, and similar for 2-morphisms. □

Proof (of Theorem 2.3.29): The first statement follows immediately from Lemma 2.3.30. Indeed, the 2-truncation of $\mathbf{Alg}_2(\mathbf{Pr})$ is simply

$$\mathbf{Alg}_2(\mathbf{Pr})_{\bullet,\bullet,0,0} := \mathrm{maps}^h(\Theta^{\bullet,\bullet,0,0}, \widehat{\mathfrak{alg}}_2(\mathbf{Pr}_{0,0}^{\boxed{\text{strong}}})_{\bullet,\bullet}) \simeq \widehat{\mathfrak{alg}}_2(\mathbf{Pr})_{\bullet,\bullet}$$

itself as it is already essentially constant, see [JS17, Remarks 3.4], and $\mathbf{Pr}_{0,0}^{\boxed{\text{strong}}} \simeq \mathbf{Pr}$. Now

$$h_2(\mathbf{Alg}_2(\mathbf{Pr})) := h_2(\mathbf{Alg}_2(\mathbf{Pr})_{\bullet,\bullet,0,0}) \simeq h_2(\widehat{\mathfrak{alg}}_2(\mathbf{Pr})_{\bullet,\bullet}) \simeq h_2(\mathfrak{alg}_2(\mathbf{Pr})_{\bullet,\bullet}) \simeq h_2(\mathbf{BrTens})$$

because completion does not affect homotopy bicategories.
The bicategory of 1, 2 and 3 morphisms:
Let $\mathcal{V}$ and $\mathcal{W}$ be two braided tensor categories, seen as objects of $\mathbf{Alg}_2(\mathbf{Pr})$ by the equivalence above. We want to identify $h_2(\mathrm{Hom}_{\mathbf{Alg}_2(\mathbf{Pr})}(\mathcal{V},\mathcal{W}))$. Its objects and morphisms are also identified (up to isomorphism) by the equivalence above. We need to understand its 2-morphisms

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-$\mathcal{W}$-central algebras and $\mathcal{M}$, $\mathcal{N}$ two $\mathcal{V}$-$\mathcal{W}$-centered $\mathcal{A}$-$\mathcal{B}$-bimodules. An element $\mathcal{H}$ of $\mathrm{Hom}_{\mathbf{Alg}_2(\mathbf{Pr})}(\mathcal{M},\mathcal{N})$ is an element of

$$\mathbf{Alg}_2(\mathbf{Pr})_{1,1,1,0} \overset{2.3.22}{\simeq} \mathbf{Alg}_2(\mathbf{Pr})_1 \underset{\widehat{\mathfrak{alg}}_2(\mathbf{Pr}_1^{\boxed{\text{strong}}})_1}{\times^h} \widehat{\mathfrak{alg}}_2(\mathbf{Pr}_1^{\boxed{\text{strong}}})_{11} \underset{\widehat{\mathfrak{alg}}_2(\mathbf{Pr}_1^{\boxed{\text{strong}}})_1}{\times^h} \mathbf{Alg}_2(\mathbf{Pr})_1$$

together with paths between its different sources and targets and $\mathcal{V},\mathcal{W},\mathcal{A},\mathcal{B},\mathcal{M}$ and $\mathcal{N}$. Therefore, $\mathcal{H}$ is a centered bimodule functor as in Lemma 2.3.31. The essential constancy condition will require that its source and target (denoted G_1 and G_2 there) are degenerate, i.e. are identity monoidal functors, and so are their sources and targets (denoted F_1 and F_2 there). Actually, we are given isomorphisms with degenerate morphisms, but because isomorphisms can be seen as bimodule we can simply compose $\mathcal{H}$ with these isomorphism and say $\mathcal{H}$ has degenerate source and target. Compatibility with $\mathcal{V}$ and $\mathcal{W}$ demands

$F_1 = \mathrm{Id}_{\mathcal{V}}$ and $F_2 = \mathrm{Id}_{\mathcal{W}}$. Compatibility with $\mathcal{A}$ and $\mathcal{B}$ demands $G_1 = \mathrm{Id}_{\mathcal{A}}$ and $G_2 = \mathrm{Id}_{\mathcal{B}}$. Compatibility with $\mathcal{M}$ and $\mathcal{N}$ demands that the source and target "sides" of $\mathcal{H}$ agree with $\mathcal{M}$ and $\mathcal{N}$.

The bicategory of 2, 3 and 4 morphisms:

Let us identify $h_2(\mathrm{Hom}_{\mathbf{Alg}_2(\mathbf{Pr})}(\mathcal{A}, \mathcal{B}))$ between two $\mathcal{V}$-$\mathcal{W}$-central algebras. Again, we know objects and morphisms up to isomorphism by the result above, and we need to identify 2-morphisms.

Fix $F_1, F_2 : \mathcal{M} \to \mathcal{N}$ two centered bimodule functors. Again we use Proposition 2.3.22, the definition of Hom spaces, and Lemma 2.3.32. A morphism between F and G is a 2-morphism in $\widehat{\mathfrak{alg}}_2(\mathbf{Pr}_{1,1}^{\boxed{\text{strong}}})_{1,1}$, i.e. a bimodule natural transformation η, with twice degenerate source and target $\alpha \simeq \mathrm{Id}_{\mathrm{Id}_{\mathcal{A}}}$ and $\beta \simeq \mathrm{Id}_{\mathrm{Id}_{\mathcal{B}}}$, so $\nu_u \simeq \mathrm{Id}_{\mathrm{Id}_{\mathcal{V}}}$ and $\nu_d \simeq \mathrm{Id}_{\mathrm{Id}_{\mathcal{W}}}$ (using compatibility with $\mathcal{V}, \mathcal{W}, \mathcal{A}$ and $\mathcal{B}$). Compatibly with $\mathcal{M}, \mathcal{N}, F_1$ and F_2 gives that up to composing with the given isomorphisms, we have $\mathcal{M} \underset{F_2}{\overset{F_1}{\Downarrow \eta}} \mathcal{N}$. ☐

We have identified all the the bicategories mentioned in Remark 2.1.40 and can now describe dualizability in $\mathbf{BrTens} := \mathbf{Alg}_2(\mathbf{Pr})$.

2.3.5 Dualizability in BrTens

Dualizability in the (pointed) factorization-algebra model of higher Morita categories is completely determined in [GS].

Theorem 2.3.33 (Theorems 5.1 and 6.1 in [GS]): *For any $\otimes$-sifted cocomplete symmetric monoidal k-category $\mathcal{S}$, the $(n+k)$-category $\mathbf{Alg}_n^{pointed}(\mathcal{S})$ has duals up to level n.*
However, the pointing prevents any higher dualizability, and every $(n+1)$-dualizable object is equivalent to the unit.

If a bimodule has an adjoint as a pointed bimodule, in particular it has an adjoint as an unpointed one, simply by forgetting the pointing of the adjoint, unit and counit. Therefore one expects at least n-dualizability in $\mathbf{Alg}_n(\mathcal{S})$.

We now turn to $n = 2$ and $\mathcal{S} = \mathbf{Pr}$. That $\mathbf{Pr}$ is $\otimes$-sifted cocomplete and has good relative tensor products is shown in [JS17, Example 8.11]. There one can check by hand that the dualizability data exhibited in [GS] gives dualizability data in **BrTens**. More generally, sufficient conditions for m-dualizability in **BrTens** were given in [BJS21, BJSS21]. Explicit subcategories with duals up to level $m = 3, 4$ are exhibited in [BJS21]. We recall some consequences of their results that will be useful in this manuscript.

Definition 2.3.34: A monoidal category in $\mathbf{Pr}$ is called *rigid finite semisimple* if it is cp-rigid, semisimple and has finitely many isomorphism classes of simple objects. It is called *fusion* if it moreover has a simple unit. Note that these correspond to the free cocompletion (or the Ind-completion which agree in this semisimple case) of the usual notions of (not cocomplete) rigid finite semisimple and fusion categories. ◇

Theorem 2.3.35 (Theorems 5.16 and 5.21 in [BJS21]): *Let $\mathcal{V} \in \mathbf{Pr}$ be a braided tensor category. Then:*

1. *$\mathcal{V}$ is 2-dualizable,*
2. *if $\mathcal{V}$ is cp-rigid, it is 3-dualizable, and*
3. *if $\mathcal{V}$ is fusion, it is 4-dualizable.*

Let $\mathcal{V}, \mathcal{W}$ be braided tensor categories and $\mathcal{A}$ a $\mathcal{V}$-$\mathcal{W}$-central algebra. Then:

1. *$\mathcal{A}$ is 1-dualizable,*
2. *if $\mathcal{V}, \mathcal{W}, \mathcal{A}$ are cp-rigid, then $\mathcal{A}$ is 2-dualizable, and*
3. *if $\mathcal{V}, \mathcal{W}, \mathcal{A}$ are fusion, then $\mathcal{A}$ is 3-dualizable.*

Invertible objects were studied in [BJSS21]. In particular, they exhibit the first examples of non-semisimple fully dualizable objects.

Definition 2.3.36: A braided tensor category is called *finite* if is is equivalent to A-*mod* for some finite dimensional algebra A. It is called *modular* if it has no non-trivial transparent object. ♢

Note that this corresponds to the Ind-completion (see e.g. [KS06] for a definition) of the usual notion of (possibly non-semisimple) modular category, or the free cocompletion of the subcategory of projective objects (though usually we also assume that our category is ribbon, which we do not here).

Theorem 2.3.37 (Theorem 3.20 in [BJSS21]): *Let $\mathcal{V} \in$ **Pr** be a (possibly non-semisimple) finite modular tensor category. Then $\mathcal{V}$ is invertible, and in particular fully dualizable, in* **BrTens**.

Chapter 3

Non-semisimple skein (3+1)-TQFTs

This chapter follows the paper [CGHP] written in collaboration with Francesco Costantino, Nathan Geer and Bertrand Patureau-Mirand. There are some changes in the exposition, but the results are the same.

Using skein theory very much in the spirit of the Reshetikhin–Turaev constructions, we define a $(3+1)$-TQFT associated with possibly non-semisimple finite unimodular ribbon tensor categories. State spaces are given by admissible skein modules, and we prescribe the TQFT on handle attachments. The reader may want to compare with the easier semisimple case described in Section 1.4.

In Sections 3.1 and 3.2, we define chromatic morphisms and gluing morphisms and give their skein properties (e.g. that the chromatic morphism can be used as a Kirby color). We define the notions of chromatic non-degenerate, chromatic compact and twist non-degenerate unimodular finite ribbon tensor categories for which our constructions below exist.

In Section 3.3, we show that a twist non-degenerate category $\mathcal{C}$ gives rise to a 3-manifold invariant. We show that this construction recovers the modified Lyubashenko invariants of [Lyu95, DGG$^+$22] and in particular the Hennings and WRT invariants. This section is independent to the next.

In Section 3.4, we build non-compact $(3+1)$-TQFTs from chromatic non-degenerate categories and $(3+1)$-TQFTs from chromatic compact categories, see Theorem 3.4.4. Here non-compact means we only consider 4-cobordisms with incoming boundary in every connected component. As a by-product, we obtain an invariant of 4-manifolds equipped with a ribbon graph in their boundary. We show in Theorem 3.4.8 that the $(3+1)$-TQFT is invertible if and only if $\mathcal{C}$ is modular.

In Section 3.5 we study some examples. We show that we recover Crane–Yetter–Kauffman TQFTs in the semisimple case. We study finite dimensional versions of quantum $\mathfrak{sl}(2)$ at roots of unity which give chromatic compact and possibly twist-degenerate examples. We conclude with a toy example in characteristic p which gives a chromatic non-degenerate category which is not chromatic compact.

3.1 Algebraic setting

We fix an algebraically closed field $\Bbbk$ and a strict finite unimodular ribbon $\Bbbk$-linear tensor category $\mathcal{C}$, see the definitions of Section 1.5.1 and [EGNO15]. By [GKP22, Corollary 5.6], it admits a non-degenerate m-trace, unique up to scalar. Finiteness implies the existence of a chromatic morphism for a non-zero projective generator, two notions we define below.

We will be interested in cases where $\mathcal{C}$ satisfied some additional non-degeneracy conditions, see Section 3.1.4.

In [CGHP], we did not require $\mathcal{C}$ to be abelian, but it appears that this generalization does not yield to new TQFTs or 3-manifold invariant. We restrict to this easier setting here.

3.1.1 Copairing

In this chapter we denote $F = \mathrm{RT}_{\mathcal{C}}$ the Reshetikhin–Turaev functor and F' the renormalized invariant of closed admissible ribbon graphs from Theorem 1.5.3. We suppose we have chosen a non-degenerate m-trace $\mathtt{t}$ on Proj, as in Section 1.5.1.

Definition 3.1.1: For any $P \in \mathrm{Proj}$, we set

$$\Omega_P = \sum_i x^i \otimes_{\Bbbk} x_i \in \mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1}) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P) \quad \text{and} \quad \Lambda_P = \sum_i x_i \circ x^i \in \mathrm{End}_{\mathcal{C}}(P), \tag{3.1}$$

where $\{x^i\}_i$ and $\{x_i\}_i$ are basis of $\mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P)$ which are dual with respect to the m-trace, that is, such that $\mathtt{t}_P(x_i \circ x^j) = \delta_{i,j}$. Clearly, Ω_P and Λ_P are independent of the choice of such dual basis. ♢

The properties of the m-trace translate to the copairings Ω_P as follows:

Lemma 3.1.2:

1. *Duality: If* $\Omega_P = \sum_i x^i \otimes x_i$, *then* $\Omega_{P^*} = \sum_i x_i^* \otimes (x^i)^* \in \mathrm{Hom}_{\mathcal{C}}(P^*, \mathbb{1}) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P^*)$.

2. *Naturality: If* $f : P \to Q$ *is a morphism in* Proj, $\Omega_P = \sum_i x^i \otimes x_i$, *and* $\Omega_Q = \sum_i y^i \otimes y_i$, *then*

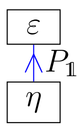
$$\sum_i x^i \otimes (f \circ x_i) = \sum_i (y^i \circ f) \otimes y_i \in \mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1}) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, Q).$$

3. *Rotation: If* $V \in \mathcal{C}$ *and* $\Omega_{P\otimes V} = \sum_i z^i \otimes z_i$ *then* $\Omega_{V \otimes P} = \sum_i \tilde{z}^i \otimes \tilde{z}_i$ *where*

$$\tilde{z}_i = (\mathrm{Id} \otimes \overrightarrow{\mathrm{ev}}_V) \circ (\mathrm{Id} \otimes z_i \otimes \mathrm{Id}) \circ (\overleftarrow{\mathrm{coev}}_V) \quad \text{and} \quad \tilde{z}^i = (\overleftarrow{\mathrm{ev}}_V) \circ (\mathrm{Id} \otimes z^i \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \overrightarrow{\mathrm{coev}}_V).$$

Proof : The duality and rotation properties follow since we apply transformations that send dual bases to dual bases. The naturality can be checked by applying $\mathtt{t}_P(x_k \circ \_) \otimes \mathtt{t}_Q(\_ \circ y^\ell)$ to both side then the equality reduces to the cyclic property of the m-trace: $\mathtt{t}_Q(f \circ x_k \circ y^\ell) = \mathtt{t}_P(x_k \circ y^\ell \circ f)$. □

We fix a projective cover of the unit $P_{\mathbb{1}}$ (which can be chosen as any indecomposable summand of $P^* \otimes P$ on which $\overleftarrow{\mathrm{ev}}_P$ is non-zero for any object $P \in \mathrm{Proj}$) and a non-zero morphism $\varepsilon : P_{\mathbb{1}} \to \mathbb{1}$. Then since the m-trace is non-degenerate, there exist $\eta : \mathbb{1} \to P_{\mathbb{1}}$ such that $\mathtt{t}_{P_{\mathbb{1}}}(\eta \circ \varepsilon) = 1$. (If $\mathcal{C}$ is semi-simple, we choose $P_{\mathbb{1}} = \mathbb{1}$ and $\varepsilon = \eta = \mathrm{Id}_{\mathbb{1}}$). Let Γ_0 be the closed ribbon graph

$\Gamma_0 =$  .

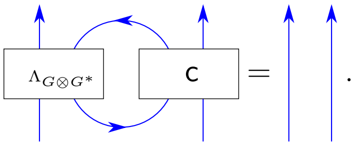
Then $F'(\Gamma_0) = \mathsf{t}_{P_{\mathbb{1}}}(\eta \circ \varepsilon) = 1$.

3.1.2 Chromatic morphisms

We define what will play the role of the Kirby color in the non-semisimple setting. We have shaped Definition 1.4.1 to be analogous to the following.

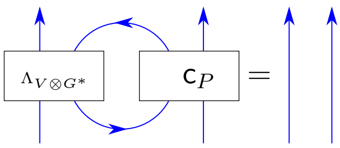
Definition 3.1.3: A *projective generator* of $\mathcal{C}$ is projective object such that any projective object is a retract of $G^{\oplus n}$ for some $n \in \mathbb{N}$. It always exists in a finite tensor category.

A *chromatic morphism* for a projective generator G is a map $\mathsf{c} \in \mathrm{End}_{\mathcal{C}}(G \otimes G)$ satisfying

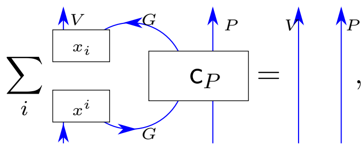


(3.2)

More generally, a *chromatic morphism based on* $P \in \mathrm{Proj}$ for a projective generator G is a map $\mathsf{c}_P \in \mathrm{End}_{\mathcal{C}}(G \otimes P)$ such that for all $V \in \mathcal{C}$, we have



that is



(3.3) ♢

where $\{x_i\}_i$ and $\{x^i\}_i$ are any dual bases.

Clearly, a chromatic morphism based on G is a chromatic morphism. Conversely, any chromatic morphism gives rise to chromatic morphisms based on projective objects:

Lemma 3.1.4 (Lemma 1.2 of [CGPVb]): *Let* $\mathsf{c} \in \mathrm{End}_{\mathcal{C}}(G \otimes G)$ *be a chromatic morphism and* $P \in \mathrm{Proj}$. *Pick any non-zero morphism* $\varepsilon_G : G \to \mathbb{1}$ *and a morphism* $e_{P,G} : P \to G \otimes P$ *such that* $\mathrm{Id}_P = (\varepsilon_G \otimes \mathrm{Id}_P) \circ e_{P,G}$ *(such morphisms always exist). Then the map*

$$\mathsf{c}_P = (\mathrm{Id}_G \otimes \varepsilon_G) \circ \mathsf{c} \circ (\mathrm{Id}_G \otimes e_{P,G}) \in \mathrm{End}_{\mathcal{C}}(G \otimes P) \tag{3.4}$$

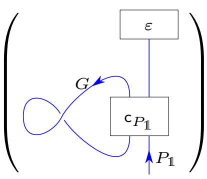
is a chromatic morphism based on P.

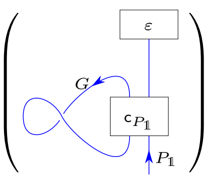
Definition 3.1.5: A *chromatic category* is a $\mathbb{k}$-linear pivotal category $\mathcal{C}$ endowed with a non-degenerate m-trace on Proj, in which there exist a non-zero projective generator and a chromatic map. ♢

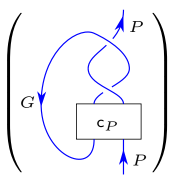
3.1.3 Gluing morphisms

We will prove the following lemma in Section 3.1.5:

Lemma 3.1.6: *There exists scalars* $\Delta_+, \Delta_- \in \mathbb{k}$ *and a family of* $\{\Delta_0^P \in \mathrm{Hom}_{\mathcal{C}}(P, P)\}_{P \in \mathrm{Proj}}$*, such that for any chromatic morphisms* $\mathsf{c}_{P_{\mathbb{1}}}$, c_P *based on* $P_{\mathbb{1}}$ *and* P *respectively, one has*

F 

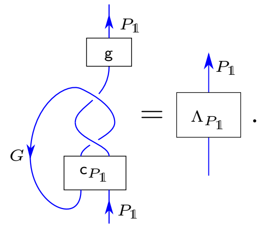
$= \Delta_+\varepsilon$, F 

$= \Delta_-\varepsilon$, *and* F 

$= \Delta_0^P$.

Definition 3.1.7: A *gluing morphism* is an endomorphism

$\mathsf{g} \in \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$ such that $\mathsf{g} \circ \Delta_0^{P_{\mathbb{1}}} = \Lambda_{P_{\mathbb{1}}}$, i.e.



◇

We also postpone to Section 3.1.5 the proof of the following:

Proposition 3.1.8: *The category $\mathcal{C}$ admits a gluing morphism $\mathsf{g} \in \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$ if and only if $\Delta_0^{P_{\mathbb{1}}} \neq 0$.*

3.1.4 Statement of main results

Definition 3.1.9: We say that

1. $\mathcal{C}$ is *twist non-degenerate* if $\Delta_+\Delta_- \neq 0$,
2. $\mathcal{C}$ is *chromatic non-degenerate* if $\Delta_0^{P_{\mathbb{1}}} \neq 0$, i.e. if $\mathcal{C}$ admits a gluing morphism,
3. $\mathcal{C}$ is *chromatic compact* if there exists a scalar $\zeta \in \Bbbk^*$, called the global dimension of $\mathcal{C}$, such that $\Delta_0^{P_{\mathbb{1}}} = \zeta\Lambda_{P_{\mathbb{1}}}$,
4. $\mathcal{C}$ is *modular* if there exists a scalar $\zeta \in \Bbbk^*$ such that for any projective P, $\Delta_0^P = \zeta\Lambda_P$. ◇

We will see in Section 3.1.6 that the notion of modular and twist non-degenerate coincide with the usual ones given for example in [DGG+22]. We called these categories "factorizable" and not modular in [CGHP], but under our standing assumption that $\mathcal{C}$ is finite, the name modular seems more appropriate. Remember that in [DGG+22] a category is called modular if it is finite and factorizable. The global dimension ζ is called modularity parameter in [DGG+22]. It does depend on the choice of the modified trace, see Proposition 3.4.7.

Clearly, modular $\implies$ chromatic compact (with the same scalar ζ) $\implies$ chromatic non-degenerate. We will also see in Lemma 3.2.4 that modular $\implies$ twist non-degenerate with $\Delta_+\Delta_- = \zeta$.

The main constructions of this chapter are the following:

1. If $\mathcal{C}$ is twist non-degenerate, there exists 3-manifold invariants $\mathrm{B}_{\mathcal{C}}$ and $\mathrm{B}'_{\mathcal{C}}$ (see Theorem 3.3.2) that generalize many quantum invariants defined through link surgery.
2. If $\mathcal{C}$ is chromatic non-degenerate, there exists a non-compact (3+1)-TQFT $\mathscr{S}_{\mathcal{C}}$ (see Theorem 3.4.4) whose vector spaces are the admissible skein modules of 3-manifolds.
3. If $\mathcal{C}$ is chromatic compact, then $\mathscr{S}_{\mathcal{C}}$ extends to a full (3+1)-TQFT (see Theorem 3.4.4).
4. $\mathcal{C}$ is modular if and only if $\mathscr{S}_{\mathcal{C}}$ is an invertible (3+1)-TQFT (see Theorem 3.4.8).

Remark 3.1.10: The name *chromatic compact* refers to the fact that $\mathscr{S}_{\mathcal{C}}$ extends from a non-compact TQFT (i.e. one defined only cobordisms whose components have non-empty source) to a full TQFT, i.e. one defined on all cobordisms. ◇

3.1.5 Existence of gluing morphisms

The proof of the existence of a gluing morphism as stated in Proposition 3.1.8 is a direct consequence of the last statement of the following lemma:

Lemma 3.1.11: *If $\mathcal{C}$ is a finite ribbon tensor category that admits a non-degenerate m-trace, we have:*

1. *For any non-zero morphism $P \xrightarrow{f} \mathbb{1}$ where P is projective, there exists an epimorphism $\widetilde{f} : P \to P_{\mathbb{1}}$ with $f = \varepsilon \circ \widetilde{f}$.*
2. *$P_{\mathbb{1}} \simeq P_{\mathbb{1}}^*$, i.e. $\mathcal{C}$ is unimodular.*
3. *$\mathrm{Hom}_{\mathcal{C}}(P_{\mathbb{1}}, \mathbb{1}) = \Bbbk\varepsilon$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P_{\mathbb{1}}) = \Bbbk\eta$.*
4. *$\Lambda_{P_{\mathbb{1}}} = \eta \circ \varepsilon$.*
5. *For any $f \in \mathrm{End}(P_{\mathbb{1}})$, f is nilpotent if and only if $\Lambda_{P_{\mathbb{1}}} \circ f = 0$.*
6. *For any non-zero endomorphism $f \in \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$, there exists $g \in \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$ with $g \circ f = \Lambda_{P_{\mathbb{1}}}$.*

PROOF : First remark that since $\Bbbk$ is algebraically closed (so $\Bbbk$ is the unique finite dimensional division $\Bbbk$-algebra) and since $P_{\mathbb{1}}$ is indecomposable, then by the Fitting Lemma (see [DK94]) we have $\mathrm{End}_{\Bbbk}(P_{\mathbb{1}}) = \Bbbk \,\mathrm{Id} \oplus J$ where J is the Jacobson radical which is a nilpotent ideal formed by the nilpotent endomorphism of $P_{\mathbb{1}}$.

1. For any nilpotent $n \in \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$, if $\varepsilon \circ n \neq 0$, then $\varepsilon \circ n$ is an epimorphism and since $P_{\mathbb{1}}$ is projective, ε factors through it: $\varepsilon = \varepsilon \circ n \circ g$ for some $g \in \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$. But $n \circ g$ belongs to J which is nilpotent thus $\varepsilon = \varepsilon \circ (n \circ g)^{\dim_{\Bbbk}(J)} = 0$ and we have a contradiction. Then the kernel of $\varepsilon \circ \_ : \mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}}) \to \mathrm{Hom}_{\mathcal{C}}(P_{\mathbb{1}}, \mathbb{1})$ contains J which is a maximal left ideal so it is equal to J. Now if $\varepsilon' : P \to \mathbb{1}$ is non-zero, then it is an epimorphism and since $P, P_{\mathbb{1}}$ are projective, there exist $P \xrightarrow{f} P_{\mathbb{1}}, P_{\mathbb{1}} \xrightarrow{g} P$ such that $\varepsilon' = \varepsilon f$, $\varepsilon = \varepsilon' g = \varepsilon f g$. In particular since fg is not nilpotent, it is an isomorphism and so f is an epimorphism which is split since $P_{\mathbb{1}}$ is projective.

2. Apply the previous to $P_{\mathbb{1}}^* \xrightarrow{\eta^*} \mathbb{1}$ implies that $P_{\mathbb{1}}$ is a direct summand of $P_{\mathbb{1}}^*$ which is indecomposable so $P_{\mathbb{1}} \simeq P_{\mathbb{1}}^*$.

3. If $\varepsilon' : P_{\mathbb{1}} \to \mathbb{1}$ is non-zero, then there is a split isomorphism $P_{\mathbb{1}} \xrightarrow{f} P_{\mathbb{1}}$ such that $\varepsilon' = \varepsilon f$. Write $f = \lambda \,\mathrm{Id} + n$ with n nilpotent and $\lambda \in \Bbbk$ then, $\varepsilon' = \lambda\varepsilon + \varepsilon n = \lambda\varepsilon$. Hence $\mathrm{Hom}(P_{\mathbb{1}}, \varepsilon) = \Bbbk\varepsilon$. Then $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P_{\mathbb{1}}) \simeq \mathrm{Hom}_{\mathcal{C}}(P_{\mathbb{1}}^*, \mathbb{1}) \simeq \mathrm{Hom}_{\mathcal{C}}(P_{\mathbb{1}}, \mathbb{1})$ also has dimension 1 so it is generated by η.

4. This follows since $\{\varepsilon\}$ and $\{\eta\}$ are dual basis.

5. From the proof of (1), we have for any $f \in \mathrm{End}(P_{\mathbb{1}})$, f is nilpotent if and only if $\varepsilon \circ f = 0$. Since η is a monomorphism, f is nilpotent if and only if $\eta \circ \varepsilon \circ f = 0$.

6. The symmetric pairing $(f, g) \mapsto \mathsf{t}_{P_{\mathbb{1}}}(fg)$ is non-degenerate on $\mathrm{End}_{\mathcal{C}}(P_{\mathbb{1}})$. From (5), $\Bbbk\Lambda_{P_{\mathbb{1}}}$ is the orthogonal of J. Recall that J is nilpotent and let $k = \max\{n \in \mathbb{N} :$

$J^n f \neq 0\}$ and let $g \in J^k$ be such that $gf \neq 0$. Then $Jgf = 0$ thus gf is orthogonal to J and $gf \in \Bbbk^* \Lambda_{P_\mathbb{1}}$. Up to rescaling g, we thus have $gf = \Lambda_{P_\mathbb{1}}$. □

3.1.6 Existence of chromatic morphisms

In this subsection we discuss existence of chromatic maps as well as some cases where $\mathcal{C}$ is chromatic compact.

In [CGPVa], a more general notion of "left" and "right" chromatic maps is defined and proven to exist for general rigid finite tensor categories. We provide a sketch of proof for the special case of interest below:

Theorem 3.1.12: *If $\mathcal{C}$ is a finite unimodular ribbon tensor category then it admits a chromatic morphism.*

Proof (Sketch of proof.): The finite tensor category $\mathcal{C}$ has a coend $\mathscr{L} = \int^{V \in \mathcal{C}} V^* \otimes V$ with dinatural transformations $i_V : V^* \otimes V \to \mathscr{L}$. The coend is a Hopf algebra object in $\mathcal{C}$ and every object $V \in \mathcal{C}$ has a structure of right $\mathscr{L}$-comodule given by $\tilde{\imath}_V = (V \xrightarrow{\overleftarrow{\mathrm{coev}}_V \otimes \mathrm{Id}} V \otimes V^* \otimes V \xrightarrow{\mathrm{Id} \otimes i_V} V \otimes \mathscr{L})$ compatible with the monoidal structure: the product $m_\mathscr{L} : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L}$ is used to define the coaction on a tensor product. Moreover, the coend is known to have (unique up to a scalar) right integrals $\lambda : \mathbb{1} \to \mathscr{L}$ and $\Lambda : \mathscr{L} \to \mathbb{1}$ with $\Lambda \circ \lambda = \mathrm{Id}_\mathbb{1}$ (since $\mathcal{C}$ is unimodular, Λ is a two-sided integral). It is shown in [CGPVa] that for a good choice of Λ, we have for any projective $P \in \mathcal{C}$:

$$(\mathrm{Id}_P \otimes \Lambda) \circ \tilde{\imath}_P = \Lambda_P \tag{3.5}$$

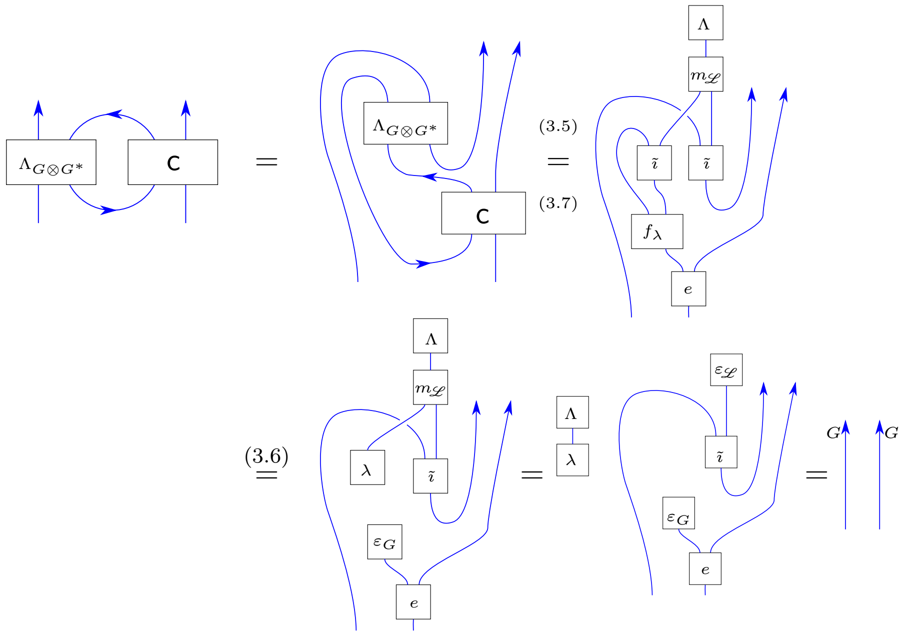
where Λ_P is defined in Equation 3.1 with the copairing of the m-trace. Let G be a projective generator, then the map i_G is an epimorphism thus the map $\lambda \circ \varepsilon_G : G \to \mathscr{L}$ factors through it and there exists a map $f_\lambda : G \to G^* \otimes G$ such that

$$\lambda \circ \varepsilon_G = i_G \circ f_\lambda \ . \tag{3.6}$$

Let $\widetilde{f_\lambda} = (\overrightarrow{\mathrm{ev}}_G \otimes \mathrm{Id}_G) \circ (\mathrm{Id}_G \otimes f_\lambda)$ and fix any map $e : G \to G \otimes G$ such that $(\varepsilon_G \otimes \mathrm{Id}_G) \circ e = \mathrm{Id}_G$. Then the map

$$\mathsf{c} = (\widetilde{f_\lambda} \otimes \mathrm{Id}_G) \circ (\mathrm{Id}_G \otimes e) \tag{3.7}$$

is a chromatic map. Indeed, the integral property of λ is that $m_\mathscr{L} \circ (\lambda \otimes \mathrm{Id}_\mathscr{L}) = \varepsilon_\mathscr{L} \otimes \lambda$ where $m_\mathscr{L}$ and $\varepsilon_\mathscr{L}$ are the product and counit of $\mathscr{L}$. Then we make a graphical proof:



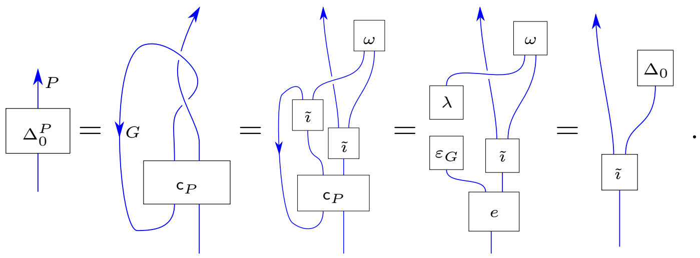
The first equality comes from the duality property of Ω (see Lemma 3.1.2); the second is expressing $\Lambda_{G\otimes G^*}$ as the coaction of Λ. This coaction on a tensor product is given by the product of the coactions. The third equality is the defining property of f_λ and the fourth is the integral property of λ. □

The coend in a ribbon tensor category $\mathcal{C}$ also gives an alternative description of the morphisms Δ_0 and $\Delta_\pm$ as follows: Let $\theta : \mathscr{L} \to \mathscr{L}$ be defined by $\theta \circ i_V = i_V \circ (\mathrm{Id}_{V^*} \otimes \theta_V)$. Then $\Delta_\pm = \varepsilon_{\mathscr{L}} \circ \theta^{\pm 1} \circ \lambda$. Let $\omega : \mathscr{L} \otimes \mathscr{L} \to \mathbb{1}$ be the Hopf pairing defined by

$$\omega \circ (i_U \otimes i_V) = (\overleftarrow{\mathrm{ev}}_U \otimes \overleftarrow{\mathrm{ev}}_V) \circ (\mathrm{Id}_{U^*} \otimes (c_{V^*,U} \circ c_{U,V^*}) \otimes \mathrm{Id}_V)$$

and let $\Delta_0 = \omega \circ (\lambda \otimes \mathrm{Id}_{\mathscr{L}}) : \mathscr{L} \to \mathbb{1}$. Then for any projective object P,

$$\Delta_0^P = (\mathrm{Id}_P \otimes \Delta_0) \circ \tilde{\imath}, \text{ indeed}$$



Thus, $\mathcal{C}$ is modular if and only if $\Delta_0 = \Lambda$ which is equivalent (see [DGG+22, Lemma 2.7]) to the non-degeneracy of ω. We thus recover the usual notion of modular and twist non-degenerate given for example in [DGG+22].

Hopf algebras. A particular example is when $\mathcal{C}$ is the category of finite dimensional left modules over a finite dimensional unimodular ribbon Hopf algebra H. By Theorem 1 of [BBG21] it has a non-degenerate left m-trace on Proj, which is also a right m-trace since H is ribbon. The module H with its left regular action is a projective generator and the map $\Lambda_{H\otimes H^*}$ is the action of the two sided cointegral. Indeed, as a Hopf algebra, $\mathscr{L}$ can be indentified with H^* where the multiplication is twisted by the braiding of $\mathcal{C}$. An explicit formula for a chromatic morphism $\mathsf{c} : H \otimes H \to H \otimes H$ is given by

$$x \otimes y \mapsto \lambda(S(y_{(1)})gx)y_{(2)} \otimes y_{(3)} \tag{3.8}$$

where g is the pivotal element, λ is the right integral, S is the antipode, and $y_{(1)}\otimes y_{(2)}\otimes y_{(3)}$ is the double coproduct of y (see [CGPT20, Lemma 6.3]). In this case the morphism Δ_0^P is given by the action of the element $\Delta_0 = \lambda \otimes \mathrm{Id}_H(R_{21}R_{12})$, where $R \in H \otimes H$ is the R-matrix of H. Note that Δ_0 being non-zero is one of the two conditions for $H - mod$ to be called 3-modular in [Lyu95, Theorem 3.7.3].

When working with Hopf algebras, it may seem more natural to express everything in terms of H and avoid mentioning $P_{\mathbb{1}}$. One can define a *gluing morphism for* H as an endomorphism $\mathsf{g}_H : H \to H$ satisfying $\mathsf{g}_H \circ \Delta_0^H = \Lambda_H$. Writing $P_{\mathbb{1}}$ as an idempotent $i \circ p$ in H, it is easy to see that the existence of a gluing morphism g_H for H implies the existence of a gluing morphism $\mathsf{g} := p \circ \mathsf{g}_H \circ i$. Similarly, the existence of g implies that of $\mathsf{g}_H := i \circ \mathsf{g} \circ p$ (there is up to automorphism a unique indecomposable summand of $P_{\mathbb{1}}$ in H which contains the unique left ideal $\Bbbk\Lambda$ isomorphic to $\mathbb{1}$). However, the existence of g_H is a priori not equivalent to Δ_0 being non-zero.

Fusion categories. Recall that the global dimension of a ribbon fusion category $\mathcal{C}$ is the sum of the squares of the dimensions of its simple objects . It is shown in [EGNO15, Theorem 7.21.12] that it is non-zero when $\Bbbk$ has characteristic zero.

Proposition 3.1.13: *A ribbon fusion (possibly non-modular) category $\mathcal{C}$ with non-zero global dimension is chromatic compact.*

Proof : We recover the definitions of Definition 1.4.1 as $P_{\mathbb{1}} = \mathbb{1}$. Let $\{S_i\}_{i\in I}$ be a set of representatives of the isomorphism classes of simple objects of $\mathcal{C}$, then $G = \oplus_{i\in I} S_i$ is a generator of $\mathrm{Proj} = \mathcal{C}$ and the quantum trace $\mathsf{t} = \mathrm{qTr}_{\mathcal{C}}$ is a non-degenerate m-trace on $\mathcal{C}$. It follows that

$$\left\{x_i = \frac{1}{\mathrm{qdim}(S_i)}\,\overleftarrow{\mathrm{coev}}_{S_i}\right\}_{i\in I} \text{ and } \left\{y_i = \overrightarrow{\mathrm{ev}}_{S_i}\right\}_{i\in I}$$

are dual bases of $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, G\otimes G^*)$ and $\mathrm{Hom}_{\mathcal{C}}(G\otimes G^*, \mathbb{1})$, respectively. Using the expansion $\Omega_{G\otimes G^*} = \sum_{i\in I} x_i \otimes_{\Bbbk} y_i$, it is straightforward to check that

$$\mathsf{c}_P = (\oplus_{i\in I}\,\mathrm{qdim}(S_i)\,\mathrm{Id}_{S_i}) \otimes \mathrm{Id}_P$$

is a chromatic morphism for G based on P (it is essentially the Kirby color tensor the identity of P).
Let $d(\mathcal{C}) = \sum_{i\in I} \mathrm{qdim}(S_i)^2$ be the global dimension of $\mathcal{C}$. As $P_{\mathbb{1}} = \mathbb{1}$, the morphism $\Delta_0^{P_{\mathbb{1}}}$ is multiplication by $d(\mathcal{C})$, so $\mathcal{C}$ is chromatic compact with $\zeta = d(\mathcal{C})$. A gluing morphism is given by $\frac{1}{d(\mathcal{C})} Id_{\mathbb{1}}$. □

In particular the parameter ζ we called global dimension coincides with the usual notion in the semisimple case and for the usual quantum trace.

Symmetric categories. It seems that having semi-simple Müger center is a good criterion for being chromatic compact in characteristic 0. On one extreme, we have seen above that if $\mathcal{C}$ has trivial Müger center (i.e. is modular) then it is chromatic compact. We consider the other extreme, the symmetric case, below:

Proposition 3.1.14: *Suppose $\mathcal{C}$ is symmetric monoidal and $char(\Bbbk) = 0$. Then the following are equivalent: $\mathcal{C}$ is chromatic non-degenerate, $\mathcal{C}$ is chromatic compact, $\mathcal{C}$ is semi-simple.*

PROOF : First, let us observe that $\mathcal{C}$ admits a fiber functor. By Deligne's theorem, see [EGNO15, Theorem 9.11.4], it is enough to check that $\mathcal{C}$ has sub-exponential growth. Find $C \in \mathbb{N}$ such that $G \otimes G \subseteq G^{\oplus C}$ and let L be the length of G. Then the length of $G^{\otimes n}$ is at most $(C.L)^n$ and $\mathcal{C}$ has sub-exponential growth.
Using Tannakian reconstruction, $\mathcal{C} \simeq H - mod^{fd}$ for some Hopf algebra H. (Note that, despite the fact $\mathcal{C}$ is symmetric, it is not clear that H is cocommutative as it might be a super-Hopf algebra). We use the chromatic map from Equation 3.8 (it does not matter which chromatic map we use by Lemma 3.2.1). We can ignore the double braiding in the definition and compute $\Delta_0^H(1) = \lambda(1)1$. If $\mathcal{C}$ is chromatic non-degenerate it has to be non-zero, hence $\lambda(1) \neq 0$. This is equivalent to semisimplicity of H^* by [Rad12] which is equivalent to semisimplicity of H in characteristic 0 by [LR88].
To conclude, we have seen in Proposition 3.1.13 that semi-simple implies chromatic compact, and it is immediate that chromatic compact implies chromatic non-degenerate. □

3.2 Skein relations and algebraic properties

Throughout this chapter, all manifolds are smooth and oriented and all diffeomorphisms are orientation preserving, unless otherwise stated.

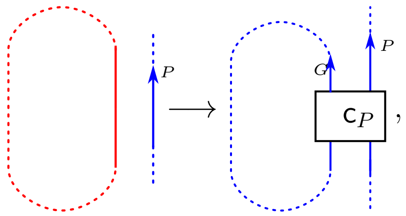
3.2.1 Blue, red and green graphs

We consider graphs whose edges are colored by one of the colors blue, red or green each representing different structures: the blue part will be a $\mathcal{C}$-colored ribbon graph with coupons in the sense of Turaev. The red part is an unoriented framed link in M which is not $\mathcal{C}$-colored. Graphs made of the disjoint union of a blue $\mathcal{C}$-colored ribbon graph and a red set of non-oriented framed circles are called *bichrome graphs*, see [CGPT20]. The green is an unoriented framed link which is used as a notation for the topology of M. This means that M is identified with the result of the S^1-surgery on the green link, see Figure 3.1.

Figure 3.1: The graph on the left is a ribbon graph T in $S^2 \times S^1$ where the green circle represents the topology of $S^2 \times S^1$. The graph on the right is a bichrome graph in S^3 which after surgery on the red circle produces the graph on the left.

Sliding a blue or red edge on an green circle should be thought as an isotopy in M. For a disconnected 3-manifolds M, we will use the symbol $\sqcup$ to separate the different components of M.

A bichrome graph in a manifold M is admissible if every connected component of M contains a blue edge colored by a projective object. A *red to blue modification* of a bichrome graph is an operation in an annulus given by



(3.9)

where c_P is any chromatic morphism based on a projective object P. Here we allow the P-colored edge to be replaced by several parallel strands with at least one colored by a projective object (indeed if $P \in \mathrm{Proj}$ and $V, W \in \mathcal{C}$ are any objects, then $V \otimes P \otimes W \in \mathrm{Proj}$).

3.2.2 Sliding for chromatic morphisms

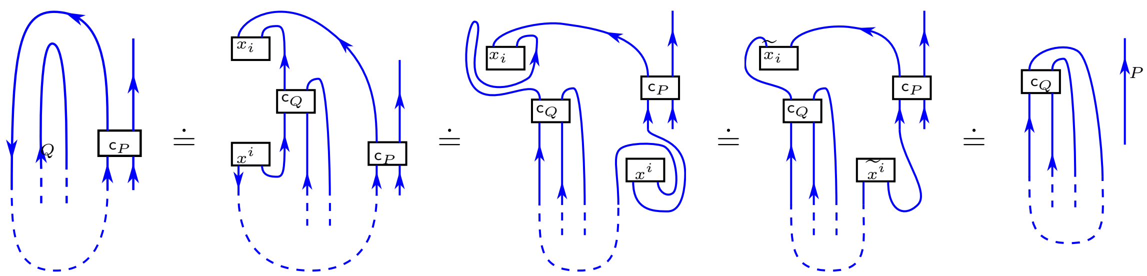
We show that the chromatic map behaves as expected, e.g. that the red-to-blue operation does not depend on the choice of a chromatic morphism, and that one can slide over red components. Remember the definition of admissible skein modules from Section 1.5.2. A bichrome graph is called admissible if its blue part is.

The following lemmas are from [CGPVb], for completeness we restate them here in a slightly different language.

Lemma 3.2.1: *Let L be an admissible bichrome graph in M and let L_1 and L_2 be two $\mathcal{C}$-ribbon graphs each obtained by using red to blue modifications to change every red component of L to blue. Then L_1 and L_2 are projective skein equivalent.*

PROOF : In this proof, we write $T \doteq T'$ if T and T' are projective skein equivalent graphs in M. We need to show that two red to blue modifications of a red circle at different places with different chromatic morphisms are projective skein equivalent. Let $\mathsf{c}_P \in \mathrm{End}_{\mathcal{C}}(G \otimes P)$ and $\mathsf{c}_Q \in \mathrm{End}_{\mathcal{C}}(G' \otimes Q)$ be two chromatic morphisms based on P and Q, respectively, where G and G' are projective generators.

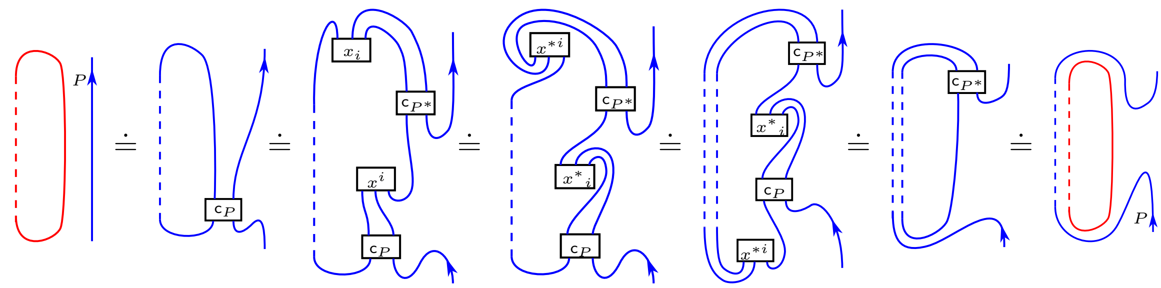
Suppose that we have two modifications of the type "red to blue" which are made on different blue strands and using opposite orientations for the red circle. Then we have (with implicit summation):



where $\widetilde{x}_i$ and $\widetilde{x}^i$ are the dual basis obtained from x_i and x^i by the rotation property of Lemma 3.1.2. If the two modifications are happening on the same side of the red circle, then by the above argument they are both equal to a third modification happening on the opposite side of the red circle. □

Lemma 3.2.2: *Let L be an admissible bichrome graph in M and let L' be the admissible bichrome graph obtained by sliding a red or blue edge of L over a red circle of L (via a Kirby II move, see Equation 3.12). If L_1 and L_2 are ribbon graphs obtained by applying a red to blue modification on each red component of L and L' respectively, then L_1 and L_2 are projective skein equivalent.*

Proof : We first consider the case of sliding a blue edge colored by $P \in \mathrm{Proj}$ on a red circle:

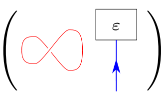
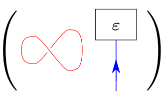
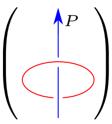


where $x^*{}_i$ and x^{*i} are the dual basis defined by $x^{*i} = (x_i)^* \circ (\phi_G \otimes \mathrm{Id}_{P^* \otimes G^*})$ and $x^*{}_i = (\phi_G^{-1} \otimes \mathrm{Id}_{P^* \otimes G^*}) \circ (x^i)^*$. Notice this implies a red circle can be made blue then slid over a red circle. Now if we want to slide a blue edge colored by $V \in \mathcal{C}$ we can fuse it with an edge colored with a projective Q to this edge creating an edge colored by $V \otimes Q$ (which is projective) and two coupons. Then we can slide the $V \otimes Q$-colored edge as in the above computation. After moving one of coupons along the red circle to the other coupon we can remove both coupons and then unslide the edge colored with Q. □

3.2.3 Properties of the morphisms of a chromatic category

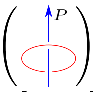
Next we prove Lemma 3.1.6.

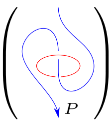
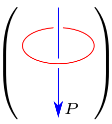
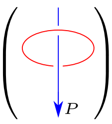
Proof (Proof of Lemma 3.1.6): By Lemma 3.2.1 we can give a meaning to the evaluation by F of F' of admissible ribbon graphs with red components as the value of any red to blue modification of it, see Equation (3.9). Thus we have that

$\Delta_+ \varepsilon = F\Big($  $\Big) \in \Bbbk\varepsilon$, $\quad \Delta_- \varepsilon = F\Big($  $\Big) \in \Bbbk\varepsilon$ and $\Delta_0^P = F\Big($  $\Big) \in \mathrm{End}_{\mathcal{C}}(P)$

does not depend on the choice of any chromatic map. □

Lemma 3.2.3: *The morphisms Δ_0^P is a natural morphism in $P \in \mathrm{Proj}$ (that is $f \circ \Delta_0^P = \Delta_0^Q \circ f$ for any $P \xrightarrow{f} Q \in \mathrm{Proj}$) and $(\Delta_0^P)^* = \Delta_0^{P^*}$ for any $P \in \mathrm{Proj}$.*

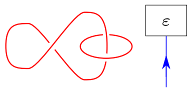
Proof : Since $\Delta_0^P = F\Big($  $\Big) \in \mathrm{End}_{\mathcal{C}}(P)$, the morphism Δ_0^P is clearly natural in $P \in \mathrm{Proj}$. Finally, since the red circle is not oriented,

$(\Delta_0^P)^* = F\Big($  $\Big) = F\Big($  $\Big) = F\Big($  $\Big) = \Delta_0^{P^*}.$

Lemma 3.2.4: *If $\mathcal{C}$ is modular, then it is twist non-degenerate and $\Delta_+\Delta_- = \zeta$.*

Proof : We have

$$\zeta\varepsilon = \quad (3.10)$$



$$= \Delta_+\Delta_-\varepsilon$$

where the left equality is obtained by making the 1-framed component blue, applying the factorizability condition and then using the defining property of the chromatic morphism. The right equality is obtained by sliding the 0-framed red unknot on the 1-framed red one. □

Proposition 3.2.5: *The dual of a gluing morphism is conjugate to a gluing morphism by any isomorphism $P_\mathbb{1} \simeq P_\mathbb{1}^*$.*

Proof : For the existence: since $\Delta_0^{P_\mathbb{1}} \neq 0$ by Lemma 3.1.11(6) there exist $\mathsf{g} \in \mathrm{End}_\mathcal{C}(P_\mathbb{1})$ such that $\Delta_0^{P_\mathbb{1}}\mathsf{g} = \Lambda_{P_\mathbb{1}}$. Since $\Delta_0^{P_\mathbb{1}}$ is central, $\mathsf{g}\Delta_0^{P_\mathbb{1}} = \Lambda_{P_\mathbb{1}}$ and taking the dual we get $(\Lambda_{P_\mathbb{1}})^* = \Lambda_{P_\mathbb{1}^*} = (\Delta_0^{P_\mathbb{1}})^*\mathsf{g}^* = (\Delta_0^{P_\mathbb{1}^*})\mathsf{g}^*$ (where the first equality is due to Lemma 3.1.2 and the last to Lemma 3.2.3). Now if we conjugate by an isomorphism $\psi : P_\mathbb{1} \to P_\mathbb{1}^*$ we get $\Lambda_{P_\mathbb{1}} = \psi \circ \Lambda_{P_\mathbb{1}^*} \circ \psi^{-1} = \Delta_0^{P_\mathbb{1}}(\psi^{-1}\mathsf{g}^*\psi)$ where the last equality follows by Lemma 3.2.3. □

Lemma 3.2.6: *The category $\mathcal{C}$ has a gluing morphism which is an isomorphism of $P_\mathbb{1}$ if and only if*

$$\Delta_0^{P_\mathbb{1}} = \zeta\Lambda_{P_\mathbb{1}} \textit{ for some scalar } \zeta \in \mathbb{k}^* \text{ (i.e. iff } \mathcal{C} \text{ is chromatic compact).}$$

In this case, $\zeta^{-1}\,\mathrm{Id}_{P_\mathbb{1}} + n$ is a gluing morphism for any nilpotent $n \in End(P_1)$.

Proof : Let g be an invertible gluing morphism. Then $\mathsf{g}^{-1} = \zeta\,\mathrm{Id} + n$ for some n nilpotent and $\zeta \in \mathbb{k}^*$. Then $\Delta_0^{P_\mathbb{1}} = \mathsf{g}^{-1}\Lambda_{P_\mathbb{1}} = \zeta\Lambda_{P_\mathbb{1}}$. □

3.3 3-manifold invariant

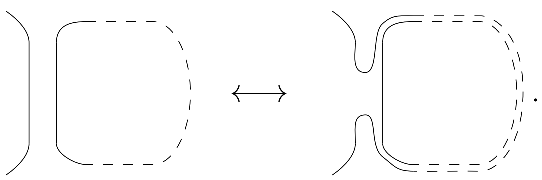
In this section we assume $\mathcal{C}$ is twist non-degenerate.

3.3.1 Surgery presentation of 3-manifolds containing ribbon graphs

It is well known that any closed 3-manifold can be represented by surgery along a link in S^3 and that two such presentations of the same manifold are related by the following two moves. A Kirby I move which replaces a link L with itself disjoint union a unknot $U_\pm$ with framing ± 1:

$$L \quad \longleftrightarrow \quad L \sqcup U_\pm \tag{3.11}$$

and a Kirby II move which replaces a component L_i of a link with a connected sum of L_i with a parallel copy of a different component L_j:



(3.12)

The result of surgery along a link L is a 3-manifold $S^3(L)$ uniquely defined up to diffeomorphism. If L and L' are related by a Kirby move, it induces a diffeomorphism $S^3(L) \xrightarrow{\sim} S^3(L')$ canonical up to isotopy.

Now given a pair (M,T) where M is a closed 3-manifold containing an admissible $\mathcal{C}$-colored ribbon graph T. We say the ribbon graph $L \cup T \subset S^3$ is a surgery presentation of (M,T) if L is a link surgery presentation representing M, and T is identified by $T \subseteq S^3 \setminus L \subseteq M$. The components of L are called the surgery components of $L \cup T$. We have the following theorem (see [CGP14]).

Theorem 3.3.1: *For $i = 1, 2$, let $L_i \cup T_i$ be a surgery presentation in S^3 of a 3-manifold M_i containing a $\mathcal{C}$-colored ribbon graph T_i. Let $f : M_1 \to M_2$ be an orientation preserving diffeomorphism such that $f(T_1) = T_2$ as $\mathcal{C}$-colored ribbon graphs. Then $L_1 \cup T_1$ and $L_2 \cup T_2$ are related by a sequence of orientation changes of the surgery components, Kirby I moves away from T_i, Kirby II moves on the surgery components and Kirby II moves obtained by sliding an edge of T_i on a component of the surgery link such that the induced diffeomorphism between M_1 and M_2 is isotopic to f.*

3.3.2 Existence of the invariant

Recall the definition of the scalars $\Delta_\pm$ given in Section 3.1.2.

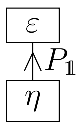
Theorem 3.3.2: *Let (M,T) be a pair where M is a closed 3-manifold containing an admissible $\mathcal{C}$-colored ribbon graph T. Let $L \cup T \subset S^3$ be a surgery presentation of (M,T). If L^{blue} is a $\mathcal{C}$-colored ribbon graph obtained by making each red component of $T \cup L$ blue using a red to blue modification then*

$$\mathrm{B}'_{\mathcal{C}}(M,T) = \frac{F'(L^{blue})}{\Delta_+^r \Delta_-^s} \tag{3.13}$$

only depends on the diffeomorphism class of (M,T), where (r,s) is the signature of the linking matrix of L and F' is as in Theorem 1.5.3.

Proof : Lemma 3.2.1 implies that any choice of making a surgery presentation blue only depends on the surgery presentation. This lemma also implies that using a red to blue modification on a unknot with ± 1 framing with any chromatic morphism produces the same the scalar $\Delta_\pm$. Thus, it is enough to show the invariant is well defined for any two surgery presentations which are related by a Kirby I or II move as in Theorem 3.3.1. Lemma 3.2.2 implies any of the Kirby II moves in Theorem 3.3.1 hold. Finally, since the category is twist non-degenerate then the normalization in Equation (3.13) implies invariance under any Kirby I move. □

When T is not necessarily admissible, and M is connected define $\mathrm{B}_{\mathcal{C}}(M,T) = \mathrm{B}'_{\mathcal{C}}(M, T \sqcup \Gamma_0)$ where

$\Gamma_0 =$ 

is contained in a ball in M. If (M,T) and (M',T') are 3-manifolds with $\mathcal{C}$-colored ribbon graphs such that T' is admissible then

$$\mathrm{B}'_{\mathcal{C}}((M,T)\sharp(M',T')) = \mathrm{B}_{\mathcal{C}}(M,T)\mathrm{B}'_{\mathcal{C}}(M',T'), \tag{3.14}$$

where $\sharp$ stands for the connected sum along balls not intersecting T nor T'.

3.3.3 Identification of the invariant

Let $\mathrm{DGGPR}_{\mathcal{C}}$ denote the renormalized Lyubashenko invariant defined in [DGG+22]. It is normalized using a choice of square root $\mathscr{D}$ of $\Delta_+\Delta_-$.

Theorem 3.3.3: *If $\mathcal{C}$ is twist non-degenerate, M is a closed connected 3-manifold and T is a admissible closed $\mathcal{C}$-colored bichrome graph inside M then*

$$\mathrm{DGGPR}_{\mathcal{C}}(M,T) = \mathscr{D}^{-1-b_1}\mathrm{B}'_{\mathcal{C}}(M,T)$$

where b_1 is the first Betti number of M.

PROOF : Let $L \cup T$ be a surgery presentation of (M,T). Let F_Λ be the extension to bichrome graphs of the RT functor [DGG+22, Proposition 3.1] and let F'_Λ be its renormalization with the m-trace [DGG+22, Theorem 3.3]. From [DGG+22, Theorem 3.8] we have

$$\mathrm{DGGPR}_{\mathcal{C}}(M,T) = \mathscr{D}^{-1-\ell}\delta^{-\sigma(L)}F'_\Lambda(L\cup T)$$

where $\sigma(L) = r - s$ is the signature linking matrix of a surgery presentation L of M, $\delta = \frac{\Delta_+}{\mathscr{D}} = \frac{\mathscr{D}}{\Delta_-}$ and ℓ is the number of components of L. The main property of the map F'_Λ is that it gives a well defined meaning to a red circle that can be made blue [DGG+22, Lemma 4.5]. Now [CGPVa, Section 3.4] shows that the red-to-blue operation using the chromatic morphism is exactly the red-to-blue operation of [DGG+22, Lemma 4.5]. Thus, with the notation of Theorem 3.3.2 it follows that $F'_\Lambda(L\cup T) = F'(L^{blue})$.
Finally, it is easy to show that $\ell = b_1 + r + s$ and $\mathscr{D}^{-1-\ell}\delta^{-\sigma(L)} = \frac{\mathscr{D}^{-1-b_1}}{\Delta_+^r\Delta_-^s}$ for a surgery presentation L. □

Similarly, comparing Equation (3.14) and [DGG+22, Proposition 3.11] one sees that $\mathrm{B}_{\mathcal{C}}(M,\emptyset)$ recovers Lyubashenko's invariant defined in [Lyu95] up to the same factor $\mathscr{D}^{-1-b_1}$.

In particular, let $\mathcal{C}$ be the category of finite dimensional left H-modules over a finite dimensional unimodular ribbon Hopf algebra with right integral λ. The modified trace on $\mathcal{C}$ is induced by λ [BBG21, Theorem 1]. Then $\mathrm{B}_{\mathcal{C}}(M,\emptyset)$ recovers the Hennings invariant defined in [Hen96] and $\mathrm{B}'_{\mathcal{C}}(M,T)$ recovers its renormalized version defined in [DGP18] up to the same factor.

If $\mathcal{C}$ is semisimple modular then $\mathrm{B}'_{\mathcal{C}}$ and $\mathrm{B}_{\mathcal{C}}$ agree and recover the Witten-Reshetikhin-Turaev invariant associated to $\mathcal{C}$ up to the same factor.

3.4 The (non-compact) (3+1)-TQFT

In this section we assume that $\mathcal{C}$ is chromatic non-degenerate.

3.4.1 Construction of TQFT and 4-dimensional invariants

We extend the functor $\mathscr{S}_{\mathcal{C}} : \mathbf{Man} \to \mathrm{Vect}$ to a functor $\mathscr{S}_{\mathcal{C}} : \mathcal{F}(\mathcal{G}^{\mathsf{nc}}) \to \mathrm{Vect}$ (respectively $\mathscr{S}_{\mathcal{C}} : \mathcal{F}(\mathcal{G}) \to \mathrm{Vect}$ if $\mathcal{C}$ is chromatic compact) by assigning to each $\mathbb{S}$-surgery a linear map between skein modules.

Let M be a closed 3-manifold. For $k = 0, ..., 4$, recall from Section 1.3.2, the cobordism $W(\mathbb{S}^{k-1})$ which is given by gluing a k-handle on $M \times [-1, 1]$. Its domain and target are related by a index k-surgery (along a framed sphere $\mathbb{S}^{k-1}$) which can be described using green circles as follows (in what follows the links L and L' are all green and describe two distinct components of M by surgery):

1. index 0-surgery: $M \to M \sqcup S^3$.

2. index 1-surgery: if the gluing $\mathbb{S}^0$ is not contained in a single component of M: $L \sqcup L' \to L \cup L'$; else : $L \to L \cup O$.

3. index 2-surgery: $L \to L \cup$ "*green knot*" arbitrarily linked with L. Alternatively, since the result of a $\mathbb{S}^1$-surgery on a 3-manifold is invertible by another $\mathbb{S}^1$ surgery, then for a well chosen representation of the domain of $W(\mathbb{S}^1)$, its target can be represented as its domain with a green knot removed.

4. index 3-surgery: if the glueing $\mathbb{S}^2$ disconnects a component of M: $L \cup L' \to L \sqcup L'$ where L and L' live in two different hemispheres of S^3; else: $L \mapsto L \setminus O$ where the green unknot bounds a disc disjoint from the other components.

5. index 4-surgery: $M \sqcup S^3 \to M$.

For $k \in \{0, ..., 4\}$, given a framed sphere $\mathbb{S}^{k-1}$ in M we define a morphism

$$\chi_{M,\mathbb{S}^{k-1}} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M(\mathbb{S}^{k-1}))$$

which will be assigned to the morphism $\mathscr{S}_{\mathcal{C}}(e_{M,\mathbb{S}^{k-1}})$ as follows.

0-handle: We only consider 0-handles when $\mathcal{C}$ is chromatic compact and so $\mathsf{g} = \zeta^{-1}\,\mathrm{Id}_{P_{\mathbb{1}}}$ is a gluing morphism. Let $\mathbb{S}^{-1} : \emptyset \hookrightarrow M$ be a framed -1-sphere. Recall Γ_0 is the ribbon graph with a unique edge from a coupon colored with η to a coupon colored by ε (see the r.h.s. of Figure 3.2). Then there exists a *birth map*:

$$\chi_{M,\mathbb{S}^{-1}} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M \sqcup S^3)$$

sending a skein in M to its disjoint union with $(S^3, \zeta\Gamma_0)$, see Figure 3.2.

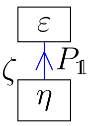
$(M,T) \mapsto (M,T) \sqcup (S^3, \zeta$  $)$

Figure 3.2: The birth map $\chi_{M,\mathbb{S}^{-1}}$ augments a skein by adding a disjoint union of S^3 containing $\zeta\Gamma_0$.

1-handle: Given a framed sphere $\mathbb{S}^0$ in M there exists a *gluing map*:

$$\chi_{M,\mathbb{S}^0} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M(\mathbb{S}^0))$$

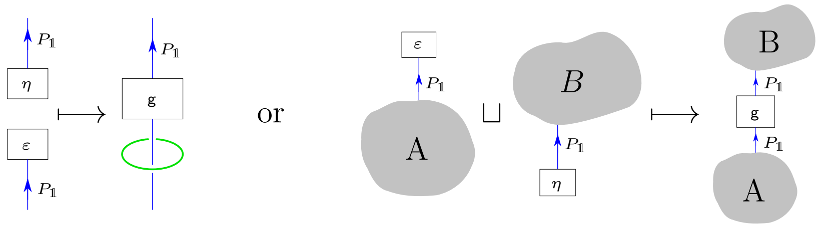


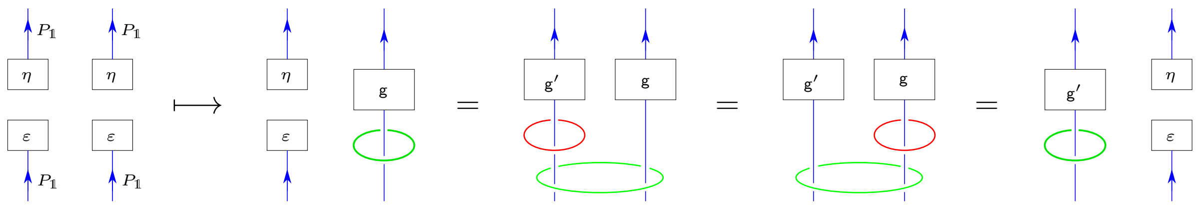
Figure 3.3: The gluing map $\chi_{M,\mathbb{S}^0}$ is depicted by two different representations depending if $\mathbb{S}^0$ is embedded in a unique connected component of M (left) or not (right).

which glues two edges terminating on coupons colored by η and ε by a gluing morphism as represented in Figure 3.3. Let us describe this morphism in more detail. Let x, y be two distinct points of a 3-manifold M. Let B_x, B_y be neighborhood of x and y both oriented and parameterized by B^3 and let $\mathbb{S}^0$ be the framed 0-sphere $B_x \sqcup B_y$. Let $M' = M \setminus (B_x \sqcup B_y) \overset{i}{\hookrightarrow} M$ be the inclusion and $C \simeq S^2 \times [0,1]$ be the cylinder such that $M(\mathbb{S}^0) = M' \cup_\partial C$. We put in this cylinder a skein Γ_{g} with a single coupon colored by any gluing morphism g and an incoming and an outgoing edge parallel to $(1,0,0) \times [0,1]$, framed in the direction $(0,0,1)$. We will say that a skein T in M is in good position with respect to $\mathbb{S}^0$ if $B_x \cap T$ consists of a planar ribbon graph in $\mathbb{R}^+ \times \mathbb{R} \times \{0\} \cap B_x$ consisting of a unique edge oriented from $(1,0,0) \in \partial B_x$ towards a coupon colored by ε and if $B_y \cap T$ consists of a planar ribbon graph in $\mathbb{R}^+ \times \mathbb{R} \times \{0\} \cap B_y$ consisting of a unique edge oriented from a coupon colored by η towards $(1,0,0) \in \partial B_y$. The map $\chi_{M,\mathbb{S}^0}$ assigns to a skein T in good position with respect to $\mathbb{S}^0$ the skein $(M', T \cap M') \cup_\partial (C, \Gamma_{\mathsf{g}})$.

Proposition 3.4.1: *The linear map $\chi_{M,\mathbb{S}^0}$ is well defined and does not depend on the ordering of $\{x, y\}$ nor on the gluing morphism* g.

PROOF : First we note that the admissible skein module is generated by skeins in M where every component of M contains a coupon colored by ε and a coupon colored by η. Indeed, consider a box containing a part of an edge colored by $P \in \mathrm{Proj}$ whose image by the RT-functor is $\overleftarrow{\mathrm{ev}}_P$ and apply Lemma 3.1.11(1) to show that a skein relation can be used to make appear a coupon colored by $\varepsilon : P_{\mathbb{1}} \to \mathbb{1}$. Let us choose an isomorphism $\psi : P_{\mathbb{1}} \to P_{\mathbb{1}}^*$ normalized so that $\eta^* \circ \psi = \varepsilon$ then a coupon colored by ε is skein equivalent to a graph with two coupons colored by ψ and η. So applying this procedure twice we can ensure the presence of a ϵ-colored coupon and of a η-colored coupon in each connected component of M.

Now, up to isotopy of the skein, the definition of $\chi_{M,\mathbb{S}^0}$ only depends a priori on the choice of the two coupons colored by ε and η, and on the choice of a gluing morphism g: we will now prove independence on these data. Let g' be an other gluing morphism and consider the element obtained by using g' instead of g and two different coupons colored with ε and η. Then we have if $\mathbb{S}^0$ is embedded in a unique connected component,



where the first and last equalities are skein equivalences given by definition of gluing morphisms and the middle one is an isotopy of the red circle in the belt 2-sphere created by gluing the 1-handle. Similarly, if the surgery is connecting two different components of M, the representation of the equivalence is similar without the green circles but with the separating belt 2-sphere represented by the horizontal plane.

The map $\chi_{M,\mathbb{S}^0}$ preserves skein relations as we can always choose coupons ε and η outside a fixed box.

Finally reversing the orientation of the sphere $\mathbb{S}^0$ that is interchanging x and y does not change the map since $\eta = \psi^{-1}\varepsilon^*$, $\varepsilon = \psi\eta^*$ and $\psi^{-1}\mathsf{g}^*\psi$ is also a gluing morphism. $\square$

2-handle: Given a framed sphere $\mathbb{S}^1$ in M there exists a *knot-surgery map*:

$$\chi_{M,\mathbb{S}^1} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M(\mathbb{S}^1))$$

adding a red circle along the meridian of the surgery knot, see the r.h.s. of Figure 3.4. Let $C = -B^2 \times S^1$ where the sign of B^2 means reversing orientation and $O_r \subset C$ be a red

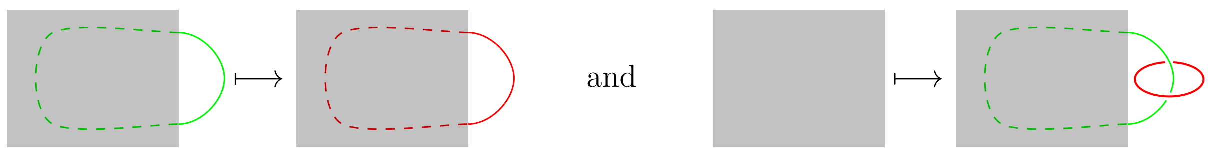


Figure 3.4: The knot-surgery map $\chi_{M,\mathbb{S}^1}$, two alternative representations: on the left we choose a representation of M where $\mathbb{S}^1$ is a meridian of a green knot; a presentation for $M(\mathbb{S}^1)$ is then obtained by forgetting the green knot in the presentation of M, but the map on skeins consists of adding a red component along that $\mathbb{S}^1$. On the right, the surgery presentation of $M(\mathbb{S}^1)$ is obtained by adding the green circle (which is $\mathbb{S}^1$) and the map on skeins consists in adding also its red meridian.

ribbon knot of the form $[-0.1, 0.1] \times \{0\} \times S^1$. Let $\mathbb{S}^1 \simeq S^1 \times B^2$ be a framed knot in M, $M' = M \setminus (S^1 \times B^2)$ and $M'' = M' \cup_\partial C$. Let $\mathscr{S}_{\mathcal{C}}(M') \xrightarrow{i} \mathscr{S}_{\mathcal{C}}(M)$ and $\mathscr{S}_{\mathcal{C}}(M') \xrightarrow{i''} \mathscr{S}_{\mathcal{C}}(M'')$ be the maps induced by the inclusions. We define $\chi_{M,\mathbb{S}^1}$ to be the map that sends a skein $i(T)$ to $i''(T) \cup O_r$. Observe that this map is defined on all $\mathscr{S}_{\mathcal{C}}(M)$ because each skein in M can be isotoped off C .

Proposition 3.4.2: *The linear map $\chi_{M,\mathbb{S}^1}$ is well defined.*

Proof : If $T_1, T_2 \in \mathscr{S}_{\mathcal{C}}(M')$ are such that $i(T_1) = i(T_2)$ then T_1 and T_2 differ by isotopies in M', slidings through meridian discs of C and skein relations which, up to isotopy, can be supposed to be supported in a box disjoint from C. Then $i''(T_1) \sqcup O_r$ and $i''(T_2) \sqcup O_r$ differ by isotopies in $i''(M')$, skein relations in $i''(M')$ and sliding of edges on the created red component O_r, which by Lemma 3.2.2 preserves the class in $\mathscr{S}_{\mathcal{C}}(M'')$. $\square$

3-handle: Given a framed sphere $\mathbb{S}^2$ in M there exists a *cutting map*:

$$\chi_{M,\mathbb{S}^2} : \mathscr{S}_{\mathcal{C}}(M) \to \mathscr{S}_{\mathcal{C}}(M(\mathbb{S}^2))$$

sending parallel strands passing through the cutting sphere S^2 to the copairing Ω, see Figure 3.5. We say that the skein is in standard position with respect to $\mathbb{S}^2$ if its intersection

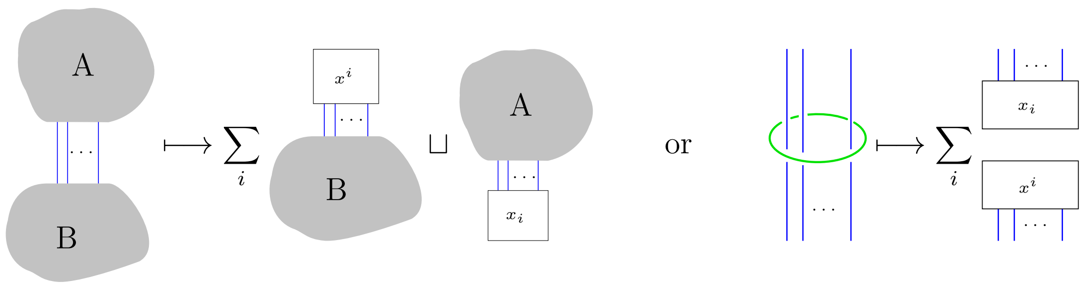


Figure 3.5: The cutting map $\chi_{M,\mathbb{S}^2}$: two representations depending if $\mathbb{S}^2$ is a separating (left) or a non-separating sphere in M (right).

consists in n parallel edges in a rectangle (i.e. a disc of the form $\alpha \times [0,1] \subset \mathbb{S}^2 \times [0,1]$ for some simple arc $\alpha \subset \mathbb{S}^2$) with at least one edge colored by a projective module (see Figure 3.5). We now consider a skein in standard position. Then the image by the RT-functor of this rectangle is the identity of P for some $P \in \mathrm{Proj}$. The cutting map $\chi_{M,\mathbb{S}^2}$ replaces the framed sphere by the sums of graphs in two balls each containing a unique coupon colored with the dual basis of $\mathrm{Hom}_{\mathcal{C}}(P, \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, P)$.

Proposition 3.4.3: *The linear map $\chi_{M,\mathbb{S}^2}$ is well defined.*

PROOF : We refer here to the proof of [CGPVb, Lemma 3.3] which is completely similar. The main idea is that the naturality of Ω implies that the images of isotopic skeins are skein equivalent. □

4-handle: Given a framed sphere $\mathbb{S}^3$ in M, the map $\chi_{M,\mathbb{S}^3}$ corresponding to filling of a 3-sphere of $M = M' \sqcup S^3$ is given by

$$(M, \Gamma) \mapsto F'(\Gamma \cap S^3)(M', \Gamma \cap M') \in \mathscr{S}_{\mathcal{C}}(M').$$

Theorem 3.4.4: *There exists a unique symmetric monoidal functor*

$$\mathscr{S}_{\mathcal{C}} : \boldsymbol{Cob}^{\mathsf{nc}} \to \mathrm{Vect}$$

extending $\mathscr{S}_{\mathcal{C}} : \boldsymbol{Man} \to \mathrm{Vect}$ *such that* $\mathscr{S}_{\mathcal{C}}(e_{\Sigma,\mathbb{S}}) = \chi_{\Sigma,\mathbb{S}}$.
*If $\mathcal{C}$ is chromatic compact, then the functor extends to a symmetric monoidal functor on **Cob**:*

$$\mathscr{S}_{\mathcal{C}} : \boldsymbol{Cob} \to \mathrm{Vect}.$$

PROOF : We only need to prove that the relation (R1)–(R5) are satisfied by $\mathscr{S}_{\mathcal{C}}$.

(R1) Since $\mathscr{S}_{\mathcal{C}}$: **Man** $\to$ Vect is functorial we have $\mathscr{S}_{\mathcal{C}}(e_{d \circ d'}) = \mathscr{S}_{\mathcal{C}}(e_d) \circ \mathscr{S}_{\mathcal{C}}(e_{d'})$. Also, since elements of $\mathscr{S}_{\mathcal{C}}(M)$ are defined by ribbon graphs up to isotopy we clearly have $\mathscr{S}_{\mathcal{C}}(e_d) = \mathrm{Id}$ if d is isotopic to Id_Σ.

(R2) Since the construction of the maps $\chi_{M,\mathbb{S}}$ are local, they are covariant under diffeomorphisms of the pair $(M, \mathbb{S})$.

(R3) Again, since the construction of the maps $\chi_{M,\mathbb{S}}$ are local, they commute for disjoint framed spheres.

(R4) The 2-3-handle cancellations reduces to the chromatic identity (3.3) as shown in Figure 3.6. Indeed since the attaching framed 2-sphere of the 3-handle intersects the belt circle of the 2-handle once, the attaching circle for the 2-handle bounds a disc in the intermediate 3-manifold. This is why we can represent the green circle in Figure 3.6 as an unknot.
The 1-2-handle cancellations reduces to the defining property of the gluing map. Indeed the sphere S^2 created by the 1-handle can't be separating since it is intersected once by the attaching $\mathbb{S}^1$ of the 2-handle. This means that we can represent the map $\chi_{M,\mathbb{S}^1}$ as in the left hand-side of Figure 3.3 and the map $\chi_{M,\mathbb{S}^2}$ is then the left hand-side of Figure 3.4 turning the green unknot into red.
The 3-4-handle cancellation relies on the fact that evaluating F' on a cut 3-ball is a skein relation.
Finally, in the compact case, the 0-1-handle cancellation is obvious since we can choose $\mathsf{g} = \zeta^{-1}\,\mathrm{Id}_{P_{\mathbb{1}}}$ as gluing morphism.

(R5) The maps $\chi_{M,\mathbb{S}}$ do not depend on the orientation of $\mathbb{S}$. □

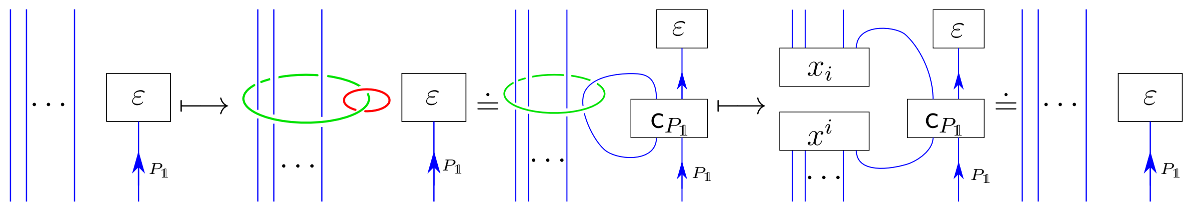


Figure 3.6: The cancellation of a 2-handle by a 3-handle.

We now extract (even in the non-compact case) two scalar invariants of 4-manifolds: $\mathscr{S}_{\mathcal{C}}(W,T)$ for manifolds with an admissible graph in the boundary and $\dot{\mathscr{S}}_{\mathcal{C}}(W)$ for connected closed 4-manifolds.

Definition 3.4.5: Let W be an oriented compact 4-manifolds with no closed components. A $\mathcal{C}$-ribbon graph $T \subset (-\partial W)$ is *admissible* if for each component M of $-\partial W$, $T \cap M$ is admissible i.e. if T represents an admissible skein of $\mathscr{S}_{\mathcal{C}}(-\partial W)$ (where the minus sign is for opposite orientation). If $T \subset (-\partial W)$ is admissible then define the invariant

$$\mathscr{S}_{\mathcal{C}}(W,T) = \mathscr{S}_{\mathcal{C}}(\widetilde{W})(T)$$

where $\widetilde{W}$ is W seen as a cobordism from $-\partial W$ to $\emptyset$. ◇

Definition 3.4.6: Let W be a connected closed 4-manifold. Define

$$\dot{\mathscr{S}}_{\mathcal{C}}(W) = \mathscr{S}_{\mathcal{C}}(\dot{W},\Gamma_0) \in \Bbbk$$

where $\dot{W} = W \setminus B^4$ is a once punctured W. ◇

If $\mathcal{C}$ is chromatic compact, by definition of the maps $\chi_{M,\mathbb{S}^0}$ (see Figure 3.2), we have for any closed connected 4-manifold W:

$$\mathscr{S}_{\mathcal{C}}(W) = \zeta\dot{\mathscr{S}}_{\mathcal{C}}(W)\,\mathrm{Id}_{\Bbbk}\,. \tag{3.15}$$

For example, $\dot{\mathscr{S}}_{\mathcal{C}}(S^4) = 1$ whereas $\mathscr{S}_{\mathcal{C}}(S^4) = \zeta\,\mathrm{Id}_{\Bbbk}$ is the global dimension.

3.4.2 Properties

Remember that in this section we are assuming that $\mathcal{C}$ is chromatic non-degenerate so that in particular it has an m-trace t, chromatic morphism c and gluing morphism g.

Proposition 3.4.7: *Let $\kappa \in \Bbbk^*$, then*

1. $\mathsf{t}' := \kappa \mathsf{t}$ *is a non-degenerate m-trace on* Proj,
2. *its associated copairing is given by* $\Omega'_P = \frac{1}{\kappa}\Omega_P$, *and* $\Gamma'_0 = \frac{1}{\kappa}\Gamma_0$,
3. $\mathsf{c}' = \kappa \mathsf{c}$ *is a chromatic morphism associated to* t',
4. $\mathsf{g}' = \frac{1}{\kappa^2}\mathsf{g}$ *is a gluing morphism, and in the compact case* $\zeta' = \kappa^2 \zeta$.

Finally the TQFT $\mathscr{S}'_{\mathcal{C}}$ *associated to* t' *satisfies* $\mathscr{S}'_{\mathcal{C}}(W) = \kappa^{\chi(W)} \mathscr{S}_{\mathcal{C}}(W)$ *where* χ *is the Euler characteristic.*

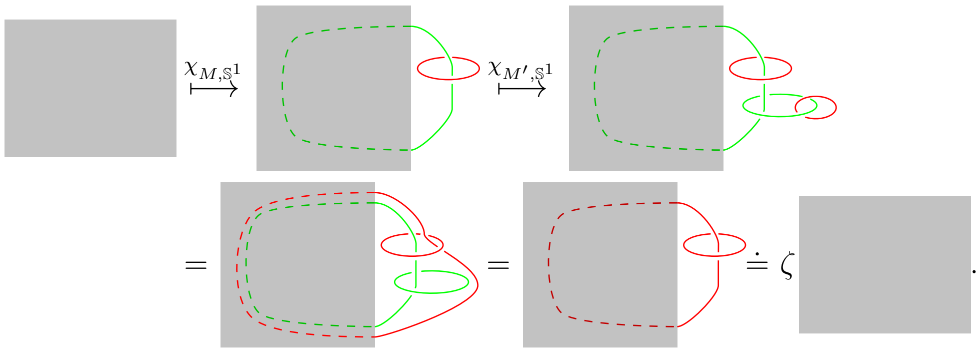
Proof : The first four points are immediate from the definitions. In the compact case, the 0-handle map becomes $\chi'_{M,\mathbb{S}^{-1}} = \kappa \chi_{M,\mathbb{S}^{-1}}$, as $\zeta' \Gamma'_0 = \kappa \zeta \Gamma_0$. The 1-handle map becomes $\chi'_{M,\mathbb{S}^0} = \frac{1}{\kappa}\chi_{M,\mathbb{S}^0}$ as it maps a $\Omega'_{P_{\mathbb{1}}}$ to a g'. The 2-handle map becomes $\chi'_{M,\mathbb{S}^0} = \kappa \chi_{M,\mathbb{S}^0}$ as $\mathsf{c}' = \kappa \mathsf{c}$. The 3-handle map becomes $\chi'_{M,\mathbb{S}^0} = \frac{1}{\kappa}\chi_{M,\mathbb{S}^0}$ as $\Omega'_P = \frac{1}{\kappa}\Omega_P$. The 4-handle map becomes $\chi'_{M,\mathbb{S}^0} = \kappa \chi_{M,\mathbb{S}^0}$ as $\mathsf{t}' = \kappa \mathsf{t}$.
Therefore for a 4-bordism W decomposed using n_i i-handles, $0 \leq i \leq 4$, one has:

$$\mathscr{S}'_{\mathcal{C}}(W) = \kappa^{n_4 - n_3 + n_2 - n_1 + n_0} \mathscr{S}_{\mathcal{C}}(W) = \kappa^{\chi(W)} \mathscr{S}_{\mathcal{C}}(W). \qquad \square$$

Theorem 3.4.8: *The TQFT* $\mathscr{S}_{\mathcal{C}}$ *is invertible if and only if* $\mathcal{C}$ *is modular.*

Proof : First we prove the necessity of the theorem: recall that $\mathscr{S}_{\mathcal{C}}(S^3) \simeq \Bbbk$ is generated by the skein (S^3, Γ_0). Let G be a projective generator with a unique indecomposable factor $P_{\mathbb{1}} \xrightarrow{i} G \xrightarrow{p} P_{\mathbb{1}}$. Then by naturality of Λ and since G contains a single copy of P_1, we have $\Lambda_G = i \Lambda_{P_{\mathbb{1}}} p$. Consider the subspace of $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ generated by graphs $\{O_f\}_{f \in \mathrm{End}_{\mathcal{C}}(G)}$ with a unique coupon colored by $f \in \mathrm{End}_{\mathcal{C}}(G)$ and a unique edge of the form $\{pt\} \times S^1$. Consider the two cobordisms $W_2, W_3 : S^2 \times S^1 \to S^3$ given by gluing a 2-handle (resp. a 3-handle) to $S^2 \times S^1 \times [0,1]$ respectively along $\{pt\} \times S^1 \times \{1\}$ and $S^2 \times \{pt\} \times \{1\}$. Then $\mathscr{S}_{\mathcal{C}}(W_2)(O_f) = \mathsf{t}_G(\Delta_0^G f) \in \Bbbk \simeq \mathscr{S}_{\mathcal{C}}(S^3)$ and $\mathscr{S}_{\mathcal{C}}(W_3)(O_f) = \mathsf{t}_G(\Lambda_G f) \in \Bbbk \simeq \mathscr{S}_{\mathcal{C}}(S^3)$. In particular, for the gluing morphism g, $\mathscr{S}_{\mathcal{C}}(W_2)(O_{i\mathsf{g}p}) = 1 = \mathscr{S}_{\mathcal{C}}(W_3)(O_{\mathrm{Id}}) = 1$ so the two maps are non-zero. If $\mathscr{S}_{\mathcal{C}}$ is invertible, $\dim_{\Bbbk}(S^2 \times S^1) = 1$ and there exists $\zeta \in \Bbbk^*$ such that $\mathscr{S}_{\mathcal{C}}(W_3) = \zeta \mathscr{S}_{\mathcal{C}}(W_2)$. Then for any $f \in \mathrm{End}_{\mathcal{C}}(G)$, $\mathsf{t}_G(\Delta_0^G f) = \mathscr{S}_{\mathcal{C}}(W_3)(O_f) = \zeta \mathscr{S}_{\mathcal{C}}(W_2) = \zeta \mathsf{t}_G(\Lambda_G f)$. Finally, by non-degeneracy of the m-trace, $\Delta_0^G = \zeta \Lambda_G$.

Now we prove the sufficiency of the theorem: we suppose $\mathcal{C}$ is modular and we show that for any connected 3-manifold M, $\dim_{\Bbbk}(\mathscr{S}_{\mathcal{C}}(M)) = 1$. This is true because the image of any 1-surgery given by a 2-handle can be inverted:



Here the first map is the image of any S^1-surgery on M and the second is the image of an appropriate second S^1-surgery; the first equality is an isotopy in the manifold obtained by sliding the second red curve along the first green curve, the second equality comes from the fact that topologically on the level of the 3-manifolds, a surgery along a meridian of a $\mathbb{S}^1$-surgery component cancels both components and the last equivalence is a skein equivalence due to the factorizability of $\mathcal{C}$.

Then we check that every cobordism induces an isomorphism. It is immediate from the definition that 0-handles and 4-handles are isomorphisms. The proof for 2-handles is given above. Note that a 1-handle followed by a 3-handle glued on the belt sphere created by the 1-handle is a scalar times the identity. Indeed, the 1-handle will introduce a gluing morphism (which is a scalar times the identity of $P_{\mathbb{1}}$ by assumption) from a pair of coupons ε and η. Then the 3-handle will cut it, turning it back to a pair of coupons ε and η. This shows that 1-handles are injective and 3-handles surjective. Because every skein module is 1-dimensional, they are also bijective. □

Theorem 3.4.9: *Assume $\mathcal{C}$ is twist non-degenerate and chromatic non-degenerate and let M be a closed connected 3-manifold. Fix any connected bordism W with $\partial W = -M$ such that the cobordism $W : M \to \emptyset$ is made by gluing 2-handles and a unique 4-handle on $M \times [0,1]$. Let (r,s) be the signature of $-W$ then for any admissible skein T in M,*

$$\mathscr{S}_{\mathcal{C}}(W,T) = \Delta_+^r \Delta_-^s \mathrm{B}'(M,T) \in \Bbbk, \tag{3.16}$$

(where B' is defined in Subsection 3.3.2).

Proof : Let $-W = B^4 \cup_{N(L)=\sqcup_{i=1}^n (\partial B^2)\times B^2} (\sqcup_{i=1}^n B^2 \times B^2) : \emptyset \to M$ where $N(L)$ is a tubular neighborhood of a n component link L in S^3. Then $L^{red} \cup T$ is a link presentation in S^3 of the pair (M,T) while $L^{green} \cup T$ represents T in the manifold M. Now $\mathscr{S}_{\mathcal{C}}(\dot{W})(M,T) = \mathscr{S}_{\mathcal{C}}(\dot{W})(L^{green} \cup T) = L^{red} \cup T \subset S^3$. Thus $\mathscr{S}_{\mathcal{C}}(W,T) = F'(L^{blue} \cup T)$ where $(L^{blue} \cup T)$ is obtained from $L^{red} \cup T$ by doing red to blue modifications. Finally (see [GS99, Proposition 4.5.11]) the linking matrix of L is the intersection form on $H^2(-W)$ so the signature of $-W$ is (r,s). □

Proposition 3.4.10: *Behavior under connected sums:*

- *The invariant of closed connected 4-manifolds $\dot{\mathscr{S}}_{\mathcal{C}}(W)$ is multiplicative under connected sum.*

- *If W is a closed connected 4-manifold and $W' : M' \to N' \in \boldsymbol{Cob}^{nc}$ (resp. $W' \in \boldsymbol{Cob}$ if $\mathcal{C}$ is chromatic compact), both non-empty, then*

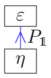
$$\mathscr{S}_{\mathcal{C}}(W \# W') = \dot{\mathscr{S}}_{\mathcal{C}}(W)\mathscr{S}_{\mathcal{C}}(W') \in \mathrm{Hom}_{\Bbbk}(\mathscr{S}_{\mathcal{C}}(M'), \mathscr{S}_{\mathcal{C}}(N')).$$

- *For non-empty 4-manifolds W, W' containing admissible graphs T, T' in their boundaries,*

$$\mathscr{S}_{\mathcal{C}}(W \# W', T \cup T') = \left| \begin{array}{l} \zeta^{-1}\mathscr{S}_{\mathcal{C}}(W,T)\mathscr{S}_{\mathcal{C}}(W,T) \text{ if } \mathcal{C} \text{ is chromatic compact,} \\ 0 \text{ else;} \end{array} \right.$$

- *If $\mathcal{C}$ is chromatic compact, for two non-empty 4-cobordisms $W : M \to N$ and $W' : M' \to N'$,*

$$\mathscr{S}_{\mathcal{C}}(W \# W') = \zeta^{-1}\mathscr{S}_{\mathcal{C}}(W) \otimes \mathscr{S}_{\mathcal{C}}(W') : \mathscr{S}_{\mathcal{C}}(M) \otimes \mathscr{S}_{\mathcal{C}}(M') \to \mathscr{S}_{\mathcal{C}}(N) \otimes \mathscr{S}_{\mathcal{C}}(N').$$

Proof : The admissible skein module $\mathscr{S}_{\mathcal{C}}(S^3)$ is one dimensional and generated by $\Gamma_0 =$  . For a closed connected 4-manifold W the twice punctured cobordism $\mathscr{S}_{\mathcal{C}}(\ddot{W}) : \mathscr{S}_{\mathcal{C}}(S^3) \to \mathscr{S}_{\mathcal{C}}(S^3)$ acts as multiplication by the scalar $\dot{\mathscr{S}}_{\mathcal{C}}(W)$. Composition corresponds to connected sum for the twice-punctured cobordisms, and to multiplication for the scalars. The second point is obtained by adding a cancelling pair of 3 and 4-handles to W'. Then connected sum with W precomposes by $\mathscr{S}_{\mathcal{C}}(\ddot{W})$ before the 4-handle, hence simply multiplies by $\dot{\mathscr{S}}_{\mathcal{C}}(W)$.
Let $P : S^3 \sqcup S^3 \to S^3$ be the three dimensional pair of pants, namely a 3-punctured S^4 which can be seen as a unique 1-handle. The cobordism $(W \dot{\#} W') : (-\partial W) \sqcup (-\partial W') \to S^3$ factors as $W \dot{\#} W' = P \circ (\dot{W} \sqcup \dot{W}')$.

The map $\mathscr{S}_{\mathcal{C}}(P) : \Bbbk \otimes \Bbbk = \Bbbk \to \Bbbk$ is a scalar morphism which sends $\Gamma_0 \otimes \Gamma_0$ to the unique graph with 3 coupons colored by η, g and ε. Since $\varepsilon \circ \mathsf{g} = 0$ unless g is invertible (i.e. $\mathcal{C}$ is chromatic compact by Lemma 3.2.6), the second case follows. Let's now assume that $\mathcal{C}$ is chromatic compact and let us use $\mathsf{g} = \zeta^{-1}\,\mathrm{Id}_{P_{\mathbb{1}}}$ for the gluing morphism. Then $\mathscr{S}_{\mathcal{C}}(W \# W', T \cup T') = \mathscr{S}_{\mathcal{C}}(W,T)\mathscr{S}_{\mathcal{C}}(W',T')F'(\mathscr{S}_{\mathcal{C}}(P)(\Gamma_0 \otimes \Gamma_0)) = \zeta^{-1}\mathscr{S}_{\mathcal{C}}(W,T)\mathscr{S}_{\mathcal{C}}(W',T')$.

For the last statement, since every object of **Cob** is dualizable we can suppose that $N = N' = \emptyset$. Then the statement follows from the previous identity since for any $T \otimes T' \in \mathscr{S}_{\mathcal{C}}(M) \otimes \mathscr{S}_{\mathcal{C}}(M') \cong \mathscr{S}_{\mathcal{C}}(-\partial(W \sqcup W'))$, we have $\mathscr{S}_{\mathcal{C}}(W \# W')(T \otimes T') = \mathscr{S}_{\mathcal{C}}(W \# W', T \cup T')$. □

Proposition 3.4.11: *The category $\mathcal{C}$ is chromatic compact if and only if*

$$\dot{\mathscr{S}}_{\mathcal{C}}(S^1 \times S^3) \neq 0\ .$$

Proof : A handle decomposition of the punctured bordism $S^1 \dot{\times} S^3 : S^3 \to \emptyset$ is given by a 1-handle followed by a 3-handle glued on its belt sphere and a closing 4-handle. The skein Γ_0 is sent to a circle with a coupon g in $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ which is then cut into the closure of $\mathsf{g} \circ \Lambda_{P_{\mathbb{1}}}$ in $\mathscr{S}_{\mathcal{C}}(S^3)$. This is non-zero if and only if g is invertible. The statement follows then by Lemma 3.2.6. □

Proposition 3.4.12: *If $\mathcal{C}$ is twist non-degenerate or if $\dot{\mathscr{S}}_{\mathcal{C}}(S^2 \times S^2) \neq 0$ then $\mathscr{S}_{\mathcal{C}}$ does not distinguish exotic pairs of cobordisms.*

Proof : Since $\dot{\mathscr{S}}_{\mathcal{C}}(\pm\mathbb{CP}^2) = \Delta_{\pm}$, the category is twist non-degenerate if and only if $\dot{\mathscr{S}}_{\mathcal{C}}(\mathbb{CP}^2)\dot{\mathscr{S}}_{\mathcal{C}}(-\mathbb{CP}^2) \neq 0$. As said in the introduction, Gompf ([Gom84]) showed that two homeomorphic compact orientable 4-manifolds (possibly with boundary) become diffeomorphic after some finite sequence of connected sums with $S^2 \times S^2$; the same is true for connected sums with complex projective planes (or their opposites) since $(S^2 \times S^2)\#\mathbb{CP}^2$ is diffeomorphic to $\mathbb{CP}^2\#\mathbb{CP}^2\#(-\mathbb{CP}^2)$. The statement then follows from Proposition 3.4.10. $\square$

Proposition 3.4.13: *Let $\mathcal{C}$ be non-semisimple and chromatic compact then $\mathscr{S}_{\mathcal{C}}(B^2 \times S^2, \Gamma_0) = 0$ (where Γ_0 is the graph of the r.h.s. Figure 3.2, contained in a ball in $\partial B^2 \times S^2$). Equivalently, the skein $\mathsf{O} \cup \Gamma_0$ is zero in $\mathscr{S}_{\mathcal{C}}(S^3)$ (where O denotes a red unknot).*

Proof : As $\mathcal{C}$ is chromatic compact, $\Delta_0^{P_{\mathbb{1}}} = \zeta\Lambda_{P_{\mathbb{1}}}$ for some $\zeta \in \mathbb{k}^*$. Therefore $\mathsf{O} \cup \Gamma_0 = \zeta\Gamma_0 \sqcup \Gamma_0$ which is zero by the skein relation which evaluates only one Γ_0 via the RT functor. This proves the second statement. The first follows by observing that $B^2 \times S^2$ is obtained by gluing a 2-handle to $S^1 \times S^2 \times [-1, 1]$ along $S^1 \times B^2 \times \{1\}$ and then filling the result by a 4-handle. Then we present $(S^1 \times S^2, \Gamma_0)$ as $\mathsf{O} \cup \Gamma_0$ and the first operation consists of changing O to O.

3.5 Examples and relations with other works

3.5.1 Semisimple case

Using the chain-mail construction of [Rob95], we can rewrite our construction in the semi-simple case as a state sum. We then recover the Crane–Yetter–Kauffman 4-manifold invariant associated with a semi-simple fusion category $\mathcal{C}$. The chain mail construction has been carried out for the state spaces in [Tha21] in characteristic 0 and we will use this description.

The state-sum 4-manifold invariant was defined for all fusion categories in [CKY97]. It was first only defined in the modular case by Crane and Yetter, and mentioned to extend to a TQFT there. It is well-known that in the modular case the TQFT is invertible, and the associated 4-manifold invariant is classical, namely only depends on the signature and Euler characteristic, see [CKY97, Proposition 6.2]. Note however that given the extra data of a boundary condition, which corresponds to the empty skein in our description, this TQFT recovers the Reshetikhin–Turaev invariants of 3-manifolds. It was shown in [BB18] that when the category $\mathcal{C}$ is not modular, i.e. has non-trivial Müger center, it is no longer true that the 4-manifold invariants depend only on the signature and the Euler characteristic, but at least also on the fundamental group. It is however still almost trivial on simply connected manifolds, see [BB18].

Theorem 3.5.1: *Let $\mathcal{C}$ be a fusion category over an algebraically closed field of characteristic 0. Choose $\mathsf{t} = \mathrm{qTr}$ the standard categorical trace. Then the TQFT $\mathscr{S}_\mathcal{C}$ coincides with the Crane–Yetter–Kauffman TQFT.*

Proof : In the semi-simple case, the admissible skein modules are the usual skein modules.
Let us describe the TQFT $\mathscr{S}_\mathcal{C}$ in this setting. Let $\{S_i\}_{i\in I}$ be a set of representatives of the isomorphism classes of simple objects of $\mathcal{C}$. We described in Proposition 3.1.13 the chromatic map $\mathsf{c}_P = (\oplus_{i\in I} \mathrm{qdim}(S_i)\,\mathrm{Id}_{S_i}) \otimes \mathrm{Id}_P$ and the gluing morphism $\mathsf{g} = \frac{1}{d(\mathcal{C})} Id_{\mathbb{1}}$, hence $\zeta = d(\mathcal{C})$.

0. A 0-handle introduces $d(\mathcal{C}) \cdot \emptyset$ in the created S^3.
1. A 1-handle on a skein disjoint from the attaching sphere multiplies by $\frac{1}{d(\mathcal{C})}$ without affecting the skein.
2. A 2-handle introduces a Kirby-colored circle along the attaching sphere.
3. A 3-handle cuts the strands passing through the canceling 2-handle represented by the green arc by introducing a copairing.
4. A 4-handle does Reshetikhin–Turaev evaluation on a skein in S^3.

This is exactly the description of [Tha21, Definition 5.11]. The only non-trivial check is for the 3-handle. Let V denote the color of the strand passing trough the green arc. In the description there, one splits V as a direct sum of simples, and only keeps the $\mathbb{1}$ components. In our construction, we choose a basis $\{f_j\}_j$ of $\mathrm{Hom}_\mathcal{C}(V, \mathbb{1})$ and the dual basis $\{f_j^\star\}_j$ of $\mathrm{Hom}_\mathcal{C}(\mathbb{1}, V)$ with respect to the m-trace. Then indeed $f_j^\star \circ f_j$ is an idempotent of V corresponding to a $\mathbb{1}$-component, and the two constructions agree.
It is shown in [Tha21, Sections 5.2 and 3.4] that the TQFT of skein modules and the construction above coincides with the Crane–Yetter TQFT. Skein modules are introduced there in Definition 5.6, and the linear maps induced by 4-manifolds in Definition 5.11. The Crane–Yetter state spaces (as outlined by Yetter, see also [BB18, Section 7.1]) are introduced in Proposition 3.50, and the linear maps induced by 4-manifolds in Definition 3.46. The fact that this recovers the Crane–Yetter invariants is proven in Theorem 3.61. The isomorphism between the skein and Crane–Yetter state spaces is given in Lemmas 5.22 and 5.24. The fact that this isomorphism is natural and respects 4-cobordisms is Theorem 5.26. □

3.5.2 The example of $\mathfrak{sl}_2$

We study the category of modules over a partially unrolled version of the small quantum group associated with $\mathfrak{sl}_2$, at roots of unity. Varying the parameters, this gives examples of possibly non-modular and possibly twist-degenerate chromatic compact categories. In particular, our construction applies and gives a plain (3+1)-TQFT $\mathscr{S}_\mathcal{C}$. We expect this TQFT to be similar to the construction of [BD23, Section 9.2] on 2-handlebodies. In particular, we expect that a result similar to [BD, Theorem 8.1] applies, and that the associated invariant of closed connected 4-manifolds only depends on the Euler characteristic, signature and (in the twist-degenerate case) spin status. The whole TQFT might be of greater interest though.

Definition 3.5.2: Let $\Bbbk = \mathbb{C}$ and m, n, r be positive integers such that $n|m$ and $r \geq 2$. Let q be a primitive $2r$-th root of unity and choose $q^{\frac{2}{mn}}$ a primitive mnr-th root of unity. Note that $(q^{\frac{2}{mn}})^{\frac{m}{n}}$ is a primitive n^2r-th root of unity. Let $H := \mathfrak{u}_q^{m,n}(\mathfrak{sl}_2)$ be the $\mathbb{C}$-algebra presented as

$$\mathbb{C}\langle E, F, \mathcal{K} | E^r = F^r = 0, \mathcal{K}^{mnr} = 1, \mathcal{K}E = q^{\frac{2}{m}} E\mathcal{K}, \mathcal{K}F = q^{-\frac{2}{m}} F\mathcal{K}, EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\rangle$$

where $K = \mathcal{K}^n$. The algebra H can be given the structure of a Hopf algebra with coproduct Δ, counit ε and antipode S defined by

$$\begin{aligned}
&\Delta(E) = 1 \otimes E + E \otimes K, && \varepsilon(E) = 0, && S(E) = -EK^{-1},\\
&\Delta(F) = K^{-1} \otimes F + F \otimes 1, && \varepsilon(F) = 0, && S(F) = -KF,\\
&\Delta(\mathcal{K}) = \mathcal{K} \otimes \mathcal{K}, && \varepsilon(\mathcal{K}) = 1, && S(\mathcal{K}) = \mathcal{K}^{-1}.
\end{aligned}$$
◊

Note that H contains a version of the small quantum group at even root of unity as the sub-Hopf-algebra generated be E, F and K. Let $\mathcal{C} = H - mod$ be the category of finite dimensional left H-modules. For $i \in \mathbb{Z}/mnr\mathbb{Z}$, denote $\mathcal{K}_i = \frac{1}{mnr} \sum_{j=0}^{mnr-1} q^{\frac{-2ij}{mn}} \mathcal{K}^j$. Then

$$\mathcal{K}\mathcal{K}_i = q^{\frac{2i}{mn}} \mathcal{K}_i, \quad \mathcal{K}_i \mathcal{K}_j = \delta_{i,j} \mathcal{K}_i, \quad \sum_{i=0}^{mnr-1} \mathcal{K}_i = 1, \quad E\mathcal{K}_i = \mathcal{K}_{i+n} E, \quad \text{and} \quad F\mathcal{K}_i = \mathcal{K}_{i-n} F.$$

Namely, $\mathcal{K}_i$ acts as the projection on the $q^{\frac{2i}{mn}}$ eigenspace of $\mathcal{K}$.

Proposition 3.5.3: *The Hopf algebra $H = \mathfrak{u}_q^{m,n}(\mathfrak{sl}_2)$ is ribbon where the R-matrix and twist are given by:*

$$R = \left(\sum_{i,j=0}^{mnr-1} q^{\frac{2ij}{n^2}} \mathcal{K}_i \otimes \mathcal{K}_j \right) \cdot \left(\sum_{k=0}^{r-1} \frac{\{1\}^{2k}}{\{k\}!} q^{\frac{k(k-1)}{2}} E^k \otimes F^k \right),$$

$$\theta = K^{r-1} \sum_{k=0}^{r-1} \frac{\{1\}^{2k}}{\{k\}!} q^{\frac{k(k-1)}{2}} S(F^k) \left(\sum_{i=0}^{mnr-1} q^{\frac{-2i^2}{n^2}} \mathcal{K}_i \right) E^k.$$

Proof : We first sketch the proof when $q = \exp(i\pi/r)$ and $q^{\frac{2}{mn}} = \exp(2i\pi/mnr)$. Then H is a sub-quotient of the topological unrolled quantum group (see [GHP22]). The R-matrix factors $R = \mathcal{H}\check{R}$ where $\check{R}$ is the quasi R-matrix. Then R is an R-matrix since $\mathcal{H}$ satisfies the following relations: $\forall x, y \in H$, $\mathcal{H}(x \otimes y)\mathcal{H}^{-1} = xK^{|y|/2} \otimes K^{|x|/2}y$ where $|x|, |y|$ are the integral weights of x and y; $\Delta \otimes \mathrm{Id}(\mathcal{H}) = \mathcal{H}_{13}\mathcal{H}_{23}$; and $\mathrm{Id} \otimes \Delta(\mathcal{H}) = \mathcal{H}_{13}\mathcal{H}_{12}$. Similarly the fact that θ is a twist follows since $\mathcal{T} = \sum_{i=0}^{mnr-1} q^{\frac{-2i^2}{n^2}} \mathcal{K}_i = m \circ (S \otimes \mathrm{Id})(\mathcal{H}_{21})$ satisfies $S(\mathcal{T}) = \mathcal{T}$.

Finally for the general case, we remark that the $\mathbb{Q}(q^{\frac{2}{mn}})$ subalgebra generated by E, F and $\mathcal{K}$ is also ribbon since it is isomorphic through a Galois isomorphism to a sub Hopf algebra of the previous case which contains the R-matrix and the twist. □

The cointegral is $\Lambda = c\mathcal{K}_0 E^{r-1} F^{r-1}$ for some scalar $c \in \Bbbk^\times$ and the right integral is $\lambda(\mathcal{K}^i E^n F^k) = \frac{mnr}{c} \delta_{i,m(1-r)} \delta_{n,r-1} \delta_{k,r-1}$. In particular $\lambda(\mathcal{K}_i F^{r-1} E^{r-1}) = \frac{1}{c} q^{\frac{2i(r-1)}{n}}$.

> **Proposition 3.5.4:** *The category $\mathcal{C} = H - mod$ is chromatic compact. It is modular if and only if $m = n$ and both n and r are odd. It is twist degenerate if and only if n is odd and r is a multiple of 4.*

PROOF : As discussed in Section 3.1.6, Δ_0^P is given by the action of the central element $\Delta_0 = (\lambda \otimes Id)(R_{21}R_{12})$. One can compute:

$$\begin{aligned}\Delta_0 &= (\lambda \otimes Id)(R_{21}R_{12}) \\ &= (\lambda \otimes Id)\Big(\sum_{i,j,p,s,k,l} \frac{\{1\}^{2k+2l}}{\{k\}!\{l\}!} q^{\frac{k(k-1)+l(l-1)}{2}} q^{\frac{2ij+2ps}{n^2}} \Bbbk_p \Bbbk_{i-nl} F^l E^k \otimes \Bbbk_s \Bbbk_{j+nl} E^l F^k\Big)\end{aligned}$$

Each summand is 0 unless $p = i - nl$ and $s = j + nl$, and after applying λ they are also 0 unless $k = l = r - 1$. We use that $\{r-1\}! = q^{r(r-1)/2} r$.

$$\begin{aligned}(\lambda \otimes Id)(R_{21}R_{12}) &= \sum_{i,j=0}^{mnr-1} \frac{\{1\}^{4(r-1)}}{cr^2} q^{\frac{2ij+2(i-n(r-1))(j+n(r-1))}{n^2}} q^{\frac{2(i-n(r-1))(r-1)}{n}} \Bbbk_{j+n(r-1)} E^{r-1}F^{r-1} \\ &= \frac{\{1\}^{4(r-1)}}{cr^2} q^{2(r-1)} \sum_{j=0}^{mnr-1} q^{\frac{-2(r-1)j}{n}} \left(\sum_{i=0}^{mnr-1} q^{\frac{4i(j+n(r-1))}{n^2}}\right) \Bbbk_{j+n(r-1)} E^{r-1}F^{r-1}\end{aligned}$$

The term in parenthesis is mnr if $j + n(r-1)$ is a multiple of $\frac{n^2 r}{gcd(n^2r,2)}$ and 0 otherwise. Let $m' = \frac{m\ gcd(n^2r,2)}{n}$. Finally,

$$(\lambda \otimes Id)(R_{21}R_{12}) = mn\frac{\{1\}^{4(r-1)}}{cr} \sum_{j=0}^{m'-1} (-1)^{\frac{-2jn(r-1)}{gcd(n^2r,2)}} \Bbbk_{\frac{jn^2r}{gcd(n^2r,2)}} E^{r-1}F^{r-1}.$$

Then g_H given by multiplication on the right by $\frac{c^2r}{mn\{1\}^{4(r-1)}}\Bbbk_0$ is a gluing morphism for H, i.e.:

$$\mathsf{g}_H \circ \Delta_0^H(1) = (\lambda \otimes \mathsf{g}_H)(R_{21}R_{12}) = \Bbbk_0 E^{r-1}F^{r-1} = \Lambda = \Lambda_H(1)$$

Write $P_{\mathbb{1}}$ as an idempotent $e_{P_{\mathbb{1}}} = i_{P_{\mathbb{1}}} \circ \pi_{P_{\mathbb{1}}}$ in H such that $\varepsilon \circ \pi_{P_{\mathbb{1}}}$ is the counit. The morphism $\mathsf{g} = \pi_{P_{\mathbb{1}}} \circ \mathsf{g}_H \circ i_{P_{\mathbb{1}}}$ is a gluing morphism by naturality of the Δ_0^P's and the Λ_P's. Hence $\mathcal{C}$ is always chromatic non-degenerate. Actually, $\varepsilon\circ = \varepsilon \circ \pi_{P_{\mathbb{1}}} \circ (- \cdot \frac{c^2r}{mn\{1\}^{4(r-1)}}\Bbbk_0) \circ i_{P_{\mathbb{1}}} = \frac{c^2r}{mn\{1\}^{4(r-1)}}\varepsilon$ as the counit is multiplicative and is 1 on $\Bbbk_0$. By Lemma 3.1.11, g is invertible, and by Lemma 3.2.6 $\mathcal{C}$ is chromatic compact.

As discussed in Section 3.1.6, $\mathcal{C}$ is modular if and only if $(\lambda \otimes Id)(R_{21}R_{12})$ is a scalar times Λ. This happens if and only if $m' = 1$, so if and only if $m = n$ and n and r are odd.

Let us check for twist non-degeneracy:

$$\begin{aligned}\Delta_- = \overline{\Delta_+} = \lambda(\theta) &= \lambda\left(K^{r-1}\sum_{k=0}^{r-1} \frac{\{1\}^{2k}}{\{k\}!} q^{\frac{k(k-1)}{2}} S(F^k)\left(\sum_{i=0}^{mnr-1} q^{\frac{-2i^2}{n^2}} \Bbbk_i\right) E^k\right) \\ &= \frac{\{1\}^{2(r-1)}}{cr} \sum_{i=0}^{mnr-1} (-1)^{r-1} q^{\frac{-2i^2}{n^2}} \lambda(K^{2r-2}F^{r-1}\Bbbk_i E^{r-1}) \\ &= \frac{\{1\}^{2(r-1)}}{(-1)^{r-1}cr} \sum_{i=0}^{mnr-1} q^{\frac{1}{n^2}\left(-2i^2+6n(r-1)(i-n(r-1))\right)} \\ &= \frac{\{1\}^{2(r-1)}}{(-1)^{r-1}cr} q^{-6(r-1)^2} \sum_{i=0}^{mnr-1} q^{\frac{2i}{n^2}(-i+3n(r-1))}\end{aligned}$$

This is a quadratic Gauss sum at a n^2r-th root of unity. They are well-studied, we are computing $G(1, -3n(r-1), n^2r)$ in the notations of [BD23, Appendix B]. It is recalled there that if $3n(r-1)$ is even, this vanishes if and only if $n^2r \equiv 2[4]$ which never happens. If $3n(r-1)$ is odd, this vanishes if and only if $4|n^2r$. Hence $\mathcal{C}$ is twist degenerate if and only if n is odd and $4|r$. □

The algebraic input in the following example is a generalization of the one used in [BD] where analogous computation was performed:

Proposition 3.5.5: *For n odd and $4|r$, the (3+1)-TQFT $\mathscr{S}_{\mathcal{C}}$ distinguishes the closed 4-manifolds $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, which have same signature, Euler characteristic and fundamental groups but different spin status. One has:*

$$\mathscr{S}_{\mathcal{C}}(S^2 \times S^2) = \frac{m^3 n \; gcd(nr,2)\{1\}^{8(r-1)}}{c^4 r^2} \quad \textit{and} \quad \mathscr{S}_{\mathcal{C}}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}) = 0$$

Proof : Both 4-manifolds can be obtained by a single 0 handle, two 2-handles and a single 4-handle. For $S^2 \times S^2$ the 2-handles form a Hopf link, whereas for $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ they are two disjoint ± 1-framed unknots. The 0-handle gives the skein $\zeta\Gamma_0$. Adding a red ± 1-framed unknot multiplies by $\Delta_\pm$, so by 0 here, and $\mathscr{S}_{\mathcal{C}}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}) = 0$. Adding a red Hopf link multiplies by $(\lambda\otimes\lambda)(R_{21}R_{12})$ which is $\frac{m^2 gcd(nr,2)\{1\}^{4(r-1)}}{c^2 r}$. The 4-handle evaluates Γ_0 to 1. So $\mathscr{S}_{\mathcal{C}}(S^2 \times S^2) = \zeta.(\lambda\otimes\lambda)(R_{21}R_{12}) = \frac{m^3 n \; gcd(nr,2)\{1\}^{8(r-1)}}{c^4 r^2}$. □

3.5.3 Characteristic p

We give an example of a category which is chromatic non-degenerate but not chromatic compact, and therefore gives a non-compact TQFT. The example we give is very simple and unlikely to give interesting 4-manifold invariant, but the TQFT already shows some very interesting features. Its associated algebra on $S^2 \times S^1$ is non-semisimple, so it does not fall under Reutter's theorem [Reu23] showing that semi-simple TQFTs cannot detect exotic structures.

The proof of Proposition 3.1.14 hints at this example. In characteristic p, one may find a cocommutative Hopf algebra H which is non-semisimple but such that H^* is semi-simple. This gives a symmetric monoidal, non-semisimple and chromatic non-degenerate category, therefore with non-semisimple Müger center.

Definition 3.5.6: Let $\Bbbk$ be an algebraically closed field of characteristic p, and $H = \Bbbk[\mathbb{Z}/p\mathbb{Z}]$. Denote α the generator of $\mathbb{Z}/p\mathbb{Z}$. Let $\mathcal{C} = H\text{-mod}^{fd}$ be the symmetric monoidal category of finite dimensional left H-modules. ♢

Proposition 3.5.7: *The category $\mathcal{C}$ is chromatic non-degenerate, but not chromatic compact. It gives a non-compact TQFT $\mathscr{S}_{\mathcal{C}}$.*

Proof : The cointegral is $\Lambda = \sum_{i=0}^{p-1} \alpha^i$, and the right integral is $\lambda = 1^*$ in the basis $(1, \alpha, \dots, \alpha^{p-1})$. We observe indeed that $\varepsilon(\Lambda) = p = 0$ whereas $\lambda(1) = 1 \neq 0$, so H^* is semi-simple whereas H is not. One computes the central element $\Delta_0 = \lambda(1)1 = 1 \in H$, thus $\Delta_0^P = \mathrm{Id}_P$ for any projective. Therefore, the gluing morphism g is given by $\Lambda_{P_{\mathbb{1}}}$ which is not invertible as $\mathcal{C}$ is non-semisimple. □

Note that $\mathsf{g} = \Lambda_{P_{\mathbb{1}}}$ means that the 1-handle map does not affect the skein. Similarly, $\mathcal{C}$ being symmetric and $\lambda(1) = 1$ implies that a homotopically-trivial red links can be ignored.
As explained in [Reu23], the vector space $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ has a natural algebra structure induced by the cobordism $\dddot{S}^3 \times S^1$ where $\dddot{S}^3$ is the thrice-punctured sphere. Note that this algebra is non-unital as the TQFT is non-compact.

Proposition 3.5.8: *The non-unital algebra $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ is non-semisimple (i.e. it is non-semisimple if one freely adjoins a unit).*

Proof : For $f : P \to P$ an endomorphism of a projective object, denote O_f the skein $\{pt\} \times S^1 \subseteq S^2 \times S^1$ colored by P with a single coupon f. The skein module of $S^2 \times S^1$ is generated by the O_f's. As the braiding and twist are trivial, the only relation is cyclicity: $O_{f\circ g} = O_{g \circ f}$ for $f : P \to Q$ and $g : Q \to P$. A handle decomposition of $\dddot{B}^3 \times S^1$ is given by a single 1-handle and a single 2-handle, both of which doesn't affect the skeins. The algebra structure is given by $O_f.O_g = O_{f \otimes g}$.
As H is a projective generator of the category, one can restrict to $P = H$ for the generators of $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$. Furthermore endomorphisms of H are right multiplications by elements of H, so since H is commutative, the cyclic relations are trivial. So $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ is isomorphic to $End_{\mathcal{C}}(H) \simeq H$ as a vector space, with basis the $O_i := O_{-.\alpha^i}$'s. To compute their product, we need to decompose $H \otimes H = \oplus_{k=0}^{p-1} H.(1 \otimes \alpha^k)$. Then $O_i.O_j$ is multiplication by $\alpha^i \otimes \alpha^j$ on $H \otimes H$. It maps $1 \otimes \alpha^k$ to $\alpha^i \otimes \alpha^{k+j}$ which is in the $k + j - i$ summand. We get $O_i.O_j = \sum_{k=0}^{p-1} \delta_{i,j} O_i = p\delta_{i,j} O_i = 0$.
If one freely adjoins a unit to $\mathscr{S}_{\mathcal{C}}(S^2 \times S^1)$ one gets the non-semisimple $(p+1)$-dimensional algebra $\mathbb{k}[O_0, O_1, \dots, O_{p-1}]/(O_i.O_j = 0)$. □

Chapter 4

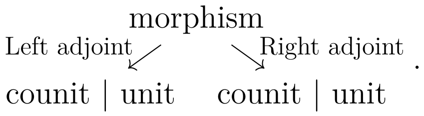
Anomalous Theories

This chapter is based on [Haï]. There are some changes, mostly that the confusion around dependence of choices of representants for $\mathcal{Z}_\mathcal{V}$ and $\mathcal{R}_\mathcal{V}$ in Section 4.3 is fixed, see Corollary 4.1.17.

The inclusion of the unit in a braided tensor category $\mathcal{V}$ induces a 1-morphism in the Morita 4-category of braided tensor categories **BrTens** described in Section 2.3. We give criteria for the dualizability of this morphism.

When $\mathcal{V}$ is a semisimple (resp. non-semisimple) modular category, we show that the unit inclusion induces under the Cobordism Hypothesis a (resp. non-compact) relative 3-dimensional topological quantum field theory. Following Jordan–Safronov, we conjecture that these relative field theories together with their bulk theories recover Witten–Reshetikhin–Turaev (resp. [DGG$^+$22]) TQFTs, in a fully extended setting. In particular, we argue that these theories can be obtained by the Cobordism Hypothesis.

In Section 4.1 we recall the definition of the oplax arrow category $\mathbf{BrTens}^{\rightarrow}$ and the various notions of dualizability for a 1-morphism. We give some general results dualizability data. Finally, we recall different versions of the cobordism hypothesis, associated with arrows, and explain how to extract a non-compact version from [Lur09b]. We define the associate notion of non-compact-n-dualizable.

In Section 4.2 we define bimodules associated with functors, and study their dualizability. We define $\mathcal{A}_\eta$ the 1-morphism in **BrTens** induced by the unit inclusion $\eta : \mathrm{Vect}_\Bbbk \to \mathcal{V}$ in Definition 4.2.2. We give its adjunctibility data explicitly, see Figure 4 using the notation 

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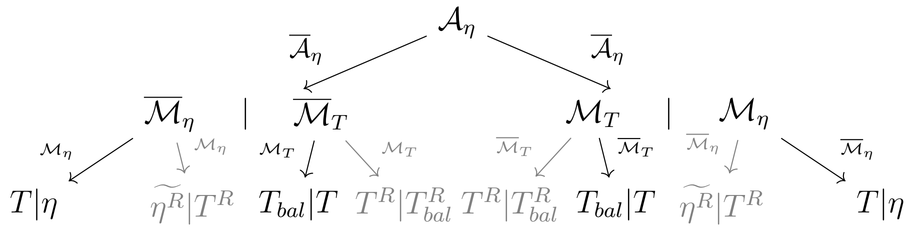


Figure 4.1: Adjunctibility data of the unit inclusion. The whole description (including gray) holds for $\mathcal{V}$ cp-rigid, see Proposition 4.2.20, and the black subset holds when $\mathcal{V}$ has enough compact-projectives, see Theorem 4.2.12.

Using this explicit dualizability data we derive the main results in Theorems 4.2.14 and

4.2.15. In particular, we obtain that the unit inclusion in a modular category induces a possibly non-compact relative 3-TQFT $\mathcal{R}_{\mathcal{V}}$. We compute its value on the circle.

In Section 4.3 we study the examples of interest to recover Witten–Reshetikhin–Turaev-type TQFTs. In Section 4.3.1 we define the category $\mathbf{Bord}_3^{filled}$ of filled cobordisms (equipped with a bounding higher manifold) and compare it to the usual "augmented" category of cobordisms $\widetilde{\mathbf{Cob}_3}$ on which WRT theories are defined. We explain how to compose $\mathcal{R}_{\mathcal{V}}$ and $\mathcal{Z}_{\mathcal{V}}$ to obtain a theory $\mathcal{A}_{\mathcal{V}} : \mathbf{Bord}_3^{filled} \to \mathbf{Tens} := \Omega\,\mathbf{BrTens}$. In Section 4.3.2 we recall the main results giving different tastes of extended TQFTs from a modular tensor category, and give conjectures to compare them. We also conjecture on sufficient data to induce orientation structures on $\mathcal{Z}_{\mathcal{V}}$ and $\mathcal{R}_{\mathcal{V}}$.

4.1 Relative and Non-compact TQFTs

In this chapter we will study the dualizability of a 1-morphism. What exact kind of dualizability we are interested in is dictated by the relative cobordism hypothesis: we want a 1-morphism that will induce a relative TQFT. It turns out that there are multiple notions of relative TQFTs, and therefore multiple interesting notions of dualizability for a 1-morphism.

Throughout, we will use the expression n-category to mean (∞, n)-category as described in Section 2.1. For $j \geq k$, we write $\circ_k$ for the composition of j-morphisms in the direction of k-morphisms. We write Id_f^k for taking k-times the identity of f.

4.1.1 Review of relative TQFTs

We recall the notions of relative TQFTs that will be our motivation. Let $\mathcal{C}$ be a symmetric monoidal n-category. We distinguish two flavors.

The first is purely topological. Lurie defines a new category $\mathbf{Bord}_n^{rel}$ of bipartite cobordisms with two different colors for the bulk and interfaces between them, see [Lur09b, Example 4.3.23]. There are in particular manifolds with only one color and without interfaces. This induces two inclusions $\mathbf{Bord}_n \to \mathbf{Bord}_n^{rel}$.

Definition 4.1.1 (Lurie): A domain wall between two theories $\mathcal{Z}_1, \mathcal{Z}_2 : \mathbf{Bord}_n \to \mathcal{C}$ is a symmetric monoidal functor $\mathbf{Bord}_n^{rel} \to \mathcal{C}$ that restricts to $\mathcal{Z}_1$ and $\mathcal{Z}_2$ on manifolds with one color. ♢

In particular, the interval with an interface point in the middle induces a morphism $\mathcal{Z}_1(pt) \to \mathcal{Z}_2(pt)$. Freed and Teleman describe a notion of relative TQFT by means of such morphisms for every values of $\mathcal{Z}_1$ and $\mathcal{Z}_2$ on manifolds of dimension strictly less than n, see [FT14]. They mention it should be equivalent.

The second notion focuses on the algebraic flavour of Freed–Teleman's description. One can drop the assumption that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are well defined on n-manifolds because these don't appear. Johnson-Freyd and Scheimbauer define three different notions of an n-category of arrows in an n-category. We will focus on the oplax one $\mathcal{C}^{\to}$.

Definition 4.1.2 (sketch, see Definition 5.14 in [JS17]): Let $\mathcal{C}$ be a symmetric monoidal n-category. The symmetric monoidal n-category $\mathcal{C}^{\to}$ of oplax arrows in $\mathcal{C}$ is defined as follows:

objects	:	triples $f = (s_f, t_f, f^\#)$ where s_f and t_f are objects of $\mathcal{C}$ and $f^\# : s_f \to t_f$ is a 1-morphism
1-morphisms $f \to g$	:	triples $h = (s_h, t_h, h^\#)$ where $s_h : s_f \to s_g$ and $t_h : t_f \to t_g$ are 1-morphisms, and $h^\# : g^\# \circ s_h \Rightarrow t_h \circ f^\#$ is a 2-morphism
		$\vdots$
k-morphisms $a \to b$	:	triples $f = (s_f, t_f, f^\#)$ where $s_f : s_a \to s_b$ and $t_f : t_a \to t_b$ are k-morphism in $\mathcal{C}$, and $f^\#$ is a $k+1$-morphism in $\mathcal{C}$ from the composition of some whiskerings of $b^\#$ and s_f to the composition of some whiskerings of t_f and $a^\#$.

It has two symmetric monoidal functor $s, t : \mathcal{C}^\to \to \mathcal{C}$.
A 1-morphism f of $\mathcal{C}$ can therefore be seen as an object of $\mathcal{C}^\to$. To avoid confusion, we will denote it $f^\flat$ when it is seen as an object of $\mathcal{C}^\to$, with $(f^\flat)^\# = f$. ♢

Definition 4.1.3 (Definition 5.16 in [JS17]): Let $\mathcal{C}$ be a symmetric monoidal n-category and $\mathcal{Z}_1, \mathcal{Z}_2 : \mathbf{Bord}_{n-1} \to \mathcal{C}$ two categorified $(n-1)$-TQFTs. An *oplax-$\mathcal{Z}_1$-$\mathcal{Z}_2$-twisted $(n-1)$-TQFT* is a symmetric monoidal functor

$$\mathcal{R} : \mathbf{Bord}_{n-1} \to \mathcal{C}^\to$$

such that $s(\mathcal{R}) = \mathcal{Z}_1$ and $t(\mathcal{R}) = \mathcal{Z}_2$. ♢

The name and strategy come from [ST11].

We will use the formalism of Johnson-Freyd and Scheimbauer in this chapter. For application, see Section 4.3, we are interested in the case where $\mathcal{Z}_1$ is the trivial theory and $\mathcal{Z}_2$ is well defined on n-manifolds. The two notions should then agree. The only argument we aware of to prove this relies on the cobordism hypothesis, and will be discussed below. If $\mathcal{Z} : \mathbf{Bord}_n \to \mathcal{C}$ is defined on n-manifolds, we will say oplax-$\mathcal{Z}$-twisted theory for oplax-Triv-$\mathcal{Z}|_{\mathbf{Bord}_{n-1}}$-twisted theory.

4.1.2 Dualizability data

Let us first recall multiple notions of dualizability and adjunctibility for morphisms in a symmetric monoidal n-category $\mathcal{C}$.
Following [Lur09b], where one often assumes $\mathcal{C}$ to have duals, one defines:

Definition 4.1.4: A k-morphism f of $\mathcal{C}$ is said *m-dualizable* if it lies in a sub-n-category with duals up to level $m+k$. It is called *fully dualizable* if it is $n-k$-dualizable. ♢

Following [JS17] one gets a few more notions. For simplicity we focus on 1-morphisms.

Definition 4.1.5: A 1-morphism $f : X \to Y$ of $\mathcal{C}$ is said *m-oplax-dualizable* if it is m-dualizable as an object $f^\flat$ of $\mathcal{C}^\to$. It is said *m-lax-dualizable* if it is m-dualizable as an object of $\mathcal{C}^\downarrow$, where $\mathcal{C}^\downarrow$ is the category of lax arrows defined in [JS17, Definition 5.14]. ♢

Definition 4.1.6: A k-morphism f is said to be *left* (resp. *right*) *adjunctible* if it has a left (resp. right) adjoint, and *adjunctible* if it has arbitrary left and right adjoints ($(f^L)^L$, $(f^R)^R$ and so on...). It is said to be *m-times* (resp. *left*, *right*) *adjunctible* if it is $m-1$-times (resp. left, right) adjunctible and every unit/counit witnessing this are themselves (resp. left, right) adjunctible. We sometimes abbreviate m-times adjunctible as m-adjunctible. ♢

Note that being (left, right) adjunctible is only a condition on the morphism while being (lax, oplax) dualizable is also a condition on its source and target.

Theorem 4.1.7 (Theorem 7.6 in [JS17]): *A 1-morphism $f : X \to Y$ of $\mathcal{C}$ is m-oplax-dualizable if and only if X and Y are both m-dualizable and f is m-times right adjunctible.*
Similarly, it is m-lax-dualizable if and only if X and Y are both m-dualizable and f is m-times left adjunctible.

Similarly, a 1-morphism $f : X \to Y$ is m-dualizable if and only if it is m-times adjunctible and its source and targets are $m+1$-dualizable. Indeed we only have to check that the subcategory generated by the dualizability data of f, X and Y still has duals up to level $m+1$. Put differently, that the various compositions of adjunctible morphisms stays adjunctible.

Redundancy in the dualizability data

The dualizability data of a morphism grows very fast: there are four units/counits for the left and right adjunctions, and this does not consider taking the right adjoint of the right adjoint and so on. In particular, checking n-adjunctibility of a morphism seems tedious. It turns out that there is a lot of redundancy in this data, especially if we are only interested in dualizability properties.

Let us begin with some notations. Let f be a k-morphism in an n-category. We say that *Radj*(f) (resp. *Ladj*(f)) exists if f has a right (resp. left) adjoint, in which case we denote this adjoint *Radj*(f) (resp. *Ladj*(f)), and the unit and counit of the adjunction *Ru*(f) and *Rco*(f) (resp. *Lu*(f) and *Lco*(f)).

We write $\underset{Rco(f)|Ru(f)}{\overset{f}{\searrow}}{}^{Radj(f)}$ (resp. $\underset{Lco(f)|Lu(f)}{{}^{Ladj(f)}\swarrow^{f}}$). Note that these are only defined up to some isomorphisms, and the notation stands for any choice.

Definition 4.1.8: We say that two k-morphisms f and g have same dualizability properties, which we denote $f \doteq g$, if for every finite sequence

$$(a_i)_{i\in\{1,\dots,m\}},\ a_i \in \{Radj,\ Ladj,\ Ru,\ Rco,\ Lu,\ Lco\},$$
$$a_m(\dots a_2(a_1(f))\dots) \text{ exists if and only if } a_m(\dots a_2(a_1(g))\dots)\text{exists},$$

and this for any choice of adjoints, units and counits. ♢

We will show that dualizability properties are preserved by isomorphisms and "higher mating" defined in Definition 4.1.10. Let us describe formally this second notion.

Proposition 4.1.9: *Let $f : x \to y$ be a k-morphism in an n-category $\mathcal{C}$ with adjoint (f^R, ε, η). Then for any other k-morphisms $g : z \to x$ and $h : z \to y$, one has an equivalence of $n-k-1$-categories of $k+1$-morphisms:*

$$\Phi^f_{g,h} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(f \circ_k g, h) & \tilde{\to} & \mathrm{Hom}_{\mathcal{C}}(g, f^R \circ_k h) \\ N : f \circ_k g \to h & \mapsto & (Id_{f^R} \circ_k N) \circ_{k+1} (\eta \circ_k Id_g) \\ k+j\text{-morphism } \alpha & \mapsto & (Id^j_{f^R} \circ_k \alpha) \circ_{k+1} (Id^{j-1}_\eta \circ_k Id^j_g) \end{array} \right\}$$

Similarly, for any $g : x \to z$ and $h : y \to z$, one gets an equivalence:

$$\Psi^f_{g,h} = (- \circ_k Id_f) \circ_{k+1} (Id_g \circ_k \eta) : \mathrm{Hom}_{\mathcal{C}}(g \circ_k f^R, h) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(g, h \circ_k f)$$

Proof : Its inverse is given by:

$$(\Phi^f_{g,h})^{-1} : \left\{ \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(g, f^R \circ_k h) & \tilde{\to} & \mathrm{Hom}_{\mathcal{C}}(f \circ_k g, h) \\ M : g \to f^R \circ_k h & \mapsto & (\varepsilon \circ_k Id_h) \circ_{k+1} (Id_f \circ_k M) \\ k+j\text{-morphism } \beta & \mapsto & (Id^{j-1}_{\varepsilon} \circ_k Id^j_h) \circ_{k+1} (Id^j_f \circ_k \beta) \end{array} \right\}$$

The composition $\Phi^f_{g,h} \circ (\Phi^f_{g,h})^{-1}$ (resp. $(\Phi^f_{g,h})^{-1} \circ \Phi^f_{g,h}$) is post- (resp. pre-) composition by a snake identity. Similarly, $(\Psi^f_{g,h})^{-1} = (Id_g \circ_k \varepsilon) \circ_{k+1} (- \circ_k Id_{f^R})$. □

Definition 4.1.10: For a $k+1$-morphism $N : f \circ_k g \to h$, we say that N and $\Phi^f_{g,h}(N)$ are *mates*. For a higher morphism α in $\mathrm{Hom}_{\mathcal{C}}(f \circ_k g, h)$, we say that α and $\Phi^f_{g,h}(\alpha)$ are *higher mates*. Similarly, for $N,\ \alpha$ in $\mathrm{Hom}_{\mathcal{C}}(g \circ_k f^R, h)$ we call N and $\Psi^f_{g,h}(N)$ mates, and α and $\Psi^f_{g,h}(\alpha)$ higher mates. More generally we say that N and M are mates (resp. α and β are higher mates) if they can be linked by a chain of matings (resp. higher matings) and isomorphisms.
For a k-morphism f, we say that g is *obtained from f by whiskering* if it can be written as a composition of f with identities of lower morphisms. Note that if α and β are higher mates, their are both obtained from the other by whiskering. ♢

Proposition 4.1.11: *Let f and g be k-morphisms in an n-category. Then:*

1. $f \doteq f$.
2. *If $f \overset{\varphi}{\simeq} g$ are isomorphic, then $f \doteq g$.*
3. *If $f = g \circ_k h$ for an isomorphism h, then $f \doteq g$.*
4. *If f and g are higher mates, then $f \doteq g$.*

Proof : What we have to prove for point 1 is that existence of higher adjoints in the adjunctibility data does not depend on the choices made in the adjunctions. It is an a inductive consequence of the following, and the fact that adjoints are unique up to isomorphism, and counits/units by pre/post composing with this isomorphism.

2. If $f \overset{\varphi}{\simeq} g$ are isomorphic, then f has a right (resp. left) adjoint if and only if g does, in which case one can choose $Radj(g) = Radj(f)$, $Ru(g) = (Id_{Radj(f)} \circ_k \varphi) \circ_{k+1} Ru(f)$ and $Rco(g) = (\varphi^{-1} \circ_k Id_{Radj(f)}) \circ_{k+1} Rco(f)$.

3. If $f = g \circ_k h$ is obtained as a composition, then f has a right (resp. left) adjoint as soon as g and h do, in which case one can choose $Radj(f) = Radj(h) \circ_k Radj(g)$, $Ru(f) = (Id_{Radj(h)} \circ_k Ru(g) \circ_k Id_h) \circ_{k+1} Ru(h)$ and $Rco(f) = (Id_g \circ_k Rco(h) \circ_k Id_{Radj(g)}) \circ_{k+1} Rco(h)$. In particular, if h is an isomorphism, then $g \simeq f \circ_k h^{-1}$, and f has a right (resp. left) adjoint if and only if g does.

4. If $f = g \circ_j h$ is obtained as a composition in the direction of j-morphisms for $j < k$, then f has a right (resp. left) adjoint as soon as g and h do, in which case one can choose $Radj(f) = Radj(g) \circ_j Radj(h)$, $Ru(f) = Ru(g) \circ_j Ru(h)$ and $Rco(f) = Rco(g) \circ_j Rco(h)$. In particular, if h is an identity of a lower morphism, then f has a right (resp. left) adjoint as soon as g does. So, if f and g are higher mates, they both can be obtained as composition of the other with identities of lower morphisms, and f has a right (resp. left) adjoint if and only if g does.

Every point follows by induction. □

We can now describe the redundancy in the dualizability data:

Proposition 4.1.12: *Let f be a k-morphism in an n-category $\mathcal{C}$, suppose that $Radj(f)$, $Radj(Rco(f))$ and $Radj(Ru(f))$ exist, then:*

1. *f is 1-adjunctible, and one can choose $Ladj(f) = Radj(f)$, $Lu(f) = Radj(Rco(f))$ and $Lco(f) = Radj(Ru(f))$.*
2. *$Rco(Ru(f)) \doteq Ru(Rco(f))$.*

Suppose moreover than $Radj(Lco(f))$ and $Radj(Lu(f))$ exist, then:

3. *f is 2-adjunctible, and $Rco(f) \doteq Radj(Radj(Rco(f)))$ and $Ru(f) \doteq Radj(Radj(Ru(f)))$.*

In particular if $f = X$ is an object in a symmetric monoidal n-category, then:

4. *X is 1-adjunctible if and only if it has a dual. It is 2-adjunctible if and only if $ev_X := Rco(X)$ and $coev_X := Ru(X)$ have right adjoints. More generally, it is m-adjunctible if and only if $Radj(Rco^k(Ru^{m-1-k}(X)))$ exist for all $0 \leq k \leq m-1$.*

Proof : Point 1 is [Lur09b, Remark 3.4.22], or [Sch14b, Lemma 20.1]. One directly checks that the right adjoints of the right counit and unit satisfy the snake relations (because taking right adjoints behaves well with composition) and exhibit $Radj(f)$ as the left adjoint of f.
Point 2 is [Lur09b, Proposition 3.4.21]. It is shown that $Rco(Ru(f))$ and $Ru(Rco(f))$ are higher mates, so in particular $Rco(Ru(f)) \doteq Ru(Rco(f))$.
Point 3 is [JS17, Lemma 7.11]. One applies point 1 twice and observes a redundancy: we have two right adjunctions for f, which are therefore isomorphic.
Point 4 is [JS17, Corollary 7.12] and point 2. It uses the fact that $\mathcal{C}$ is symmetric, and therefore the right and left adjoints of an object agree. Point 3 applies automatically, and point 1 enables to move right adjunctibility to left adjunctibility properties. Using point 2, we know that Ru and Rco commute as far as existence of adjoints is concerned, so there are only m different $m-1$-morphisms whose adjunctibility should be checked, $Rco^k(Ru^{m-1-k}(X))$, $0 \leq k \leq m-1$. □

Oplax dualizability data

We investigate the proof of Theorem 4.1.7 and explain how to get from adjunctibility data in $\mathcal{C}$ to dualizability data in $\mathcal{C}^{\rightarrow}$.

Theorem 4.1.13 (Johnson-Freyd–Scheimbauer): *Let $f = (s_f, t_f, f^{\#}) : a = (s_a, t_a, a^{\#}) \to b = (s_b, t_b, b^{\#})$ be a k-morphism in $\mathcal{C}^{\rightarrow}$ so $s_f : s_a \to s_b$ and $t_f : t_a \to t_b$ are k-morphism in $\mathcal{C}$, and $f^{\#}$ is a $k+1$-morphism in $\mathcal{C}$ from the composition of some whiskerings of $b^{\#}$ and s_f to the composition of some whiskerings of t_f and $a^{\#}$. Then:*

f has a right adjoint in $\mathcal{C}^{\rightarrow}$ if and only if s_f, t_f and $f^{\#}$ have right adjoints in $\mathcal{C}$.

In this case:

- $Radj(f) = (Radj(s_f), Radj(t_f), g)$ *where* g *is a mate of* $Radj(f^{\#})$,
- $Ru(f) = (Ru(s_f), Ru(t_f), u)$ *where* u *is a higher mate of* $Rco(f^{\#})$, *and*
- $Rco(f) = (Rco(s_f), Rco(t_f), v)$ *where* v *is a higher mate of* $Ru(f^{\#})$.

In particular, if we only look at the right dualizability data, and only take right adjoints once, then:

$$\forall i,j \in \mathbb{N},\ Radj(Rco^i(Ru^j(f))) \text{ exists if and only if } Radj(Rco^i(Ru^j(s_f))),\ Radj(Rco^i(Ru^j(t_f))) \text{ and } Radj(Ru^i(Rco^j(f^{\#}))) \text{ exist.}$$

Proof : The description of the right adjunctibility of a morphism in $\mathcal{C}^{\to}$ is [JS17, Proposition 7.13], in the oplax case.
For the last statement, we use that higher mates have same dualizability properties. Note that we can only take right adjoints once, because mates do not. ☐

Example 4.1.14 ($k = 0$): An object $f = (X, Y, A : X \to Y)$ of $\mathcal{C}^{\to}$ is dualizable if and only if X and Y are dualizable, and A has a right adjoint $Radj(A)$. Then:

- $f^* = (X^*, Y^*, Radj(A)^* := (Id_{Y^*} \otimes ev_X) \circ (Id^*_Y \otimes Radj(A) \otimes Id_{X^*}) \circ (coev_Y \otimes Id_{X^*}))$,
- $coev_f = (coev_X, coev_Y, (Rco(A) \otimes Id_{Id_{Y^*}}) \circ_1 Id_{coev_Y})$, and
- $ev_f = (ev_X, ev_Y, Id_{ev_X} \circ_1 (Ru(A) \otimes Id_{Id_{X^*}}))$.

A surprising consequence of this is that if f is 2-dualizable, the right counit and unit of A are biadjoints up to isomorphisms and mating. A drawing for this is given in Figure 4.3. ♢

4.1.3 Cobordism Hypotheses

The cobordism hypothesis described in Section 2.2.2 generalizes in various directions. We recall relative versions that describes relative TQFTs, and a non-compact version that describes partially-defined TQFTs.

The relative Cobordism Hypothesis

Lurie proposes a result classifying his notion of domain wall.

Conjecture 4.1.15 (Theorem 4.3.11 and Example 4.3.23 in [Lur09b]): *Let* $\mathcal{C}$ *be a symmetric monoidal n-category with duals and* $X, Y \in \mathcal{C}$. *There is a bijection between isomorphism classes of framed domain walls between* $\mathcal{Z}_X$ *and* $\mathcal{Z}_Y$ *and isomorphism classes of 1-morphisms* $f : X \to Y$, *given by evaluation at the interval with an interface point in the middle. Here* $\mathcal{Z}_X$ *and* $\mathcal{Z}_Y$ *are the fully extended framed TQFTs associated with* X *and* Y, *which are fully dualizable as we assumed that* $\mathcal{C}$ *has duals.*
In particular if one drops the assumption that $\mathcal{C}$ *has duals, then a 1-morphism* $f : X \to Y$ *induces a framed domain wall as soon as it is fully dualizable.*

There is an oriented version asking that f preserves orientation structures.
On the other hand, [JS17]'s notions of a twisted quantum field theory are already classified by the usual Cobordism Hypothesis. Note however that [JS17, Definition 5.16] is surprisingly strict because it demands that the source and target of the functor $\mathcal{R} : \mathbf{Bord}_{n-1} \to \mathcal{C}$ agree strictly with $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Equivalently, we could have asked that $\mathcal{R}$ comes equipped with isomorphisms $s(\mathcal{R}) \simeq \mathcal{Z}_1$ and $t(\mathcal{R}) \simeq \mathcal{Z}_2$. In both cases, it is clear that the Cobordism Hypothesis does not apply on the nose. The fix is easy.

Definition 4.1.16: Let $\mathcal{C}$ be a symmetric monoidal n-category and $X, Y \in \mathcal{C}$. Denote $(\mathcal{C}^{\to})^{\sim}_{X,Y}$ the homotopy pullback

$$\begin{array}{ccc} (\mathcal{C}^{\to})^{\sim}_{X,Y} & \longrightarrow & (\mathcal{C}^{\to})^{\sim} \\ \big\downarrow & \lrcorner h & \big\downarrow{\scriptstyle s,t} \\ * & \xrightarrow{X,Y} & (\mathcal{C}^{\sim})^{\times 2} \end{array} .$$

Similarly, for $\mathcal{Z}_1, \mathcal{Z}_2 : \mathbf{Bord}^{fr}_{n-1} \to \mathcal{C}$ denote $\mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C}^{\to})_{\mathcal{Z}_1,\mathcal{Z}_2}$ the homotopy pullback

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C}^{\to})_{\mathcal{Z}_1,\mathcal{Z}_2} & \longrightarrow & \mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C}^{\to}) \\ \big\downarrow & \lrcorner h & \big\downarrow{\scriptstyle s,t} \\ * & \xrightarrow{\mathcal{Z}_1,\mathcal{Z}_2} & (\mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C}))^{\times 2} \end{array}$$

called the space of framed oplax-$\mathcal{Z}_1$-$\mathcal{Z}_2$-twisted-$(n-1)$-TQFTs.
Note that both are also strict pullbacks as taking source and target induces a fibration of spaces. ♢

Corollary 4.1.17 (of the Cobordism Hypothesis): *Let $\mathcal{C}$ be a symmetric monoidal n-category and $X, Y \in \mathcal{C}$. Choose $\mathcal{Z}_X, \mathcal{Z}_Y : \mathbf{Bord}^{fr}_{n-1} \to \mathcal{C}$ two TQFTs associated with X and Y by the cobordism hypothesis. Evaluation at the point induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C}^{\to})_{\mathcal{Z}_X,\mathcal{Z}_Y} \simeq (\mathcal{C}^{\to})^{\sim}_{X,Y} .$$

Proof : The cobordism hypothesis on $\mathcal{C}$ and $\mathcal{C}^{\to}$ gives a commutative diagram of horizontal equivalences

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C}^{\to}) & \xrightarrow{ev_{pt}} & (\mathcal{C}^{\to})^{\sim} \\ \big\downarrow{\scriptstyle s,t} & & \big\downarrow{\scriptstyle s,t} \\ (\mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C}))^{\times 2} & \xrightarrow{ev_{pt} \times ev_{pt}} & (\mathcal{C}^{\sim})^{\times 2} \\ {\scriptstyle \mathcal{Z}_X,\mathcal{Z}_Y}\big\uparrow & & {\scriptstyle X,Y}\big\uparrow \\ * & \longrightarrow & * \end{array}$$

inducing an equivalence between homotopy pullbacks. □

Remark 4.1.18: There is an oriented version as well. The maps $s, t : \mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C}^{\to}) \to \mathrm{Fun}^{\otimes}(\mathbf{Bord}^{fr}_{n-1}, \mathcal{C})$ are $SO(n-1)$-equivariant because $SO(n-1)$ acts on the source $\mathbf{Bord}^{fr}_{n-1}$. Therefore the maps $s, t : (\mathcal{C}^{\to})^{\sim} \to \mathcal{C}^{\sim}$ are also equivariant, and descend to maps between the $SO(n-1)$-homotopy-fixed-points $s, t : (\mathcal{C}^{\to})^{\sim, SO(n-1)} \to \mathcal{C}^{\sim, SO(n-1)}$.

Given two objects $X, Y \in \mathcal{C}$ equipped with $SO(n-1)$-homotopy-fixed point structure, one can reproduce exactly the whole paragraph above and define $(\mathcal{C}^{\to})^{\sim,SO(n-1)}_{X,Y}$ as a pullback. We get

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_{n-1}, \mathcal{C}^{\to})_{\mathcal{Z}_X,\mathcal{Z}_Y} \simeq (\mathcal{C}^{\to})^{\sim,SO(n-1)}_{X,Y}$$

by the same proof, using the oriented cobordism hypothesis. $\diamond$

Remark 4.1.19: To appear results of Will Stewart show that if we assume that the source and target objects X and Y are fully dualizable then a morphism $f : X \to Y$ is $(n-1)$-oplax dualizable if and only if it is $(n-1)$-dualizable. In particular, if we restrict the notion of oplax twisted TQFTs to the case where the "twisting" theories $\mathcal{Z}_1$ and $\mathcal{Z}_2$ extend to $\mathbf{Bord}_n$, then this notion, using the cobordism hypothesis twice, is equivalent to Lurie's notion of domain walls. $\diamond$

Non-compact TQFTs

To study non-semisimple variants of Witten–Reshetikhin–Turaev TQFTs, we will be interested in theories defined on a restricted class of cobordisms, namely where top-dimensional cobordisms have non-empty outgoing boundary in every connected component.
Lurie's sketch of proof of the cobordism hypothesis is done by induction on the handle indices allowed. One starts with only opening balls, then allows more and more complex cobordisms. Eventually one allows every cobordisms but closing balls, namely cobordisms with outgoing boundary in every connected component. Finally one allows every cobordism, and obtain a TQFT. We call it a non-compact TQFT when we stop at this ante-last step. Lurie's proof then gives an algebraic criterion classifying these.
We follow [Lur09b, Section 3.4] and state the results there in a form fitted for our use. It should be noted that the proofs of the statements below are not very formal.

Definition 4.1.20: Let $\mathbf{Bord}_n^{fr,nc} \subseteq \mathbf{Bord}_n^{fr}$ denote the subcategory where n-dimensional bordisms have non-empty outgoing boundary in every connected component.
A *framed fully extended non-compact n-TQFT* with values in a symmetric monoidal n-category $\mathcal{C}$ is a symmetric monoidal functor $\mathcal{Z} : \mathbf{Bord}_n^{fr,nc} \to \mathcal{C}$. $\diamond$

Lurie defines in [Lur09b, Definition 3.4.9] an n-category $\mathcal{F}_k$ of $\leq n$-dimensional bordisms where all n-manifolds are equipped with a decomposition into handles of index $\leq k$. Here bordisms are actually equipped with a framed function without certain kinds of critical points.
We denote $\alpha_k^m = D^k \times D^{m-k} : S^{k-1} \times D^{m-k} \to D^k \times S^{m-k-1}$ the m-dimensional index k handle attachment, seen as an m-morphism in $\mathbf{Bord}_m^{fr}$, or in $\mathcal{F}_k$ if $m = n$. Let $x = S^{k-2} \times D^{n-k}$, $y = D^{k-1} \times S^{n-k-1}$ seen as $n-2$-morphisms $\emptyset \to S^{k-2} \times S^{n-k-1}$ in $\mathbf{Bord}_{n-1}^{fr}$. Note that for $1 \leq k \leq n$, $\alpha_{k-1}^{n-1} : x \to y$ and $\alpha_{n-k}^{n-1} : y \to x$. Then, α_{k-1}^n can be seen (up to higher mating) as a morphism $Id_x \to \alpha_{n-k}^{n-1} \circ \alpha_{k-1}^{n-1}$ and α_k^n as a morphism $\alpha_{k-1}^{n-1} \circ \alpha_{n-k}^{n-1} \to Id_y$, and they form a unit/counit pair in $\mathcal{F}_k$, see [Lur09b, Claim 3.4.17].

Namely, $Radj(\alpha_{k-1}^{n-1}) = \alpha_{n-k}^{n-1}$, $Ru(\alpha_{k-1}^{n-1}) = \alpha_{k-1}^n$ and $Rco(\alpha_{k-1}^{n-1}) = \alpha_k^n$, or

$$\begin{array}{ccc} \alpha_{k-1}^{n-1} & & \\ & \searrow & \alpha_{n-k}^{n-1} \\ & \alpha_k^n \mid \alpha_{k-1}^n & \end{array}.$$

By induction, $Rco^k(Ru^{m-k}(pt)) = \alpha_k^m$.

Conjecture 4.1.21 (Index-k cobordism hypothesis, Lurie): *A symmetric monoidal functor $\mathcal{Z}_0 : \mathbf{Bord}_{n-1}^{fr} \to \mathcal{C}$ extends to $\mathcal{Z} : \mathcal{F}_k \to \mathcal{C}$, $1 \leq k \leq n$, if and only if the images of every $n-1$-dimensional handle of index $\leq k-1$ is right adjunctible. This extension is essentially unique.*

Proof (Sketch): For $k = 0$, one can extend $\mathcal{Z}_0 : \mathbf{Bord}_{n-1}^{fr} \to \mathcal{C}$ with any n-morphism $\mathcal{Z}(\alpha_0^n) : 1 \to \mathcal{Z}_0(S^{n-1})$, see [Lur09b, Claim 3.4.13] (note that Lurie works in the unoriented case there, and demands on $O(n)$-equivariant morphism, and we look at the framed case). Now, for $1 \leq k \leq n$, a symmetric monoidal functor $\mathcal{Z}_0 : \mathcal{F}_{k-1} \to \mathcal{C}$ extends to $\mathcal{Z} : \mathcal{F}_k \to \mathcal{C}$ if and only if α_{k-1}^n is mapped to a unit of an adjunction between α_{k-1}^{n-1} and α_{n-k}^{n-1}, see [Lur09b, Proposition 3.4.19]. In this case, the extension is essentially unique, and α_k^n is mapped to the counit of the adjunction.
For $k = 1$, this gives little choice for the n-morphism $\mathcal{Z}(\alpha_0^n)$, it has to be the unit of an adjunction. Then, α_1^n will be sent to the counit.
For $k \geq 2$, we want $\mathcal{Z}(\alpha_{k-1}^n)$, which is so far defined as the counit of the adjunction between $\mathcal{Z}(\alpha_{k-2}^{n-1})$ and $\mathcal{Z}(\alpha_{n-k+1}^{n-1})$, to be also the unit of the adjunction between $\mathcal{Z}(\alpha_{k-1}^{n-1})$ and $\mathcal{Z}(\alpha_{n-k}^{n-1})$. This in particular implies that the $n-1$-dimensional handle of index $k-1$ is right adjunctible, as stated in the conjecture. For the converse, we use [Lur09b, Proposition 3.4.20] which states that provided the adjunction exists, α_{k-1}^n must map to the unit. This exploits some redundancy in the dualizability data, namely Proposition 4.1.12 point 2. □

Definition 4.1.22: Let $\mathcal{C}$ be a symmetric monoidal n-category. An object X in $\mathcal{C}$ is said (n,k)-dualizable if it is $n-1$-dualizable and the k following $n-1$-morphisms $Ru^{n-1}(X)$, $Rco(Ru^{n-2}(X))$, …, $Rco^{k-1}(Ru^{n-k}(X))$ have right adjoints. We say X is non-compact-n-dualizable if it is $(n, n-1)$-dualizable.
For example, for $n = 3$, $k = 2$, we want X to have a dual $(X^*, ev_X, coev_X)$, both its evaluation and coevaluation maps to have right adjoints (ev_X^R, a, b) and $(coev_X^R, c, d)$, and the unit and counit of the right adjunction of the coevaluation to have right adjoints c^R and d^R. ♢

Conjecture 4.1.23 (Non-compact Cobordism Hypothesis): *Let $\mathcal{C}$ be a symmetric monoidal n-category. There is a bijection between isomorphism classes of framed fully extended non-compact n-TQFTs with values in $\mathcal{C}$ and isomorphism classes of non-compact-n-dualizable objects of $\mathcal{C}$, given by evaluation at the point.*

Proof (Sketch): We apply the index-k cobordism hypothesis for $k = n-1$, and obtain a symmetric monoidal functor $\mathcal{F}_{n-1} \to \mathcal{C}$. This is not exactly what we want, as $\mathcal{F}_{n-1}$ is indeed $\mathbf{Bord}_n^{fr,nc}$ but with the extra data of a framed function whose critical point have index $< n$. Lurie proves that the forgetful functor $\mathcal{F}_n = \mathbf{Bord}_n^{ff} \to \mathbf{Bord}_n^{fr}$ is an equivalence, see [Lur09b, Section 3.5]. The same proof should apply to $\mathcal{F}_{n-1} \to \mathbf{Bord}_n^{fr,nc}$. □

4.2 Dualizability of the unit inclusion

Remember the definition of $\mathbf{Alg}_2(\mathbf{Pr})$ from Section 2.3 and its explicit description under the name $\mathbf{BrTens}$. Let $\mathcal{V} \in \mathbf{BrTens}$ be a braided tensor category. We consider the inclusion of the unit $\eta : \mathrm{Vect}_{\Bbbk} \to \mathcal{V}$. It is a braided monoidal functor and we define an associated $\mathrm{Vect}_{\Bbbk}$-$\mathcal{V}$-central algebra $\mathcal{A}_\eta$, which is simply the category $\mathcal{V}$ seen as the regular right $\mathcal{V}$-module, see Definition 4.2.2. We study the dualizability of this 1-morphism in **BrTens**. First, we recall some context and develop some properties of bimodules induced by functors. Then we describe all the dualizability data explicitly and give criteria for dualizability.

Let us recall previously known results about the dualizability of $\mathcal{A}_\eta$. The following is [GS, Theorem 5.1], [BJS21, Theorem 5.16] and [BJS21, Theorem 5.21] respectively.

Theorem 4.2.1: *The 1-morphism $\mathcal{A}_\eta$ is always 1-dualizable. It is 2-dualizable as soon as $\mathcal{V}$ is cp-rigid, and 3-dualizable as soon as $\mathcal{V}$ is fusion.*

Note that the requirement fusion can easily be relaxed to rigid finite semisimple (without the assumption that the unit is simple), see the proof of Theorem 4.2.16.

4.2.1 Bimodules induced by functors

We give basic definitions and facts about bimodules induced by (braided) monoidal functors, and show how to compute their adjoints.

Definition and coherence

We show that the notion of bimodules induced by functors behaves as expected in **BrTens**. Namely, the Morita category, whose morphisms are bimodules, extends the category whose morphisms are functors.

Definition 4.2.2: Let $\mathcal{A}$ and $\mathcal{B}$ be two objects of **BrTens**. A braided monoidal functor $F : \mathcal{A} \to \mathcal{B}$ induces an $\mathcal{A}$-$\mathcal{B}$-central algebra $\mathcal{A}_F$ which is given by $\mathcal{B}$ as a monoidal category on which $\mathcal{A}$ acts on the top using $F(-) \otimes -$ and $\mathcal{B}$ acts on the bottom using $- \otimes -$. More formally its structure of $\mathcal{A}$-$\mathcal{B}$-central algebra is given by:

$$\begin{array}{rcl} \mathcal{A} \boxtimes \mathcal{B}^{\sigma op} & \to & Z(\mathcal{B}) \\ (A, B) & \mapsto & (F(A) \otimes B, (Id_{F(A)} \otimes \sigma^{-1}_{B,-}) \circ (\sigma_{-,F(A)} \otimes Id_B)) \end{array}$$

where σ is the braiding in $\mathcal{B}$. It is braided monoidal because F is braided monoidal.
It also induces a $\mathcal{B}$-$\mathcal{A}$-central algebra $\overline{\mathcal{A}}_F$ which is also given by $\mathcal{B}$ as a monoidal category on which $\mathcal{A}$ acts on the bottom using $- \otimes F(-)$ and $\mathcal{B}$ acts on the top using $- \otimes -$.
When the functor F is understood, we may write ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$ for $\mathcal{A}_F$ and ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}}$ for $\overline{\mathcal{A}}_F$ $\diamondsuit$

Figure 4.2: The 1-morphisms $\mathcal{A}_F$ and $\overline{\mathcal{A}}_F$

Proposition 4.2.3: *The above induced-central-algebra construction preserves composition. Given two braided monoidal functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, one has $\mathcal{A}_G \circ \mathcal{A}_F \simeq \mathcal{A}_{G \circ F}$ and $\overline{\mathcal{A}}_F \circ \overline{\mathcal{A}}_G \simeq \overline{\mathcal{A}}_{G \circ F}$.*

Proof : We want to prove that ${}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}} \underset{\mathcal{B}}{\boxtimes} {}_{\mathcal{B}}\mathcal{C}_{\mathcal{C}} \simeq {}_{\mathcal{A}}\mathcal{C}_{\mathcal{C}}$. This is true on the underlying categories as $\mathcal{B} \underset{\mathcal{B}}{\boxtimes} \mathcal{C} \overset{\Phi}{\simeq} \mathcal{C}$ with equivalence given on pure tensors by $\Phi(B \boxtimes C) = G(B) \otimes C$. This assignment is balanced as G is monoidal:

$$\Phi((B \otimes B') \boxtimes C) = G(B \otimes B') \otimes C \simeq G(B) \otimes G(B') \otimes C = \Phi(B \boxtimes (G(B') \otimes C)).$$

It is monoidal (the monoidal structure on the relative tensor product is described in [BJS21, Definition-Proposition 3.6]) by:

$$\Phi(B\boxtimes C)\otimes\Phi(B'\boxtimes C') = G(B)\otimes C\otimes G(B')\otimes C' \underset{\sim}{\xrightarrow{\sigma_{C,B'}}} G(B)\otimes G(B')\otimes C\otimes C' \simeq \Phi((B\boxtimes C)\otimes(B'\boxtimes C')).$$

The bottom action of $\mathcal{C}$ is unchanged, and the top action of $\mathcal{A}$ is preserved by Φ:

$$A\triangleright(B\boxtimes C) := (A\triangleright\mathbb{1})\otimes(B\boxtimes C) = (F(A)\otimes B)\boxtimes C \overset{\Phi}{\mapsto} G(F(A))\otimes G(B)\otimes C = A\triangleright\Phi(B\boxtimes C).$$

Finally, let us show that Φ preserves the central structure. The central structure in the composed bimodule $\mathcal{A}_F \underset{\mathcal{B}}{\boxtimes} \mathcal{A}_G$ is given by:

$$(B\boxtimes C)\triangleleft A := (B\boxtimes C)\otimes(F(A)\boxtimes 1_{\mathcal{C}}) = (B\otimes F(A))\boxtimes C \underset{\sim}{\xrightarrow{\sigma^{\mathcal{B}}_{B,F(A)}\boxtimes Id_C}} (F(A)\otimes B)\boxtimes C = A\triangleright(B\boxtimes C)$$

This maps under Φ, using that G is braided monoidal, to $\sigma^{\mathcal{C}}_{G(B),G(F(A))} \otimes Id_C$. And indeed, the following diagram, where the horizontal arrows are the central structures and the vertical arrow monoidality of Φ, commutes:

$$\begin{array}{ccc}
\Phi(B \boxtimes C) \otimes \Phi(\mathbb{1} \triangleleft A) & \xrightarrow{\sigma^{\mathcal{C}}_{(G(B)\otimes C),G(F(A))}} & \Phi(A \triangleright \mathbb{1}) \otimes \Phi(B \boxtimes C) \\
{\scriptstyle Id_{G(B)} \otimes \sigma^{\mathcal{C}}_{C,G(F(A))}} \downarrow & & \downarrow {\scriptstyle Id} \\
\Phi((B \boxtimes C) \otimes (\mathbb{1} \triangleleft A)) & \xrightarrow{\sigma^{\mathcal{C}}_{G(B),G(F(A))} \otimes Id_C} & \Phi((A \triangleright \mathbb{1}) \otimes (B \boxtimes C))
\end{array}$$

The $\overline{\mathcal{A}}$ case is similar. □

Definition 4.2.4: Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{A}$-$\mathcal{B}$-central algebras, i.e. 1-morphisms of **BrTens**. A bimodule monoidal functor $F : \mathcal{C} \to \mathcal{D}$ preserving the $\mathcal{A}$-$\mathcal{B}$-central structures induces an $\mathcal{A}$-$\mathcal{B}$-centered $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{M}_F$ which is given by $\mathcal{D}$ as a category on which $\mathcal{C}$ acts on the left using $F(-) \otimes -$ and $\mathcal{D}$ act on the right using $- \otimes -$. The $\mathcal{A}$-$\mathcal{B}$-centered structure on $\mathcal{M}_F$ is induced by the $\mathcal{A}$-$\mathcal{B}$-central structure of $\mathcal{D}$, and the fact that F is a bimodule functor:

$$F(A \triangleright \mathbb{1}_{\mathcal{C}} \triangleleft B) \otimes M \simeq (A \triangleright 1_{\mathcal{D}} \triangleleft B) \otimes M \overset{\sigma^{\mathcal{D}}}{\simeq} M \otimes (A \triangleright 1_{\mathcal{D}} \triangleleft B)$$

It also induces an $\mathcal{A}$-$\mathcal{B}$-centered $\mathcal{D}$-$\mathcal{C}$-bimodule $\overline{\mathcal{M}}_F$ which is again given by $\mathcal{D}$ as a monoidal category on which $\mathcal{C}$ acts on the right using $- \otimes F(-)$ and $\mathcal{D}$ act on the left using $- \otimes -$.
When the functor F is understood, we may write ${}_{\mathcal{C}}\mathcal{D}_{\mathcal{D}}$ for $\mathcal{M}_F$ and ${}_{\mathcal{D}}\mathcal{D}_{\mathcal{C}}$ for $\overline{\mathcal{M}}_F$ ◇

Proposition 4.2.5: *The above induced-bimodule construction preserves:*

1. *horizontal composition:*
   Given two $\mathcal{A}$-$\mathcal{B}$-bimodule monoidal functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ preserving central structures, one has $\mathcal{M}_G \circ \mathcal{M}_F \simeq \mathcal{M}_{G\circ F}$ and $\overline{\mathcal{M}}_F \circ \overline{\mathcal{M}}_G \simeq \overline{\mathcal{M}}_{G\circ F}$,

2. *vertical composition:*
   Given $\mathcal{C}$ and $\mathcal{D}$ two $\mathcal{A}_1$-$\mathcal{A}_2$-central algebras, $\mathcal{C}'$ and $\mathcal{D}'$ two $\mathcal{A}_2$-$\mathcal{A}_3$ central algebras, $F : \mathcal{C} \to \mathcal{D}$ an $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule monoidal functor and $F' : \mathcal{C}' \to \mathcal{D}'$ an $\mathcal{A}_2$-$\mathcal{A}_3$-bimodule monoidal functor preserving central structures, one has $\mathcal{M}_F \underset{\mathcal{A}_2}{\boxtimes} \mathcal{M}_{F'} \simeq \mathcal{M}_{F \underset{\mathcal{A}_2}{\boxtimes} F'}$ and $\overline{\mathcal{M}}_F \underset{\mathcal{A}_2}{\boxtimes} \overline{\mathcal{M}}_{F'} \simeq \overline{\mathcal{M}}_{F \underset{\mathcal{A}_2}{\boxtimes} F'}$.

PROOF : The first point is similar to the last proposition. We proved that ${}_{\mathcal{C}}\mathcal{D}_{\mathcal{D}} \underset{\mathcal{D}}{\boxtimes} {}_{\mathcal{D}}\mathcal{E}_{\mathcal{E}} \overset{\Phi}{\simeq} {}_{\mathcal{C}}\mathcal{E}_{\mathcal{E}}$, as bimodules. Recall from [BJS21, Definition-Proposition 3.13] that the centered structure on the composition of bimodules $\mathcal{D} \underset{\mathcal{D}}{\boxtimes} \mathcal{E}$ is given by the composition of the centered structure and a balancing. In our case on some A, D, E, this is:

$$D \boxtimes (E \otimes A) \overset{Id_D \boxtimes \sigma^{\mathcal{E}}_{E,A}}{\underset{\sim}{\longrightarrow}} D \boxtimes (A \otimes E) \simeq (D \otimes A) \boxtimes E \overset{\sigma^{\mathcal{D}}_{D,A} \boxtimes Id_E}{\underset{\sim}{\longrightarrow}} (A \otimes D) \boxtimes E$$

which maps by Φ to $(G(\sigma^{\mathcal{D}}_{D,A}) \otimes Id_E) \circ (Id_{G(D)} \otimes \sigma^{\mathcal{E}}_{E,A})$. The centered structure of ${}_{\mathcal{C}}\mathcal{E}_{\mathcal{E}}$ is given by $\sigma^{\mathcal{E}}_{G(D)\otimes E,A}$. They coincide as G preserves central structures.
The second point is not surprising either. We want ${}_{\mathcal{C}}\mathcal{D}_{\mathcal{D}} \underset{\mathcal{A}_2}{\boxtimes} {}_{\mathcal{C}'}\mathcal{D}'_{\mathcal{D}'} \simeq {}_{\mathcal{C} \underset{\mathcal{A}_2}{\boxtimes} \mathcal{C}'}\mathcal{D} \underset{\mathcal{A}_2}{\boxtimes} \mathcal{D}'_{\mathcal{D} \underset{\mathcal{A}_2}{\boxtimes} \mathcal{D}'}$, which is true on the underlying categories. Because F and F' are bimodule functors, the functor $F \boxtimes F' : \mathcal{C} \boxtimes \mathcal{C}' \to \mathcal{D} \boxtimes \mathcal{D}' \twoheadrightarrow \mathcal{D} \underset{\mathcal{A}_2}{\boxtimes} \mathcal{D}'$ is $\mathcal{B}$-balanced and descends to the relative tensor product $\mathcal{C} \underset{\mathcal{A}_2}{\boxtimes} \mathcal{C}'$. We then see that the left $\mathcal{C} \underset{\mathcal{A}_2}{\boxtimes} \mathcal{C}'$-action is the one induced by $F \boxtimes F'$ on the relative tensor product, namely action by $F \underset{\mathcal{A}_2}{\boxtimes} F'$. The centered structures are both given by the central structure of $\mathcal{D} \underset{\mathcal{A}_2}{\boxtimes} \mathcal{D}'$ and coincide. □

Dualizability

Given a braided monoidal functor $F : \mathcal{A} \to \mathcal{B}$, we will prove that both adjoints of $\mathcal{A}_F$ are given by $\overline{\mathcal{A}}_F$. For the right adjunction, the counit should go:

$$\mathcal{A}_F \circ \overline{\mathcal{A}}_F = {}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}} \underset{\mathcal{A}}{\boxtimes} {}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}} \to Id_{\mathcal{B}} = {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}.$$

We actually have a functor going this way, the tensor product T in $\mathcal{B}$, which is $\mathcal{A}$-balanced and descends to the relative tensor product. We denote it $T_{bal} : \mathcal{B} \underset{\mathcal{A}}{\boxtimes} \mathcal{B} \to \mathcal{B}$, and it is indeed a $\mathcal{B}$-$\mathcal{B}$-bimodule monoidal functor. The central structures on both sides are given by braiding in $\mathcal{B}$, which is preserved by T. Hence we can construct a $\mathcal{B}$-$\mathcal{B}$-centered ${}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}} \underset{\mathcal{A}}{\boxtimes} {}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}}$-${}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}$-bimodule $\mathcal{M}_{T_{bal}}$ using Definition 4.2.4.
The unit should go:

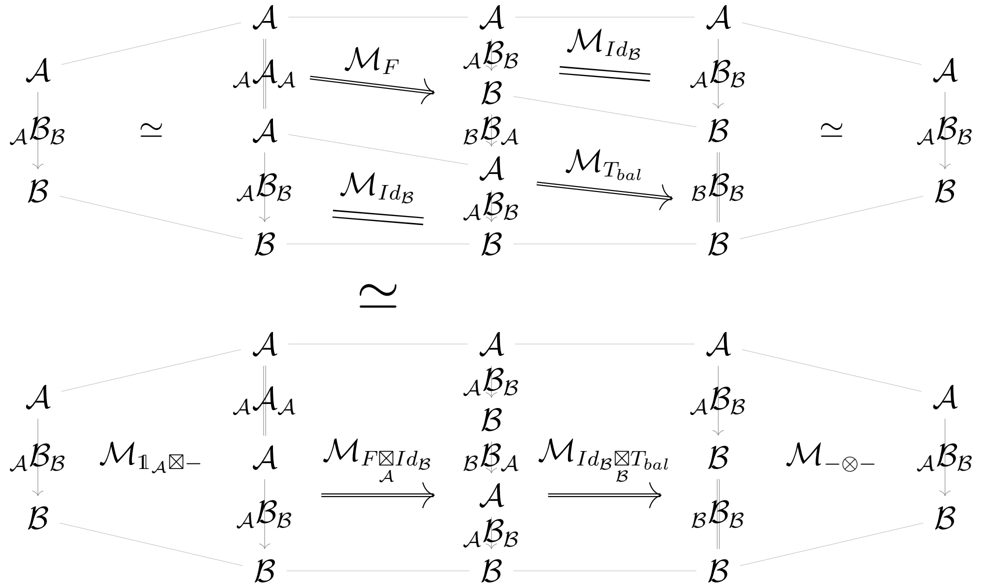
$$Id_{\mathcal{A}} = {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}} \to \overline{\mathcal{A}}_F \circ \mathcal{A}_F = {}_{\mathcal{A}}\mathcal{B}_{\mathcal{B}} \underset{\mathcal{B}}{\boxtimes} {}_{\mathcal{B}}\mathcal{B}_{\mathcal{A}} \simeq {}_{\mathcal{A}}\mathcal{B}_{\mathcal{A}}.$$

Again we have a functor $F : \mathcal{A} \to \mathcal{B}$ which is an $\mathcal{A}$-$\mathcal{A}$-module monoidal functor. The central structure on the left is given by braiding in $\mathcal{A}$, and on the right by braiding in $\mathcal{B}$. The first is sent on the latter because F is braided monoidal, and the central structures

are preserved. Therefore we also have an $\mathcal{A}$-$\mathcal{A}$-centered ${}_\mathcal{A}\mathcal{A}_\mathcal{A}$-${}_\mathcal{A}\mathcal{B}_\mathcal{A}$-bimodule $\mathcal{M}_F$.
Note also that the identity of $\mathcal{A}_F$ is the bimodule induced by $Id_\mathcal{B}$ seen as an $\mathcal{A}$-$\mathcal{B}$-bimodule monoidal functor.

Proposition 4.2.6: *The 1-morphism $\mathcal{A}_F$ has right adjoint given by $\overline{\mathcal{A}}_F$, with counit $\mathcal{M}_{T_{bal}}$ and unit $\mathcal{M}_F$. Its left adjoint is also given by $\overline{\mathcal{A}}_F$, with counit $\overline{\mathcal{M}}_F$ and unit $\overline{\mathcal{M}}_{T_{bal}}$.*

Proof : We directly check the snake. We repeatedly use Proposition 4.2.5:



which is the bimodule induced by the composition:

$$\begin{array}{ccccccccc} \mathcal{B} & \longrightarrow & \mathcal{A}\underset{\mathcal{A}}{\boxtimes}\mathcal{B} & \longrightarrow & \mathcal{B}\underset{\mathcal{B}}{\boxtimes}\mathcal{B}\underset{\mathcal{A}}{\boxtimes}\mathcal{B} & \longrightarrow & \mathcal{B}\underset{\mathcal{B}}{\boxtimes}\mathcal{B} & \longrightarrow & \mathcal{B} \\ X & \mapsto & (\mathbb{1}_\mathcal{A}, X) & \mapsto & (\mathbb{1}_\mathcal{B}, \mathbb{1}_\mathcal{B}, X) & \mapsto & (\mathbb{1}_\mathcal{B}, X) & \mapsto & X \end{array}$$

which is indeed the identity.
Every other snake identity is very similar, with functors going in the other direction for the left adjunction. □

Proposition 4.2.7: *Let $F : \mathcal{C} \to \mathcal{D}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule monoidal functor. The bimodule $\mathcal{M}_F$ has right adjoint given by $\overline{\mathcal{M}}_F$, with counit $T_{bal} : \mathcal{D}\underset{\mathcal{C}}{\boxtimes}\mathcal{D} \to \mathcal{D}$ seen as a $\mathcal{D}$-$\mathcal{D}$-bimodule functor and unit F seen as a $\mathcal{C}$-$\mathcal{C}$-bimodule functor.*

Proof : The proof is the same as above, except that the horizontal morphisms are now the functors instead of the bimodules induced by the functors. The snake identities read:

$$(Id_\mathcal{D}\underset{\mathcal{D}}{\boxtimes} T_{bal})\circ(F\underset{\mathcal{C}}{\boxtimes} Id_\mathcal{D}) \simeq Id_{\mathcal{M}_F} \quad \text{and} \quad (T_{bal}\underset{\mathcal{D}}{\boxtimes} Id_\mathcal{D})\circ(Id_\mathcal{D}\underset{\mathcal{C}}{\boxtimes} F) \simeq Id_{\overline{\mathcal{M}}_F} \tag{4.1}$$

as has been used above. Here $Id_\mathcal{D}$ is seen alternatively as a $\mathcal{C}$-$\mathcal{D}$-bimodule functor and as a $\mathcal{D}$-$\mathcal{C}$-bimodule functor. □

We would like to apply Proposition 4.1.12.1, to have the left adjoint of $\mathcal{M}_F$. We need F and T_{bal} to have right adjoints in **BrTens**. There is a well-known sufficient condition for this.

Proposition 4.2.8 ([BJS21, Proposition 4.2 and Corollary 4.3]): *Let $F : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{A}$-$\mathcal{B}$-centered $\mathcal{C}$-$\mathcal{D}$-bimodule functor, so a 3-morphism in* **BrTens**. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ have enough cp, that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are cp-rigid, and that F preserves cp. Then $F^R : \mathcal{N} \to \mathcal{M}$ is an $\mathcal{A}$-$\mathcal{B}$-centered $\mathcal{C}$-$\mathcal{D}$-bimodule functor, and is the right adjoint of F in* **BrTens**.

All we need to check is that both F and T_{bal} preserve cp.

Lemma 4.2.9: *Let $\mathcal{M}$ and $\mathcal{N}$ be right and left modules over $\mathcal{C}$ and $F : \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{P}$ be a cocontinuous $\mathcal{C}$-balanced functor. Suppose $\mathcal{M}$ and $\mathcal{N}$ have enough cp, $\mathcal{C}$ is cp-rigid and F preserves cp. Then the induced functor $F_{bal} : \mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N} \to \mathcal{P}$ preserves cp.*
In particular, if $\mathcal{A}$ and $\mathcal{B}$ are cp-rigid, then $T_{bal} : \mathcal{B} \underset{\mathcal{A}}{\boxtimes} \mathcal{B} \to \mathcal{B}$ preserves cp.

Proof : Following the proof of closure under composition of 1-morphisms [BJS21, Section 4.2], the cp objects of $\mathcal{M} \underset{\mathcal{C}}{\boxtimes} \mathcal{N}$ are generated by pure tensors of cp objects. These are sent to cp objects in $\mathcal{P}$.
For the second point, T_{bal} is induced by T which preserves cp as $\mathcal{B}$ is cp-rigid. □

We can summarize the result as follows:

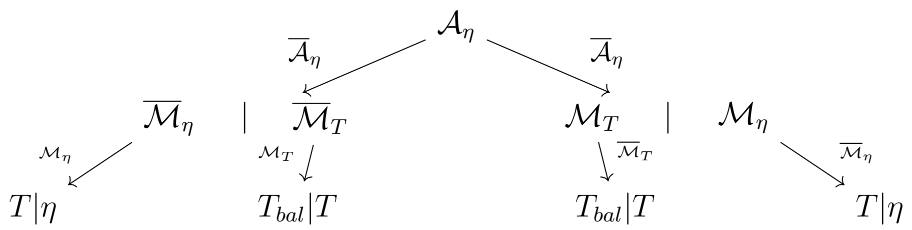
Proposition 4.2.10: *Let $F : \mathcal{C} \to \mathcal{D}$ be an $\mathcal{A}$-$\mathcal{B}$-bimodule monoidal functor which preserves cp, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are cp-rigid. The bimodule $\mathcal{M}_F$ has left adjoint given by $\overline{\mathcal{M}}_F$, with counit F^R seen as a $\mathcal{C}$-$\mathcal{C}$-bimodule functor and unit T_{bal}^R seen as a $\mathcal{D}$-$\mathcal{D}$-bimodule functor.*

4.2.2 Unit inclusion

We give explicitly the dualizability data of the 1-morphism induced by the unit inclusion in a braided tensor category $\mathcal{V}$, and criteria for dualizability when $\mathcal{V}$ has enough cp.

Definition 4.2.11: Let $\mathcal{V} \in$ **BrTens** be an E_2-algebra in **Pr**. We denote by $T : \mathcal{V} \boxtimes \mathcal{V} \to \mathcal{V}$ its monoidal structure, and $\eta : \mathrm{Vect}_{\Bbbk} \to \mathcal{V}$ the inclusion of the unit. The functor η is braided monoidal and induces a $\mathrm{Vect}_{\Bbbk}$-$\mathcal{V}$-central algebra $\mathcal{A}_\eta$, namely a 1-morphism in **BrTens**. Remember that we denote by $\mathcal{A}_\eta^\flat \in \mathbf{BrTens}^\to$ the associated object in the oplax arrow category. ♢

Theorem 4.2.12: *The 1-morphism $\mathcal{A}_\eta$ is both twice left and twice right adjunctible, with adjunctibility data as displayed:*



where $T_{bal} : \mathcal{V} \underset{\mathcal{V}\boxtimes\mathcal{V}}{\boxtimes} \mathcal{V} \to \mathcal{V}$ is induced by T on the relative tensor product

Proof : We use the results of Section 4.2.1. By Proposition 4.2.6, the 1-morphism $\mathcal{A}_\eta$ has left and right adjoints given by $\overline{\mathcal{A}}_\eta$, with units and counits as displayed in the second line above, with $\eta : \mathrm{Vect}_\Bbbk \to \mathcal{V}$ now seen as a $\mathrm{Vect}_\Bbbk$-$\mathrm{Vect}_\Bbbk$-bimodule monoidal functor, and $T : \mathcal{V} \underset{\mathrm{Vect}_\Bbbk}{\boxtimes} \mathcal{V} \to \mathcal{V}$ the tensor product balanced over $\mathrm{Vect}_\Bbbk$ so not balanced.
Then by Proposition 4.2.7 each of these bimodules has either a left or a right adjoint, with units and counits as displayed, with $T_{bal} : \overline{\mathcal{M}}_T \underset{\mathcal{V}\boxtimes\mathcal{V}}{\boxtimes} \mathcal{M}_T = \mathcal{V} \underset{\mathcal{V}\boxtimes\mathcal{V}}{\boxtimes} \mathcal{V} \to \mathcal{V}$ induced by T. □

Corollary 4.2.13: *The object $\mathcal{A}_\eta^\flat$ is 2-dualizable in* $\mathbf{BrTens}^{\to}$*, and :*
$Ru(Ru(\mathcal{A}_\eta^\flat))$ has a right adjoint if and only if both T_{bal} and $Ru(Ru(\mathcal{V}))$ do.
$Rco(Ru(\mathcal{A}_\eta^\flat))$ has a right adjoint if and only if both T and $Rco(Ru(\mathcal{V}))$ do.
$Rco(Rco(\mathcal{A}_\eta^\flat))$ has a right adjoint if and only if both η and $Rco(Rco(\mathcal{V}))$ do.

Proof : For 2-dualizability, we use the criterion of [JS17, Theorem 7.6], we know that $\mathcal{V}$ is 2-dualizable by [GS, Theorem 5.1] and $\mathcal{A}_\eta$ is twice right adjunctible by the theorem above. The rest is Theorem 4.1.13 on the right dualizability data of $\mathcal{A}_\eta$. □

Theorem 4.2.14: *Suppose that $\mathcal{V}$ has enough cp, then $\mathcal{A}_\eta^\flat$ is 3-dualizable if and only if $\mathcal{V}$ the free cocompletion of a small rigid braided monoidal category.*

Proof : The heart of the proof is to notice that T appears in the dualizability data, and by [BJS21, Proposition 4.1] when $\mathcal{V}$ has enough cp, it is cp-rigid if and only if T has a bimodule cocontinuous right adjoint.
If $\mathcal{A}_\eta^\flat$ is 3-dualizable then $Ru(Ru(\mathcal{A}_\eta^\flat))$, $Rco(Ru(\mathcal{A}_\eta^\flat))$ and $Rco(Rco(\mathcal{A}_\eta^\flat))$ have right adjoints, so T_{bal}, T and η have bimodule cocontinuous right adjoints. The functors T and η preserving cp mean that they are well-defined on $\mathscr{V} := \mathcal{V}^{cp}$ and endow it with a monoidal structure, and $\mathscr{V}$ is rigid as $\mathcal{V}$ is cp-rigid. Therefore $\mathcal{V}$ is the free cocompletion of a small rigid braided monoidal category.
On the other hand if $\mathcal{V}$ is the free cocompletion of a small rigid braided monoidal category then it is cp-rigid and hence 3-dualizable, [BJS21, Theorem 5.16]. The functors T and η, and also T_{bal} by Lemma 4.2.9, preserve cp, and have bimodule cocontinuous right adjoints by Proposition 4.2.8. We get that $\mathcal{A}_\eta$ is 3-times right adjunctible and its source and targets are 3-dualizable, so $\mathcal{A}_\eta^\flat$ 3-dualizable by [JS17, Theorem 7.6]. □

Theorem 4.2.15: *Suppose that $\mathcal{V}$ has enough cp, then $\mathcal{A}_\eta^\flat$ is non-compact-3-dualizable if and only if $\mathcal{V}$ is cp-rigid.*

Proof : If $\mathcal{V}$ is cp-rigid, then $\mathcal{V}$ is 3-dualizable and T and T_{bal} have right adjoints in **BrTens**. By Corollary 4.2.13, $\mathcal{A}_\eta^\flat$ is non-compact-3-dualizable.
Suppose now $\mathcal{A}_\eta^\flat$ non-compact-3-dualizable, then T has a bimodule cocontinuous right adjoint, and $\mathcal{V}$ is cp-rigid. □

Theorem 4.2.16: *Let $\mathcal{V}$ be a braided tensor category with enough cp. Then the following are equivalent:*

1. *$\mathcal{A}_\eta$ is 3-dualizable,*
2. *$\mathcal{A}_\eta$ is 3-adjunctible, and*
3. *$\mathcal{V}$ is rigid finite semisimple.*

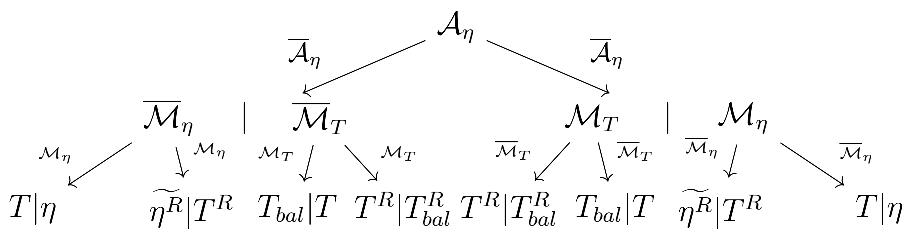
PROOF : The implication $1 \Rightarrow 2$ is immediate: for a 1-morphism 3-dualizable demands 3-adjunctible and 4-dualizablility of the source and target.
The implication $3 \Rightarrow 1$ is essentially [BJS21, Theorem 5.21]. If $\mathcal{V}$ is fusion, then $\mathcal{V}$ and $\mathcal{A}_\eta$ lie $BrFus$ which has duals. Now fusion demands simplicity of the unit, which may not be the case here. This is easily solved by noticing that coproduct agrees with product in **Pr** and ought to be called direct sum [BCJ15, Remark 2.5], and that braided rigid finite semisimple categories are direct sums of fusion categories [EGNO15, Section 4.3].
Let us prove $2 \Rightarrow 3$. If $\mathcal{A}_\eta$ is 3-adjunctible then $\mathcal{M}_\eta$ and $\overline{\mathcal{M}}_\eta$, which are respectively $Ru(\mathcal{A}_\eta)$ and $Lco(\mathcal{A}_\eta)$ by Theorem 4.2.14, must be 2-adjunctible. Hence their composite $\mathcal{M}_\eta \underset{\mathcal{V}}{\boxtimes} \overline{\mathcal{M}}_\eta$ has to be 2-adjunctible in the symmetric monoidal 2-category $\Omega\Omega\,\mathbf{BrTens} \simeq \mathbf{Pr}$. This composition is just $\mathcal{V} \underset{\mathcal{V}}{\boxtimes} \mathcal{V} \simeq \mathcal{V}$ as a category, and by our assumption that it has enough cp, it actually lies in the full subcategory $\mathrm{Bimod}_{\Bbbk} \subseteq \mathbf{Pr}$. By [BDSV, Theorem A.22], the 2-dualizable objects of $\mathrm{Bimod}_{\Bbbk}$ are finite semisimple categories. We already saw that $\mathcal{V}$ has to be cp-rigid, so $\mathcal{V}^{cp}$ is rigid finite semisimple, and so is $\mathcal{V} \simeq \mathrm{Free}(\mathcal{V}^{cp})$.□

Remark 4.2.17: A very similar result one categorical dimension down, in $\mathbf{Alg}_1(Rex_{\mathbb{C}})$, is proven in [FT21, Theorem B]. The proof is similar too, but we couldn't directly use their result on $\mathcal{M}_\eta$ as we work in $\mathrm{Bimod}_{\Bbbk}$ and not in $Rex_{\mathbb{C}}$. ◇

Remark 4.2.18: Both results need full adjunctibility of $\mathcal{A}_\eta$: oplax dualizability does not imply semisimplicity (take the free cocompletion of a non-semisimple ribbon category in Theorem 4.2.14). Semisimplicity is not needed for 4-dualizability either, as proven in [BJSS21]. However, if we assume that $\mathcal{V}$ is 4-dualizable and $\mathcal{A}_\eta$ is 3-oplax-dualizable, which is the case of interest for Section 4.3, then to-appear work of Will Stewart shows that $\mathcal{A}_\eta$ is 3-adjunctible. This has an interesting consequence: the free cocompletion of a ribbon category which is not semisimple cannot be 4-dualizable. Indeed if it were Stewart's result would apply and $\mathcal{V}$ would have to be semisimple. This justifies that, given a non-semisimple ribbon tensor category as in Chapter 3, we want to work with its Ind-completions, and not its free cocompletion. ◇

Remark 4.2.19: Being dualizable for a morphism is both a condition on its adjunctibility and on the dualizability of its source and target. However, we saw in the proof of Theorem 4.2.14 that $\mathcal{A}_\eta$ 3-right-adjunctible $\Leftrightarrow$ $\mathcal{A}_\eta$ 3-oplax-dualizable, and in the Theorem above that $\mathcal{A}_\eta$ 3-adjunctible $\Leftrightarrow$ $\mathcal{A}_\eta$ 3-dualizable. This phenomenon seems to be specific to the unit inclusion. ◇

Proposition 4.2.20: *Suppose that $\mathcal{V}$ is cp-rigid, then $\mathcal{A}_\eta$ is 2-adjunctible with the following adjunctibility data in* **BrTens***:*



where $\widetilde{\eta^R}$ is the essentially unique cocontinuous functor that agrees with η^R on cp objects.

Proof : The snake for T^R and $\widetilde{\eta^R}$ comes from the following. Denote $\mathscr{V} = \mathcal{V}^{cp}$.
T^R is the coend $T^R(\mathbb{1}_\mathcal{V}) = \int^{(V,W)\in\mathscr{V}^{\otimes 2}} (V \boxtimes W) \otimes \mathrm{Hom}_\mathcal{V}(V \otimes W, \mathbb{1}_\mathcal{V}) \simeq \int^{V\in\mathscr{V}} V \boxtimes V^*$, and more generally $T^R(X) \simeq \int^{V\in\mathscr{V}} (X \otimes V) \boxtimes V^* \simeq \int^{V\in\mathscr{V}} V \boxtimes (V^* \otimes X)$. For X cp, the snake goes

$$\begin{aligned}(\widetilde{\eta^R} \underset{\mathrm{Vect}_\mathbb{k}}{\boxtimes} Id_\mathcal{V}) \circ (Id_\mathcal{V} \underset{\mathcal{V}}{\boxtimes} T^R)(X) &\simeq \int^{V\in\mathscr{V}} \widetilde{\eta^R}(X \otimes V) \boxtimes V^* \\ &= \int^{V\in\mathscr{V}} \mathrm{Hom}(\mathbb{1}_\mathcal{V}, X \otimes V) \otimes V^* \\ &\simeq \int^{V\in\mathscr{V}} \mathrm{Hom}(V^*, X) \otimes V^* \simeq X\end{aligned}$$

The part with T^R and T^R_{bal} is given by Proposition 4.2.10. Indeed T, and hence T_{bal}, preserves cp as $\mathcal{V}$ is cp-rigid.
The fact that this is sufficient for 2-adjunctibility is [JS17, Lemma 7.11]. □

Remark 4.2.21: Using Theorem 4.2.12 we can also see that $\mathcal{A}_\eta$ is always 2-lax-dualizable, and it is 3-times left adjunctible if and only if η, T and T_{bal} have left adjoints in **BrTens**. Using the proposition above, we can also get another characterisation of adjunctibility: every morphism appearing there must have a right adjoint. If $\mathcal{V}$ has enough cp, then $\mathcal{A}_\eta$ is 3-adjunctible if and only if $\mathcal{V}$ is cp-rigid and η, η^R, T^R and T^R_{bal} preserve cp. ◇

We studied the unit inclusion, but the same arguments work in greater generality. We loose that they are criteria though, because T no longer appears in the dualizability data, only some balanced version does.

Theorem 4.2.22: *Let $F : \mathcal{V} \to \mathcal{W}$ be a braided monoidal functor between two objects of* **BrTens**. *Then the object $\mathcal{A}_F^\flat \in$* **BrTens**$^\to$ *induced by the 1-morphism $\mathcal{A}_F$ is 2-dualizable. It is non-compact-3-dualizable as soon as $\mathcal{V}$ and $\mathcal{W}$ are cp-rigid. In this case, it is 3-dualizable if and only if F preserves cp.*

Proof : We know that $Radj(\mathcal{A}_F) = \overline{\mathcal{A}}_F$ with $Ru(\mathcal{A}_F) = \mathcal{M}_F$ and $Rco(\mathcal{A}_F) = \mathcal{M}_{T_{\mathcal{V}-bal}}$ by Proposition 4.2.6, where $T_{\mathcal{V}-bal} : \mathcal{W} \underset{\mathcal{V}}{\boxtimes} \mathcal{W} \to \mathcal{W}$ is induced by the monoidal structure on $\mathcal{W}$.
Then, $Radj(\mathcal{M}_{T_{\mathcal{V}-bal}}) = \overline{\mathcal{M}}_{T_{\mathcal{V}-bal}}$ with $Ru(\mathcal{M}_{T_{\mathcal{V}-bal}}) = T_{\mathcal{V}-bal}$ and $Rco(\mathcal{M}_{T_{\mathcal{V}-bal}}) = T_{2bal}$ by Proposition 4.2.7, where $T_{2bal} : \mathcal{W} \underset{\mathcal{W}\underset{\mathcal{V}}{\boxtimes}\mathcal{W}}{\boxtimes} \mathcal{W} \to \mathcal{W}$ is induced by the monoidal structure on $\mathcal{W}$.
Similarly, $Radj(\mathcal{M}_F) = \overline{\mathcal{M}}_F$ with $Ru(\mathcal{M}_F) = F$ and $Rco(\mathcal{M}_F) = T_{\mathcal{V}-bal}$.
We know by Theorem 4.1.13 that the existence and right adjunctibility of $Ru(Ru(\mathcal{A}_F^\flat))$,

$Rco(Ru(\mathcal{A}_F^\flat))$ and $Rco(Rco(\mathcal{A}_F^\flat))$ is equivalent to that of respectively T_{2bal}, $T_{\mathcal{V}-bal}$ and F, and of the same units/counits of the source and target. So $\mathcal{A}_F^\flat$ is non-compact-3-dualizable if and only if $T_{\mathcal{V}-bal}$ and T_{2bal} have right adjoints in **BrTens**, and both $\mathcal{V}$ and $\mathcal{W}$ are non-compact-3-dualizable. This is true as soon as $\mathcal{V}$ and $\mathcal{W}$ are cp-rigid by Lemma 4.2.9 and [BJS21, Theorem 5.6].
It is 3-dualizable if and only if F, $T_{\mathcal{V}-bal}$ and T_{2bal} have right adjoints and $\mathcal{V}$ and $\mathcal{W}$ are 3-dualizable. If $\mathcal{V}$ and $\mathcal{W}$ are cp-rigid, this is true if and only if F preserves cp. ☐

4.2.3 The relative theory on the circle

We compute the value on the circle of the relative TQFT $\mathcal{R}_\mathcal{V}$ induced by $\mathcal{A}_\eta^\flat$ under the cobordism hypothesis, for any $\mathcal{V}$. Namely, we write $S^1_{nb} = ev_{pt} \circ coev_{pt}$, compute the images of ev_{pt} and $coev_{pt}$ under $\mathcal{R}_\mathcal{V}$, which are $ev_{\mathcal{A}_\eta^\flat}$ and $coev_{\mathcal{A}_\eta^\flat}$, and compose them. Note that it is S^1 with non-bounding framing that we are computing. We need the symmetric monoidal structure of $\mathcal{C}$ to compose $ev_X : \mathbb{1} \to X \otimes X^*$ and $coev_X : X \otimes X^* \simeq X^* \otimes X \to \mathbb{1}$. We know that the evaluation and coevaluation for $\mathcal{A}_\eta^\flat$ are mates of the unit and counit for the right adjunction of $\mathcal{A}_\eta$, namely $\mathcal{M}_\eta$ and $\mathcal{M}_T$. It might sound surprising that one can compose them, but indeed up to whiskering and mating they are composeable, see Figure 4.3.

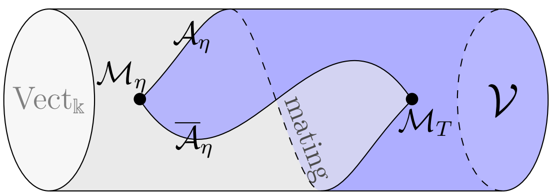
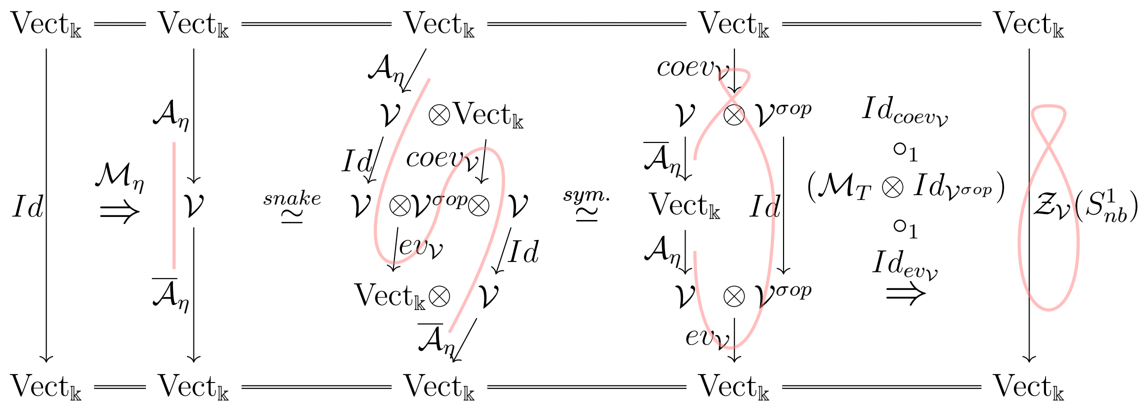


Figure 4.3: The unit and the counit compose up to mating

We know from [BJSS21, Theorem 2.19] that the evaluation and coevaluation for $\mathcal{V}$ are respectively ${}_{\mathcal{V}\boxtimes\mathcal{V}^{\sigma op}}\mathcal{V}_{\mathrm{Vect}_\Bbbk}$ and ${}_{\mathrm{Vect}_\Bbbk}\mathcal{V}_{\mathcal{V}^{\sigma op}\boxtimes\mathcal{V}}$. Then, Example 4.1.14 gives:

$$\mathcal{R}_\mathcal{V}(ev_{pt}) = \begin{array}{ccc} \mathrm{Vect}_\Bbbk & \xrightarrow{\mathcal{A}_\eta \otimes (\overline{\mathcal{A}}_\eta)^*} & \mathcal{V} \otimes \mathcal{V}^{\sigma op} \\ {\scriptstyle Id}\downarrow & \overset{\mathcal{M}_\eta}{\Nearrow} & \downarrow{\scriptstyle ev_\mathcal{V}} \\ \mathrm{Vect}_\Bbbk & \xrightarrow[\mathrm{Vect}_\Bbbk]{} & \mathrm{Vect}_\Bbbk \end{array} \quad \text{and } \mathcal{R}_\mathcal{V}(coev_{pt}) = \begin{array}{ccc} \mathrm{Vect}_\Bbbk & \xrightarrow{\mathrm{Vect}_\Bbbk} & \mathrm{Vect}_\Bbbk \\ {\scriptstyle Id}\downarrow & \overset{(\mathcal{M}_T \otimes Id_{\mathcal{V}^{\sigma op}}) \circ_1 Id_{coev_\mathcal{V}}}{\Nearrow} & \downarrow{\scriptstyle coev_\mathcal{V}} \\ \mathrm{Vect}_\Bbbk & \xrightarrow[\mathcal{A}_\eta \otimes (\overline{\mathcal{A}}_\eta)^*]{} & \mathcal{V} \otimes \mathcal{V}^{\sigma op} \end{array}$$

Their composition is vertical stacking and gives that $\mathcal{R}_\mathcal{V}(S^1_{nb})$ is:



Note that every bimodule above is induced by a functor as displayed here:

$$\begin{array}{ccccccccc} \mathrm{Vect}_{\Bbbk} & \xrightarrow{\eta} & \mathcal{V}\underset{\mathcal{V}}{\boxtimes}\mathcal{V} \simeq (\mathcal{V}\otimes\mathcal{V}) \underset{\mathcal{V}\otimes\mathcal{V}^{\sigma op}\otimes\mathcal{V}}{\boxtimes} (\mathcal{V}\otimes\mathcal{V}) & \xrightarrow{\sim} & \mathcal{V}\underset{\mathrm{Vect}_{\Bbbk}\otimes\mathcal{V}^{\sigma op}}{\boxtimes}\mathcal{V} & \xrightarrow{Id_{\mathcal{V}}\boxtimes(T\otimes Id)\boxtimes Id_{\mathcal{V}}} & \mathcal{V}\underset{\mathcal{V}\otimes\mathcal{V}^{\sigma op}}{\boxtimes}\mathcal{V} \\ k & \longmapsto & \mathbb{1}\boxtimes\mathbb{1} \longmapsto (\mathbb{1}\otimes\mathbb{1})\boxtimes(\mathbb{1}\otimes\mathbb{1}) & \longmapsto & \mathbb{1}\boxtimes\mathbb{1} & \longmapsto & \mathbb{1}\boxtimes\mathbb{1} \end{array}$$

So $\mathcal{R}_{\mathcal{V}}(S^1_{nb})$ is induced by the monoidal functor given by inclusion of the unit in $\mathcal{Z}_{\mathcal{V}}(S^1_{nb})$.

4.3 Non-semisimple WRT relative to CY

We can now state the conjectures which are the main motivation for the study above. The main idea is that the Witten–Reshetikhin–Turaev theories and their non-semisimple variants can be obtained in a fully extended setting from a 3D theory relative to an invertible 4D anomaly. In particular, they are defined in a setting where the cobordism hypothesis applies, and can be rebuilt out of their value at the point. These would be a (not necessarily semisimple) modular tensor category for the invertible 4-TQFT and the 1-morphism induced by the inclusion of the unit for the relative 3-TQFT. As exposed above, in the non-semisimple case the unit inclusion is only partially dualizable, and induces a non-compact TQFT.
These conjectures follow ideas of Walker [Walb], Freed and Teleman [Fre] in the semisimple case, of Jordan and Safronov in the non-semisimple case. We do not know of a formal statement in the existing literature and propose one here.

4.3.1 Bulk+Relative=Anomalous

Remember that the WRT theories, and their non-semisimple variants, are defined on a category of cobordisms equipped with some extra structure. They morally come from the data of a bounding higher manifold. 3-manifolds come equipped with an integer, which corresponds to the signature of the bounding 4-manifold, and surfaces come equipped with a Lagrangian in their first cohomology group, which corresponds to the data of the contractible curves in a bounding handlebody. In this setting, this extra structure is used to resolve an anomaly, and is due to Walker. We describe below how this kind of extra structure arises in the setting of relative field theories.

Definition 4.3.1: The $(n-1)$-*category of filled bordisms*

$$\mathbf{Bord}_{n-1}^{filled} \subseteq \mathbf{Bord}_n^{\rightarrow}$$

is the sub-$(n-1)$-category of bordisms that map to the empty under the target functor $\mathbf{Bord}_n^{\rightarrow} \to \mathbf{Bord}_n$ and to $\mathbf{Bord}_{n-1}$ under the source functor. These are k-bordisms,

$k \leq n-1$, equipped with a bounding $k+1$-bordism which we call the filling. We denote

$$\mathrm{Hollow} : \mathbf{Bord}_{n-1}^{filled} \to \mathbf{Bord}_{n-1}$$

the functor that forgets the filling, namely the source functor.
The $(n-1)$*-category of non-compact filled bordisms*

$$\mathbf{Bord}_{n-1}^{nc,filled} \subseteq \mathbf{Bord}_n^{\to}$$

is the sub-$(n-1)$-category of bordisms that map to the empty under the target functor and to $\mathbf{Bord}_{n-1}^{nc}$ under the source functor. ♢

Definition 4.3.2: An *n-relative pair* $(\mathcal{Z}, \mathcal{R})$ is the data of:

an n-TQFT $\mathcal{Z} : \mathbf{Bord}_n \to \mathcal{C}$

an oplax-$\mathcal{Z}$-twisted-$(n-1)$-TQFT $\mathcal{R} : \mathbf{Bord}_{n-1} \to \mathcal{C}^{\to}$, namely an oplax transformation $\mathrm{Triv} \Rightarrow \mathcal{Z}|_{\mathbf{Bord}_{n-1}}$.

It is called a *non-compact n-relative pair* if $\mathcal{R}$ is a non-compact theory. ♢

Given an n-relative pair $(\mathcal{Z}, \mathcal{R})$ one has two symmetric monoidal functors $\mathbf{Bord}_{n-1}^{filled} \to \mathcal{C}^{\to}$. One is given by applying functoriality of $(-)^{\to}$ on $\mathcal{Z}$, namely applying $\mathcal{Z}$ to any diagram in $\mathbf{Bord}_n$ to get a diagram of the same shape in $\mathcal{C}$. It has trivial target and gives an oplax transformation

$$\mathcal{Z}^{\to \mathbb{1}} : \mathcal{Z}|_{\mathbf{Bord}_{n-1}} \circ \mathrm{Hollow} \Rightarrow \mathrm{Triv}$$

between functors $\mathbf{Bord}_{n-1}^{filled} \to \mathcal{C}$.
The other one is given by applying the relative field theory on the hollowed out bordism, it is an oplax transformation

$$\mathcal{R} \circ \mathrm{Hollow} : \mathrm{Triv} \Rightarrow \mathcal{Z}|_{\mathbf{Bord}_{n-1}} \circ \mathrm{Hollow} \quad .$$

Definition 4.3.3: The *anomalous* $(n-1)$*-theory* $\mathcal{A}$ induced by the n-relative pair $(\mathcal{Z}, \mathcal{R})$ is the composition $\mathcal{Z}^{\to \mathbb{1}} \circ (\mathcal{R} \circ \mathrm{Hollow})$ of these two oplax transformations. It gives an oplax ransformation $\mathrm{Triv} \Rightarrow \mathrm{Triv}$ which by [JS17, Theorem 7.4 and Remark 7.5] is equivalent to a symmetric monoidal functor

$$\mathcal{A} : \mathbf{Bord}_{n-1}^{filled} \to (\Omega\mathcal{C})^{odd\ opp} ,$$

where *odd opp* means we take opposite of k-morphisms for k odd, and $\Omega\mathcal{C} := \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is the delooping $(n-1)$-category.
If $(\mathcal{Z}, \mathcal{R})$ is a non-compact n-relative pair, the same construction on the appropriate subcategories gives an anomalous theory $\mathcal{A} : \mathbf{Bord}_{n-1}^{nc,filled} \to (\Omega\mathcal{C})^{odd\ opp}$. ♢

For comparison with WRT theories, we will need to restrict to a once extended theory, namely look at endomorphisms of the trivial in $\mathbf{Bord}_{n-1}^{filled}$, and to check that the anomalous theory descends to the quotient where one only remembers signatures and Lagrangians out of the fillings. We will also move this odd opposite to the source category.

Definition 4.3.4: The *bicategory of simply filled 3-2-1-cobordisms* $\mathbf{Cob}_{321}^{filled}$ is the subcategory of $h_2(\Omega\,\mathbf{Bord}_3^{filled,\ odd\ opp})$ where circles can only be filled by disks, and surfaces by handlebodies. Taking the opposite orientation for 1- and 2-manifolds (which will have the effect of switching the source and target of a 3-bordism), one can identify this bicategory as:

$$\mathbf{Cob}_{321}^{filled} \simeq \begin{cases} \text{objects} & : & (\sqcup^n S^1, \sqcup^n \mathbb{D}^2 : \sqcup^n S^1 \to \emptyset),\ n \in \mathbb{N} \\ \text{1-morphisms} & : & (\Sigma : \sqcup^{n_1} S^1 \to \sqcup^{n_2} S^1, H : \emptyset \to (\sqcup^{n_1}\overline{\mathbb{D}^2}) \cup \Sigma \cup (\sqcup^{n_2}\mathbb{D}^2)) \\ \text{2-morphisms} & : & (M : \Sigma_1 \to \Sigma_2, W : H_1 \cup M \cup \overline{H_2} \to \emptyset) \end{cases}$$

The analogous definition in the non-compact case $\mathbf{Cob}_{321}^{nc,filled} \subseteq h_2(\Omega\mathbf{Bord}_3^{nc,filled,\ odd\ opp})$ will require 3-bordism to have non-empty *incoming* boundary in every connected component, as source and targets of 3-manifolds are switched. To facilitate comparison with the existing literature, we also require that all surfaces have non-empty incoming boundary, although in our setting this is purely artificial. ♢

This is to be compared with:

Definition 4.3.5: The bicategory $\widetilde{\mathbf{Cob}}_{321}$ (resp. $\widetilde{\mathbf{Cob}}_{321}^{nc}$) is the bicategory of circles, surfaces bordisms (resp. surface bordisms with non-empty incoming boundary) equipped with a Lagrangian subspace in their first homology group, and 3-bordisms (resp. 3-bordisms with non-empty incoming boundary) equipped with an integer. Composition is given by usual composition on the underlying bordisms, plus:

taking the sum of the Lagrangian subspaces for composition of surfaces,

adding the integers plus some Maslov index for composition of 3-bordisms,

just adding the integers for composition of 3-bordisms in the direction of 1-morphisms.

See [De 21, Section 3] and references therein for a precise definition. The bordisms there are decorated by objects of a ribbon category, and we are looking at the subcategory where every decoration is empty. The category $\widetilde{\mathbf{Cob}}_{321}^{nc}$ corresponds to admissible bordisms there. ♢

Proposition 4.3.6: *The assignment*

$$\pi_{321} : \begin{cases} \boldsymbol{Cob}_{321}^{filled} & \to & \widetilde{\boldsymbol{Cob}}_{321} \\ (\sqcup^n S^1, \sqcup^n \mathbb{D}^2) & \mapsto & \sqcup^n S^1 \\ (\Sigma, H) & \mapsto & (\Sigma, \ker(i_* : H_1(\Sigma) \to H_1(H))) \\ (M, W) & \mapsto & (M, \sigma(W)) \end{cases}$$

is a symmetric monoidal functor.

Proof : For composition of 1-morphisms we want to show that the kernel of a gluing is the sum of the kernels. One inclusion is immediate and the other one follows by dimensions since both are Lagrangians, see [De 17, Propositions B.6.5 and B.6.6].
For composition of 2-morphisms we use Wall's theorem, see [De 17, Theorem B.6.1] for a statement in our context.
For composition of 2-morphisms in the direction of 1-morphisms we use that filled surfaces only glue on disks, hence filled 3-manifolds on 3-balls, so the signature of the filling simply adds. □

Similarly, one can restrict to non-compact cobordisms and get

$$\pi_{321}^{nc} : \mathbf{Cob}_{321}^{nc,filled} \to \widetilde{\mathbf{Cob}}_{321}^{nc}\ .$$

If we restrict $\widetilde{\mathbf{Cob}}_{321}$ to surfaces equipped with Lagrangians that are induced by some handlebody, these functors are essentially surjective, hence the name.

4.3.2 Conjectures

We want to relate the Witten–Reshetikhin–Turaev theories and their non-semisimple variants to the ones induced by the cobordism hypothesis. We want to say that the anomalous theory induced the relative pair $(\mathcal{Z}_{\mathcal{V}}, \mathcal{R}_{\mathcal{V}})$ factors through $\widetilde{\mathbf{Cob}}_{321}$ and recovers WRT and DGGPR theories.
A closer look at the WRT construction from [Tur94] shows that Witten–Reshetikhin–Turaev theories extend to the circle. Once-extended 3-TQFTs are classified in the preprint [BDSV, Theorem 3], and the following result can be obtained from it (in our case the unit is simple). We give the statement of [De 17] restricted to trivially decorated bordisms.

Theorem 4.3.7 (Theorem 1.1.1 in [De 17]): *For a semisimple modular tensor category $\mathscr{V}$ with a chosen square root of its global dimension, the Witten–Reshetikhin–Turaev TQFT extends to the circle as a symmetric monoidal functor*

$$\mathrm{WRT}_{\mathscr{V}} : \widetilde{\boldsymbol{Cob}}_{321} \to \widehat{\mathrm{Cat}}_{\Bbbk}$$

where $\widehat{\mathrm{Cat}}_{\Bbbk}$ is the category of Cauchy-complete categories.

Similarly, restricting the statement of [De 21] to trivially decorated bordisms:

Theorem 4.3.8 (Theorem 1.1 in [De 21]): *For a non-semisimple modular tensor category $\mathscr{V}$ with a chosen square root of its global dimension, the non-semisimple TQFT from [DGG+22] extends to the circle as a symmetric monoidal functor*

$$\mathrm{DGGPR}_{\mathscr{V}} : \widetilde{\boldsymbol{Cob}}^{nc}_{321} \to \widehat{\mathrm{Cat}}_{\Bbbk}$$

On the other hand, using the Cobordism Hypothesis:

Theorem 4.3.9 (Brochier–Jordan–Safronov–Snyder): *For a semisimple or non-semisimple modular tensor category $\mathscr{V}$, its Ind-cocompletion $\mathcal{V} \in \mathbf{BrTens}$ is 4-dualizable and induces under the Cobordism Hypothesis a 4-TQFT $\mathcal{Z}_{\mathcal{V}} : \mathbf{Bord}_4^{fr} \to \mathbf{BrTens}$.*

The main result of this chapter can be stated in this context.

Theorem 4.3.10: *For a semisimple modular tensor category $\mathscr{V}$, the arrow $\mathcal{A}_\eta^\flat \in \mathbf{BrTens}^{\to}$ induced by the unit inclusion $\eta : \mathrm{Vect}_{\Bbbk} \to \mathcal{V} := Ind(\mathscr{V})$ is 3-dualizable and induces under the Cobordism Hypothesis a framed oplax-$\mathcal{Z}_{\mathcal{V}}$-twisted 3-TQFT*

$$\mathcal{R}_{\mathcal{V}} : \mathbf{Bord}_3^{fr} \to \mathbf{BrTens}^{\to} \ .$$

For a non-semisimple modular tensor category $\mathscr{V}$, $\mathcal{A}_\eta^\flat$ is not 3-dualizable but is non-compact-3-dualizable and induces under the non-compact Cobordism Hypothesis a framed non-compact oplax-$\mathcal{Z}_{\mathcal{V}}$-twisted 3-TQFT

$$\mathcal{R}_{\mathcal{V}} : \mathbf{Bord}_3^{fr,nc} \to \mathbf{BrTens}^{\to} \ .$$

Proof : If $\mathscr{V}$ is semisimple, $\mathcal{V} = Ind(\mathscr{V}) = Free(\mathscr{V})$ and Theorem 4.2.14 applies. If $\mathscr{V}$ is not semisimple, the unit is not projective in $\mathscr{V}$, nor in $\mathcal{V} = Ind(\mathscr{V})$, so $\mathcal{A}_\eta^\flat$ is not 3-dualizable. But $\mathcal{V}$ is cp-rigid and Theorem 4.2.15 applies. □

To compare the two sides, we need all theories to be oriented. We assume the following:

> **Conjecture 4.3.11:** *The ribbon structure of $\mathscr{V}$ induces an $SO(3)$-homotopy-fixed-point structure on $\mathcal{V}$.*
> *The ribbon structure of η induces an $SO(3)$-homotopy-fixed-point structure on $\mathcal{A}_\eta^\flat$.*

The first statement is expected by experts. The second one follows [Lur09b, Example 4.3.23]. Note that in the second statement we actually mean an $SO(3)$-homotopy-fixed-point structure compatible with the one on $\mathcal{V}$, as in Remark 4.1.18.

Remark 4.3.12: The fact that the anomalous theory $\mathcal{A}_\mathcal{V}$ would factor through $\widetilde{\mathbf{Cob}}_{321}$ is not too surprising. As was pointed to me by Pavel Safronov, we know from [BJSS21] that $\mathcal{V}$ is not only 4-dualizable, but invertible, and hence 5-dualizable. But **BrTens** has no non-trivial 5-morphisms, and hence the 5-theory induced by $\mathcal{V}$ is trivial on 5-bordisms. This means that $\mathcal{Z}_\mathcal{V}$ should give the same value on cobordant 4-manifolds. If this story can be made oriented, it means it depends only on the signature of 4-manifolds.
It was observed by Walker [Walb, Chapter 9] in the semisimple case that there is a scalar choice of ways to extend $\mathcal{Z}_\mathcal{V}$ from $\mathbf{Bord}_3^{or}$ to $\mathbf{Bord}_4^{or}$, namely $\mathcal{Z}_\mathcal{V}(B^4)$, and that exactly two of these scalars yield to a theory which is cobordant-invariant on 4-manifolds. These scalars are exactly the two square roots of the global dimension among which one has to choose when defining WRT theories. This motivates the following conjecture. In the non-semisimple case, it is supported by the fact that the constructions of the (3+1)-TQFTs of Chapter 3 need exactly the choice of a modified trace. ◇

> **Conjecture 4.3.13:** *A choice of modified trace on $\mathscr{V}$ induces an $SO(4)$-homotopy-fixed-point structure on $\mathcal{V}$.*
> *A modified trace induces an $SO(5)$-homotopy-fixed-point structure on $\mathcal{V}$ if and only if $\mathscr{S}_\mathscr{V}(S^4) = 1$ with this choice of modified trace in the construction of Chapter 3.*

In particular, we conjecture that every modular tensor category has an $SO(5)$-homotopy-fixed-point structure. Indeed let $\mathscr{V}$ be a modular tensor category and choose t a modified trace on $\mathscr{V}$ (which exists by [GKP22, Corollary 5.6]). Choose a square root $\mathscr{D}$ of its global dimension $\zeta = \Delta_+\Delta_- = \mathscr{S}_{\mathscr{V},\mathsf{t}}(S^4)$ as defined in Chapter 3. Then the modified trace $\mathscr{D}^{-1}\mathsf{t}$ satisfies $\mathscr{S}_{\mathscr{V},\mathscr{D}^{-1}\mathsf{t}}(S^4) = 1$ by Proposition 3.4.7.

> **Corollary 4.3.14 (of conjectures):** *Both $\mathcal{Z}_\mathcal{V}$ and $\mathcal{R}_\mathcal{V}$ give oriented TQFTs by the oriented cobordism hypothesis.*

We now assume that this corollary is true, that a the choice of modified and square root of global dimension has been made, and that $\mathcal{Z}_\mathcal{V}$ and $\mathcal{R}_\mathcal{V}$ are oriented.
In the semisimple case, the relative pair

$$(\mathcal{Z}_\mathcal{V} : \mathbf{Bord}_4 \to \mathbf{BrTens}, \mathcal{R}_\mathcal{V} : \mathbf{Bord}_3 \to \mathbf{BrTens}^\to)$$

induces an anomalous theory

$$\mathcal{A}_{\mathcal{V}} : \mathbf{Bord}_3^{filled,\ odd\ opp} \to \mathbf{Tens} := \Omega\mathbf{BrTens}\ .$$

Its restriction on $\mathbf{Cob}_{321}^{filled}$ gives a 2-functor

$$\mathcal{A}_{\mathcal{V}}^{321} : \mathbf{Cob}_{321}^{filled} \to \Omega\mathbf{Tens} \simeq \mathbf{Pr}\ .$$

Conjecture 4.3.15: *For a semisimple modular tensor category $\mathscr{V}$, the anomalous theory induced by $(\mathcal{Z}_{\mathcal{V}}, \mathcal{R}_{\mathcal{V}})$ recovers the Witten–Reshetikhin–Turaev theory. Namely:*

$$\begin{array}{ccc} \boldsymbol{Cob}_{321}^{filled} & \xrightarrow{\mathcal{A}_{\mathcal{V}}^{321}} & \mathbf{Pr} \\ {\scriptstyle \pi_{321}}\downarrow & & \uparrow\downarrow{\scriptstyle \mathrm{Free}} \\ \widetilde{\boldsymbol{Cob}}_{321} & \xrightarrow{\mathrm{WRT}_{\mathscr{V}}} & \widehat{\mathrm{Cat}}_{\Bbbk} \end{array}$$

commutes up to isomorphism.

In the non-semisimple case, the relative pair

$$(\mathcal{Z}_{\mathcal{V}} : \mathbf{Bord}_4 \to \mathbf{BrTens}, \mathcal{R}_{\mathcal{V}} : \mathbf{Bord}_3^{nc} \to \mathbf{BrTens}^{\to})$$

induces an anomalous theory

$$\mathcal{A}_{\mathcal{V}} : \mathbf{Bord}_3^{nc, filled,\ odd\ opp} \to \mathbf{Tens} := \Omega\mathbf{BrTens}\ .$$

Its restriction on $\mathbf{Cob}_{321}^{nc,filled}$ gives a 2-functor

$$\mathcal{A}_{\mathcal{V}}^{321} : \mathbf{Cob}_{321}^{nc,filled} \to \Omega\mathbf{Tens} \simeq \mathbf{Pr}\ .$$

Conjecture 4.3.16: *For a non-semisimple modular tensor category $\mathscr{V}$, the non-compact anomalous theory induced by $(\mathcal{Z}_{\mathcal{V}}, \mathcal{R}_{\mathcal{V}})$ recovers the De Renzi–Gainutdinov–Geer–Patureau-Mirand–Runkel theory. Namely:*

$$\begin{array}{ccc} \boldsymbol{Cob}_{321}^{nc,filled} & \xrightarrow{\mathcal{A}_{\mathcal{V}}^{321}} & \mathbf{Pr} \\ {\scriptstyle \pi_{321}^{nc}}\downarrow & & \uparrow\downarrow{\scriptstyle \mathrm{Free}} \\ \widetilde{\boldsymbol{Cob}}_{321}^{nc} & \xrightarrow{\mathrm{DGGPR}_{\mathscr{V}}} & \widehat{\mathrm{Cat}}_{\Bbbk} \end{array}$$

commutes up to isomorphism.

We know how to check these conjectures on the circle. We have $\mathrm{WRT}_{\mathscr{V}}(S^1) = \mathscr{V}$ whose free cocompletion is equivalent to $\mathcal{V}$ because $\mathscr{V}$ is semisimple. Similarly, $\mathrm{DGGPR}_{\mathscr{V}}(S^1) = \mathrm{Proj}(\mathscr{V})$ whose free cocompletion is equivalent to $\mathcal{V}$. On the other side, we know that in dimension two $\mathcal{Z}_{\mathcal{V}}$ coincides with factorization homology, and we computed $\mathcal{R}_{\mathcal{V}}(S^1)$ in Section 4.2.3. So:

$$\mathcal{A}_{\mathcal{V}}^{321}(S^1, \mathbb{D}^2) = \mathcal{R}_{\mathcal{V}}(S^1) \underset{\mathcal{Z}_{\mathcal{V}}(S^1)}{\boxtimes} \mathcal{Z}_{\mathcal{V}}(\mathbb{D}^2) \simeq {}_{\mathrm{Vect}_{\Bbbk}}\mathcal{Z}_{\mathcal{V}}(S^1) \underset{\mathcal{Z}_{\mathcal{V}}(S^1)}{\boxtimes} \mathcal{V}_{\mathrm{Vect}_{\Bbbk}} \simeq {}_{\mathrm{Vect}_{\Bbbk}}\mathcal{V}_{\mathrm{Vect}_{\Bbbk}}$$

Computing the values of the theories induced by the Cobordism Hypothesis on higher dimensional bordisms comes down to computing some adjoints in **BrTens** and compose them in various ways. This will be carried out in future work.

Corollary 4.3.17 (of conjectures): *Both* $\mathrm{WRT}_{\mathscr{V}}$ *and* $\mathrm{DGGPR}_{\mathscr{V}}$ *extend to* S^0.

Proof : Indeed, the anomalous theory $\mathcal{A}_{\mathcal{V}}$ is really defined as a functor between the 3-categories $\mathbf{Bord}_3^{filled} \to \mathbf{Tens}$ (or $\mathbf{Bord}_3^{nc,filled} \to \mathbf{Tens}$ in the non-semisimple case). The two points S^0 are bordant, by a cap, and therefore give an object $(S^0, \cap) \in \mathbf{Bord}_3^{filled}$ (or $\mathbf{Bord}_3^{nc,filled}$).
It is easy to compute the value of the anomalous theory on this object, namely $\mathcal{A}_{\mathcal{V}}(S^0, \cap) = \mathcal{R}_{\mathcal{V}}(S^0) \circ \mathcal{Z}_{\mathcal{V}}(\cap) = (\mathcal{A}_\eta \boxtimes (\overline{\mathcal{A}}_\eta)^*) \underset{\mathcal{V}\boxtimes\mathcal{V}^{\sigma op}}{\boxtimes} \mathcal{V} \simeq \mathcal{V}$ seen as a $\mathrm{Vect}_{\Bbbk}$-$\mathrm{Vect}_{\Bbbk}$-central algebra. □

Remark 4.3.18: This is to be compared with results of [DSS20] which shows that $\mathrm{WRT}_{\mathscr{V}}$ extends to the point if and only if $\mathscr{V} \simeq Z(\mathscr{C})$ is a Drinfeld center, in which case the point is mapped to $\mathscr{C}$. In the modular case, the Drinfeld center $Z(\mathscr{C})$ is isomorphic to $\mathscr{C} \otimes \mathscr{C}^{\sigma op}$, and the two descriptions agree on S^0. Therefore it appears that $\mathrm{WRT}_{\mathscr{V}}$ always extends to S^0, and extends to the point if and only if one can find a "square root" for its value on S^0. This is also related to ongoing work of Freed, Teleman and Scheimbauer.
Note however that the statement above is a bit informal, because it is really $\mathrm{Free} \circ \mathrm{WRT}_{\mathscr{V}} \circ \pi_{321}$ that extends to S^0, so WRT indeed but with different source and target. In particular, the results of [DSS20] do not apply directly in this context. ◇

Chapter 5

Stated versus internal skein algebras

This chapter is based on [Haï22].

We give an explicit correspondence between stated skein algebras, which are defined via explicit relations on stated tangles in [Lê18], and internal skein algebras, which are defined as internal endomorphism algebras in free cocompletions of skein categories in [BBJ18a, GJS23].

In Section 5.1 we recall the structure on stated skein algebras and skein categories induced by a boundary edge. The first gives an $\mathcal{O}_{q^2}(SL_2)$-comodule algebra structure on stated skein algebras, which enables one to express their gluing properties [CL22]. The second gives a structure of module categories on skein categories, which enables one to express gluing properties too, and to define [BBJ18a, GJS23]'s internal skein algebras.

In Section 5.2, we explicitly give the natural isomorphism St exhibiting the stated skein algebra as the (left) internal endomorphism algebra of the empty set in the skein category associated with $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)\text{–}comod^{fin} \simeq \mathcal{U}_{q^2}(\mathfrak{sl}_2)\text{–}mod^{fin}$, without relying on excision properties. On an object of the Temperley-Lieb category, this natural isomorphism simply expresses that assigning states to a tangle defines an $\mathcal{O}_{q^2}(SL_2)$-comodule morphism from some tensor product of the fundamental representation to the stated skein algebra.

In Section 5.3, we complete the picture. We extend in a straightforward manner the definition of internal skein algebras to surfaces with multiple boundary edges, labelled either left or right. The left and right actions differ by rotating the picture by 180 degrees, which is homotopic to the identity by the half twist . We give an explicit way to compare internal skein algebras obtained from right and left edges using the half twist, which induces a braided opposite algebra structure, see Proposition 5.3.6. In Section 5.3.4, we describe the action of the half twist in the context of stated skein algebras, i.e. on $\mathcal{O}_{q^2}(SL_2)\text{–}comod^{fin}$. There are multiple choices, but as we explain ours is imposed by conventions from stated skein algebras. We extend the main theorem to surfaces with multiple boundary edges, and appropriate braided opposites inserted, see Theorem 5.3.28. Finally we compare excision properties. We prove excision properties of internal skein algebras for any ribbon category in Theorem 5.3.32. We prove that this recovers the usual statements for stated skein algebras for our choice of half-twist. This sheds light on the choice in the definition of stated skein algebras. Indeed it is shown in [Lê18, Section 3.4] there is essentially no choice for the boundary skein relations, if we want the cutting morphism to be well-defined. We claim there is choice in how to write this cutting morphism, which corresponds to the choice of the half-twist, see Remark 5.3.34.

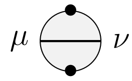
5.1 Boundary structure and gluing properties

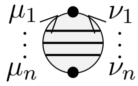
We recall the extra structure on stated skein algebras [CL22] and skein categories [Coo23] that come from a boundary edge. This second structure allows us to define internal skein algebras [BBJ18a, GJS23].

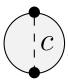
Let $\Bbbk$ be either $\mathbb{Q}(q^{\frac{1}{2}})$ or $\mathbb{C}$ with $q^{\frac{1}{2}} \in \mathbb{C}^\times$ generic, i.e. not a root of unity.

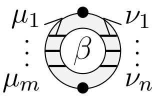
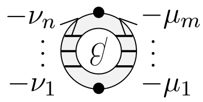
5.1.1 Stated skein algebras

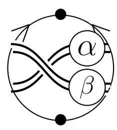
The key player for stated skein algebras will be the stated skein algebra of the bigon. Indeed, one can cut a bigon out of a boundary edge of any stated skein algebra, giving an extra comodule structure for every boundary edge.

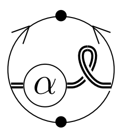
Example 5.1.1: The bigon B is the marked surface $(\mathbb{D}, \{\pm i\})$, the disk with two marked points. The algebra $\mathscr{S}(B)$ is generated as an algebra by the ${}_\mu\beta_\nu =$  , $\mu, \nu \in \{\pm\}$,

and has basis the ${}_{\vec{\mu}}\beta_{\vec{\nu}} =$  where $\vec{\mu} = (\mu_1, \dots, \mu_n)$ and $\vec{\nu} = (\nu_1, \dots, \nu_n)$ are decreasing sequences of signs.

It is a bialgebra with coproduct given by cutting along the "unique" arc joining the two marked points , $\Delta = \rho_c : \mathscr{S}(B) \to \mathscr{S}(B \sqcup B) \simeq \mathscr{S}(B) \otimes \mathscr{S}(B)$. Coassociativity comes from the second part of Theorem 1.1.10. The counit $\varepsilon : \mathscr{S}(B) \to \Bbbk$ is defined on the basis by $\varepsilon({}_{\vec{\mu}}\beta_{\vec{\nu}}) = \delta_{\vec{\mu},\vec{\nu}}$.

It is a Hopf algebra with antipode $S\Big($  $\Big) =$  $\cdot \dfrac{C(\vec{\nu})}{C(\vec{\mu})}$ where $C(\vec{\nu}) := \prod_{i=1}^{n} C(\nu_i)$.

It is coquasitriangular with co-R-matrix $R(\alpha \otimes \beta) = \varepsilon\Big($  $\Big)$, see [CL22, Theorem 3.5].

It is coribbon with coribbon functional $\theta(\alpha) = \varepsilon\Big($  $\Big)$. ♢

> **Proposition 5.1.2 (Thm. 4.1 in [Lê18], Sec. 2.2 in [KQ], Thm. 3.4 [CL22]):** *One has an isomorphism of coribbon Hopf algebras $\mathscr{S}(B) \simeq \mathcal{O}_{q^2}(SL_2)$ given on the generators by ${}_+\beta_+ \mapsto a$, ${}_-\beta_- \mapsto d$, ${}_+\beta_- \mapsto b$ and ${}_-\beta_+ \mapsto c$.*

Definition 5.1.3: Consider a marked surface $\mathfrak{S}$ and a boundary edge e of $\mathfrak{S}$. Let c be an ideal arc parallel to e inside $\mathring{\mathfrak{S}}$, see Figure 5.1. Cutting along c splits $\mathfrak{S}$ into a bigon (between e and c) and a surface canonically homeomorphic to $\mathfrak{S}$. Therefore the splitting morphism along c goes

$$\Delta = \rho_c : \mathscr{S}(\mathfrak{S}) \to \mathscr{S}(\mathfrak{S} \sqcup B) \simeq \mathscr{S}(\mathfrak{S}) \otimes \mathscr{S}(B)$$

and endows $\mathscr{S}(\mathfrak{S})$ with right $\mathcal{O}_{q^2}(SL_2)$-comodule structure. It is compatible with its algebra structure, namely $\mathscr{S}(\mathfrak{S})$ is an $\mathcal{O}_{q^2}(SL_2)$-comodule-algebra, see [CL22, Proposition 4.1]. ♢

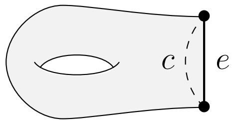
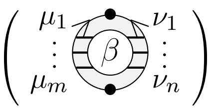
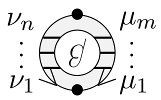
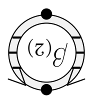
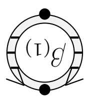


Figure 5.1: The comodule structure on stated skein algebras

Definition 5.1.4: Let $rot : B \to B$ be the homeomorphism of marked surfaces given by the planar 180 degree rotation. It induces an algebra isomorphism

$rot_* : \mathscr{S}(B) \to \mathscr{S}(B)$, with rot_*  $=$  .

It reverses the coproduct, namely

$\Delta \circ rot_*(\beta) =$  $\otimes$  $= (rot_* \otimes rot_*) \circ \Delta^{op}(\beta)$,

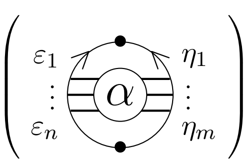
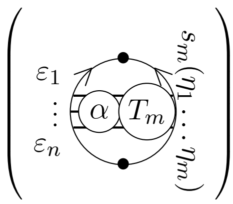
and preserves the counit, because $\varepsilon(rot_*({}_{\vec{\mu}}\beta_{\vec{\nu}})) = \varepsilon({}_{\vec{\nu}}\beta_{\vec{\mu}}) = \delta_{\vec{\nu},\vec{\mu}} = \delta_{\vec{\mu},\vec{\nu}}$.
On $\mathcal{O}_{q^2}(SL_2)$, it is given by $r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. ♢

Remark 5.1.5: If one sees the edge e at the left instead of the right of the surface, one gets a structure of left $\mathcal{O}_{q^2}(SL_2)$-comodule. One can easily get from one to another by rotating the whole picture, see Figure 5.2. Namely, the left coaction Δ_l is obtained from the right coaction Δ_r by rotating the bigon by 180 degrees. In [CL22, Proposition 4.1] one gets

$$\Delta_l = fl \circ (Id_{\mathscr{S}(\mathfrak{S})} \otimes rot_*) \circ \Delta_r \ ,$$

where fl denotes the flip of tensors.
Actually, one gets such a structure for each boundary edge of $\mathfrak{S}$, and if $\mathfrak{S}$ has n right boundary edges and m left, $\mathscr{S}(\mathfrak{S})$ is an $(\mathcal{O}_{q^2}(SL_2)^{\otimes n}, \mathcal{O}_{q^2}(SL_2)^{\otimes m})$-bicomodule algebra.♢

Remark 5.1.6: The structure forms on $\mathscr{S}(B)$, such as the co-R-matrix or the coribbon functional, are often defined using the counit on some transformation of the tangle. This has a direct interpretation on how this form then acts on comodules:

Let $\varphi : \mathscr{S}(B) \to \Bbbk$ be given by some φ  $= \varepsilon$ , where

T_m, $m \in \mathbb{N}$, is a family of tangles with m right and m left boundary points and s_m :

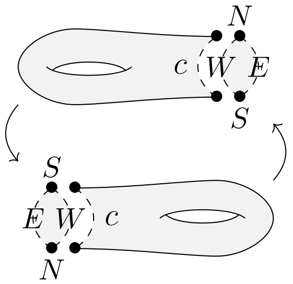
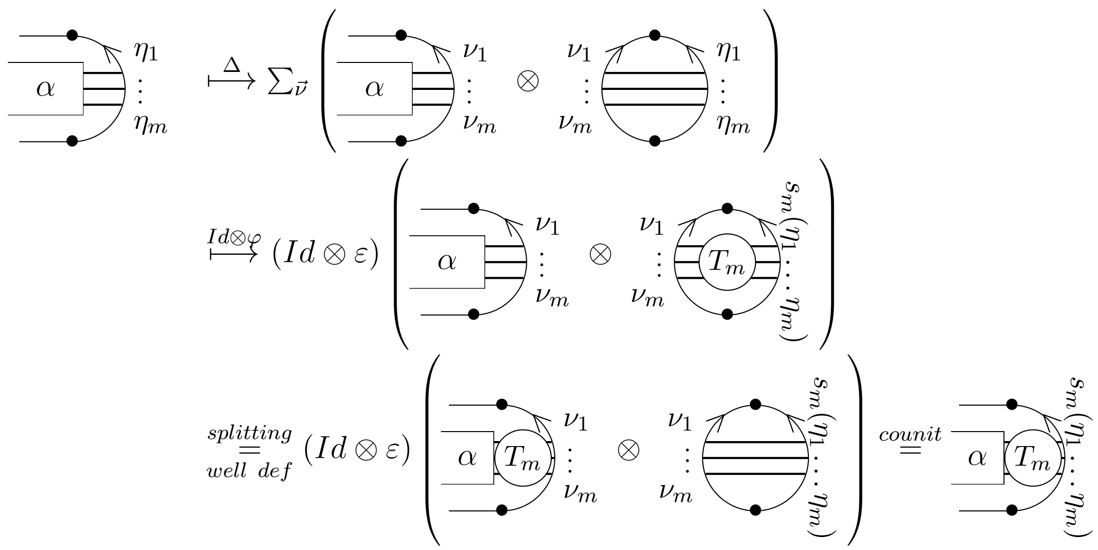


Figure 5.2: From left to right boundary edges for stated skein algebras

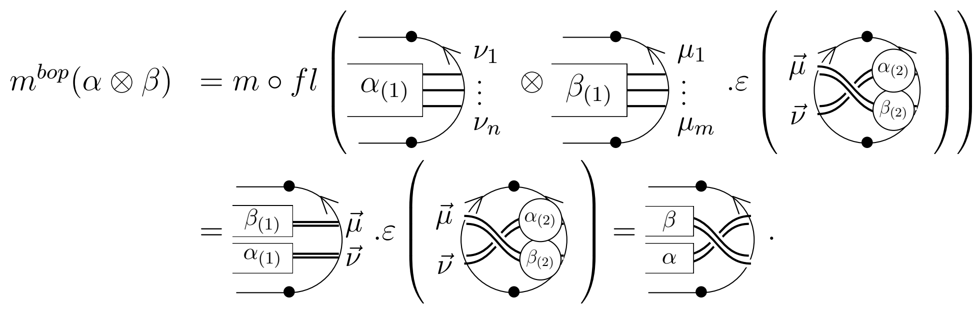
$\{\pm\}^m \to \Bbbk\langle\{\pm\}^m\rangle$. Here, we allowed s_m to have values in formal linear combination of m-tuples of states because in the definition of the co-R-matrix for example one needs coefficients depending on the states. What we mean by a tangle α with state a formal linear combination of states $\sum_i \lambda_i \vec{\eta}_i$ is the linear sum of stated tangles $\alpha_{\sum_i \lambda_i \vec{\eta}_i} := \sum_i \lambda_i \alpha_{\vec{\eta}_i}$. Note that the T_m's and the s_m's should satisfy extra conditions for this to be well-defined on $\mathscr{S}(B)$. Then for a marked surface $\mathfrak{S}$ with a right edge e, the map

$$\Phi_{\mathscr{S}(\mathfrak{S})} : \mathscr{S}(\mathfrak{S}) \xrightarrow{\Delta} \mathscr{S}(\mathfrak{S}) \otimes \mathscr{S}(B) \xrightarrow{Id\otimes\varphi} \mathscr{S}(\mathfrak{S})$$

is given by:



The co-R-matrix isn't exactly of this form, but the same kind of computation applies. Remember that the braiding on $\mathscr{S}(\mathfrak{S}) \otimes \mathscr{S}(\mathfrak{S})$ is defined using the coaction on each $\mathscr{S}(\mathfrak{S})$, the co-$R$-matrix on the $\mathscr{S}(B)^{\otimes 2}$ part thus obtained, and then flipping the two factors, namely $c_{\mathscr{S}(\mathfrak{S}),\mathscr{S}(\mathfrak{S})} = fl \circ R_{24} \circ (\Delta_{\mathscr{S}(\mathfrak{S})} \otimes \Delta_{\mathscr{S}(\mathfrak{S})})$. The braided opposite product on $\mathscr{S}(\mathfrak{S})$ is defined as $m^{bop} := m \circ c_{\mathscr{S}(\mathfrak{S}),\mathscr{S}(\mathfrak{S})}$ and it has a nice geometric depiction, namely:



◇

Endowed with extra boundary structure, stated skein algebras satisfy a form of excision. The stated skein algebra of a gluing is obtained as a from of relative tensor product of the stated skein algebras of the two initial surfaces.

Definition 5.1.7: Let H be a Hopf algebra and M an (H,H)-bicomodule, with coproducts denoted by $\Delta_1 : M \to M \otimes H$ and $\Delta_2 : M \to H \otimes M$. The 0-th Hochschild cohomology of M is the subalgebra of M defined as $HH^0(M) := \{x \in M \ / \ \Delta_1(x) = fl \circ \Delta_2(x)\}$.$\diamondsuit$

Let $\mathfrak{S}'$ be a marked surface and c_1 and c_2 respectively a right and a left boundary edges of $\mathfrak{S}'$ as in Figure 5.3. Then $\mathscr{S}(\mathfrak{S}')$ has a structure of $(\mathcal{O}_{q^2}(SL_2), \mathcal{O}_{q^2}(SL_2))$-bicomodule. We consider $\mathfrak{S} = \mathfrak{S}'/_{c_1=c_2}$ the marked surface obtained by gluing c_1 to c_2, and c the ideal arc formed by $c_1 = c_2$ in $\mathfrak{S}$. We have $\mathfrak{S}' = Cut_c(\mathfrak{S})$, and Theorem 1.1.10 gives an injective algebra morphism $\rho_c : \mathscr{S}(\mathfrak{S}) \to \mathscr{S}(\mathfrak{S}')$.

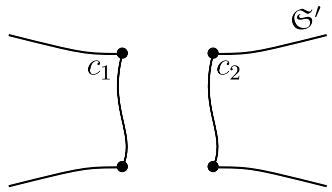


Figure 5.3: Gluing surfaces

Theorem 5.1.8 (Section 2.3 in [KQ], Theorem 4.8 in [CL22]): *Consider a marked surface $\mathfrak{S} = \mathfrak{S}'/_{c_1=c_2}$ obtained as a gluing. Then the splitting morphism $\rho_c : \mathscr{S}(\mathfrak{S}) \to \mathscr{S}(\mathfrak{S}')$ maps isomorphically $\mathscr{S}(\mathfrak{S})$ onto $HH^0(\mathscr{S}(\mathfrak{S}'))$.*

Remark 5.1.9: The formula is slightly nicer if $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two marked surfaces, c_1 is a right boundary edge of $\mathfrak{S}_1$, c_2 a left boundary edge of $\mathfrak{S}_2$ and $\mathfrak{S}' = \mathfrak{S}_1 \sqcup \mathfrak{S}_2$. Then $\mathscr{S}(\mathfrak{S}') \simeq \mathscr{S}(\mathfrak{S}_1) \otimes_{\Bbbk} \mathscr{S}(\mathfrak{S}_2)$ (this is the vector space tensor product) and the 0-th Hochschild cohomology of $\mathscr{S}(\mathfrak{S}')$ corresponds to the cotensor product

$$\mathscr{S}(\mathfrak{S}_1)\square_H\mathscr{S}(\mathfrak{S}_2) := \{x \in \mathscr{S}(\mathfrak{S}_1) \otimes \mathscr{S}(\mathfrak{S}_2) \ / \ \Delta_1 \otimes \mathrm{Id}_2(x) = \mathrm{Id}_1 \otimes \Delta_2(x)\}$$

of $\mathscr{S}(\mathfrak{S}_1)$ and $\mathscr{S}(\mathfrak{S}_2)$ over $\mathscr{S}(B)$. $\diamondsuit$

5.1.2 Skein categories

Remember the definition of the coribbon Hopf algebra $\mathcal{O}_{q^2}(SL_2)$, the description of its category of right comodules and its links with left $\mathcal{U}_{q^2}(\mathfrak{sl}_2)$-modules from Section 1.2.4.

Remark 5.1.10 (Remark 1.7 in [Coo23]): For a general surface Σ, its skein category is not monoidal because there is no notion of horizontal juxtaposition, which we use in $\mathbb{R}^2$. However, if $A = C \times [0,1]$ for a 1-manifold C, the category $Sk_{\mathcal{V}}(A)$ is monoidal with tensor product induced by $A \sqcup A \overset{[0,\frac{1}{3}]\sqcup[\frac{2}{3},1]}{\hookrightarrow} A$. $\diamondsuit$

Remark 5.1.11 (Remarks 1.6 and 1.18 in [Coo23]): An orientation-preserving smooth embedding $f : \Sigma_1 \to \Sigma_2$ induces a functor $Sk_{\mathcal{V}}(f) : Sk_{\mathcal{V}}(\Sigma_1) \to Sk_{\mathcal{V}}(\Sigma_2)$. It maps an object s, which is a bunch of coloured points in Σ_1, to $f(s)$, and a ribbon graph $T \subseteq \Sigma_1 \times [0,1]$ to $(f \times Id)(T)$. This defines a symmetric monoidal functor

$$Sk_{\mathcal{V}} : (\mathbf{Man}_2^{or}, \sqcup) \to (\mathrm{Cat}_{\Bbbk}, \otimes_{\Bbbk}) \ .$$

An isotopy of smooth embeddings $\lambda : \Sigma_1 \times [0,1] \to \Sigma_2$ between $f = \lambda_0$ and $g = \lambda_1$ induces a natural isomorphism $rib_\lambda : Sk_\mathcal{V}(f) \Rightarrow Sk_\mathcal{V}(g)$ where $rib_{\lambda,s} : f(s) \to g(s)$ is the braid in $\Sigma_2 \times [0,1]$ drawn by $\{(\lambda_t(s), t), t \in [0,1]\}$. Homotopic isotopies give isotopic ribbon graphs, and this extends to a symmetric monoidal functor of $(\infty,1)$-categories

$$Sk_\mathcal{V} : (\mathrm{Mfld}_2^{or}, \sqcup) \to (\mathrm{Cat}_\Bbbk, \otimes_\Bbbk) \ .$$

It is shown in [Coo23] that this functor coincides with factorisation homology [AFT17] with coefficients in $\mathcal{V}$. ◇

Remark 5.1.12 (Section 1.3 of [Coo23]): As for stated skein algebras, whenever we talk of a boundary component $C \subseteq \Sigma$ below, we mean thickened boundary component, i.e. equipped with a trivialization of its neighborhood. This notion distinguished left and right boundary components, equipped respectively with an embedding $C \times [0,2) \hookrightarrow \Sigma$ and $C \times (-1,1] \hookrightarrow \Sigma$. We denote $A := C \times [0,1]$. By retracting Σ away from its boundary, we have a (canonical given the data above) embedding $A \sqcup \Sigma \to \Sigma$.
This embedding induces an action

$$\triangleright : Sk_\mathcal{V}(A) \otimes Sk_\mathcal{V}(\Sigma) \to Sk_\mathcal{V}(\Sigma)$$

which endows $Sk_\mathcal{V}(\Sigma)$ with a structure of left or right $Sk_\mathcal{V}(A)$-module category. ◇

Skein categories satisfy a form of excision, namely the skein category of a gluing is obtained as a relative tensor product of the skein categories of the initial surfaces.
Let Σ_1 and Σ_2 be two surfaces and C a curve which is both a right boundary component of Σ_1 and a left boundary component of Σ_2. We denote $\Sigma_1 \cup_A \Sigma_2$ the collar gluing of the surfaces along the two embeddings of A.

Theorem 5.1.13 (Theorem 1.22 in [Coo23]): *The skein category of a collar gluing is obtained as the relative tensor product*

$$Sk_\mathcal{V}(\Sigma_1 \cup_A \Sigma_2) \simeq Sk_\mathcal{V}(\Sigma_1) \underset{Sk_\mathcal{V}(A)}{\otimes} Sk_\mathcal{V}(\Sigma_2)$$

of the right $Sk_\mathcal{V}(A)$-module $Sk_\mathcal{V}(\Sigma_1)$ and the left $Sk_\mathcal{V}(A)$-module $Sk_\mathcal{V}(\Sigma_2)$.

Proof (sketch): Here relative tensor product may be taken to be either Tambara's relative tenbsor product, or the usual 2-colimit definition, which are equivalent by [Coo23, Theorem 2.27].
One has an explicit description of the Tambara relative tensor product of two module categories, by formally adding a balancing natural isomorphism ι. For s_1, a, s_2 some sets of colored points respectively in Σ_1, A, Σ_2, the image of the balancing isomorphism $\iota_{s_1,a,s_2} : (s_1 \triangleleft a, s_2) \to (s_1, a \triangleright s_2)$ in $Sk_\mathcal{V}(\Sigma_1 \cup_A \Sigma_2)$ is depicted in Figure 5.4. □

Corollary 5.1.14: *Let $s_1 \in Sk_\mathcal{V}(\Sigma_1)$ and $s_2 \in Sk_\mathcal{V}(\Sigma_2)$. Then any morphism*

$$\alpha \in \mathrm{Hom}_{Sk_\mathcal{V}(\Sigma_1 \cup_A \Sigma_2)}(s_1 \sqcup s_2, \emptyset)$$

can be decomposed in a (linear combination of) pair(s)

$$\alpha_1 \in \mathrm{Hom}_{Sk_\mathcal{V}(\Sigma_1)}(s_1, \emptyset \triangleleft a) \qquad , \qquad \alpha_2 \in \mathrm{Hom}_{Sk_\mathcal{V}(\Sigma_2)}(a \triangleright s_2, \emptyset)$$

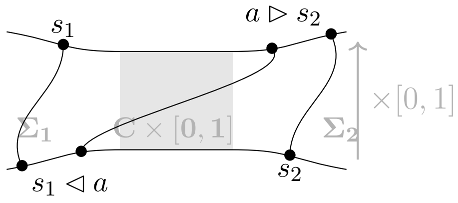
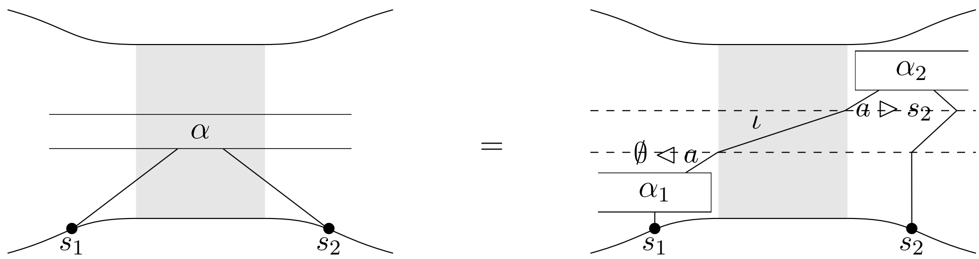


Figure 5.4: The skein category of the gluing is balanced

for some $a \in Sk_{\mathcal{V}}(A)$, with $\alpha = (Id_\emptyset, \alpha_2) \circ \iota_{\emptyset,a,s_2} \circ (\alpha_1, Id_{s_2})$.
This decomposition is unique up to balancing, namely if α_2 can be written $\beta_2 \circ (\gamma \rhd Id_{s_2})$, with $\beta_2 \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma_2)}(b \rhd s_2, \emptyset)$ and $\gamma \in \mathrm{Hom}_{Sk_{\mathcal{V}}(A)}(a, b)$, for some $b \in Sk_{\mathcal{V}}(A)$, then $(\alpha_1, \beta_2 \circ (\gamma \rhd Id_{s_2})) \sim ((Id_\emptyset \lhd \gamma) \circ \alpha_1, \beta_2)$.

PROOF : On a drawing one wants to decompose α as:



which is easily done by pushing the ribbon graph happening in the middle region inside say Σ_2 leaving only straight lines (namely, ι's) behind.
The relation $(\alpha_1, \beta_2 \circ (\gamma \rhd Id_{s_2})) \sim ((Id_\emptyset \lhd \gamma) \circ \alpha_1, \beta_2)$ is true by sliding γ along the straight lines of ι, and this is the only relation by the theorem above. □

Note that the asymmetry in this description is purely artificial, and one could have chosen a cup or a cap instead of a slanted line to link the left and right actions.

5.1.3 Internal skein algebras

When a surface has a boundary edge, one can push a little disk inside the surface from this boundary edge. This process induces an action of the skein category of the disk on the skein category of the surface. Namely, $Sk_{\mathcal{V}}(\Sigma)$ is a $Sk_{\mathcal{V}}(\mathbb{R}^2)$-module category, see [Coo23, Sections 3.2]. In this situation there is a notion of internal Hom objects that encode entirely in $Sk_{\mathcal{V}}(\mathbb{R}^2)$ the behavior of objects of $Sk_{\mathcal{V}}(\mathbb{R}^2)$ seen in $Sk_{\mathcal{V}}(\Sigma)$ by the action. Namely for fixed $V, W \in Sk_{\mathcal{V}}(\Sigma)$ one has $\mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(X \rhd V, W) \simeq \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathbb{R}^2)}(X, \underline{\mathrm{Hom}}(V, W))$ naturally in X, see [EGNO15, Section 7.9]. However, such an object does not always exist in $Sk_{\mathcal{V}}(\mathbb{R}^2)$, and actually lives in its free cocompletion. The internal skein algebra A_Σ of the surface is the internal endomorphism algebra of the empty set $\underline{\mathrm{Hom}}(\emptyset, \emptyset)$, see [GJS23] or [BBJ18a] together with [Coo23]. This means one can understand ribbon graphs in $Sk_{\mathcal{V}}(\Sigma)$ with boundary points on the bottom and near the boundary edge as morphisms in (the free cocompletion of) $Sk_{\mathcal{V}}(\mathbb{R}^2)$ with target A_Σ.
Note that in [Coo23], [BBJ18a] and [GJS23] one uses a right $Sk_{\mathcal{V}}(\mathbb{R}^2)$-action and we use a left $Sk_{\mathcal{V}}(\mathbb{R}^2)$-action here. See Section 5.3 for more details.

Definition 5.1.15 (See Section 7.9 of [EGNO15]): Let $\mathcal{M}$ be a $\mathcal{A}$-module category and M_1, M_2 in $\mathcal{M}$. The $\mathcal{A}$-*internal* Hom $\underline{\mathrm{Hom}}(M_1, M_2)$ of M_1 and M_2 is the object of $\mathcal{A}$ representing $\mathrm{Hom}_{\mathcal{M}}(- \triangleright M_1, M_2) : \mathcal{A}^{op} \to \mathrm{Vect}_{\Bbbk}$. It comes equipped with a natural isomorphism

$$\eta : \mathrm{Hom}_{\mathcal{M}}(- \triangleright M_1, M_2) \tilde{\to} \mathrm{Hom}_{\mathcal{A}}(-, \underline{\mathrm{Hom}}(M_1, M_2)) \ .$$

It is unique up to canonical isomorphism when it exists.
When all involved internal Hom objects exist, this natural isomorphism defines

$$ev_{M_1,M_2} : \underline{\mathrm{Hom}}(M_1, M_2) \triangleright M_1 \to M_2$$

and

$$c : \underline{\mathrm{Hom}}(M_2, M_3) \otimes \underline{\mathrm{Hom}}(M_1, M_2) \to \underline{\mathrm{Hom}}(M_1, M_3) \ .$$

In particular, an internal endomorphism $\underline{\mathrm{End}}(M) := \underline{\mathrm{Hom}}(M, M)$ is an algebra object in $\mathcal{A}$, with unit the morphism $1_{\mathcal{A}} \to \underline{\mathrm{End}}(M)$ associated with Id_M. ♢

The functor $\mathrm{Hom}_{\mathcal{M}}(- \triangleright M_1, M_2) : \mathcal{A}^{op} \to \mathrm{Vect}_{\Bbbk}$ cannot always be represented in $\mathcal{A}$. However, it is always an object of its free cocompletion.

Proposition 5.1.16: *Let $\mathcal{A} \in \mathrm{Cat}_{\Bbbk}$ be a monoidal $\Bbbk$-linear category, $\mathcal{M}$ an $\mathcal{A}$-module category and M_1, M_2 two objects of $\mathcal{M}$. The presheaf*

$$F = \mathrm{Hom}_{\mathcal{M}}(- \triangleright M_1, M_2) \in \mathrm{Free}(\mathcal{A})$$

is the internal Hom *object of M_1 and M_2 (seen as objects of* $\mathrm{Free}(\mathcal{M})$ *by the Yoneda embedding) with respect to the* $\mathrm{Free}(\mathcal{A})$*-module structure.*

Proof : For the "small" objects $A \in \mathcal{A} \hookrightarrow \mathrm{Free}(\mathcal{A})$, the isomorphism

$$\mathrm{Hom}_{\mathrm{Free}(\mathcal{A})}(A, F) \simeq F(A) := \mathrm{Hom}_{\mathcal{M}}(A \triangleright M_1, M_2)$$

is given by the Yoneda Lemma.
Now any object $X \in \mathrm{Free}(\mathcal{A})$ is obtained as a colimit of such small objects, $X = \mathrm{colim}_i\ A_i$, $A_i \in \mathcal{A}$, by the co-Yoneda Lemma. Then it is straightforward to check that:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Free}(\mathcal{A})}(X, F) &\simeq \lim_i \mathrm{Hom}_{\mathrm{Free}(\mathcal{A})}(A_i, F) \simeq \lim_i \mathrm{Hom}_{\mathrm{Free}(\mathcal{M})}(A_i \triangleright M_1, M_2) \\ &\simeq \mathrm{Hom}_{\mathrm{Free}(\mathcal{M})}(\mathrm{colim}_i(A_i \triangleright M_1), M_2) \simeq \mathrm{Hom}_{\mathrm{Free}(\mathcal{M})}(X \triangleright M_1, M_2).\end{aligned}$$

Here we kept the notation $\triangleright$ for its essentially unique cocontinuous extention to free cocompletions. □

We now apply the internal Hom object constructions to the case of skein categories. We choose a $\Bbbk$-linear ribbon category $\mathcal{V}$ and denote its free cocompletion $\mathcal{E}$.

Definition 5.1.17: We consider an oriented surface with boundary Σ, with a "red" arc chosen on its boundary, seen at the left. This arc can be thickened in the surface, which gives a thick left embedding $(0,1) \to \partial\Sigma$. In particular one has an embedding of surfaces $(0,1) \times [0,1] \sqcup \Sigma \hookrightarrow \Sigma$ as in Figure 5.5.
By Remark 5.1.12, this gives a structure of left $Sk_{\mathcal{V}}(\mathbb{R}^2)$-module category on $Sk_{\mathcal{V}}(\Sigma)$. We denote

$$\triangleright : Sk_{\mathcal{V}}(\mathbb{R}^2) \otimes Sk_{\mathcal{V}}(\Sigma) \to Sk_{\mathcal{V}}(\Sigma)$$

the action functor.
We denote $SK_{\mathcal{V}}(-) := \mathrm{Free}(Sk_{\mathcal{V}}(-))$ and still denote the action functor on free cocompletions by

$$\triangleright : \mathcal{E} \boxtimes SK_{\mathcal{V}}(\Sigma) \to SK_{\mathcal{V}}(\Sigma) \ .$$

For M_1 and M_2 two objects of $Sk_{\mathcal{V}}(\Sigma)$, one has an internal Hom object $\underline{\mathrm{Hom}}(M_1, M_2) \in \mathcal{E}$. ♢

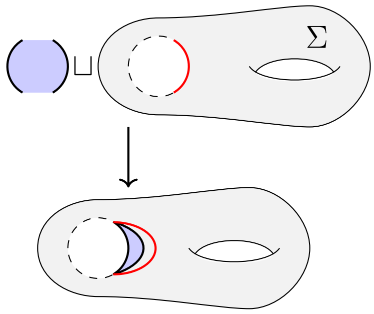


Figure 5.5: Skein categories of punctured surfaces are module categories

Definition 5.1.18: Let Σ be a surface with boundary with a chosen thickened arc on its boundary. The *internal skein algebra* $A_\Sigma := \underline{\mathrm{Hom}}(\emptyset, \emptyset)$ is the $\mathcal{E}$-internal endomorphism algebra of $\emptyset \in Sk_{\mathcal{V}}(\Sigma) \subseteq SK_{\mathcal{V}}(\Sigma)$.
Explicitly, A_Σ comes equipped with a natural isomorphism

$$\sigma : \mathrm{Hom}_{SK_{\mathcal{V}}(\Sigma)}(- \triangleright \emptyset, \emptyset) \overset{\sim}{\Rightarrow} \mathrm{Hom}_{\mathcal{E}}(-, A_\Sigma) \ .$$

This isomorphism is determined by its restriction on $\mathcal{V}$, in which case the left hand side agrees with Hom spaces in $Sk_{\mathcal{V}}(\Sigma)$. ♢

Proposition 5.1.19: *The inclusion $\mathcal{O}_{q^2}(SL_2)$–comod$^{fin} \hookrightarrow \mathcal{O}_{q^2}(SL_2)$–comod is a free cocompletion.*
In particular, if $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$–comod^{fin} and $\mathcal{E} = \mathcal{O}_{q^2}(SL_2)$–comod in the definition above, then A_Σ is an $\mathcal{O}_{q^2}(SL_2)$-comodule algebra.

Proof : Remember from Section 1.2.4 that (at generic q which we assume throughout) the category $\mathcal{O}_{q^2}(SL_2)$–*comod* is semisimple and its simples are finite dimensional. Hence any $\mathcal{O}_{q^2}(SL_2)$-comodule is a direct sum of these, and any colimit in $\mathcal{O}_{q^2}(SL_2)$–*comod*^{fin} is simply a direct sum.
The usual monoidal structure on $\mathcal{O}_{q^2}(SL_2)$–*comod* is the free cocompletion of the monoidal structure on $\mathcal{O}_{q^2}(SL_2)$–*comod*^{fin} as it extends it and commutes with direct sums in both factors. □

One could replace $\mathcal{O}_{q^2}(SL_2)$–*comod*^{fin} with its full subcategory TL, as every $\mathcal{O}_{q^2}(SL_2)$-comodule is a quotient of direct sums of objects of TL. Similarly, the inclusion $Sk_{TL}(\Sigma) \hookrightarrow \mathrm{Free}(Sk_{\mathcal{O}_{q^2}(SL_2)\text{–}comod^{fin}}(\Sigma))$ is also a free cocompletion, see [Coo23, Theorems 2.28 and 3.27].

Remark 5.1.20: The defining properties of A_Σ describes morphisms from $V \triangleright \emptyset$ to $\emptyset$ in $Sk_\mathcal{V}(\Sigma)$, so where the boundary points of tangles are at the bottom. Using duality, see [EGNO15, Proposition 7.1.6], we can describe morphisms from $W \triangleright \emptyset$ to $V \triangleright \emptyset$:

$$\mathrm{Hom}_{Sk_\mathcal{V}(\Sigma)}(W \triangleright \emptyset, V \triangleright \emptyset) \simeq \mathrm{Hom}_\mathcal{E}(W, V \otimes A_\Sigma).$$

When Σ is connected, every object of $Sk_\mathcal{V}(\Sigma)$ is isomorphic to one of the form $V \triangleright \emptyset$, and the above natural isomorphism suggests that the algebra $A_\Sigma \in \mathcal{E}$ is enough to fully describe $SK_\mathcal{V}(\Sigma)$. ◇

Theorem 5.1.21 ([BBJ18a, Theorem 5.14]): *Suppose that Σ is connected, then there is an equivalence of categories*

$$\begin{array}{ccc} SK_\mathcal{V}(\Sigma) & \tilde{\to} & mod_\mathcal{E} - A_\Sigma \\ M & \mapsto & \underline{\mathrm{Hom}}(\emptyset, M) \end{array}$$

between the free cocompletion of the skein category of Σ and the category of right modules over A_Σ in $\mathcal{E}$.
For M of the form $V \triangleright \emptyset$, $V \in \mathcal{E}$, which is always the case for $M \in Sk_\mathcal{V}(\Sigma)$, this functor is given by $V \triangleright \emptyset \mapsto V \otimes A_\Sigma$.

Proof : For the last statement, one has $\underline{\mathrm{Hom}}(\emptyset, V \triangleright \emptyset) \simeq V \otimes A_\Sigma$ by Remark 5.1.20. This is the general idea of the proof, as morphisms of A_Σ-modules from $W \otimes A_\Sigma$ to $V \otimes A_\Sigma$ are equivalent to morphisms in $\mathcal{E}$ from W to $V \otimes A_\Sigma$, which are equivalent by the above Remark to morphisms from $W \triangleright \emptyset$ to $V \triangleright \emptyset$ in $SK_\mathcal{V}(\Sigma)$.
For the details, one uses Barr-Beck reconstruction, or more precisely its reformulation in [BBJ18a, Theorem 4.6]. One has to check that $\emptyset \in SK_\mathcal{V}(\Sigma)$ is a progenerator.
It is projective: $act_\emptyset^R = \underline{\mathrm{Hom}}(\emptyset, -) : \begin{cases} SK_\mathcal{V}(\Sigma) & \to & [\mathcal{V}^{op}, \mathrm{Vect}_\Bbbk] \simeq \mathcal{E} \\ M & \mapsto & \mathrm{Hom}_{SK_\mathcal{V}(\Sigma)}(- \triangleright \emptyset, M) \end{cases}$ is cocontinuous because $- \triangleright \emptyset : \mathcal{V} \to Sk_\mathcal{V}(\Sigma) \subseteq SK_\mathcal{V}(\Sigma)$ takes values in compact projective objects.
It is a generator: $act_\emptyset^R$ being faithful is equivalent to $- \triangleright \emptyset : \mathcal{E} \to SK_\mathcal{V}(\Sigma)$ being dominant by [BBJ18a, Remark 4.9], which is the case because $- \triangleright \emptyset$ is essentially surjective on $Sk_\mathcal{V}(\Sigma)$ which generate $SK_\mathcal{V}(\Sigma)$ under colimits.
One gets right modules over A_Σ because we considered left module categories, see [BBJ18a, Remark 4.7]. □

Remark 5.1.22: We are in the same context as [BBJ18a] and [GJS23]. In [GJS23, Definition 2.18], the internal skein algebra is defined similarly as $\mathrm{Hom}_{Sk_\mathcal{V}(\Sigma)}(\emptyset \triangleleft -, \emptyset) \in \mathrm{Free}(\mathcal{V})$ for $\mathcal{V}$ a $\Bbbk$-linear ribbon category whose unit $1_\mathcal{V}$ is simple, which is the case for $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$*–comod*^{fin}.
In [BBJ18a, Definition 5.3], the moduli algebra A_Σ is defined to be the endomorphism algebra of the distinguished object $\mathcal{O}_{\mathcal{E},\Sigma}$ of the factorization homology over Σ of a presentable abelian balanced $\Bbbk$-linear category $\mathcal{E}$ generated under filtered colimits by rigid objects, with respect to the $\mathcal{E}$-module structure. The factorization homology of $\mathcal{V}$ is computed by $Sk_\mathcal{V}$, see [Coo23, Theorem 2.28], and the factorization homology of $\mathcal{E} = \mathrm{Free}(\mathcal{V})$ by $SK_\mathcal{V} = \mathrm{Free}(Sk_\mathcal{V})$, see [Coo23, Theorem 3.27]. The distinguished object is $\emptyset \in Sk_\mathcal{V}(\Sigma) \subseteq SK_\mathcal{V}(\Sigma)$, and $\mathcal{O}_{q^2}(SL_2)$*–comod* is abelian and generated under filtered colimit by rigid objects $\mathcal{O}_{q^2}(SL_2)$*–comod*^{fin}.
There is one difference with our chapter though: we consider a left $\mathcal{E}$-action, and [BBJ18a,

GJS23] consider a right $\mathcal{E}$-action, with adapted definitions of internal Hom objects. In our description, it would mean seeing the red arc on the right instead of on the left, thus $Sk_{\mathcal{V}}(\mathbb{R}^2)$ acting from the right. This gives a braided opposite product, and we need left actions to have the same product than the one on $\mathscr{S}(\Sigma)$. Right actions and more generally multiple right/left actions and how they interact will be studied in Section 5.3. $\diamond$

5.2 The relation

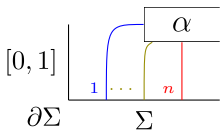
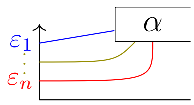
We show in this section that the stated skein algebra of a surface with a single boundary edge is isomorphic to A_Σ. We consider the algebra A_Σ from last section with $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)\text{–}comod^{fin}$ and $\mathcal{E} = \mathcal{O}_{q^2}(SL_2)\text{–}comod$ at generic q, and prove that $\mathscr{S}(\Sigma) \simeq A_\Sigma$ as $\mathcal{O}_{q^2}(SL_2)$-comodule algebras. Actually since A_Σ is only defined up to canonical isomorphism one may take an equality here, so we prove that $\mathscr{S}(\Sigma)$ satisfies the defining properties of A_Σ, namely that it is the internal endomorphism algebra of the empty set in $SK_{\mathcal{V}}(\Sigma)$ with respect to the $\mathcal{E}$-module structure. This result is not new and was stated in a weaker form in [LY22, Theorem 4.4], [LS, Theorem 9.1] and [GJS23, Remark 2.21], namely as algebras in $\mathrm{Vect}_{\Bbbk}$. However, in these references one considers right $\mathcal{E}$-actions and therefore the internal skein algebra is isomorphic to the braided opposite of the stated skein algebra. The full result can still be recovered using [Fai, Theorem 5.3] or [Kora], which give an isomorphism between the opposite of the stated skein algebras and Alekseev–Grosse–Schomerus-algebras, which are themselves isomorphic to internal skein algebras by [BBJ18a]. Our approach here is more direct, more explicit and uses only skein theory.

We need a natural isomorphisms

$$\mathrm{St}_W : \mathrm{Hom}_{SK_{\mathcal{V}}(\Sigma)}(W \triangleright \emptyset, \emptyset) \Rightarrow \mathrm{Hom}_{\mathcal{E}}(W, \mathscr{S}(\Sigma))$$

for $W \in \mathcal{E} = \mathcal{O}_{q^2}(SL_2)\text{–}comod$. In the case where $W = V^{\otimes n} \in TL$ is a tensor product of standard corepresentations, and the element on the left hand side is a tangle α with n boundary points, we want a morphism $V^{\otimes n} \to \mathscr{S}(\Sigma)$ associated to it. This is done by setting the entries, elements of $V^{\otimes n}$, as states of the tangle α. Recall that V has basis v_+, v_- and we identify the states $+$ and $-$ with these elements. As in Remark 5.1.6, we allow formal linear combination of states for stated tangles. In this context, the defining relations of stated skein algebras correspond exactly to naturality conditions, see Proposition 5.2.3.
This gives the idea of how to deal with the objects of the full subcategory $TL \subseteq \mathcal{V}$ of objects of the form $V^{\otimes n}$, and this extends to $\mathcal{E} \simeq \mathrm{Free}(TL)$ by Proposition 5.1.16. Note that one still has an action $\triangleright : TL \otimes Sk_{TL}(\Sigma) \to Sk_{TL}(\Sigma)$ which is the restriction of $\triangleright : \mathcal{V} \otimes Sk_{\mathcal{V}}(\Sigma) \to Sk_{\mathcal{V}}(\Sigma)$.

Definition 5.2.1: Let $W = V^{\otimes n} \in TL$ and $\alpha \in \mathrm{Hom}_{Sk_{TL}(\Sigma)}(W \triangleright \emptyset, \emptyset)$. By Theorem 1.2.42, α can be represented by a (linear combination of) tangle(s), still denoted by α, with n ordered boundary points near the boundary edge, which is well defined up to isotopy and Kauffman-bracket relations.

Graphically, we set $\mathrm{St}_W \Bigg($  $\Bigg) (v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_n}) :=$  $, \varepsilon_i \in \{\pm\}$.

For brevity we denote this last stated tangle by ${}_{\vec{\varepsilon}}\alpha$ and $v_{\vec{\varepsilon}} = v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_n}$. Note that in this figure there are two implicit sums, α is a linear combination of tangles and an element of $V^{\otimes n}$ is a linear combination of $v_{\vec{\varepsilon}}$'s. They will remain implicit in the following. In the context of stated skein algebras one needs the boundary points of the tangle α to be above the boundary arc though they are on the bottom at its right in the context of morphisms in $Sk_{TL}(\Sigma)$. One wants to simply push the boundary points through the boundary edge and up above it, but without braiding the strands with one another. Therefore we use a global diffeomorphism of $\Sigma \times [0,1]$. Consider an isotopy of the identity on $\Sigma \times [0,1]$ which is trivial far from the corner and in it pushes the bottom boundary through and up above the edge. It results in the global automorphism ψ_{hv} of $\Sigma \times [0,1]$ which maps a tangle "from skein categories" to one "from stated skein algebras" and preserves the order as desired, namely the bottom points on the leftmost will end up above. We will still denote the modified tangle $\psi_{hv}(\alpha)$ by α. Now, it only needs states to give an element of $\mathscr{S}(\Sigma)$.

We set $\mathrm{St}_W(\alpha) := \begin{cases} V^{\otimes n} & \to & \mathscr{S}(\Sigma) \\ v_{\vec{\varepsilon}} & \mapsto & {}_{\vec{\varepsilon}}\alpha; \end{cases}$ so where ${}_{\vec{\varepsilon}}\alpha$ is the tangle $\psi_{hv}(\alpha)$ with states $\varepsilon_1, \cdots, \varepsilon_n$ from top to bottom. It is well defined because the tangle representing α is well defined up to isotopy and Kauffman-bracket relations, which are quotiented out in $\mathscr{S}(\Sigma)$. ♢

Proposition 5.2.2: *Given a tangle α, the map*

$$\mathrm{St}_W(\alpha) : V^{\otimes n} \to \mathscr{S}(\Sigma)$$

is an $\mathcal{O}_{q^2}(SL_2)$-comodule morphism.

Proof : Note that we still see $\mathscr{S}(\Sigma)$ as a right $\mathcal{O}_{q^2}(SL_2)$-comodule even though we draw the edge at the left. Its comodule structure is given by $\Delta({}_{\vec{\varepsilon}}\alpha) = \sum_{\vec{\eta} \in \{\pm\}^n} {}_{\vec{\eta}}\alpha \otimes {}_{\vec{\eta}}\beta_{\vec{\varepsilon}}$, and here ${}_{\vec{\eta}}\beta_{\vec{\varepsilon}}$ is a product $\prod_i {}_{\eta_i}\beta_{\varepsilon_i}$ in $\mathscr{S}(B)$. The comodule structure on $V^{\otimes n}$ is given by $\Delta(v_{\vec{\varepsilon}}) = \sum_{\vec{\eta}} v_{\vec{\eta}} \otimes \prod_i x_{\eta_i, \varepsilon_i}$ where $x_{\eta,\varepsilon}$ is the v_η part of $\Delta(v_\varepsilon)$. Namely, $x_{+,+} = a$, $x_{+,-} = b$, $x_{-,+} = c$ and $x_{-,-} = d$. Under the isomorphism $\mathcal{O}_{q^2}(SL_2) \simeq \mathscr{S}(B)$, $x_{\eta,\varepsilon} \mapsto {}_{\eta}\beta_{\varepsilon}$.

Thus $\Delta(\mathrm{St}_W(\alpha)(v_{\vec{\varepsilon}})) = \Delta({}_{\vec{\varepsilon}}\alpha) = \sum_{\vec{\eta}} \mathrm{St}_W(\alpha)(v_{\vec{\eta}}) \otimes {}_{\vec{\eta}}\beta_{\vec{\varepsilon}} = (\mathrm{St}_W(\alpha) \otimes Id_{\mathscr{S}(B)})(\Delta(v_{\vec{\varepsilon}}))$. □

Proposition 5.2.3: *The maps*

$$\mathrm{St}_W : \mathrm{Hom}_{Sk_{TL}(\Sigma)}(W \triangleright \emptyset, \emptyset) \to \mathrm{Hom}_{\mathcal{O}_{q^2}(SL_2)\text{–}comod}(W, \mathscr{S}(\Sigma))$$

for $W \in TL$ define a natural transformation

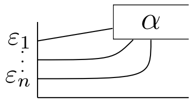
$$\mathrm{St} : \mathrm{Hom}_{Sk_{TL}(\Sigma)}(- \triangleright \emptyset, \emptyset) \Rightarrow \mathrm{Hom}_{\mathcal{E}}(-, \mathscr{S}(\Sigma))$$

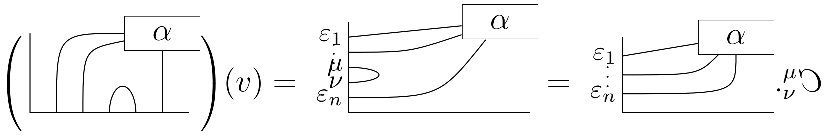
between functors $TL^{op} \to \mathrm{Vect}_{\Bbbk}$.

Proof : For $g \in \mathrm{Hom}_{TL}(V^{\otimes n}, V^{\otimes m})$, $\alpha \in \mathrm{Hom}_{Sk_{TL}(\Sigma)}(V^{\otimes m} \triangleright \emptyset, \emptyset)$ and $v_{\vec{\varepsilon}} \in V^{\otimes n}$, one needs to check that $\mathrm{St}_{V^{\otimes m}}(\alpha)(g(v_{\vec{\varepsilon}})) = \mathrm{St}_{V^{\otimes n}}(\alpha \circ (g \triangleright \mathrm{Id}_{\emptyset}))(v_{\vec{\varepsilon}})$. Morphisms of TL are generated under composition and juxtaposition by identities, caps and cups, so one only needs to prove the result for these last two.

We will need some explicit computations of these caps and cups. Recall from Remark 1.2.22, or [Tin, Theorem 4.2], that to do so one uses the isomorphism $\varphi : \begin{cases} V & \to & V^* \\ v_+ & \mapsto & -q^{\frac{5}{2}} v_-^* \\ v_- & \mapsto & q^{\frac{1}{2}} v_+^* \end{cases}$

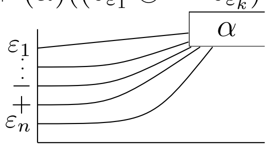
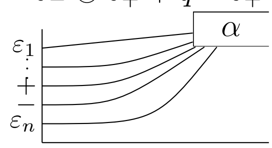
of Definition 1.2.31 and sets $\cap = \curvearrowleft \circ (\varphi \otimes \mathrm{Id}_V)$ and $\cup = (\mathrm{Id}_V \otimes \varphi^{-1}) \circ \curvearrowleft$, where $\curvearrowleft$ and $\curvearrowleft$ are the usual ev and $coev$ in $\mathrm{Vect}_{\Bbbk}^{fin}$.
Let $g = |\,|\,| \cap |\,|\,| = Id_V^{\otimes k} \otimes \cap \otimes Id_V^{\otimes n-k} : V^{\otimes n+2} \to V^{\otimes n}$, $\alpha \in \mathrm{Hom}_{Sk_{TL}(\Sigma)}(V^{\otimes n} \rhd \emptyset, \emptyset)$ and $v = (v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_k}) \otimes v_\mu \otimes v_\nu \otimes (v_{\varepsilon_{k+1}} \otimes \cdots \otimes v_{\varepsilon_n}) \in V^{\otimes n+2}$. We want to compare:

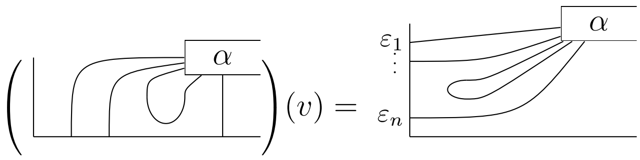
$$\mathrm{St}_{V^{\otimes n}}(\alpha)(g(v)) = \mathrm{St}_{V^{\otimes n}}(\alpha)(v_{\vec{\varepsilon}} \,.\, \cap (v_\mu \otimes v_\nu)) =$$

$$. \cap (v_\mu \otimes v_\nu) \qquad \text{and}$$

$$\mathrm{St}_{V^{\otimes n+2}}(\alpha \circ (g \rhd Id_\emptyset))(v) = \mathrm{St}_{V^{\otimes n+2}}\left(\right)(v) =$$


One simply needs to check that the coefficient ${}^{\mu}_{\nu}\circlearrowleft$ from the left boundary skein relations of Proposition 1.1.7 coincides with $\cap(v_\mu \otimes v_\nu)$ and indeed $\cap(v_+ \otimes v_+) = ev(-q^{\frac{5}{2}} v_-^* \otimes v_+) = 0 = {}^{+}_{+}\circlearrowleft$, $\cap(v_- \otimes v_-) = ev(q^{\frac{1}{2}} v_+^* \otimes v_-) = 0 = {}^{-}_{-}\circlearrowleft$, $\cap(v_+ \otimes v_-) = ev(-q^{\frac{5}{2}} v_-^* \otimes v_-) = -q^{\frac{5}{2}} = {}^{+}_{-}\circlearrowleft$ and $\cap(v_- \otimes v_+) = ev(q^{\frac{1}{2}} v_+^* \otimes v_+) = q^{\frac{1}{2}} = {}^{-}_{+}\circlearrowleft$.
Now let $g = |\,|\,| \cup |\,|\,| = Id_V^{\otimes k} \otimes \cup \otimes Id_V^{\otimes n-k} : V^{\otimes n} \to V^{\otimes n+2}$, $\alpha \in \mathrm{Hom}_{Sk_{TL}(\Sigma)}(V^{\otimes n+2} \rhd \emptyset, \emptyset)$ and $v = v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_n} \in V^{\otimes n}$. One can directly compute $\cup(1) = (Id_V \otimes \varphi^{-1}) \circ coev(1) = (Id_V \otimes \varphi^{-1})(v_- \otimes v_-^* + v_+ \otimes v_+^*) = -q^{-\frac{5}{2}} v_- \otimes v_+ + q^{-\frac{1}{2}} v_+ \otimes v_-$.
We want to compare:

$$\mathrm{St}_{V^{\otimes n+2}}(\alpha)(g(v)) = \mathrm{St}_{V^{\otimes n+2}}(\alpha)((v_{\varepsilon_1} \otimes \cdots v_{\varepsilon_k}) \otimes (-q^{-\frac{5}{2}} v_- \otimes v_+ + q^{-\frac{1}{2}} v_+ \otimes v_-) \otimes (v_{\varepsilon_{k+1}} \otimes \cdots v_{\varepsilon_n}))$$

$$= -q^{-\frac{5}{2}}$$

$$+ q^{-\frac{1}{2}}$$

and

$$\mathrm{St}_{V^{\otimes n}}(\alpha \circ (g \rhd Id_\emptyset))(v) = \mathrm{St}_{V^{\otimes n}}\left(\right)(v) =$$


They are equal by the last left boundary skein relation of Proposition 1.1.7. □

Recall from Proposition 5.1.16 that the $\mathcal{E}$-internal endomorphism of $\emptyset$ is an object X equipped with an isomorphism

$$\mathrm{Hom}_{Sk_{TL}(\Sigma)}(- \rhd \emptyset, \emptyset) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(-, X)$$

in $[TL^{op}, \mathrm{Vect}_{\Bbbk}]$. We only need to show that St is an isomorphism.

Theorem 5.2.4: *The natural transformation* St *is a natural isomorphism and exhibits* $\mathscr{S}(\Sigma)$ *as the internal endomorphism object of the empty set. Namely one can take* $A_\Sigma = \mathscr{S}(\Sigma)$ *as* $\mathcal{O}_{q^2}(SL_2)$*-comodule.*

Proof : We exhibit an inverse to St. Let $W = V^{\otimes n} \in TL$, which decomposes as a direct sum of simples W_i. Let $f : W \to \mathscr{S}(\Sigma)$ be a morphism in $\mathcal{E}$, and denote by f_i its restriction on W_i. We want a morphism $\mathrm{St}_W^{-1}(f) \in \mathrm{Hom}_{Sk_{TL}(\Sigma)}(W \rhd \emptyset, \emptyset)$, which is equivalent to a collection of morphisms $\mathrm{St}_W^{-1}(f_i) \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(W_i \rhd \emptyset, \emptyset)$. Each f_i is determined by its value on a single element $w_i \in W_i$ (pick a highest weight element for example), it extends

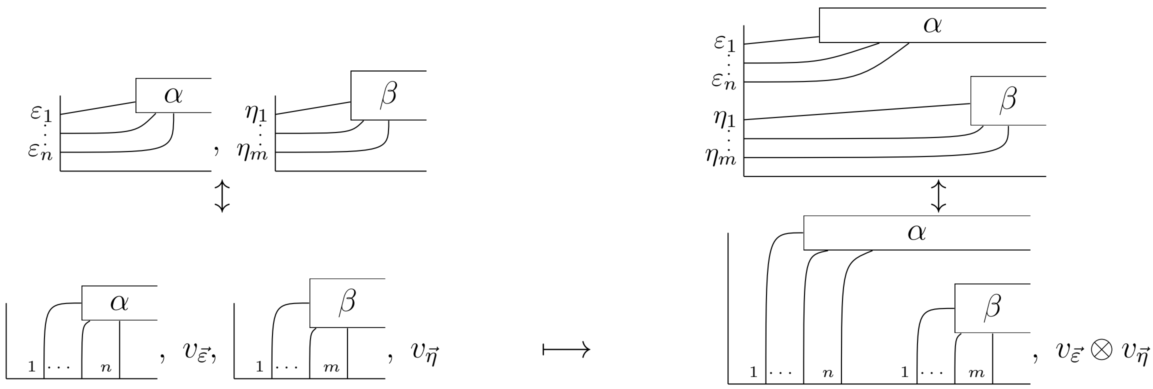
to all W_i by applying the $\mathcal{U}_{q^2}(\mathfrak{sl}_2)$-action (recall from Proposition 1.2.39 that $\mathcal{O}_{q^2}(SL_2)$-comodules correspond to $\mathcal{U}_{q^2}(\mathfrak{sl}_2)$-modules). Choose any stated tangle a_i representing the element $f_i(w_i) \in \mathscr{S}(\Sigma)$. Denote by α_i its underlying tangle, $n_i = |\partial\alpha_i|$ its number of boundary points and $\vec{\varepsilon_i}$ its states. This representative is well defined up to the boundary skein relations, the usual skein relations and isotopy. The assignment $w_i \mapsto v_{\vec{\varepsilon_i}}$ extends to a unique $\mathcal{O}_{q^2}(SL_2)$-morphism $g_i : W_i \to V^{\otimes n_i}$ by applying the $\mathcal{U}_{q^2}(\mathfrak{sl}_2)$-action. We then set $\mathrm{St}_W^{-1}(f_i) = \alpha_i \circ (g_i \rhd Id_\emptyset)$. Note that here α_i denotes the tangle α_i seen as a morphism in $Sk_\mathcal{V}(\Sigma)$, so we actually mean $\psi_{hv}^{-1}(\alpha_i)$, the same tangle but with boundary points at the bottom instead of the left. We have to check that this definition does not depend on the representative a_i. Usual skein relations and isotopy do not change α_i seen as a morphism in the skein category. The boundary skein relations are equivalent to naturality using Proposition 5.2.3. Namely, another representative a_i' has to be of the form $\alpha_i' = \alpha_i \circ (g \rhd Id_\emptyset)$, for some $g \in TL$, with states $\vec{\varepsilon_i}'$ such that $g(\vec{\varepsilon_i}') = \vec{\varepsilon_i}$. Therefore the g_i' for a_i' will be such that $g_i = g \circ g_i'$. Then we simply check that $\alpha_i \circ (g_i \rhd Id_\emptyset) = \alpha_i \circ (g \rhd Id_\emptyset) \circ (g_i' \rhd Id_\emptyset) = \alpha_i' \circ (g_i' \rhd Id_\emptyset)$, and $St_W^{-1}(f_i)$ is well defined.
For simplicity we assume that $\mathrm{St}_W^{-1}(f_i)$ and g_i are actually defined on all W but are 0 except on W_i, namely we precompose by the projection $W \twoheadrightarrow W_i$, and thus we set $\mathrm{St}_W^{-1}(f) = \sum_i \mathrm{St}_W^{-1}(f_i)$.
It is now easy to check that St_W^{-1} is the inverse of St_W. For $v_{\vec{\varepsilon}} \in W$, suppose $v_{\vec{\varepsilon}} \in W_{i_0} \subseteq W$ lies in a simple, or decompose it in the W_i's, and write $v_{\vec{\varepsilon}} = X \cdot w_{i_0}$, $X \in \mathcal{U}_{q^2}(\mathfrak{sl}_2)$. One has $g_{i_0}(v_{\vec{\varepsilon}}) = X \cdot v_{\vec{\varepsilon}_{i_0}}$ and for $j \neq i_0,\ g_j(v_{\vec{\varepsilon}}) = 0$. Thus:

$$\begin{aligned}
\mathrm{St}_W(\mathrm{St}_W^{-1}(f))(v_{\vec{\varepsilon}}) &= \mathrm{St}_W(\textstyle\sum_i \alpha_i \circ (g_i \rhd Id_\emptyset))(v_{\vec{\varepsilon}}) = \sum_i \mathrm{St}_W(\alpha_i \circ (g_i \rhd Id_\emptyset))(v_{\vec{\varepsilon}}) \\
&\overset{5.2.3}{=} \textstyle\sum_i \mathrm{St}_W(\alpha_i)(g_i(v_{\vec{\varepsilon}})) = \mathrm{St}_W(\alpha_{i_0})(X \cdot v_{\vec{\varepsilon}_{i_0}}) \\
&\overset{5.2.2}{=} X \cdot \mathrm{St}_W(\alpha_{i_0})(v_{\vec{\varepsilon}_{i_0}}) = X \cdot a_{i_0} = X \cdot f_{i_0}(w_{i_0}) = f_{i_0}(v_{\vec{\varepsilon}}) = f(v_{\vec{\varepsilon}})
\end{aligned}$$

Symmetrically, let $\alpha \in \mathrm{Hom}_{Sk_{TL}(\Sigma)}(W \rhd \emptyset, \emptyset)$ and $v_{\vec{\varepsilon}} \in W$. Set $f = \mathrm{St}_W(\alpha) : V^{\otimes|\partial\alpha|} \to \mathscr{S}(\Sigma)$, in the definition of $\mathrm{St}_W^{-1}(f)$ one has $\alpha_i = \alpha$ and $v_{\vec{\varepsilon_i}} = w_i$, so g_i is the inclusion $W_i \hookrightarrow W$. If $v_{\vec{\varepsilon}} \in W_i \subseteq W$, one has $\mathrm{St}_W^{-1}(\mathrm{St}_W(\alpha)) = \sum_i \alpha \circ (g_i \rhd Id_\emptyset) = \alpha \circ (Id_W \rhd Id_\emptyset) = \alpha$. □

Proposition 5.2.5: *The algebra structure inherited from the internal endomorphism object structure on $\mathscr{S}(\Sigma)$ coincides with its usual algebra structure. Namely $A_\Sigma = \mathscr{S}(\Sigma)$ as $\mathcal{O}_{q^2}(SL_2)$-comodule algebras.*

Proof : Recall that the product on $\mathscr{S}(\Sigma)$ is given by stacking the left tangle a above the right one b, and the product on A_Σ is defined by evaluation and composition maps on internal Hom objects, in Definition 5.1.15. Graphically, the stated tangles a and b, which we see as morphisms α and β in $Sk_\mathcal{V}(\Sigma)$ with prescribed inputs $v_{\vec{\varepsilon}}$ and $v_{\vec{\eta}}$, map to the morphism $\alpha \circ (Id_{V^{\otimes|\partial\alpha|}} \rhd \beta)$ in $Sk_\mathcal{V}(\Sigma)$ with prescribed input $v_{\vec{\varepsilon}} \otimes v_{\vec{\eta}}$, which we see as the stated tangle a above b:



The evaluation map $ev_{\emptyset,\emptyset} : \mathscr{S}(\Sigma) \triangleright \emptyset \to \emptyset$ is the image under St^{-1} of $Id_{\mathscr{S}(\Sigma)}$. We have not constructed St^{-1} on all $\mathcal{E}$ above, but only on TL, and it extends by cocontinuity in Proposition 5.1.16. The comodule $\mathscr{S}(\Sigma)$ decomposes as simples as $\mathscr{S}(\Sigma) = \bigoplus_{\alpha \in \mathcal{B}^+} \mathcal{U}_{q^2}(\mathfrak{sl}_2) \cdot \alpha$ where $\mathcal{B}^+$ is the set of $\mathfrak{o}$-ordered simple stated tangles with only + states, see [CL22, Theorem 4.6 b)]. These stated tangles with only + states are simply a way to represent canonically a tangle without state information, and again in the following we will see α as a morphism in $\mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(V^{\otimes|\partial\alpha|}, \emptyset)$, which is actually $\psi_{hv}^{-1}(\alpha)$.
We denote by $W_\alpha = \mathcal{U}_{q^2}(\mathfrak{sl}_2) \cdot \alpha$ and $g_\alpha : W_\alpha \to V^{\otimes|\partial\alpha|}$ the inclusion mapping α to its states $v_{\overrightarrow{+\cdots+}}$. Then:

$$ev_{\emptyset,\emptyset} = \mathrm{St}^{-1}(Id_{\mathscr{S}(\Sigma)}) = \oplus_{\alpha \in \mathcal{B}^+} \mathrm{St}^{-1}(W_\alpha \hookrightarrow \mathscr{S}(\Sigma)) = \oplus_{\alpha \in \mathcal{B}^+} \alpha \circ (g_\alpha \triangleright Id_\emptyset).$$

The composition map $c : \mathscr{S}(\Sigma) \otimes \mathscr{S}(\Sigma) \to \mathscr{S}(\Sigma)$ is the image under St of the morphism $ev_{\emptyset,\emptyset} \circ (Id_{\mathscr{S}(\Sigma)} \triangleright ev_{\emptyset,\emptyset}) : (\mathscr{S}(\Sigma) \otimes \mathscr{S}(\Sigma)) \triangleright \emptyset \to \mathscr{S}(\Sigma) \triangleright \emptyset \to \emptyset$. This morphism is the double sum:

$$\oplus_{\alpha \in \mathcal{B}^+} \oplus_{\beta \in \mathcal{B}^+} (\alpha \circ (g_\alpha \triangleright Id_\emptyset)) \circ (Id_{\mathscr{S}(\Sigma)} \triangleright (\beta \circ (g_\beta \triangleright Id_\emptyset))) = \oplus_{\alpha \in \mathcal{B}^+} \oplus_{\beta \in \mathcal{B}^+} \alpha \circ (Id_{V^{\otimes|\partial\alpha|}} \triangleright \beta) \circ (g_\alpha \triangleright g_\beta \triangleright Id_\emptyset).$$

The product is obtained by applying St to this morphism. For $a, b \in \mathscr{S}(\Sigma)$, write $a = X \cdot \alpha$ and $b = Y \cdot \beta$ with $X, Y \in \mathcal{U}_{q^2}(\mathfrak{sl}_2)$ and $\alpha, \beta \in \mathcal{B}^+$. Thus a has states $v_{\vec{\varepsilon}} = X \cdot v_{\overrightarrow{+\cdots+}}$ and b has states $v_{\vec{\eta}} = Y \cdot v_{\overrightarrow{+\cdots+}}$. By naturality:

$$\begin{aligned}
c(a \otimes b) \quad &:= \quad \mathrm{St}_{\mathscr{S}(\Sigma) \otimes \mathscr{S}(\Sigma)}(ev_{\emptyset,\emptyset} \circ (Id_{\mathscr{S}(\Sigma)} \triangleright ev_{\emptyset,\emptyset}))(a \otimes b) \\
&= \quad \oplus_{\alpha' \in \mathcal{B}^+} \oplus_{\beta' \in \mathcal{B}^+} \mathrm{St}(\alpha' \circ (Id_{V^{\otimes|\partial\alpha'|}} \triangleright \beta'))((g_{\alpha'} \otimes g_{\beta'})(a \otimes b)) \\
&= \quad \mathrm{St}(\alpha \circ (Id_{V^{\otimes|\partial\alpha|}} \triangleright \beta))((g_\alpha \otimes g_\beta)(a \otimes b)) = \mathrm{St}(\alpha \circ (Id_{V^{\otimes|\partial\alpha|}} \triangleright \beta))(v_{\vec{\varepsilon}} \otimes v_{\vec{\eta}})
\end{aligned}$$

which is precisely the usual product of a and b in $\mathscr{S}(\Sigma)$. □

5.3 Multi-edges

We define internal skein algebras for surfaces with more than one boundary, and possibly left or right boundary edges. We show they are isomorphic to stated skein algebras when $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$–$comod^{fin}$, and re-prove their excision properties using excision properties of skein categories.

5.3.1 Right internal skein algebras

In order to extend the definition of internal skein algebras to surfaces with multiple boundary edges, we would need a notion of left and right action to be able to glue surfaces together, such that internal skein algebras satisfy excision properties, just like stated skein algebras. One subtlety though is that one is only allowed to talk about right $\mathcal{O}_{q^2}(SL_2)$-comodules in the context of internal skein algebras, to stay in the category $\mathcal{E}$ (as opposed to the stated skein algebra context).

Definition 5.3.1 (Section 3.2 in [Coo23]): Let Σ be a surface with a chosen boundary interval, which we see at the right of the surface. One can make a construction similar to Definition 5.1.17 to have a right action functor

$$\triangleleft : Sk_{\mathcal{V}}(\Sigma) \otimes Sk_{\mathcal{V}}(\mathbb{R}^2) \to Sk_{\mathcal{V}}(\Sigma) \ .$$

It differs the one $\triangleright$ of Definition 5.1.17 only by rotating the disk by 180 degrees.
The right moduli algebra A_Σ^R of [BBJ18a, Section 5.2], or right internal skein algebra of [GJS23], is the internal endomorphism algebra of the empty set in $Sk_{\mathcal{V}}(\Sigma)$ with respect to this $Sk_{\mathcal{V}}(\mathbb{R}^2)^{\otimes-op}$-module structure. ♢

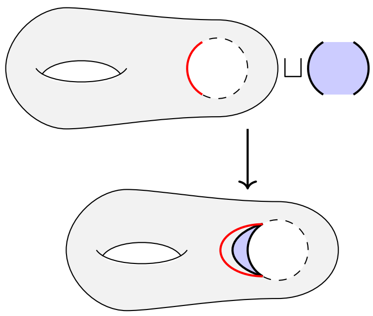


Figure 5.6: The right action of a disk on a punctured surface

Definition 5.3.2: Denote *rot* the diffeomorphism of the disk given by 180° rotation. By Remark 5.1.11 it induces an automorphism

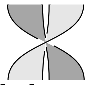
$$(-)^{\text{ht}} := Sk_{\mathcal{V}}(rot) : Sk_{\mathcal{V}}(\mathbb{R}^2) \to Sk_{\mathcal{V}}(\mathbb{R}^2)$$

which squares to the identity. For $X \in Sk_{\mathcal{V}}(\mathbb{R}^2)$, we call $X^{\text{ht}} := Sk_{\mathcal{V}}(rot)(X)$ the half-twisted X. One easily checks that $(-)^{\text{ht}}$ is anti-monoidal, namely $(X \otimes Y)^{\text{ht}} = Y^{\text{ht}} \otimes X^{\text{ht}}$, because *rot* reverses left-right order.
The diffeomorphism *rot* is isotopic to the identity by rotating from 0 to 180°. This isotopy induces a natural isomorphism

$$\text{ht} : Id_{Sk_{\mathcal{V}}(\mathbb{R}^2)} \stackrel{\sim}{\Rightarrow} (-)^{\text{ht}}$$

called the half twist, which squares to the twist (the 360° rotation). ♢

Remember from Remark 5.1.11 that ht is given on n blackboard framed points on the real axis by n parallelly half-twisted vertical strands, namely drawn on the half twisted ribbon  . Naturality, namely $\text{ht}_W \circ f \circ \text{ht}_V^{-1} = f^{\text{ht}}$, expresses the fact that one can untwist a top half twist and a bottom anti-half-twist by half twisting the middle. For $f = \text{ht}_V$ one

gets $\mathrm{ht}_{V^{\mathrm{ht}}} = \mathrm{ht}_V^{\mathrm{ht}}$. Note too that $\mathrm{ht}_{V\otimes W} = (\mathrm{ht}_W \otimes \mathrm{ht}_V) \circ c_{V,W}$ by  = .
Now, One can easily relate the left and right actions of the disk by rotating the whole picture, i.e.

$$\triangleright = \triangleleft \circ fl \circ ((-)^{\mathrm{ht}} \otimes Id_{Sk_{\mathcal{V}}(\Sigma)})$$

and

$$\triangleleft \circ fl = \triangleright \circ ((-)^{\mathrm{ht}} \otimes Id_{Sk_{\mathcal{V}}(\Sigma)})$$

where fl is the flip of tensors. And indeed a left action turns into a right action under an anti-monoidal functor. Moreover, the natural isomorphism $\mathrm{ht} : Id_{Sk_{\mathcal{V}}(\mathbb{R}^2)} \overset{\sim}{\Rightarrow} (-)^{\mathrm{ht}}$ gives a natural isomorphism

$$\mathrm{ht} \triangleright - := \triangleright \circ (\mathrm{ht} \otimes Id_{Sk_{\mathcal{V}}(\Sigma)}) : \triangleright \overset{\sim}{\Rightarrow} \triangleleft \circ fl \ .$$

Remark 5.3.3: One can give an explicit relation between left and right internal skein algebras. Internal skein algebras are only defined up to isomorphism, so this description is just one choice. Actually, we will give another one below. Consider the internal skein algebra A_Σ defined as in Section 5.1.3 by seeing the red arc at the left, with a natural isomorphism $\sigma : \mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(- \triangleright \emptyset, \emptyset) \overset{\sim}{\Rightarrow} \mathrm{Hom}_{\mathcal{E}}(-, A_\Sigma)$. Then one has natural isomorphisms

$$\mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(\emptyset \triangleleft V, \emptyset) = \mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(V^{\mathrm{ht}} \triangleright \emptyset, \emptyset) \overset{\sigma_{V^{\mathrm{ht}}}}{\to} \mathrm{Hom}_{\mathcal{E}}(V^{\mathrm{ht}}, A_\Sigma) \overset{(-)^{\mathrm{ht}}}{\to} \mathrm{Hom}_{\mathcal{E}}(V, A_\Sigma^{\mathrm{ht}}) \ .$$

Namely, $A_\Sigma^R \simeq A_\Sigma^{\mathrm{ht}}$ as object of $SK_{\mathcal{V}}(\mathbb{R}^2)$, with natural isomorphism $(\sigma_{(-)^{\mathrm{ht}}})^{\mathrm{ht}}$. ◇

Remark 5.3.4: Note that the half twist is defined on $Sk_{\mathcal{V}}(\mathbb{R}^2)$ and not on $\mathcal{V}$, which is its full subcategory of objects with only one coloured point, with blackboard framing. The half twist will map such an object to a point with anti-blackboard framing, so it does not stabilise $\mathcal{V}$. The equivalence of categories $Sk_{\mathcal{V}}(\mathbb{R}^2) \simeq \mathcal{V}$ preserves properties of the half twist only up to natural isomorphism, and depends on the choice of a quasi-inverse of the inclusion. The one described in Remark 1.2.20 will map the half twist on $Sk_{\mathcal{V}}(\mathbb{R}^2)$ to the identity on $\mathcal{V}$, but if we had chosen to restore the framing counter-clockwise it would map it to the full twist. In $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$–$comod^{fin}$ an actual half twist exists, see [ST09], and we will study it below. However, in general we will prefer a construction that uses the half twist only on the disk inserted in $Sk_{\mathcal{V}}(\Sigma)$, where it is well defined, and not on $\mathcal{V}$. In particular, the above description $A_\Sigma^R \simeq A_\Sigma^{\mathrm{ht}}$ holds in $SK_{\mathcal{V}}(\mathbb{R}^2)$ but has an unclear meaning in $\mathcal{E}$ (it depends on the choice of some quasi-inverses). ◇

Definition 5.3.5: There is another explicit relation between left and right internal skein algebras using $\mathrm{ht} \triangleright -$ to relate the right action to the left one. Set

$$\sigma^R = \sigma \circ (\mathrm{ht} \triangleright \emptyset) : \mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(\emptyset \triangleleft -, \emptyset) \overset{\sim}{\Rightarrow} \mathrm{Hom}_{\mathcal{E}}(-, A_\Sigma)$$

i.e. for $V \in \mathcal{V}$ and $\alpha \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(\emptyset \triangleleft V, \emptyset)$, we have $\alpha \circ (\mathrm{ht}_V \triangleright Id_\emptyset) \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(V \triangleright \emptyset, \emptyset)$ and we set

$$\sigma_V^R(\alpha) := \sigma_V(\alpha \circ (\mathrm{ht}_V \triangleright Id_\emptyset)) \ .$$ ◇

Proposition 5.3.6: *The natural isomorphism σ^R exhibits A_Σ as the right internal skein algebra. The algebra structure $m^R : A_\Sigma \otimes A_\Sigma \to A_\Sigma$ inherited from this right internal endomorphism object structure differs from the left one $m : A_\Sigma \otimes A_\Sigma \to A_\Sigma$ by a braiding: $m^R = m \circ c_{A_\Sigma \otimes A_\Sigma}$.*
In other words, the right internal skein algebras introduced in [GJS23] and [BBJ18a] are the braided opposites of the left ones introduced in Section 5.1.3.

Proof : The σ_V^R's form a natural isomorphism: let $f \in \mathrm{Hom}_{\mathcal{V}}(V, W)$ and $\alpha \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\Sigma)}(\emptyset \triangleleft W, \emptyset)$, one checks:

$$\begin{aligned} \sigma_V^R(\alpha \circ (Id_\emptyset \triangleleft f)) = \ & \sigma_V^R(\alpha \circ (f^{\mathrm{ht}} \triangleright Id_\emptyset)) := \sigma_V(\alpha \circ (f^{\mathrm{ht}} \triangleright Id_\emptyset) \circ (\mathrm{ht}_V \triangleright Id_\emptyset)) \\ = \ & \sigma_V(\alpha \circ (\mathrm{ht}_W \triangleright Id_\emptyset) \circ (f \triangleright Id_\emptyset)) = \sigma_W^R(\alpha) \circ f \end{aligned}$$

by using naturality of σ and of $\mathrm{ht} \triangleright -$. For $f \in \mathrm{Hom}_{\mathcal{E}}(V, A_\Sigma)$ one has $(\sigma_V^R)^{-1}(f) = \sigma_V^{-1}(f) \circ (\mathrm{ht}_V^{-1} \triangleright Id_\emptyset)$.
Therefore (A_Σ, σ^R) is the internal endomorphism object of the empty set in $SK_{\mathcal{V}}(\Sigma)$ with respect to the right $SK_{\mathcal{V}}(\mathbb{R}^2)$-action by Proposition 5.1.16. We now study its algebra structure.
The evaluation map is:

$$ev^R := (\sigma_{A_\Sigma}^R)^{-1}(Id_{A_\Sigma}) = \sigma_{A_\Sigma}^{-1}(Id_{A_\Sigma}) \circ (\mathrm{ht}_{A_\Sigma}^{-1} \triangleright Id_\emptyset) = ev \circ (\mathrm{ht}_{A_\Sigma}^{-1} \triangleright Id_\emptyset) \in \mathrm{Hom}_{SK_{\mathcal{V}}(\Sigma)}(\emptyset \triangleleft A_\Sigma, \emptyset).$$

The product, or composition map, is:

$$\begin{aligned} m^R := \ & \sigma_{A_\Sigma \otimes A_\Sigma}^R(ev^R \circ (ev^R \triangleleft Id_{A_\Sigma})) \\ = \ & \sigma_{A_\Sigma \otimes A_\Sigma}(ev \circ (\mathrm{ht}_{A_\Sigma}^{-1} \triangleright Id_\emptyset) \circ Id_{A_\Sigma^{\mathrm{ht}}} \triangleright (ev \circ (\mathrm{ht}_{A_\Sigma}^{-1} \triangleright Id_\emptyset)) \circ (\mathrm{ht}_{A_\Sigma \otimes A_\Sigma} \triangleright Id_\emptyset)) \\ = \ & \sigma_{A_\Sigma \otimes A_\Sigma}(ev \circ (Id_{A_\Sigma} \triangleright ev) \circ (\mathrm{ht}_{A_\Sigma}^{-1} \otimes \mathrm{ht}_{A_\Sigma}^{-1} \triangleright Id_\emptyset) \circ (\mathrm{ht}_{A_\Sigma \otimes A_\Sigma} \triangleright Id_\emptyset)) \\ = \ & \sigma_{A_\Sigma \otimes A_\Sigma}(ev \circ (Id_{A_\Sigma} \triangleright ev) \circ (c_{A_\Sigma \otimes A_\Sigma} \triangleright Id_\emptyset)) = m \circ c_{A_\Sigma \otimes A_\Sigma}. \end{aligned}$$

The units $\sigma_{1_{\mathcal{V}}}^R(Id_\emptyset) := \sigma_{1_{\mathcal{V}}}(Id_\emptyset \circ (\mathrm{ht}_{1_{\mathcal{V}}} \triangleright Id_\emptyset)) = \sigma_{1_{\mathcal{V}}}(Id_\emptyset)$ coincide. □

Remark 5.3.7: In $SK_{\mathcal{V}}(\mathbb{R}^2)$, one has $\sigma_V^R(\alpha) := \sigma_V(\alpha \circ (\mathrm{ht}_V \triangleright Id_\emptyset)) = \sigma_{V^{\mathrm{ht}}}(\alpha) \circ \mathrm{ht}_V$. If one post-composes with $\mathrm{ht}_{A_\Sigma^{\mathrm{ht}}}^{-1}$ this is exactly $(\sigma_{V^{\mathrm{ht}}})^{\mathrm{ht}}$. Hence the two descriptions of right internal skein algebras we gave, $(A_\Sigma^{\mathrm{ht}}, \sigma_{(-)^{\mathrm{ht}}}{}^{\mathrm{ht}})$ in Remark 5.3.3 and $(A_\Sigma, \sigma \circ (\mathrm{ht} \triangleright -))$ in Proposition 5.3.6, are isomorphic (as they should) by $\mathrm{ht}_{A_\Sigma^{\mathrm{ht}}}^{-1} : A_\Sigma \to A_\Sigma^{\mathrm{ht}}$. The product on A_Σ^{ht} is given by:

$$\begin{aligned} \mathrm{ht}_{A_\Sigma^{\mathrm{ht}}}^{-1} \circ m^R \circ \mathrm{ht}_{A_\Sigma^{\mathrm{ht}}} \otimes \mathrm{ht}_{A_\Sigma^{\mathrm{ht}}} & = \mathrm{ht}_{A_\Sigma^{\mathrm{ht}}}^{-1} \circ m \circ c_{A_\Sigma \otimes A_\Sigma} \circ \mathrm{ht}_{A_\Sigma^{\mathrm{ht}}} \otimes \mathrm{ht}_{A_\Sigma^{\mathrm{ht}}} \\ & = \mathrm{ht}_{A_\Sigma^{\mathrm{ht}}}^{-1} \circ m \circ \mathrm{ht}_{A_\Sigma \otimes A_\Sigma} = m^{\mathrm{ht}}. \end{aligned}$$ ◇

5.3.2 Multi-edges internal skein algebras

We extend the definition of internal skein algebras to the multi-edge context, and define them as internal endomorphism algebras of the empty set in the skein category with multiple boundary actions, as expected. We check that they still describe skein categories well-enough.

Definition 5.3.8: Let $\mathfrak{S}$ be a marked surface with n boundary edges labelled either as left (numbered 1 to k) or as right (numbered $k+1$ to n) edges. Each left (resp. right) boundary edge induces a left $Sk_{\mathcal{V}}(\mathbb{R}^2)$-action (resp. left $Sk_{\mathcal{V}}(\mathbb{R}^2)^{\otimes-op}$-action) on $Sk_{\mathcal{V}}(\mathfrak{S})$, which all commute, so one has a left $Sk_{\mathcal{V}}(\mathbb{R}^2)^{\otimes k} \otimes (Sk_{\mathcal{V}}(\mathbb{R}^2)^{\otimes-op})^{\otimes n-k}$-action $\triangleright$ on $Sk_{\mathcal{V}}(\mathfrak{S})$. We denote its components by

$$\triangleright_i \text{ or } \triangleleft_i \circ fl : Sk_{\mathcal{V}}(\mathbb{R}^2) \otimes Sk_{\mathcal{V}}(\mathfrak{S}) \to Sk_{\mathcal{V}}(\mathfrak{S}),\ 1 \le i \le n\ ,$$

though we forget the indices when they are understood. When there are missing components they will be implicitly filled by $1_{\mathcal{V}}$. We may also write $(V_1, \dots, V_k) \triangleright \emptyset \triangleleft (V_{k+1}, \dots, V_n)$ instead of $(V_1, \dots, V_n) \triangleright \emptyset$.
The internal skein algebra $A_{\mathfrak{S}}$ is the internal endomorphism object of the empty set in $Sk_{\mathcal{V}}(\mathfrak{S})$ with respect to the $Sk_{\mathcal{V}}(\mathbb{R}^2)^{\otimes k} \otimes (Sk_{\mathcal{V}}(\mathbb{R}^2)^{\otimes-op})^{\otimes n-k}$-action. It is an algebra object in $\mathcal{E}^{\boxtimes n} \simeq \mathrm{Free}(\mathcal{V}^{\otimes n}) \simeq \mathrm{Free}(Sk_{\mathcal{V}}(\mathbb{R}^2)^{\otimes n})$, where $\mathcal{E}^{\boxtimes n}$ has opposite tensor products on the last $n-k$ components. We denote this monoidal structure by $\otimes$. We denote by $\otimes_i$ the tensor product on coordinate i, and adopt the same convention as with $\triangleright$ filling with missing $1_{\mathcal{V}}$'s and writing $(V_1, \dots, V_k) \otimes W \otimes (V_{k+1}, \dots, V_n)$ instead of $(V_1, \dots, V_n) \otimes W$.
Explicitly, $A_{\mathfrak{S}}$ comes equipped with natural isomorphisms

$$\sigma_{\vec{V}} : \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(\vec{V} \triangleright \emptyset, \emptyset) \tilde{\Rightarrow} \mathrm{Hom}_{\mathcal{E}^{\boxtimes n}}(\vec{V}, A_{\mathfrak{S}})$$

for $\vec{V} = (V_1, \dots, V_n) \in \mathcal{V}^{\otimes n}$.

Remark 5.3.9: Objects and morphisms of tensor product of categories (e.g. $\mathcal{V}^{\otimes n}$) are sometimes denoted by tensor product of objects and morphisms (e.g. $V_1 \otimes \cdots \otimes V_n$ and $f_1 \otimes \cdots \otimes f_n$). To avoid confusion with the monoidal structures on the categories (e.g. $\otimes$ on $\mathcal{V}$), we prefer to use commas (e.g. $(V_1, \dots, V_n)$ and $(f_1, \dots, f_n)$). ◇

Remark 5.3.10: A legitimate worry about this extended definition of internal skein algebras is that when one has multiple boundary actions on a same connected component one cannot keep track of where did an object come from. Namely for $V \in \mathcal{V}$ and c_1, c_2 two boundary edges on a same connected component of $\mathfrak{S}$, one has an isomorphism $V \triangleright_1 \emptyset \to V \triangleright_2 \emptyset$ which sounds surprising because $(V, 1_{\mathcal{V}})$ and $(1_{\mathcal{V}}, V)$ are hardly isomorphic in $\mathcal{V}^{\otimes 2}$. The internal skein algebra actually keeps track of such identifications, and one has an isomorphism $(V, 1_{\mathcal{V}}) \otimes A_{\mathfrak{S}} \simeq (1_{\mathcal{V}}, V) \otimes A_{\mathfrak{S}}$. ◇

Note that the above definition makes sense for $n = 0$, where we want endomorphisms of the empty set in $Sk_{\mathcal{V}}(\mathfrak{S})$ to be described by morphisms $k \to A_{\mathfrak{S}}$ in $\mathcal{E}^{\boxtimes 0} = \mathrm{Vect}_{\Bbbk}$. For $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)\text{–}comod^{fin}$ one gets $A_{\mathfrak{S}} = \mathring{\mathscr{S}}(\mathfrak{S})$ is the usual skein algebra. It is no longer true, however, that all objects of $Sk_{\mathcal{V}}(\mathfrak{S})$ are described as modules over $A_{\mathfrak{S}}$, because the (trivial) action of $\mathcal{E}^{\boxtimes 0}$ on $\emptyset$ is no longer dominant.

Definition 5.3.11: Let $\mathfrak{S}$ be a marked surface and $\mathcal{V}$ a ribbon category. The reduced skein category $Sk^{red}_{\mathcal{V}}(\mathfrak{S})$ is the full subcategory of $Sk_{\mathcal{V}}(\mathfrak{S})$ spanned by objects of the form $\vec{V} \triangleright \emptyset$, namely in the image of the action of $Sk_{\mathcal{V}}(\mathbb{R}^2)^{\otimes n}$ on the empty set.
It is equivalent to $Sk_{\mathcal{V}}(\mathfrak{S})$ if $\mathfrak{S}$ has at least one boundary edge per connected component.◇

Remark 5.3.12: One can still apply Remark 5.1.20, slightly modified because for right edges the left adjoint of $- \triangleleft V$ is given by acting by the left dual $- \triangleleft {}^*V$. For $\vec{V} = (V_1, \dots, V_n) \in \mathcal{V}^{\otimes n}$ write $\vec{V}^* = (V_i^* \text{ or } {}^*V_i)_{1 \le i \le n}$ with right duals for left edges and left duals for right edges. Then, as the notation suggests, $\vec{V}^*$ is the left dual of $\vec{V}$ in $\mathcal{V}^{\otimes n}$ for the monoidal structure $\otimes$, and $\vec{V} \triangleright -$ has left adjoint $\vec{V}^* \triangleright -$. For $\vec{W} \in \mathcal{V}^{\otimes n}$, one has natural isomorphisms:

$$\begin{array}{rcl} \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(\vec{W} \triangleright \emptyset, \vec{V} \triangleright \emptyset) & \simeq & \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(\vec{V}^* \triangleright (\vec{W} \triangleright \emptyset), \emptyset) \\ & \overset{\sigma_{\vec{V}^* \otimes \vec{W}}}{\to} & \mathrm{Hom}_{\mathcal{E}^{\boxtimes n}}(\vec{V}^* \otimes \vec{W}, A_{\mathfrak{S}}) \ \simeq \ \mathrm{Hom}_{\mathcal{E}^{\boxtimes n}}(\vec{W}, \vec{V} \otimes A_{\mathfrak{S}}). \end{array}$$

◇

Theorem 5.3.13: *There is an equivalence of categories*

$$\begin{array}{ccc} SK_{\mathcal{V}}^{red}(\mathfrak{S}) & \tilde{\to} & mod_{\mathcal{E}} - A_{\mathfrak{S}} \\ M & \mapsto & \underline{\mathrm{Hom}}(\emptyset, M) \end{array}$$

between the free cocompletion $SK_{\mathcal{V}}^{red}(\mathfrak{S})$ of the reduced skein category and the category of right $A_{\mathfrak{S}}$-modules in $\mathcal{E}^{\boxtimes n}$ (with monoidal structure $\otimes$).
For M of the form $\vec{V} \triangleright \emptyset$, one has $\underline{\mathrm{Hom}}(\emptyset, \vec{V} \triangleright \emptyset) \simeq \vec{V} \otimes A_{\mathfrak{S}}$.

Proof : We follow the proof of Theorem 5.1.21, namely we use [BBJ18a, Theorem 4.6] on $\emptyset \in SK_{\mathcal{V}}^{red}(\mathfrak{S})$. It is projective by the same arguments and $- \triangleright \emptyset : \mathcal{E}^{\boxtimes n} \to SK_{\mathcal{V}}^{red}(\mathfrak{S})$ is dominant by construction so $act_{\emptyset}^R$ is faithful and $\emptyset$ is a generator. For the last statement, one has $\underline{\mathrm{Hom}}(\emptyset, \vec{V} \triangleright \emptyset) \simeq \vec{V} \otimes A_{\mathfrak{S}}$ by Remark 5.3.12. □

5.3.3 Relation for multiple left edges

Let $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$–$comod^{fin}$, $\mathcal{E} = \mathcal{O}_{q^2}(SL_2)$–$comod \simeq \mathrm{Free}(\mathcal{V})$ and $\mathfrak{S}$ be a marked surface with all boundary edges labelled left. We show that $A_{\mathfrak{S}} \simeq \mathscr{S}(\mathfrak{S})$ as $\mathcal{O}_{q^2}(SL_2)^{\otimes n}$-comodule-algebras.

Proposition 5.3.14: *There is an equivalence of categories $\mathcal{E}^{\boxtimes n} \simeq \mathcal{O}_{q^2}(SL_2)^{\otimes n}$–comod.*

Proof : The category $\mathcal{O}_{q^2}(SL_2)^{\otimes n}$–*comod* is semi-simple with simples tensor products of simples $\mathcal{O}_{q^2}(SL_2)$-comodules. It implies that the cocontinuous extension of $\otimes^n : \mathcal{V}^{\otimes n} \to \mathcal{O}_{q^2}(SL_2)^{\otimes n}$–*comod* to $\mathcal{E}^{\boxtimes n} \to \mathcal{O}_{q^2}(SL_2)^{\otimes n}$–*comod* is an equivalence. □

In particular $\mathrm{Free}(TL^{\otimes n}) \simeq \mathrm{Free}(TL)^{\boxtimes n} \simeq \mathcal{E}^{\boxtimes n} \simeq \mathcal{O}_{q^2}(SL_2)^{\otimes n}$–*comod*.

Theorem 5.3.15: *Let $\mathfrak{S}$ be a marked surface with all boundary edges labelled left, then $A_{\mathfrak{S}} \simeq \mathscr{S}(\mathfrak{S})$ as $\mathcal{O}_{q^2}(SL_2)^{\otimes n}$-comodule-algebras.*

Proof : We give a natural isomorphism St exhibiting $\mathscr{S}(\mathfrak{S})$ as the internal endomorphism object of $\emptyset \in SK_{\mathcal{V}}(\mathfrak{S})$ with respect to the $\mathcal{O}_{q^2}(SL_2)^{\otimes n}$–*comod*-module structure. For $X \in \mathcal{O}_{q^2}(SL_2)^{\otimes n}$–*comod*, we want

$$\mathrm{St}_X : \mathrm{Hom}_{SK_{\mathcal{V}}(\mathfrak{S})}(X \rhd \emptyset, \emptyset) \to \mathrm{Hom}_{\mathcal{O}_{q^2}(SL_2)^{\otimes n}\text{–}comod}(X, \mathscr{S}(\mathfrak{S})) \ .$$

Let $X \in TL^{\otimes n}$ (which is enough by Proposition 5.1.16) and $\alpha \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(X \rhd \emptyset, \emptyset)$ represented by a tangle, we set

$$\mathrm{St}_X(\alpha) : \left\{ \begin{array}{ccc} X & \to & \mathscr{S}(\mathfrak{S}) \\ v_{\vec{\varepsilon}_1}^1 \otimes \cdots \otimes v_{\vec{\varepsilon}_n}^n & \mapsto & {}_{\vec{\varepsilon}_1 \ldots \vec{\varepsilon}_n}\alpha \end{array} \right.$$

where ${}_{\vec{\varepsilon}_1\dots\vec{\varepsilon}_n}\alpha$ is the tangle α with endpoints pushed over the boundary edges and states the ε^i_j over the i-th edge and in j-th position from top to bottom, as in Section 5.2 but with more than one edge.
It is an $\mathcal{O}_{q^2}(SL_2)^{\otimes n}$-comodule morphism because it is an $\mathcal{O}_{q^2}(SL_2)$-comodule morphism on each coordinate by the same calculations as in Proposition 5.2.2. It is natural in $TL^{\otimes n}$ because it is a natural in each coordinate by the same calculations as in Proposition 5.2.3.
It is an isomorphism by the same arguments as in Theorem 5.2.4. Namely, let $W = V_1 \otimes \cdots \otimes V_n \in TL^{\otimes n}$ and $f : W \to \mathscr{S}(\mathfrak{S})$, split $W = \oplus_i W_i$ and $f = \oplus f_i$ with $W_i = V_{i,1} \otimes \cdots \otimes V_{i,n}$ simple and choose $w_i \in W_i \setminus \{0\}$. Choose a_i representing $f_i(w_i)$ and denote by α_i its underlying tangle and $\vec{\varepsilon}_{i,1}, \dots, \vec{\varepsilon}_{i,n}$ its states. Include $W_i \overset{g_i}{\hookrightarrow} V^{\otimes n_{i,1}} \otimes \cdots \otimes V^{\otimes n_{i,n}}$ by mapping w_i to $v_{\vec{\varepsilon}_{i,1}} \otimes \cdots \otimes v_{\vec{\varepsilon}_{i,n}}$, and set

$$\mathrm{St}^{-1}_{W_i}(f_i) = \alpha_i \circ (g_i \triangleright Id_\emptyset) \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(W_i \triangleright \emptyset, \emptyset) \ .$$

Then the inverse of St is given by

$$\mathrm{St}^{-1}_{W}(f) = \oplus_i \mathrm{St}^{-1}_{W_i}(f_i) \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(W \triangleright \emptyset, \emptyset)$$

and does not depend on the choice of representative.
As in Proposition 5.2.5, because every boundary edge of $\mathfrak{S}$ is labelled left, the product inherited from the internal endomorphism object structure is still given by α with prescribed inputs $v_{\vec{\varepsilon}_{i,1}} \otimes \cdots \otimes v_{\vec{\varepsilon}_{i,n}}$ times β with prescribed inputs $v_{\vec{\eta}_{i,1}} \otimes \cdots \otimes v_{\vec{\eta}_{i,n}}$ equals $\alpha \circ (Id \triangleright \beta)$ with prescribed inputs $(v_{\vec{\varepsilon}_{i,1}} \otimes v_{\vec{\eta}_{i,1}}) \otimes \cdots \otimes (v_{\vec{\varepsilon}_{i,n}} \otimes v_{\vec{\eta}_{i,n}})$ which is the usual product on $\mathscr{S}(\mathfrak{S})$. □

5.3.4 The half twist on $\mathcal{O}_{q^2}(SL_2)$–*comod*

In the last subsection, we only allowed left $Sk_{\mathcal{V}}(\mathbb{R}^2)$-actions. We study here how to change from left to right actions using the half twist in the case $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)\text{–}comod^{fin}$.

Remark 5.3.16: As we saw, a half twist on $\mathcal{V}$ is usually not necessary to the general study of internal skein algebras, but it is needed to relate them to stated skein algebras when there are right edges, and to mirror their excision properties. When one sees a boundary edge at the right instead of the left, it has very different effects on both sides. For stated skein algebras, it does not change the vector space, nor the algebra structure, but switches the right comodule structure to a left one using rot_*. For internal skein algebras, it does not change the vector space, one keeps right comodules (A^R_Σ is still an object of $\mathcal{E}$) though slightly changed: it is half-twisted, and the algebra structure is opposed. To make both sides agree, one needs to switch the comodule structure of the internal skein algebra while taking the opposite of its algebra structure. This is done very naturally by using S. Therefor one expects that the half twist on $\mathcal{O}_{q^2}(SL_2)$–*comod* should be the difference between switching the comodule structure using rot_* and switching it using S. This is Proposition 5.3.26, but we give a more complete and algebraic definition below. ◇

Definition 5.3.17 (Section 4.1 in [ST09], for categories of comodules):
A half-coribbon Hopf algebra is a coribbon Hopf algebra H equipped with a half-coribbon functional, i.e. a map $t : H \to \Bbbk$ such that:
(1) t is invertible by convolution: $\exists\ t^{-1} : H \to \Bbbk$ such that

$$t(a_{(1)})t^{-1}(a_{(2)}) = t^{-1}(a_{(1)})t(a_{(2)}) = \varepsilon(a),$$

(2) t squares to the twist: $t(a_{(1)})t(a_{(2)}) = \theta(a)$ and
(3) compatibility with product: $t(a.b) = t(b_{(1)})t(a_{(1)})R(a_{(2)} \otimes b_{(2)})$. ♢

Definition 5.3.18: The half-coribbon functional induces a half twist ht on the category *H–comod* by

$$\mathrm{ht}_V : V \overset{\Delta}{\to} V \otimes H \overset{Id\otimes t}{\longrightarrow} V \ .$$

It is an isomorphism of vector space with $\mathrm{ht}_V^{-1} : V \overset{\Delta}{\to} V \otimes H \overset{Id\otimes t^{-1}}{\longrightarrow} V$. The half-coribbon functional is not supposed to be central though, and this means that ht_V is not an H-comodule morphism. There is a unique comodule structure on (the target) V that makes it a comodule morphism, namely

$$\Delta^{\mathrm{ht}} := (\mathrm{ht}_V \otimes Id_H) \circ \Delta \circ \mathrm{ht}_V^{-1} \ .$$

We denote by V^{ht} the vector space V equipped with the coaction Δ^{ht}. Now, $\mathrm{ht}_V : V \to V^{\mathrm{ht}}$ is an isomorphism of H-comodules.
One has a functor

$$(-)^{\mathrm{ht}} : H\text{–}comod \to H\text{–}comod$$

which sends an object V to the half-twisted V^{ht} and a morphism $f : V \to W$ to $f^{\mathrm{ht}} = \mathrm{ht}_W \circ f \circ \mathrm{ht}_V^{-1} : V^{\mathrm{ht}} \to W^{\mathrm{ht}}$. It is defined so that $\mathrm{ht} : Id \Rightarrow (-)^{\mathrm{ht}}$ is a natural isomorphism.♢

As maps of vector spaces, one simply has:

$$\begin{aligned} f^{\mathrm{ht}} &= (Id_W \otimes t) \circ \Delta_W \circ f \circ (Id_V \otimes t^{-1}) \circ \Delta_V \\ &= (Id_W \otimes t) \circ \Delta_W \circ (Id_W \otimes t^{-1}) \circ (f \otimes Id_H) \circ \Delta_V \\ &= (Id_W \otimes t) \circ \Delta_W \circ (Id_W \otimes t^{-1}) \circ \Delta_W \circ f = \mathrm{ht}_W \circ \mathrm{ht}_W^{-1} \circ f = f \end{aligned}$$

In particular $\mathrm{ht}_{V^{\mathrm{ht}}} = \mathrm{ht}_V^{\mathrm{ht}} = \mathrm{ht}_V$. The square of t is θ so $\mathrm{ht}_{V^{\mathrm{ht}}} \circ \mathrm{ht}_V = \theta_V$, and θ is central so $(V^{\mathrm{ht}})^{\mathrm{ht}} = V$, and $(-)^{\mathrm{ht}} \circ (-)^{\mathrm{ht}} = \mathrm{Id}_{H\text{–}comod}$.
Regarding the monoidal structure, let V and W be two H-comodules. Remember that the braiding is defined as $c_{V,W} : V \otimes W \overset{R_{24}\circ(\Delta_V \otimes \Delta_W)}{\longrightarrow} V \otimes W \overset{fl}{\to} W \otimes V$. The third condition gives that

$$\mathrm{ht}_{V\otimes W} = \mathrm{ht}_V \otimes \mathrm{ht}_W \circ (fl \circ c_{V,W})$$

so $fl \circ \mathrm{ht}_{V\otimes W} = \mathrm{ht}_W \otimes \mathrm{ht}_V \circ c_{V,W}$. In particular, $fl : (V \otimes W)^{\mathrm{ht}} \to W^{\mathrm{ht}} \otimes V^{\mathrm{ht}}$ is an H-comodule isomorphism.

Definition 5.3.19: For $\mathcal{V}$ a ribbon category, let ht–$\mathrm{Rib}_{\mathcal{V}}$ be the full subcategory of $\mathrm{Rib}_{\mathcal{V}}(\mathbb{R}^2)$ spanned by objects of the form $[n] \subseteq \mathbb{R}^2$ but now allowing either blackboard or anti-blackboard framing for every point. This subcategory is still ribbon and is stable by the functor $(-)^{\mathrm{ht}}$, and equipped with $\mathrm{ht} : Id \Rightarrow (-)^{\mathrm{ht}}$. It also contains the category $\mathrm{Rib}_{\mathcal{V}}$ of blackboard framed points. ♢

Theorem 5.3.20 (Theorem 4.11 in [ST09]): *Let H be a half-coribbon Hopf algebra. There is a unique monoidal functor*

$$\mathrm{ht–RT} : \mathrm{ht–Rib}_{H\text{–}comod^{fin}} \to H\text{–}comod^{fin}$$

extending RT *and commuting with both* $(-)^{\mathrm{ht}}$ *and* ht*, so preserving the "half-ribbon structure".*

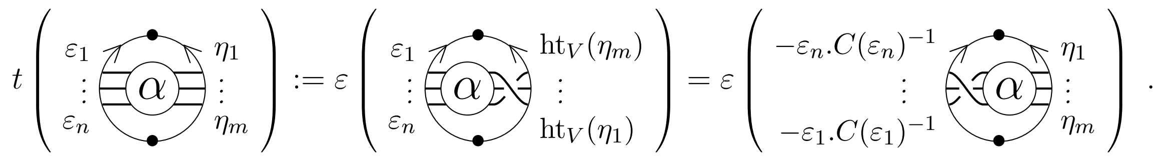
Remark 5.3.21: This half-twisted Reshetikhin–Turaev functor also gives an equivalence of categories $Sk_{\mathcal{V}}(\mathbb{R}^2) \simeq \mathcal{V}$ but this time with much nicer properties regarding the half twist. In the usual Reshetikhin–Turaev functor one only prescribes where to send points with blackboard framing and well-placed on the real line. For framed points not of this form, one has to choose an isomorphism with one of these, like in Remark 1.2.20, but these choices are quite arbitrary. Then the half twist sends blackboard framed points to anti-blackboard framed points which are re-identified with blackboard framed points via these arbitrary isomorphisms. With the half-twisted Reshetikhin–Turaev functor one also prescribes where to send anti-blackboard framed points, so one controls closely what happens with the half twist, namely the half twist on $Sk_{\mathcal{V}}(\mathbb{R}^2)$ is mapped to the half twist on $\mathcal{V}$.
Note however that unlike on $Sk_{\mathcal{V}}(\mathbb{R}^2)$, the half twist on $\mathcal{V}$ is not strictly anti-monoidal (indeed it is the identity on underlying vector spaces) but $(X \otimes Y)^{\mathrm{ht}} \simeq Y^{\mathrm{ht}} \otimes X^{\mathrm{ht}}$ is simply given by the flip of tensors. A bit like the R-matrix, the half twist gives the difference between the monoidal structure on *H–comod* and the symmetric one on $\mathrm{Vect}_{\Bbbk}$. Formally, this error lies in the fact that the inclusion of $\mathcal{V}$ in $Sk_{\mathcal{V}}(\mathbb{R}^2)$ is only monoidal up to natural isomorphism, and this isomorphism, given in Remark 1.2.21, maps by the half twist to the flip of tensors. ◇

In the case of $\mathcal{O}_{q^2}(SL_2)$, we can define a half-coribbon functional on the generators by $t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -q^{\frac{5}{2}} \\ q^{\frac{1}{2}} & 0 \end{pmatrix}$. This tells in particular how the half twist acts on the standard corepresentation V, namely

$$\mathrm{ht}_V(v_+) = q^{\frac{1}{2}} v_- = C(-)^{-1} v_- \text{ and } \mathrm{ht}_V(v_-) = -q^{\frac{5}{2}} v_+ = C(+)^{-1} v_+ \ .$$

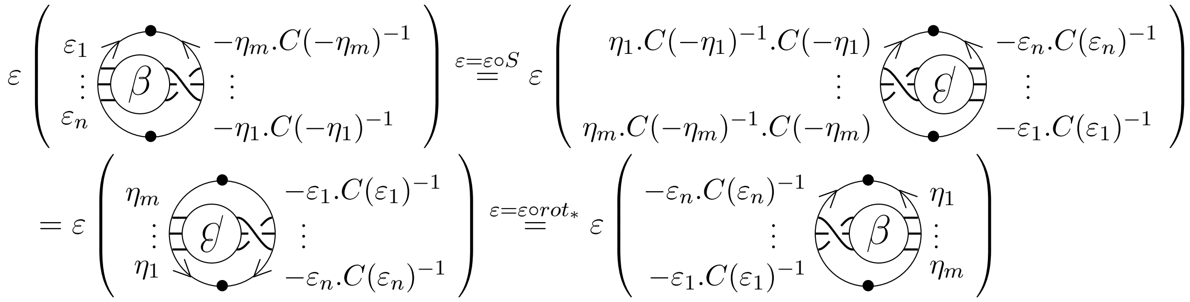
For states $\eta \in \{\pm\}$, we will write $\mathrm{ht}_V(\eta) = -\eta . C(-\eta)^{-1}$.
Note that [ST09] introduces another half-coribbon element corresponding to the matrix $\begin{pmatrix} 0 & q^{\frac{3}{2}} \\ -q^{\frac{3}{2}} & 0 \end{pmatrix}$, but our choice is imposed by conventions from stated skein algebras.
To define it on all $\mathcal{O}_{q^2}(SL_2)$ we prefer a geometric description on $\mathscr{S}(B)$. We would like to give the same definition as for the coribbon functional θ with a half twist instead of a full twist, but in the definition of stated skein algebras one only allows upward-framed boundary points, which would map to downward-framed points after the half twist. Still, we know how to do the "global" half twist on many strands (without twisting the framing), and we only need to add a "local" half twist on each strand, which are implicitly coloured by V, on which we know how the half twist acts.

Proposition 5.3.22: *The coribbon Hopf algebra $\mathscr{S}(B)$ is half-coribbon with half-coribbon functional*

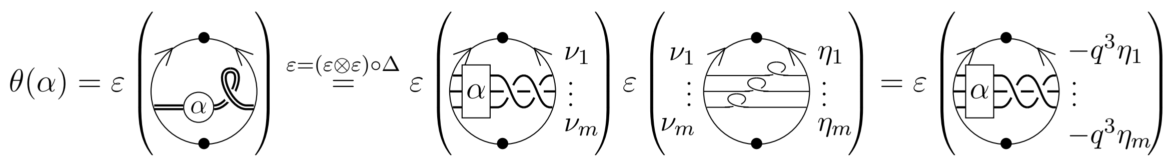


By Remark 5.1.6, $\mathrm{ht}_{\mathscr{S}(B)}(\alpha) = (Id \otimes t) \circ \Delta(\alpha)$ is the stated tangle represented in the middle, and by a left version of Remark 5.1.6, $(t \otimes Id) \circ \Delta(\alpha)$ is the stated tangle represented in the right.

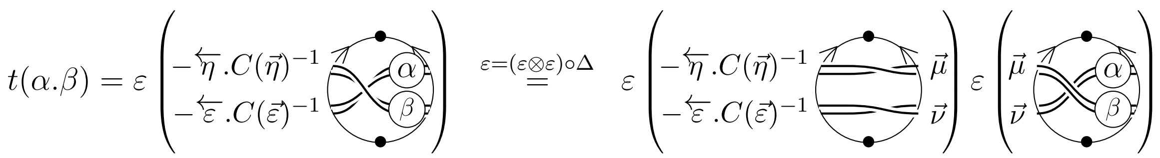
Proof : These two formulations prove that t is well defined on $\mathscr{S}(B)$ as it respects the boundary relations on the left edge by the first and on the right one by the second. So we begin by proving that these two formulations actually coincide. Let $\beta \in \mathscr{S}(B)$, then :



where the second equality is only a change of picture representation, not of stated tangles, coming from switching the orientation of the edges, see [BW11, Section 3.5].
The convolution inverse t^{-1} of t is obtained the same way as the middle term but with the inverse half twist and ht_V^{-1} on states. Indeed by Remark 5.1.6, $(Id \otimes t^{-1}) \circ \Delta(\alpha)$ is α with an inverse half twist at the right and ht_V^{-1} on right states. Thus $(t \otimes t^{-1}) \circ \Delta$ is the counit of α with an inverse half twist and a half twist at the right, and $\mathrm{ht}_V \circ \mathrm{ht}_V^{-1}$ on right states, namely the counit of α. Similarly, $(t^{-1} \otimes t) \circ \Delta = \varepsilon$.
One directly checks that $\mathrm{ht}_V \circ \mathrm{ht}_V = \theta_V = -q^3 Id_V$ on the standard corepresentation. Then $(Id \otimes t) \circ \Delta(\alpha)$ is α with a half twist at the right and ht_V on right states, and $(t \otimes t) \circ \Delta(\alpha)$ is the counit of α with a full twist (without framing twist) at the right and θ_V on right states. This is exactly the full twist by separating the unframed full twist and the full twists on framings:



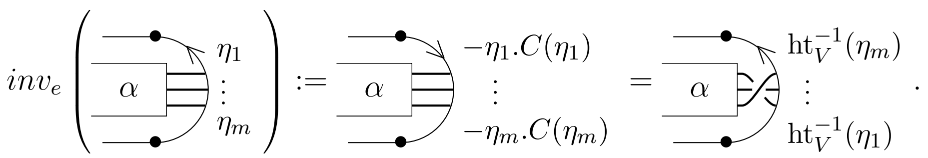
Finally,



$$\overset{\varepsilon \circ m = \varepsilon\otimes\varepsilon}{=} t(\beta_{(1)}).t(\alpha_{(1)}).R(\alpha_{(2)} \otimes \beta_{(2)})$$

□

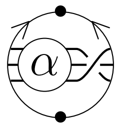
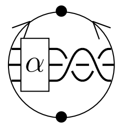
Definition 5.3.23: Let $\mathfrak{S}$ be a marked surface and e a boundary edge, with orientation induced by the one of $\mathfrak{S}$. The inversion along the edge e is the morphism of k-vector-spaces $inv_e : \mathscr{S}(\mathfrak{S}) \to \mathscr{S}(\mathfrak{S})$ given on a stated tangle α by ordering the heights according to the orientation of e, then switching height order vertically, then taking opposite states and some coefficients, namely:



It is well-defined by [CL22, Proposition 2.7]. Note that inv_e is neither an algebra morphism nor a comodule morphism. ♢

Proposition 5.3.24: *Let e_r be the right edge of the bigon and $\alpha \in \mathscr{S}(B)$, then $t(\alpha) = \varepsilon \circ inv_{e_r}^{-1}(\alpha)$.*
In particular, by Remark 5.1.6, for $\mathfrak{S}$ a marked surface with a right edge e, the half twist acts on $\mathscr{S}(\mathfrak{S})$ as inv_e^{-1}.

Proof : Indeed,

$$t \circ inv_{e_r}(\alpha) = t \left(\right.$$  $$\left. \begin{matrix} \mathrm{ht}_V^{-1}(\eta_m) \\ \vdots \\ \mathrm{ht}_V^{-1}(\eta_1) \end{matrix} \right) = \varepsilon \left(\right.$$  $$\left. \begin{matrix} \mathrm{ht}_V \circ \mathrm{ht}_V^{-1}(\eta_1) \\ \vdots \\ \mathrm{ht}_V \circ \mathrm{ht}_V^{-1}(\eta_m) \end{matrix} \right) = \varepsilon(\alpha).$$

□

Remark 5.3.25: In [CL22, Section 3.4] the counit is defined as $\varepsilon = i_* \circ inv_{e_r}$, where i is the inclusion of the bigon in the monogon. Surprisingly enough, the half coribbon functional is then $t = i_*$ and is simpler to write. This suggests that there is a half twist built in the construction of stated skein algebras. We claim this comes from the passage from right to left comodule structure. ◇

If A is a right comodule over a Hopf algebra H, it is naturally a left comodule with $\Delta^L = fl \circ (Id_A \otimes S) \circ \Delta$, and we will consider these two as the "same" comodule. Indeed, one has an isomorphism of categories

$$(-)^L : H\text{–}comod \to comod\text{–}H$$

which is the identity on vector spaces, switches the action as above, and is the identity on morphisms. However, when passing from right to left edges – and comodule structures – on stated skein algebras, one uses another way to see a right comodule A as a left, namely with $\Delta^l = fl \circ (Id_A \otimes rot_*) \circ \Delta$. Again one has

$$(-)^l : H\text{–}comod \to comod\text{–}H$$

with the identity on morphisms. So we have two functors $(-)^L$ and $(-)^l$ and we claim that the difference between them is precisely a half twist:

Proposition 5.3.26: *One has $(-)^l = (-)^L \circ (-)^{\mathrm{ht}}$ and $(-)^L = (-)^l \circ (-)^{\mathrm{ht}}$.*
Equivalently, the map $\mathrm{ht}_A^l : A^l \to A^L$ is an isomorphism of left $\mathcal{O}_{q^2}(SL_2)$-comodules.

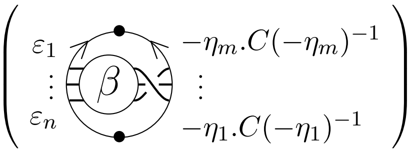
Proof : All these functors are the identity on morphisms and only change the comodule structure. The map $\mathrm{ht}_A^l : A^l \to (A^{\mathrm{ht}})^l$ is just ht_A as a map of vector spaces and is an isomorphism of vector spaces. The comodule structure on $(A^{\mathrm{ht}})^l$ is the unique so that ht_A^l is a comodule morphism. We show that it is a comodule morphism $A^l \to A^L$, and hence that $A^L = (A^{\mathrm{ht}})^l$. Let $a \in A$, one compares

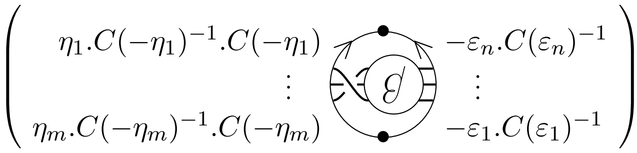
$$\Delta^L \circ \mathrm{ht}_A(a) = \Delta^L(a_{(1)} \otimes t(a_{(2)})) = Sa_{(2)}.t(a_{(3)}) \otimes a_{(1)}$$

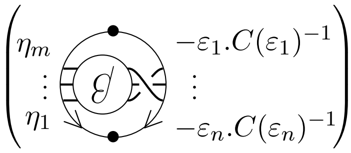
and

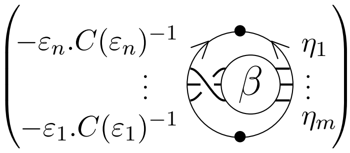
$$(Id \otimes \mathrm{ht}_A) \circ \Delta^l = (Id \otimes \mathrm{ht}_A)(rot_*(a_{(2)}) \otimes a_{(1)}) = rot_*(a_{(3)}).t(a_{(2)}) \otimes a_{(1)} \ .$$

We show directly that for $\beta \in \mathscr{S}(B)$, we have $S\beta_{(1)}.t(\beta_{(2)}) = rot_*(\beta_{(2)}).t(\beta_{(1)})$:

$S \circ (Id \otimes t) \circ \Delta(\beta) = S$ 

$=$ 

$=$ 

$\overset{rot_*^2=Id}{=} rot_*$ 

$= rot_* \circ (t \otimes Id) \circ \Delta(\beta)$

□

> **Proposition 5.3.27:** *Given two right $\mathcal{O}_{q^2}(SL_2)$-comodules A and B one has*
>
> $$(A \otimes B)^{inv} = HH^0(A^L \otimes B) = HH^0((A^{\mathrm{ht}})^l \otimes B) \ .$$

Proof : The first equality is true in any Hopf algebra by a direct computation, and the second is just the proposition above. □

5.3.5 The general relation

We can now express the full correspondence between stated skein algebras and internal skein algebras, with both right and left boundary edges.
On a single edge, by Remark 5.3.3 one gets $A^R_\Sigma = A^{\mathrm{ht}}_\Sigma = \mathscr{S}(\Sigma)^{\mathrm{ht}}$ equipped with the natural isomorphism $(\sigma_{(-)^{\mathrm{ht}}})^{\mathrm{ht}}$, using the half twist in $\mathcal{O}_{q^2}(SL_2)$–*comod*. More precisely, for $\alpha \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(\emptyset \lhd V, \emptyset)$ one has

$$\sigma_{V^{\mathrm{ht}}}(\alpha)^{\mathrm{ht}} = \mathrm{ht}^{-1}_{A^{\mathrm{ht}}_\Sigma} \circ \sigma_{V^{\mathrm{ht}}}(\alpha) \circ \mathrm{ht}_V = \mathrm{ht}^{-1}_{A^{\mathrm{ht}}_\Sigma} \circ \sigma_V(\alpha \circ (\mathrm{ht}_V \rhd \emptyset)) = \mathrm{ht}^{-1}_{A^{\mathrm{ht}}_\Sigma} \circ \sigma^R_V(\alpha) \ .$$

As in Remark 5.3.7 its algebra structure is

$$\begin{aligned} \mathrm{ht}^{-1}_{A^{\mathrm{ht}}_\Sigma} \circ m^R \circ \mathrm{ht}_{A^{\mathrm{ht}}_\Sigma} \otimes \mathrm{ht}_{A^{\mathrm{ht}}_\Sigma} &= \mathrm{ht}^{-1}_{A^{\mathrm{ht}}_\Sigma} \circ m \circ c_{A_\Sigma \otimes A_\Sigma} \circ \mathrm{ht}_{A^{\mathrm{ht}}_\Sigma} \otimes \mathrm{ht}_{A^{\mathrm{ht}}_\Sigma} \\ &= \mathrm{ht}^{-1}_{A^{\mathrm{ht}}_\Sigma} \circ m \circ fl \circ \mathrm{ht}_{A_\Sigma \otimes A_\Sigma} = m^{\mathrm{ht}} \circ fl \end{aligned}$$

so m^{op} as maps of vector spaces.
Note that because S is an anti-algebra morphism, the functor $(-)^L$ is (almost strictly) anti-monoidal (like the half twist, it is the identity on vector spaces but is anti-monoidal on the comodule structure) namely $fl : (V \otimes W)^L \to W^L \otimes V^L$ is an isomorphism of left H-comodules. Thus a right H-comodule algebra A induces a left H-comodule algebra A^L with $A^L \otimes A^L \overset{fl}{\simeq} (A \otimes A)^L \overset{m}{\to} A^L$, namely with product m^{op}.
When one has multiple edges one can switch the i-th right $\mathcal{O}_{q^2}(SL_2)$-comodule structure to a left using either S or rot_* and we denote the associated functors by $(-)^{L_i}$ and $(-)^{l_i}$, one can take opposite product on the i-th coordinate which we denote by m^{op_i}, and there are half twists on each coordinates, which we denote by ht_i.

Theorem 5.3.28: *Let $\mathfrak{S}$ be a marked surface with n boundary edges labelled either as left (numbered 1 to k) or as right (numbered $k+1$ to n) edges. There is an isomorphism of $(\mathcal{O}_{q^2}(SL_2)^{\otimes k}, \mathcal{O}_{q^2}(SL_2)^{\otimes n-k})$-bicomodules algebras*

$$A_{\mathfrak{S}}^{L_{k+1},\dots,L_n} \simeq \mathscr{S}(\mathfrak{S}) \ .$$

Proof : To avoid confusion we denote the stated skein algebra of the marked surface $\mathfrak{S}$ by $\mathscr{S}^R(\mathfrak{S})$ when it is seen as a right $\mathcal{O}_{q^2}(SL_2)^{\otimes n}$-comodule and by $\mathscr{S}(\mathfrak{S}) = \mathscr{S}^R(\mathfrak{S})^{l_{k+1},\dots,l_n}$ when it is seen as an $(\mathcal{O}_{q^2}(SL_2)^{\otimes k}, \mathcal{O}_{q^2}(SL_2)^{\otimes n-k})$-bicomodule. We denote by m its product, which is the same in both cases.
By Theorem 5.3.15, $\mathscr{S}^R(\mathfrak{S})$ is the internal skein algebra of $\mathfrak{S}$ with every edge labelled as left. Now by Remark 5.3.3 on coordinates $k+1,\dots,n$ one may take

$$A_{\mathfrak{S}} := \mathscr{S}^R(\mathfrak{S})^{\mathrm{ht}_{k+1},\dots,\mathrm{ht}_n}$$

as an object in $\mathcal{E}^{\boxtimes n}$ (which has skew monoidal structure $\otimes$) with algebra structure $m^{op_{k+1},\dots,op_n}$.
Thus by Proposition 5.3.26 and [CL22, Proposition 4.1],

$$A_{\mathfrak{S}}^{L_{k+1},\dots,L_n} := (\mathscr{S}^R(\mathfrak{S})^{\mathrm{ht}_{k+1},\dots,\mathrm{ht}_n})^{L_{k+1},\dots,L_n} = \mathscr{S}^R(\mathfrak{S})^{l_{k+1},\dots,l_n} = \mathscr{S}(\mathfrak{S})$$

as $(\mathcal{O}_{q^2}(SL_2)^{\otimes k}, \mathcal{O}_{q^2}(SL_2)^{\otimes n-k})$-bicomodules.
The algebra structure on $A_{\mathfrak{S}}^{L_{k+1},\dots,L_n}$ is $(m^{op_{k+1},\dots,op_n})^{op_{k+1},\dots,op_n} = m$. □

Remark 5.3.29: A nice miracle with stated skein algebras is that the quantum group $\mathcal{O}_{q^2}(SL_2)$, which is used to define the tangle invariants used to define stated skein algebras, is re-obtained as the stated skein algebra of the bigon. One can see why this should be true in internal skein algebras. By Definition 5.3.8, the internal skein algebra of the bigon is an object $A_B \in \mathcal{O}_{q^2}(SL_2)^{\otimes 2}$*–comod* together with a natural isomorphism

$$\mathrm{Hom}_{SK(B)}((X,Y)\triangleright\emptyset,\emptyset) = \mathrm{Hom}_{\mathcal{O}_{q^2}(SL_2)\text{–}comod}(X\otimes Y,\Bbbk) \tilde{\to} \mathrm{Hom}_{\mathcal{O}_{q^2}(SL_2)^{\otimes 2}\text{–}comod}(X\otimes Y, A_B)$$

for $X, Y \in \mathcal{O}_{q^2}(SL_2)$*–comod*. We set $A_B = \mathcal{O}_{q^2}(SL_2)$ with usual first right comodule structure $\Delta_1 = \Delta$ and with second comodule strucure its left one switched using L_2^{-1} namely $\Delta_2 = fl \circ (S^{-1} \otimes Id) \circ \Delta$. The demanded isomorphism is given by $f \mapsto \tilde{f}$ where

$$\tilde{f}(x\otimes y) = x_{(2)}.f(x_{(1)}\otimes y) = S(y_{(2)}).f(x\otimes y_{(1)}) \ .$$

Its inverse is given by $\tilde{f} \mapsto \varepsilon \circ \tilde{f}$. ◇

Remark 5.3.30: Despite this theorem, it is still annoying that in the simplest case one wants to see the boundary at the right for stated skein algebras and at the left for internal skein algebras. This should be solvable by considering the category of left (instead of right) $\mathcal{O}_{q^2}(SL_2)$-comodules as coefficients, so it is a minor issue. ◇

5.3.6 Excision properties of multi-edges internal skein algebras

Let $\mathfrak{S}_1 \hookleftarrow C \hookrightarrow \mathfrak{S}_2$ be a right and a left thick embeddings in two marked surfaces and $\mathfrak{S}$ their collar gluing. Namely, C embeds as a sequence of k right boundary edges $\vec{c}_1$ of

$\partial\mathfrak{S}_1$ and as k left boundary edges $\vec{c}_2$ of $\partial\mathfrak{S}_2$, and $\mathfrak{S}$ is the gluing $\mathfrak{S}_1 \cup_{\vec{c}_1=\vec{c}_2} \mathfrak{S}_2$. We show how to compute $A_{\mathfrak{S}}$ from $A_{\mathfrak{S}_1}$ and $A_{\mathfrak{S}_2}$.
The general idea goes as follows. In the case where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ both have a single boundary edge, so $k = 1$, one wants to describe endomorphisms α of the empty set in $Sk_{\mathcal{V}}(\mathfrak{S})$. By Corollary 5.1.14 they are described by a morphism $\alpha_1 : \emptyset \to \emptyset \lhd V$ in $Sk_{\mathcal{V}}(\mathfrak{S}_1)$ and a morphism $\alpha_2 : V \rhd \emptyset \to \emptyset$ in $Sk_{\mathcal{V}}(\mathfrak{S}_2)$, linked by an isomorphism $\iota_V : \emptyset \lhd V \to V \rhd \emptyset$, which is just a slanted skein crossing over C in $Sk_{\mathcal{V}}(\mathfrak{S})$, see the idea of proof of Theorem 5.1.13. One can reconstruct α as $\alpha = (Id_\emptyset, \alpha_2) \circ \iota_V \circ (\alpha_1, Id_\emptyset)$. The morphisms $\alpha_{1,2}$ are well defined up to balancing, namely naturality of ι. Now, by definition of $A_{\mathfrak{S}_1}$ and $A_{\mathfrak{S}_2}$, they are described by some $f_1 \in \mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{V}}, A_{\mathfrak{S}_1} \otimes V)$ and $f_2 \in \mathrm{Hom}_{\mathcal{E}}(V, A_{\mathfrak{S}_2})$. Composing them mimicking the reconstruction of α gives a morphism $f = (Id_{A_{\mathfrak{S}_1}} \otimes f_2) \circ f_1 : 1_{\mathcal{V}} \to A_{\mathfrak{S}_1} \otimes A_{\mathfrak{S}_2}$, namely an invariant inside $A_{\mathfrak{S}_1} \otimes A_{\mathfrak{S}_2}$. This suggests $A_{\mathfrak{S}} \simeq (A_{\mathfrak{S}_1} \otimes A_{\mathfrak{S}_2})^{inv}$, which we will prove below.
Now we need to define what we mean by invariants of a tensor product in any ribbon category $\mathcal{V}$.

Definition 5.3.31: Let $\mathcal{V}$ be a ribbon category, $\mathcal{E} = \mathrm{Free}(\mathcal{V})$ and $n \geq 2$. For $1 \leq i < j \leq n-1$ we denote the tensor product of coordinates i and j by:

$$\otimes_{i,j} : \begin{cases} \mathcal{V}^{\otimes n} & \to & \mathcal{V}^{\otimes n-1} \\ (V_1, \dots, V_n) & \mapsto & (V_1, \dots, V_{i-1}, V_i \otimes V_j, V_{i+1}, \dots, V_{j-1}, V_{j+1}, \dots, V_n) \end{cases}.$$

For $\vec{k}_1 < \vec{k}_2$ two sequences of k distinct indices we denote the tensor product of coordinates $\vec{k}_1$ with coordinates $\vec{k}_2$ by $\otimes_{\vec{k}_1,\vec{k}_2} : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n-k}$. It extends to $\otimes_{\vec{k}_1,\vec{k}_2} : \mathcal{E}^{\boxtimes n} \to \mathcal{E}^{\boxtimes n-k}$ by cocontinuity.
We denote the unit on i-th coordinate by:

$$\eta_i : \begin{cases} \mathcal{V}^{\otimes n-2} & \to & \mathcal{V}^{\otimes n-1} \\ (V_1, \dots, \overset{\vee}{V_i}, \dots, \overset{\vee}{V_j}, \dots, V_n) & \mapsto & (V_1, \dots, 1_{\mathcal{V}}, \dots, \overset{\vee}{V_j}, \dots, V_n) \end{cases}$$

and the unit on $\vec{k}_1$-th coordinates as $\eta_{\vec{k}_1} : \mathcal{V}^{\otimes n-2k} \to \mathcal{V}^{\otimes n-k}$.
Let $X \in \mathcal{E}^{\boxtimes n}$, we want to define its $(\vec{k}_1, \vec{k}_2)$-invariants $X^{inv_{\vec{k}_1,\vec{k}_2}} \in \mathcal{E}^{\boxtimes n-2k}$. One only needs to describe morphisms from any $\vec{V} \in \mathcal{V}^{\otimes n-2k}$ to it. We set:

$$X^{inv_{\vec{k}_1,\vec{k}_2}}(\vec{V}) = \mathrm{Hom}_{\mathcal{E}^{\boxtimes n-2k}}(\vec{V}, X^{inv_{\vec{k}_1,\vec{k}_2}}) := \mathrm{Hom}_{\mathcal{E}^{\boxtimes n-k}}(\eta_{\vec{k}_1}(\vec{V}), \otimes_{\vec{k}_1,\vec{k}_2}(X)).$$

For $X \in \mathcal{E}^{\boxtimes n_1} \boxtimes \mathcal{E}^{\boxtimes \vec{k}_1}$ and $Y \in \mathcal{E}^{\boxtimes \vec{k}_2} \boxtimes \mathcal{E}^{\boxtimes n_2}$ we write $X \otimes_{\vec{k}_1,\vec{k}_2} Y := \otimes_{\vec{k}_1,\vec{k}_2}(X, Y)$. Then:

$$\begin{aligned} (X,Y)^{inv_{\vec{k}_1,\vec{k}_2}}((V_{\vec{n}_1}, V_{\vec{n}_2})) := & \ \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}, V_{\vec{n}_2}), X \otimes_{\vec{k}_1,\vec{k}_2} Y) \\ = & \ \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}) \otimes_{\vec{k}_1,\vec{k}_2} (1_{\mathcal{V}^{\otimes k}}, V_{\vec{n}_2}), X \otimes_{\vec{k}_1,\vec{k}_2} Y). \end{aligned}$$

◇

For $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$*–comod*^{fin} we get a notion of invariants for bicomodules $(X_i, X_j) \in \mathcal{V}^{\otimes 2}$ where we first "merge" the two comodule structures (by the product, in the definition of the tensor product) and then take invariants in the usual sense, namely maps $k \to X_i \otimes X_j$.

Theorem 5.3.32: *Let $\mathfrak{S}_1$ be a marked surface with $n_1 + k$ boundary edges with a sequence of k right boundary edges $\vec{c}_1$ (numbered $\vec{k}_1 = \{n_1+1, \dots, n_1+k\}$) and $\mathfrak{S}_2$ a marked surface with $n_2 + k$ boundary edges with a sequence of k left boundary edges*

$\vec{c}_2$ (numbered $\vec{k}_2 = \{n_1 + k + 1, \dots, n_1 + 2k\}$). We write $\vec{n}_1 = \{1, \dots, n_1\}$ and $\vec{n}_2 = \{n_1 + 2k + 1, \dots, n_1 + 2k + n_2\}$ the indices of the other edges of $\mathfrak{S}_1$ and $\mathfrak{S}_2$.
Let $\mathfrak{S} = \mathfrak{S}_1 \cup_{\vec{c}_1 = \vec{c}_2} \mathfrak{S}_2$, then one has an isomorphism $A_{\mathfrak{S}} \simeq (A_{\mathfrak{S}_1}, A_{\mathfrak{S}_2})^{inv_{\vec{k}_1, \vec{k}_2}}$ in $\mathcal{E}^{\boxtimes n_1+n_2}$.
Note that one has two thick embeddings $\mathfrak{S}_1 \hookleftarrow C \hookrightarrow \mathfrak{S}_2$ where $C = \sqcup^k(0,1)$ and $\mathfrak{S}$ is their collar gluing.

PROOF : We describe a natural isomorphism

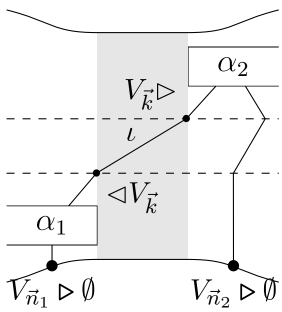
$$\begin{aligned}\sigma : \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(- \triangleright \emptyset, \emptyset) \Rrightarrow \ & \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2}}(-, (A_{\mathfrak{S}_1}, A_{\mathfrak{S}_2})^{inv_{\vec{k}_1, \vec{k}_2}}) \\ & := \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2+k}}(\eta_{\vec{k}_1}(-), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1, \vec{k}_2} A_{\mathfrak{S}_2})\end{aligned} .$$

We write $(\sigma_1)_{(V_{\vec{n}_1}, V_{\vec{k}_1})} : \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S}_1)}(V_{\vec{n}_1} \triangleright \emptyset, \emptyset \triangleleft V_{\vec{k}_1}) \Rrightarrow \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1} V_{\vec{k}_1})$ for $V_{\vec{n}_1} \in \mathcal{V}^{\otimes n_1}$ and $V_{\vec{k}_1} \in \mathcal{V}^{\otimes k}$, obtained from the defining natural isomorphism of $A_{\mathfrak{S}_1}$ by Remark 5.3.12. We write $\sigma_2 : \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S}_2)}(- \triangleright \emptyset, \emptyset) \Rrightarrow \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_2+k}}(-, A_{\mathfrak{S}_2})$ the defining natural isomorphism of $A_{\mathfrak{S}_2}$.

Step 1 (decomposition in $\mathfrak{S}$). Let $\vec{V} = (V_{\vec{n}_1}, V_{\vec{n}_2}) \in \mathcal{V}^{\otimes n_1+n_2}$ and $\alpha \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(\vec{V} \triangleright \emptyset, \emptyset)$ a morphism from $V_{\vec{n}_1} \triangleright \emptyset$ in $\mathfrak{S}_1$ and $V_{\vec{n}_2} \triangleright \emptyset$ in $\mathfrak{S}_2$ to the empty set in $\mathfrak{S}$. By Corollary 5.1.14, α decomposes into a pair (α_1, α_2) as

$$\alpha = (Id_{\emptyset}, \alpha_2) \circ \iota_{\emptyset, V_{\vec{k}}, V_{\vec{n}_2} \triangleright \emptyset} \circ (\alpha_1, Id_{V_{\vec{n}_2} \triangleright \emptyset})$$

with $\alpha_1 \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S}_1)}(V_{\vec{n}_1} \triangleright \emptyset, \emptyset \triangleleft_{\vec{k}_1} V_{\vec{k}})$ and $\alpha_2 \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S}_2)}(V_{\vec{k}} \triangleright_{\vec{k}_2} (V_{\vec{n}_2} \triangleright \emptyset), \emptyset)$ for some $V_{\vec{k}} \in Sk_{\mathcal{V}}(C \times (0,1)) \simeq \mathcal{V}^{\otimes k}$, with an implicit sum. Graphically,



This decomposition is unique up to balancing, namely if α_2 can be written $\beta_2 \circ (\gamma \triangleright Id_{V_{\vec{n}_2} \triangleright \emptyset})$, with $\beta_2 \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S}_2)}(W_{\vec{k}} \triangleright (V_{\vec{n}_2} \triangleright \emptyset), \emptyset)$ and $\gamma \in \mathrm{Hom}_{Sk_{\mathcal{V}}(C \times (0,1))}(V_{\vec{k}}, W_{\vec{k}})$ for some $W_{\vec{k}}$ in $Sk_{\mathcal{V}}(C \times (0,1))$, then:

$$(\alpha_1, \beta_2 \circ (\gamma \triangleright Id_{V_{\vec{n}_2} \triangleright \emptyset})) \sim ((Id_{\emptyset} \triangleleft \gamma) \circ \alpha_1, \beta_2).$$

Step 2 (re-composition in $\mathbb{R}^2$). The morphism α_1 is described by a morphism

$$f_1 = (\sigma_1)_{(V_{\vec{n}_1}, V_{\vec{k}})}(\alpha_1) \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1} V_{\vec{k}})$$

and α_2 is described by

$$f_2 = (\sigma_2)_{(V_{\vec{k}}, V_{\vec{n}_2})}(\alpha_2) \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_2+k}}((V_{\vec{k}}, V_{\vec{n}_2}), A_{\mathfrak{S}_2}) .$$

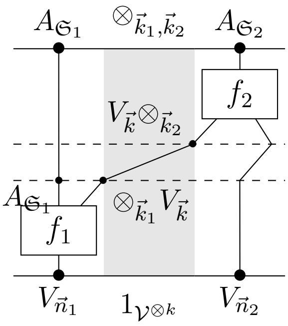
These morphisms are well-defined (depend only on α) up to balancing, namely if $\alpha_2 = \beta_2 \circ (\gamma \triangleright Id_{V_{\vec{n}_2} \triangleright \emptyset})$ with β_2 described by $g_2 = \sigma_2(\beta_2) \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_2+k}}((W_{\vec{k}}, V_{\vec{n}_2}), A_{\mathfrak{S}_2})$, then by naturality of σ_1 and σ_2, $f_2 = g_2 \circ (\gamma, Id_{V_{\vec{n}_2}})$ and the above relation becomes

$$(f_1, g_2 \circ (\gamma, Id_{V_{\vec{n}_2}})) \sim ((Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1} \gamma) \circ f_1, g_2) .$$

Thus the map

$$\sigma_{\vec{V}}(\alpha) := (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} f_2) \circ (f_1, Id_{V_{\vec{n}_2}}) \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}, V_{\vec{n}_2}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\mathfrak{S}_2})$$

is well defined, because this relation is killed. Graphically,



Step 3 (naturalilty). Naturality is quite obvious from the picture: one can insert morphisms from below. For $g_1 \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1}}(W_{\vec{n}_1}, V_{\vec{n}_1})$ and $g_2 \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_2}}(W_{\vec{n}_2}, V_{\vec{n}_2})$, by naturality of σ_1 and σ_2, one has $\sigma_1(\alpha_1 \circ (g_1 \triangleright Id_\emptyset)) = f_1 \circ g_1$ and $\sigma_2(\alpha_2 \circ (g_2 \triangleright Id_\emptyset)) = f_2 \circ g_2$. Now $\alpha \circ ((g_1, g_2) \triangleright Id_\emptyset)$ splits in Step 1 as

$$\begin{aligned}(Id_\emptyset, \alpha_2) \circ \iota_{\emptyset, V_{\vec{k}}, V_{\vec{n}_2} \triangleright \emptyset} \circ (\alpha_1, Id_{V_{\vec{n}_2} \triangleright \emptyset}) \circ (g_1 \triangleright Id_\emptyset, g_2 \triangleright Id_\emptyset) \\ = (Id_\emptyset, \alpha_2 \circ (g_2 \triangleright Id_\emptyset)) \circ \iota_{\emptyset, V_{\vec{k}}, W_{\vec{n}_2} \triangleright \emptyset} \circ (\alpha_1 \circ (g_1 \triangleright Id_\emptyset), Id_{W_{\vec{n}_2} \triangleright \emptyset}).\end{aligned}$$

Thus:

$$\sigma(\alpha \circ ((g_1, g_2) \triangleright Id_\emptyset)) = (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} (f_2 \circ g_2)) \circ ((f_1 \circ g_1), Id_{W_{\vec{n}_2}}) = \sigma(\alpha) \circ (g_1, g_2).$$

We now construct an inverse to σ by the same steps in reverse order:
Step 2^{-1} (decomposition in $\mathbb{R}^2$). We want to decompose a morphism

$$f \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2+k}}(\eta_{\vec{k}_1}(\vec{V}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\mathfrak{S}_2}) \quad \text{as} \quad f = (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} f_2) \circ (f_1, Id_{V_{\vec{n}_2}})$$

with $f_1 \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1} V_{\vec{k}})$ and $f_2 \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_2+k}}((V_{\vec{k}}, V_{\vec{n}_2}), A_{\mathfrak{S}_2})$.
This is easy in $\mathcal{V}^{\otimes n_1+n_2+k}$, as all maps split on each coordinates. For $\vec{A}_1 = (A_{\vec{n}_1}, A_{\vec{k}_1})$ in $\mathcal{V}^{\otimes n_1+k}$ and $\vec{A}_2 = (A_{\vec{k}_2}, A_{\vec{n}_2})$ in $\mathcal{V}^{\otimes n_2+k}$, a morphism

$$f \in \mathrm{Hom}_{\mathcal{V}^{\otimes n_1+n_2+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}, V_{\vec{n}_2}), \vec{A}_1 \otimes_{\vec{k}_1,\vec{k}_2} \vec{A}_2)$$

is, up to a linear combination, of the form $(g_{\vec{n}_1}, g_{\vec{k}_1}, g_{\vec{n}_2})$ with

$$g_{\vec{n}_1} : V_{\vec{n}_1} \to A_{\vec{n}_1},\ g_{\vec{k}_1} : 1_{\mathcal{V}^{\otimes k}} \to A_{\vec{k}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\vec{k}_2} \text{ and } g_{\vec{n}_2} : V_{\vec{n}_2} \to A_{\vec{n}_2} .$$

Then, set $V_{\vec{k}} = A_{\vec{k}_2}$,

$$f_1 = g_{\vec{n}_1} \otimes g_{\vec{k}_1} \in \mathrm{Hom}_{\mathcal{V}^{\otimes n_1+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}), \vec{A}_1 \otimes_{\vec{k}_1} V_{\vec{k}})$$

and

$$f_2 = Id_{V_{\vec{k}}} \otimes g_{\vec{n}_2} \in \mathrm{Hom}_{\mathcal{V}^{\otimes n_2+k}}((V_{\vec{k}}, V_{\vec{n}_2}), \vec{A}_2) .$$

One has

$$f = (Id_{\vec{A}_1} \otimes_{\vec{k}_1,\vec{k}_2} f_2) \circ (f_1, Id_{V_{\vec{n}_2}})$$

as required.
This decomposition is unique up to balancing, if

$$f = (Id_{\vec{A}_1} \otimes_{\vec{k}_1,\vec{k}_2} f_2') \circ (f_1', Id_{V_{\vec{n}_2}})$$

one can split f_2', which has to coincide with f_2 on $\vec{n}_2$ coordinates, and is some $\gamma : W_{\vec{k}} \to A_{\vec{k}_2}$ on $\vec{k}_2$ coordinates (which are now $\vec{k}_1$ coordinates after the $\otimes_{\vec{k}_1,\vec{k}_2}$). Similarly, f_1' coincides with f_1 on $\vec{n}_1$ coordinates, and is some $\delta : 1_{\mathcal{V}} \to A_{\vec{k}_1} \otimes W_{\vec{k}}$ on $\vec{k}_1$ coordinates. On $\vec{k}_1$ coordinates one has $(Id_{A_{\vec{k}_1}} \otimes_{\vec{k}_1} \gamma) \circ \delta = g_{\vec{k}_1}$, so the only relation is

$$((-, (Id_{A_{\vec{k}_1}} \otimes_{\vec{k}_1} \gamma) \circ \delta), (Id_{V_{\vec{k}}}, -)) \sim ((-, \delta), (\gamma, -)) \ .$$

Now, $A_{\mathfrak{S}_1}$ and $A_{\mathfrak{S}_2}$ are not objects of $\mathcal{V}^{\otimes n_1+k}$ and $\mathcal{V}^{\otimes n_2+k}$, but are obtained as canonical colimits of such objects, $A_{\mathfrak{S}_1} = \operatorname{colim}_i \vec{A}_{1,i}$ and $A_{\mathfrak{S}_2} = \operatorname{colim}_j \vec{A}_{2,j}$, so

$$A_{\mathfrak{S}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\mathfrak{S}_2} = \operatorname{colim}_{i,j} \vec{A}_{1,i} \otimes_{\vec{k}_1,\vec{k}_2} \vec{A}_{2,j}$$

by cocontinuity. The object $\eta_{\vec{k}_1}(\vec{V}) = (V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}, V_{\vec{n}_2})$ is compact projective in $\mathcal{E}^{\boxtimes n_1+n_2+k}$ therefore

$$\operatorname{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2+k}}(\eta_{\vec{k}_1}(\vec{V}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\mathfrak{S}_2}) = \operatorname{colim}_{i,j} \operatorname{Hom}_{\mathcal{V}^{\otimes n_1+n_2+k}}(\eta_{\vec{k}_1}(\vec{V}), \vec{A}_{1,i} \otimes_{\vec{k}_1,\vec{k}_2} \vec{A}_{2,j}) \ .$$

A morphism $f \in \operatorname{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2+k}}(\eta_{\vec{k}_1}(\vec{V}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\mathfrak{S}_2})$ factorises through a single (actually, a linear combination of) $\vec{A}_{1,i} \otimes_{\vec{k}_1,\vec{k}_2} \vec{A}_{2,j}$ as:

$$f : \eta_{\vec{k}_1}(\vec{V}) \xrightarrow{f_{i,j}} \vec{A}_{1,i} \otimes_{\vec{k}_1,\vec{k}_2} \vec{A}_{2,j} \xrightarrow{can_{1,i} \otimes_{\vec{k}_1,\vec{k}_2} can_{2,j}} A_{\mathfrak{S}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\mathfrak{S}_2}.$$

There it splits as $f_{i,j} = (Id_{\vec{A}_{1,i}} \otimes_{\vec{k}_1,\vec{k}_2} f_2) \circ (f_1, Id_{V_{\vec{n}_2}})$, and

$$f = (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} (can_{2,j} \circ f_2)) \circ ((can_{1,i} \otimes_{\vec{k}_1} Id_{V_{\vec{k}}}) \circ f_1, Id_{V_{\vec{n}_2}}).$$

This $f_{i,j}$ is unique up to the relations in the above colimit, namely for

$$h_1 \otimes_{\vec{k}_1,\vec{k}_2} h_2 : \vec{A}_{1,i} \otimes_{\vec{k}_1,\vec{k}_2} \vec{A}_{2,j} \to \vec{A}_{1,i'} \otimes_{\vec{k}_1,\vec{k}_2} \vec{A}_{2,j'}$$

over $A_{\mathfrak{S}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\mathfrak{S}_2}$ one has

$$f_{i',j'} = (h_1 \otimes_{\vec{k}_1,\vec{k}_2} h_2) \circ f_{i,j}$$

and

$$f = (can_{1,i'} \otimes_{\vec{k}_1,\vec{k}_2} can_{2,j'}) \circ f_{i',j'} \ .$$

Split h_2 as $(h_{\vec{k}_2}, h_{\vec{n}_2})$, then f decomposes through $f_{i',j'}$ as:

$$f = (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} (can_{2,j'} \circ (Id_{V_{\vec{k}}'}, h_{\vec{n}_2}) \circ f_2)) \circ ((can_{1,i'} \otimes_{\vec{k}_1} Id_{V_{\vec{k}}'}) \circ (h_1 \otimes_{\vec{k}_1} h_{\vec{k}_2}) \circ f_1, Id_{V_{\vec{n}_2}}).$$

Set

$$F_1 = (can_{1,i} \otimes_{\vec{k}_1} Id_{V_{\vec{k}}}) \circ f_1 = ((can_{1,i'} \circ h_1) \otimes_{\vec{k}_1} Id_{V_{\vec{k}}}) \circ f_1$$

and

$$F_2 = can_{2,j'} \circ (Id_{V_{\vec{k}}'}, h_{\vec{n}_2}) \circ f_2 = can_{2,j'} \circ (Id_{V_{\vec{k}}'}, h_{\vec{n}_2} \circ g_{\vec{n}_2}).$$

The first decomposition was

$$(F_1, can_{2,j} \circ f_2) = (F_1, can_{2,j'} \circ (h_{\vec{k}_2}, h_{\vec{n}_2}) \circ (Id_{V_{\vec{k}}}, g_{\vec{n}_2})) = (F_1, F_2 \circ (h_{\vec{k}_2}, Id_{V_{\vec{n}_2}}))$$

and the second is

$$((can_{1,i'} \otimes_{\vec{k}_1} Id_{V_{\vec{k}}'}) \circ (h_1 \otimes_{\vec{k}_1} h_{\vec{k}_2}) \circ f_1, F_2) = ((Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1} h_{\vec{k}_2}) \circ F_1, F_2)$$

so the only relation is

$$(F_1, F_2 \circ (h_{\vec{k}_2}, Id_{V_{\vec{n}_2}})) \sim ((Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1} h_{\vec{k}_2}) \circ F_1, F_2)$$

Step 1^{-1} (re-composition in $\mathfrak{S}$). We decompose f using last step as

$$f = (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} f_2) \circ (f_1, Id_{V_{\vec{n}_2}})$$

with $f_1 \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+k}}((V_{\vec{n}_1}, 1_{\mathcal{V}^{\otimes k}}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1} V_{\vec{k}})$ and $f_2 \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_2+k}}((V_{\vec{k}}, V_{\vec{n}_2}), A_{\mathfrak{S}_2})$. They are described by morphisms

$$\alpha_1 := (\sigma_1)^{-1}_{(V_{\vec{n}_1}, V_{\vec{k}})}(f_1) \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S}_1)}(V_{\vec{n}_1} \triangleright \emptyset, \emptyset \triangleleft_{\vec{k}_1} V_{\vec{k}})$$

and

$$\alpha_2 := (\sigma_2)^{-1}_{(V_{\vec{k}}, V_{\vec{n}_2})}(f_2) \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S}_2)}(V_{\vec{k}} \triangleright_{\vec{k}_2} (V_{\vec{n}_2} \triangleright \emptyset), \emptyset) \ .$$

The above relation becomes

$$(\alpha_1, \beta_2 \circ (\gamma \triangleright Id_{V_{\vec{n}_2} \triangleright \emptyset})) \sim ((Id_\emptyset \triangleleft \gamma) \circ \alpha_1, \beta_2) \ .$$

The morphism

$$\sigma^{-1}_{\vec{V}}(f) := (Id_\emptyset, \alpha_2) \circ \iota_{\emptyset, V_{\vec{k}}, V_{\vec{n}_2} \triangleright \emptyset} \circ (\alpha_1, Id_{V_{\vec{n}_2} \triangleright \emptyset})$$

is well defined, because this relation is killed.

Step 4 (isomorphism). One easily checks that σ^{-1} defined this way is an inverse to σ. Let $\alpha \in \mathrm{Hom}_{Sk_{\mathcal{V}}(\mathfrak{S})}(\vec{V} \triangleright \emptyset, \emptyset)$ that decomposes as

$$\alpha = (Id_\emptyset, \alpha_2) \circ \iota_{\emptyset, V_{\vec{k}}, V_{\vec{n}_2} \triangleright \emptyset} \circ (\alpha_1, Id_{V_{\vec{n}_2} \triangleright \emptyset}) \ ,$$

then $\sigma_{\vec{V}}(\alpha) := (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} f_2) \circ (f_1, Id_{V_{\vec{n}_2}})$ is already decomposed with $\alpha_1 = (\sigma_1)^{-1}_{(V_{\vec{n}_1}, V_{\vec{k}})}(f_1)$ and $\alpha_2 = (\sigma_2)^{-1}_{(V_{\vec{k}}, V_{\vec{n}_2})}(f_2)$, so:

$$\sigma^{-1}_{\vec{V}}(\sigma_{\vec{V}}(\alpha)) := (Id_\emptyset, \alpha_2) \circ \iota_{\emptyset, V_{\vec{k}}, V_{\vec{n}_2} \triangleright \emptyset} \circ (\alpha_1, Id_{V_{\vec{n}_2} \triangleright \emptyset}) = \alpha.$$

Similarly, let $f \in \mathrm{Hom}_{\mathcal{E}^{\boxtimes n_1+n_2+k}}(\eta_{\vec{k}_1}(\vec{V}), A_{\mathfrak{S}_1} \otimes_{\vec{k}_1,\vec{k}_2} A_{\mathfrak{S}_2})$ that decomposes as

$$f = (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} f_2) \circ (f_1, Id_{V_{\vec{n}_2}})$$

then $\sigma^{-1}_{\vec{V}}(f) := (Id_\emptyset, \alpha_2) \circ \iota_{\emptyset, V_{\vec{k}}, V_{\vec{n}_2} \triangleright \emptyset} \circ (\alpha_1, Id_{V_{\vec{n}_2} \triangleright \emptyset})$ is already decomposed with $f_1 = (\sigma_1)_{(V_{\vec{n}_1}, V_{\vec{k}})}(\alpha_1)$ and $f_2 = (\sigma_2)_{(V_{\vec{k}}, V_{\vec{n}_2})}(\alpha_2)$, so:

$$\sigma_{\vec{V}}(\sigma^{-1}_{\vec{V}}(f)) := (Id_{A_{\mathfrak{S}_1}} \otimes_{\vec{k}_1,\vec{k}_2} f_2) \circ (f_1, Id_{V_{\vec{n}_2}}) = f$$

and σ^{-1} is indeed an inverse to σ. □

Remark 5.3.33: When $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$*–comod*^{fin}, with $A_{\mathfrak{S}} \simeq \mathscr{S}(\mathfrak{S})$, one obtains the same excision properties as in Theorem 5.1.8. One uses repeatedly Theorem 5.1.8 on $\mathfrak{S}_1 \sqcup \mathfrak{S}_2$ on each couple of boundary edges to glue. This gives

$$\begin{array}{rl} \mathscr{S}(\mathfrak{S}) \simeq & HH^0_{\vec{k_1},\vec{k_2}}(\mathscr{S}(\mathfrak{S}_1) \otimes \mathscr{S}(\mathfrak{S}_2)) \\ & := \{x \in \mathscr{S}(\mathfrak{S}_1) \otimes \mathscr{S}(\mathfrak{S}_2) \ / \ \forall 1 \leq i \leq k, \ \Delta_{n_1+i}(x) = fl \circ \Delta^l_{n_1+n_2+k+i}(x)\} \end{array} .$$

By Proposition 5.3.27 on all couples of edges to glue one gets

$$HH^0_{\vec{k_1},\vec{k_2}}((\mathscr{S}(\mathfrak{S}_1), \mathscr{S}(\mathfrak{S}_2))) = (\mathscr{S}^R(\mathfrak{S}_1)^{\mathrm{ht}_{\vec{k}_1}}, \mathscr{S}(\mathfrak{S}_2))^{inv_{\vec{k}_1,\vec{k_2}}} .$$

By Theorem 5.3.28 $\mathscr{S}^R(\mathfrak{S}_1)^{\mathrm{ht}_{\vec{k}_1}}$ is the internal skein algebra of $\mathfrak{S}_1$, and one obtains exactly the formulation of Theorem 5.3.32.
Note that we described how to glue two surfaces along many edges at once and Theorem 5.1.8 describes how to glue only two edges but possibly of the same surface. The two forms of excision are equivalent, in one way by applying it repeatedly as above and in the other way by gluing a bigon to the two edges of the surface that one wants to glue together. ◇

Remark 5.3.34: This remark answers a natural question arising at the sight of the cutting property of stated skein algebras: why is it not a coevaluation one sees on newly created states when one cuts along an ideal arc? Indeed in the definition one uses $\sum_{\vec{\mu}} v_{\vec{\mu}} \otimes v_{\vec{\mu}}$ though the coevaluation would give $coev(1) = \sum_{\vec{\mu}} v_{\vec{\mu}} \otimes v^*_{\vec{\mu}} \overset{Id\otimes\varphi^{-1}}{\longmapsto} \sum_{\vec{\mu}} v_{\vec{\mu}} \otimes v_{-\overleftarrow{\mu}} C(-\vec{\mu})$ in particular matching + states to − states. The answer is that it is indeed given by a coevaluation, but the stated skein algebra of the surface at the right is not the good object: one must take its half-twisted version. Then the half twist re-exchanges + signs to − signs and kills the coefficients appearing. In particular we see that there has been a choice in the way the splitting morphism of stated skein algebras is defined, and that this choice seems to determine both the half twist and the identification $V \simeq V^*$. This is to be put in light with the unicity of stated skein coefficients proved in [Lê18, Section 3.4].◇

Remark 5.3.35: Internal skein algebras are defined for any ribbon category $\mathcal{V}$, and coincide with stated skein algebras when $\mathcal{V} = \mathcal{O}_{q^2}(SL_2)$*–comod*^{fin}. Stated skein algebras for SL_n were very recently introduced in [LS], and one can expect to prove they coincide with internal skein algebras for $\mathcal{V} = \mathcal{O}_{q^n}(SL_n)$*–comod*^{fin} for generic q with the very same proof. The authors actually showed it for surfaces with a single boundary interval using excision properties with respect to gluing patterns from both theories. The constructions and arguments of this chapter work more generally with any semisimple coribbon Hopf algebras H, using the equivalence H*–comod* $\simeq$ Free(H*–comod*^{fin}), and are actually [GJS23]'s candidate for the generalisation of stated skein algebras. The results of this section show that this generalisation extends to multiple markings, and that one obtains excision properties immediately.
Internal skein algebras are defined more generally in [BBJ18a] for any E_2-algebra $\mathcal{A} \in \mathbf{Pr}$ under the name moduli algebras, and the skein-theoretic description holds when $\mathcal{A}$ is the free cocompletion of a ribbon category. As both moduli algebras and stated skein algebras can be defined integrally, or at roots of unity, it would be very interesting to understand how they compare in greater generality. So far, there is no skein-theoretic description of the factorization homology used in the construction of moduli algebras, but it seems credible that with extra work one could rewrite this whole theory in these integral or non-semisimple contexts. ◇

Conclusion and future research

As is, I believe, usual, once a mathematical result or insight has been known for long enough, it has a tendency to look more and more natural and easy. Looking back at the statements and constructions that took me years to get right, they can sometimes all seem somewhat straightforward. Luckily, the research presented above opens to many future directions which are all still unclear, changing, confusing and, therefore, exciting.

Define and study examples of non-semisimple (3+1)-TQFTs

This is an ongoing project with Francesco Costantino, Nathan Geer and Bertrand Patureau-Mirand.

Few interesting examples of 4-TQFTs are known. Invertible theories associated with modular tensor categories only depend on the signature and Euler characteristic of the 4-manifolds. Crane–Yetter–Kauffman theories associated with ribbon fusion categories are expected to only capture an additional dependence on the fundamental group. On simply connected 4-manifolds it is shown to only depend on signature and Euler characteristic. For the examples given in Chapter 3, we expect to capture an additional dependence on spin status. It is unknown whether the construction we described could detect subtle 4-manifold topology.

Reutter showed that "semisimple" 4-TQFTs cannot detect exotic pairs, but his definition of semisimple is quite broad and encompasses most of the examples of Chapter 3, and in particular once-extended 4-TQFTs. According to our expectation that our (3+1)-TQFTs are fully extended, the only remaining candidates are those that do not give rise to fully-defined TQFTs, but only to non-compact ones. In other words, we are looking for a chromatic non-degenerate category which is not chromatic compact. We exhibited such an example and we crucially need it to be in positive characteristic. We showed that the resulting TQFT is indeed far from being semisimple in Reutter's sense. The example we give is symmetric and hence uninteresting, but it suggests that one should look at non-trivial examples in positive characteristic.

One could look for examples whose Müger center is our symmetric example. One could construct a ribbon category as representations of a semi-direct product of an interesting quantum group by a finite group of order non prime with the characteristic. For this, we need a quantum group with interesting automorphisms. In the classical example of $\mathcal{U}_q(\mathfrak{g})$ for a simple complex Lie algebra $\mathfrak{g}$, Hopf algebra automorphisms are described by automorphisms of Dynkin diagrams. In characteristic 2, we could look at type A_2, so $\mathfrak{g} = \mathfrak{sl}_3$, which has a $\mathbb{Z}/2$-action. In characteristic 3, we could look at type D_4 which has a $\mathfrak{S}_3$-action. This second example could be more interesting, because $\mathfrak{S}_3$ has a non-invertible irreducible representation. In the semisimple case, this is shown to be related to having a dependence on π_1 instead of on H_1 in [BB18].

In another direction, super quantum groups have a representation theory which is rich enough to have the potential to give very interesting examples. The representation category of $SL(2|1)$ is not modular and non-semisimple, and this is a promising example to study.

Describe a fully-extended version of non-semisimple WRT

We want to show the conjectures in Chapter 4.

The first and most difficult step, before discussing the relative part, is to give a skein theoretic description of the fully extended 4-TQFT associated with a modular tensor category $\mathscr{V}$. We know natural guesses. The description in dimension 3 and 4 would follow exactly the construction of Chapter 3, in particular admissible skein modules in dimension 3. In dimensions 2 and lower one expects some admissible skein categories.

Note that in the semisimple case this guess in much older, but is not proven, even as a once-extended theory. The only known part is that factorization homology form a fully extended 2-TQFT, and that skein categories compute factorization homology in the semisimple case.

It is a joint project with Jennifer Brown to show that admissible skein categories compute factorization homology in the non-semisimple case.

For the rest of the TQFT, the two approaches described in the introduction still hold. For the top-down approach, one could check that the description we give do define a fully extended 4-TQFT. This is difficult, as defining a 4-functor is a lot of data, and that mapping into Haugseng's combinatorial (and not geometric) model one would need to choose and keep track of a lot of isomorphisms.

For the bottom-up approach, one wants to use the cobordism hypothesis and identify the induced values. The second part is tractable, one has to compute many adjoints. The first relies on the non-proven cobordism hypothesis. It is well-accepted and one could be happy to assume it. However, we are still left with a difficult problem, which is to find orientation structures, i.e. one would have to show the conjectures on orientation structures we gave. This is an ongoing project with David Jordan and Patrick Kinnear.

The second step is to carry out a similar argument, one dimension lower but in a more complicated arrow category, for the boundary condition. We believe that this second step will be easier. Similarly, we can guess what the values on any manifold will be, and essentially the boundary condition always include the empty skein. This is why it is not defined on closed 3-manifolds in the non-semisimple case. Then one should check directly that the composition of the boundary condition with the 4-TQFT on a bounding manifold recovers WRT and DGGPR, as claimed.

One could also give a fully-extended version of [BCGP16] TQFTs, and more generally of non-semisimple "G-relative" TQFTs. On top of the relative layer described above in the WRT case, one would need to see BCGP theories as a relative to the classical theory associated with their Müger center $Rep(G)$. Such "twice relative" theories are quite new and both the algebraic and the topological descriptions remain a challenge. An ambitious application is to describe invariants of 3-manifolds equipped with a flat G-connection. This is an ongoing project with Patrick Kinnear and David Jordan.

This fully extended description of non-semisimple WRT TQFTs brings these constructions into a framework where we have a standard set of tools to study them. For example, the cobordism hypothesis with singularity entirely describes domain walls and higher-codimensional defects between different WRT theories. The relative nature of these theories also gives the appropriate notion for these domain walls: of course one should also describe how they interact with the bulk CY theories.

The very explicit requirements on input data (a 4-dualizable object $\mathcal{V}$ together with a non-compact-3-dualizable morphism $\mathbf{1} \to \mathcal{V}$) makes it very clear how to generalize these constructions to different contexts and dimensions. One can use the same technique to define and study derived skein modules associated with 3-dualizable E_2-algebras in say dg-categories. Note that David Ayala recently defined a notion of derived skein modules using "beta" factorization homology which we could compare. One could also apply the same techniques to 4-dualizable E_2-algebras in bicategories. One would expect to recover the "lasagna skein tehory" of [MWW22] associated with Khovanov homology. Indeed they have constructions very similar to stated skein modules for 4-manifolds, and can define analogs of skein categories associated with 3-manifolds. One can also define and study internal skein algebras analogous to [BBJ18a, GJS23] in their context. The results of [BBJ18a, BBJ18b] may give new results in this context.

The use of relative TQFTs to describe skein theories can be extrapolated outside the finite realm. In particular, for representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ at generic q, we do not have a WRT theory, but we still have a $(3+\varepsilon)$ skein TQFT, and a boundary theory to it. I realized with Adrien Brochier that this should be enough to give values to link complements in S^3 and in connected sums of $S^2 \times S^1$. This could give the missing TQFT-theoretic explanation of the link invariants associated with 3-dualizable objects. It is also an avenue for generalization of such link invariants. This may be the right setting to study holonomicity properties of quantum invariants.

Study stated skein algebras via internal skein algebras

The first example of a possible application of Chapter 5 is that gluing properties of internal skein algebras for cutting along a circle are well-known [BBJ18b]. One should be able to express them for stated skein algebras. This idea was suggested to me by Adam Sikora at an AIM workshop, and we indeed could guess a formula.

Now, properties of stated skein algebras for SL_2 are rather well-known, and we know of no other application there. The story changes entirely when we change SL_2. Stated skein algebras were defined by Lê and Sikora for SL_n, and there less is known. In particular the splitting morphism is not known to be injective for $n \geq 4$, see [LS, Conjecture 7.12]. On the other side, gluing properties of internal skein algebras are proven in great generality in Chapter 5. Relating SL_n stated skein algebras to internal skein algebras could be very fruitful, and should be very similar.

Note that our relation only holds for q generic and working over a field. The correct assumptions under which the generalization should be straightforward is that one wants to work over the category of representations of a semisimple Hopf algebra over a field. It would be very interesting to generalize our result outside this context.

Given a non-semisimple ribbon category $\mathscr{V}$ (still over a field), its "moduli algebra" is still defined in [BBJ18a] using the Ind-completion of $\mathscr{V}$. However, Cooke's skein description of factorization homology does not hold anymore and the name "internal skein algebra" from [GJS23] would be abusive. As mentioned, it is an ongoing project with Jennifer Brown to extend Cooke's result to non-semisimple settings. Then, one could carry out the same comparison in the non-semisimple setting.

Extending this comparison to stated skein modules of 3-manifolds would also be interesting. In the case of SL_2 at q generic, it is rather straightforward. But recent results of [CL] show that many surprising phenomenons happen at q a root of unity. It could be fruitful to study them with the algebraic tools of [BBJ18a, BBJ18b].

Finally, it would be interesting to recover the quantum trace map of [BW11] using skein theory in a theoretical setting that makes generalizations easier. This quantum trace map have been well studied and is known to come from a cluster structure on stated skein algebras [Lê18, LY22], related to Fock-Goncharov coordinates. Current work of Brown, Jordan, Schrader and Shapiro is giving a defect-skein-theoretic framework to study them. In the SL_2 case, we expect to recover reduced stated skein algebras, and the quantum trace map is expected to be related to a defect cobordism between decorated surfaces. Using these surfaces and cobordism with defects, one would give an intrinsic characterization of the cluster structures on stated skein algebras. This is an ongoing project with Jennifer Brown.

Bibliography

[Abe80] Eiichi Abe. *Hopf algebras*, volume 74 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge-New York, 1980. Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka.

[AF] David Ayala and John Francis. The cobordism hypothesis. arXiv:1705.02240.

[AF92] Frank W. Anderson and Kent R. Fuller. *Rings and categories of modules*, volume 13 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[AFT17] David Ayala, John Francis, and Hiro Lee Tanaka. Factorization homology of stratified spaces. *Selecta Math. (N.S.)*, 23(1):293–362, 2017. arXiv:1409.0848.

[ALZ15] Henning H. Andersen, Gustav I. Lehrer, and Ruibin Zhang. Cellularity of certain quantum endomorphism algebras. *Pacific J. Math.*, 279(1-2):11–35, 2015. arXiv:1303.0984.

[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[Ara14] Dimitri Ara. Higher quasi-categories vs higher Rezk spaces. *J. K-Theory*, 14(3):701–749, 2014.

[Ati88] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186, 1988.

[Bar05] Clarck Barwick. *(∞, n)-Cat as a closed model category*. PhD thesis, University of Pennsylvania, 2005.

[BB18] Manuel Bärenz and John Barrett. Dichromatic state sum models for four-manifolds from pivotal functors. *Comm. Math. Phys.*, 360(2):663–714, 2018.

[BBG21] Anna Beliakova, Christian Blanchet, and Azat M. Gainutdinov. Modified trace is a symmetrised integral. *Selecta Math. (N.S.)*, 27(3):Paper No. 31, 51, 2021. arXiv:1801.00321.

[BBJ18a] David Ben-Zvi, Adrien Brochier, and David Jordan. Integrating quantum groups over surfaces. *Journal of Topology*, 11(4):874–917, 2018. arXiv:1501.04652v5.

[BBJ18b] David Ben-Zvi, Adrien Brochier, and David Jordan. Quantum character varieties and braided module categories. *Selecta Math. (N.S.)*, 24(5):4711–4748, 2018. arXiv:1606.04769.

[BCGP16] Christian Blanchet, Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Non-semi-simple TQFTs, Reidemeister torsion and Kashaev's invariants. *Adv. Math.*, 301:1–78, 2016. arXiv:1404.7289.

[BCJ15] Martin Brandenburg, Alexandru Chirvasitu, and Theo Johnson-Freyd. Reflexivity and dualizability in categorified linear algebra. *Theory Appl. Categ.*, 30:Paper No. 23, 808–835, 2015. arXiv:1409.5934.

[BD] Anna Beliakova and Marco De Renzi. Refined Bobtcheva-Messia Invariants of 4-Dimensional 2-Handlebodies . arXiv:2205.11385.

[BD95] John C. Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36(11):6073–6105, 1995. arXiv:q-alg/9503002.

[BD23] Anna Beliakova and Marco De Renzi. Kerler–Lyubashenko Functors on 4-Dimensional 2-Handlebodies. *International Mathematics Research Notices*, page rnac039, 2023. arXiv:2105.02789.

[BDSV] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. Modular categories as representations of the 3-dimensional bordism 2-category. arXiv:1509.06811.

[BHMV95] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, 34(4):883–927, 1995.

[BJS21] Adrien Brochier, David Jordan, and Noah Snyder. On dualizability of braided tensor categories. *Compositio Mathematica*, 157(3):435–483, 2021. arXiv:1804.07538.

[BJSS21] Adrien Brochier, David Jordan, Pavel Safronov, and Noah Snyder. Invertible braided tensor categories. *Algebr. Geom. Topol.*, 21(4):2107–2140, 2021. arXiv:2003.13812.

[BK] Benjamin Balsam and Alexande Jr. Kirillov. Turaev-Viro invariants as an extended TQFT. arXiv:1004.1533.

[BR20] Julia E. Bergner and Charles Rezk. Comparison of models for (∞, n)-categories, II. *J. Topol.*, 13(4):1554–1581, 2020.

[BW96] John W. Barrett and Bruce W. Westbury. Invariants of piecewise-linear 3-manifolds. *Trans. Amer. Math. Soc.*, 348(10):3997–4022, 1996.

[BW11] Francis Bonahon and Helen Wong. Quantum traces for representations of surface groups in $\mathrm{SL}_2(\mathbb{C})$. *Geom. Topol.*, 15(3):1569–1615, 2011. arXiv:1003.5250.

[Cam20] Alexander Campbell. A homotopy coherent cellular nerve for bicategories. *Adv. Math.*, 368:107138, 67, 2020.

[CFS95] J. Scott Carter, Daniel E. Flath, and Masahico Saito. *The classical and quantum 6j-symbols*, volume 43 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1995.

[CGHP] Francesco Costantino, Nathan Geer, Benjamin Haïoun, and Bertrand Patureau-Mirand. Skein (3+1)-TQFTs from non-semisimple ribbon categories. arXiv:2306.03225.

[CGP] Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Admissible Skein Modules. arXiv:2302.04493.

[CGP14] Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand. Quantum invariants of 3-manifolds via link surgery presentations and non-semi-simple categories. *J. Topol.*, 7(4):1005–1053, 2014.

[CGPT20] Francesco Costantino, Nathan Geer, Bertrand Patureau-Mirand, and Vladimir Turaev. Kuperberg and Turaev-Viro invariants in unimodular categories. *Pacific J. Math.*, 306(2):421–450, 2020. arXiv:1809.07991.

[CGPVa] Francesco Costantino, Nathan Geer, Bertrand Patureau-Mirand, and Alexis Virelizier. Chromatic maps for finite tensor categories. arXiv:2305.14626.

[CGPVb] Francesco Costantino, Nathan Geer, Bertrand Patureau-Mirand, and Alexis Virelizier. Non compact (2+1)-TQFTs from non-semisimple spherical categories. arXiv:2302.04509.

[CKY97] Louis Crane, Louis H. Kauffman, and David N. Yetter. State-sum invariants of 4-manifolds. *J. Knot Theory Ramifications*, 6(2):177–234, 1997. arXiv:hep-th/9409167.

[CL] Francesco Costantino and Thang T. Q. Lê. Stated skein modules of 3-manifolds and TQFT. arXiv:2206.10906.

[CL22] Francesco Costantino and Thang T. Q. Lê. Stated skein algebras of surfaces. *J. Eur. Math. Soc. (JEMS)*, 24(12):4063–4142, 2022. arXiv:1907.11400.

[Coo23] Juliet Cooke. Excision of skein categories and factorisation homology. *Adv. Math.*, 414:Paper No. 108848, 51, 2023. arXiv:1910.02630.

[CS19] Damien Calaque and Claudia Scheimbauer. A note on the (∞, n)-category of cobordisms. *Algebr. Geom. Topol.*, 19(2):533–655, 2019. arXiv:1509.08906.

[CY93] Louis Crane and David Yetter. A categorical construction of 4d topological quantum field theories. In *Quantum topology*, volume 3 of *Ser. Knots Everything*, pages 120–130. World Sci. Publ., River Edge, NJ, 1993. arXiv:hep-th/9301062.

[De 17] Marco De Renzi. Construction of extended topological quantum field theories. 2017. PhD thesis, Universitè Sorbonne Paris Citè, https://theses.hal.science/tel-02003025.

[De 21] Marco De Renzi. Extended TQFTs from non-semisimple modular categories. *Indiana Univ. Math. J.*, 70(5):1769–1811, 2021. arXiv:2103.04724.

[DGG+22] Marco De Renzi, Azat M. Gainutdinov, Nathan Geer, Bertrand Patureau-Mirand, and Ingo Runkel. 3-Dimensional TQFTs from non-semisimple modular categories. *Selecta Math. (N.S.)*, 28(2):Paper No. 42, 60, 2022. arXiv:1912.02063.

[DGP18] Marco De Renzi, Nathan Geer, and Bertrand Patureau-Mirand. Renormalized Hennings invariants and $2+1$-TQFTs. *Comm. Math. Phys.*, 362(3):855–907, 2018. arXiv:1707.08044.

[DK94] Yurij A. Drozd and Vladimir V. Kirichenko. *Finite-dimensional algebras*. Springer-Verlag, Berlin, 1994. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab.

[DKS] Renaud Detcherry, Efstratia Kalfagianni, and Adam S. Sikora. Kauffman bracket skein modules of small 3-manifolds. arXiv:2305.16188.

[DSS20] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. Dualizable tensor categories. *Mem. Amer. Math. Soc.*, 268(1308):vii+88, 2020. arXiv:1312.7188.

[Dug] D. Dugger. Sheaves and homotopy theory. 1998, incomplete draft, https://ncatlab.org/nlab/files/cech.pdf.

[Dus02] John W. Duskin. Simplicial matrices and the nerves of weak n-categories. I. Nerves of bicategories. volume 9, pages 198–308. 2001/02. CT2000 Conference (Como).

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[EK71] Robert D. Edwards and Robion C. Kirby. Deformations of spaces of imbeddings. *Ann. of Math. (2)*, 93:63–88, 1971.

[Fai] Matthieu Faitg. Holonomy and (stated) skein algebras in combinatorial quantization. arXiv:2003.08992.

[Fre] Daniel S. Freed. Remarks on fully extended 3-dimensional topological field theories. 2011, Talk at String-Math, https://web.ma.utexas.edu/users/dafr/stringsmath_np.pdf.

[FT14] Daniel S. Freed and Constantin Teleman. Relative quantum field theory. *Comm. Math. Phys.*, 326(2):459–476, 2014. arXiv:1212.1692.

[FT21] Daniel S. Freed and Constantin Teleman. Gapped boundary theories in three dimensions. *Comm. Math. Phys.*, 388(2):845–892, 2021. arXiv:2006.10200.

[FV15] Domenico Fiorenza and Alessandro Valentino. Boundary conditions for topological quantum field theories, anomalies and projective modular functors. *Comm. Math. Phys.*, 338(3):1043–1074, 2015. arXiv:1409.5723.

[GHP22] Nathan Geer, Ngoc Phu Ha, and Bertrand Patureau-Mirand. Modified graded Hennings invariants from unrolled quantum groups and modified integral. *J. Pure Appl. Algebra*, 226(3):Paper No. 106815, 46, 2022. arXiv:2006.12050.

[GJS23] Sam Gunningham, David Jordan, and Pavel Safronov. The finiteness conjecture for skein modules. *Invent. Math.*, 232(1):301–363, 2023. arXiv:1908.05233.

[GKP11] Nathan Geer, Jonathan Kujawa, and Bertrand Patureau-Mirand. Generalized trace and modified dimension functions on ribbon categories. *Selecta Math. (N.S.)*, 17(2):453–504, 2011. arXiv:1001.0985.

[GKP13] Nathan Geer, Jonathan Kujawa, and Bertrand Patureau-Mirand. Ambidextrous objects and trace functions for nonsemisimple categories. *Proc. Amer. Math. Soc.*, 141(9):2963–2978, 2013.

[GKP22] Nathan Geer, Jonathan Kujawa, and Bertrand Patureau-Mirand. M-traces in (non-unimodular) pivotal categories. *Algebr. Represent. Theory*, 25(3):759–776, 2022. arXiv:1809.00499.

[Gom84] Robert E. Gompf. Stable diffeomorphism of compact 4-manifolds. *Topology Appl.*, 18(2-3):115–120, 1984.

[GP] Daniel Grady and Dmitri Pavlov. The geometric cobordism hypothesis . arXiv:2111.01095.

[GP18] Nathan Geer and Bertrand Patureau-Mirand. The trace on projective representations of quantum groups. *Lett. Math. Phys.*, 108(1):117–140, 2018. arXiv:1610.09129.

[GPT09] Nathan Geer, Bertrand Patureau-Mirand, and Vladimir Turaev. Modified quantum dimensions and re-normalized link invariants. *Compos. Math.*, 145(1):196–212, 2009. arXiv:0711.4229.

[GS] Owen Gwilliam and Claudia Scheimbauer. Duals and Adjoints in Higher Morita Categories . arXiv:1804.10924.

[GS99] Robert E. Gompf and András I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.

[Haï] Benjamin Haïoun. Unit inclusion in a (non-semisimple) braided tensor category and (non-compact) relative TQFTs. arXiv:2304.12167.

[Haï22] Benjamin Haïoun. Relating stated skein algebras and internal skein algebras. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 18:Paper No. 042, 39, 2022. arXiv:2104.13848.

[Hau] Rune Haugseng. Some remarks on higher Morita categories. arXiv:2309.09761.

[Hau17] Rune Haugseng. The higher Morita category of $\mathbb{E}_n$-algebras. *Geom. Topol.*, 21(3):1631–1730, 2017. arXiv:1412.8459.

[Hen96] Mark Hennings. Invariants of links and 3-manifolds obtained from Hopf algebras. *J. London Math. Soc. (2)*, 54(3):594–624, 1996.

[Hen17] André G. Henriques. What Chern-Simons theory assigns to a point. *Proc. Natl. Acad. Sci. USA*, 114(51):13418–13423, 2017. arXiv:1503.06254.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Joh21] Theo Johnson-Freyd. Heisenberg-picture quantum field theory. In *Representation theory, mathematical physics, and integrable systems*, volume 340 of *Progr. Math.*, pages 371–409. Birkhäuser/Springer, Cham, [2021] ©2021. arXiv:1508.05908.

[Jon85] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 1985.

[JS17] Theo Johnson-Freyd and Claudia Scheimbauer. (Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories. *Adv. Math.*, 307:147–223, 2017. arXiv:1502.06526.

[JT07] André Joyal and Myles Tierney. Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 277–326. Amer. Math. Soc., Providence, RI, 2007.

[Juh18] András Juhász. Defining and classifying TQFTs via surgery. *Quantum Topol.*, 9(2):229–321, 2018. arXiv:1408.0668.

[Kan57] Daniel M. Kan. On c. s. s. complexes. *Amer. J. Math.*, 79:449–476, 1957.

[Kas95] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kau87] Louis H. Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.

[Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].

[Kin] Patrick Kinnear. Skein module dimensions of mapping tori of the 2-torus. arXiv:2304.07332.

[KL01] Thomas Kerler and Volodymyr V. Lyubashenko. *Non-semisimple topological quantum field theories for 3-manifolds with corners*, volume 1765 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[Kora] Julien Korinman. Finite presentations for stated skein algebras and lattice gauge field theory. arXiv:2012.03237.

[Korb] Julien Korinman. Stated skein algebras and their representations. arXiv:2105.09563.

[KQ] Julien Korinman and Alexandre Quesney. Classical shadows of stated skein representations at roots of unity. arXiv:1905.03441.

[KS97] Anatoli Klimyk and Konrad Schmüdgen. *Quantum groups and their representations*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[KT22] Alexander Kirillov, Jr. and Ying Hong Tham. Factorization homology and 4D TQFT. *Quantum Topol.*, 13(1):1–54, 2022. arXiv:2002.08571.

[Lê18] Thang T. Q. Lê. Triangular decomposition of skein algebras. *Quantum Topol.*, 9(3):591–632, 2018. arXiv:1609.04987.

[LR88] Richard G. Larson and David E. Radford. Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple. *J. Algebra*, 117(2):267–289, 1988.

[LS] Thang T. Q. Lê and Adam S. Sikora. Stated SL(n)-Skein Modules and Algebras. arXiv:2201.00045.

[Lur] Jacob Lurie. Higher algebra. `https://people.math.harvard.edu/~lurie/papers/HA.pdf`.

[Lur09a] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. arXiv:math/0608040.

[Lur09b] Jacob Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009. `https://www.math.ias.edu/~lurie/papers/cobordism.pdf`, arXiv:0905.0465.

[LY22] Thang T. Q. Lê and Tao Yu. Quantum traces and embeddings of stated skein algebras into quantum tori. *Selecta Math. (N.S.)*, 28(4):Paper No. 66, 48, 2022. arXiv:2012.15272.

[Lyu95] Volodymyr V. Lyubashenko. Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity. *Comm. Math. Phys.*, 172(3):467–516, 1995. arXiv:hep-th/9405167.

[Maj95] Shahn Majid. *Foundations of quantum group theory*. Cambridge University Press, Cambridge, 1995.

[Mil65] John Milnor. *Lectures on the h-cobordism theorem*. Princeton University Press, Princeton, NJ, 1965. Notes by L. Siebenmann and J. Sondow.

[MM92] Paul P. Martin and David S. McAnally. On commutants, dual pairs and nonsemisimple algebras from statistical mechanics. In *Infinite analysis, Part A, B (Kyoto, 1991)*, volume 16 of *Adv. Ser. Math. Phys.*, pages 675–705. World Sci. Publ., River Edge, NJ, 1992.

[Mul] Greg Muller. Skein algebras and cluster algebras of marked surfaces. arXiv:1204.0020.

[MWW22] Scott Morrison, Kevin Walker, and Paul Wedrich. Invariants of 4-manifolds from Khovanov-Rozansky link homology. *Geom. Topol.*, 26(8):3367–3420, 2022. arXiv:1907.12194.

[Prz91] Józef H. Przytycki. Skein modules of 3-manifolds. *Bull. Polish Acad. Sci. Math.*, 39(1-2):91–100, 1991.

[Rad12] David E. Radford. *Hopf algebras*, volume 49 of *Series on Knots and Everything*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[Ram17] Julia Ramos González. On the tensor product of large categories. 2017. PhD thesis, University of Antwerp.

[Reu23] David Reutter. Semisimple four-dimensional topological field theories cannot detect exotic smooth structure. *J. Topol.*, 16(2):542–566, 2023. arXiv:2001.02288.

[Rez01] Charles Rezk. A model for the homotopy theory of homotopy theory. *Trans. Amer. Math. Soc.*, 353(3):973–1007, 2001. arXiv:math/9811037.

[Rob95] Justin Roberts. Skein theory and Turaev-Viro invariants. *Topology*, 34(4):771–787, 1995.

[Rom] Jack Romö. Homotopy Bicategories of Complete 2-fold Segal Spaces. arXiv:2311.11983.

[RT91] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.

[Sch14a] Claudia Scheimbauer. *Factorization Homology as a Fully Extended Topological Field Theory*. PhD thesis, ETH ZÜRICH, 2014. `http://www.scheimbauer.at/ScheimbauerThesis.pdf`.

[Sch14b] Christopher J. Schommer-Pries. Dualizability in low-dimensional higher category theory. In *Topology and field theories*, volume 613 of *Contemp. Math.*, pages 111–176. Amer. Math. Soc., Providence, RI, 2014. arXiv:1308.3574.

[Sch14c] Christopher J. Schommer-Pries. Dualizability in low-dimensional higher category theory. In *Topology and field theories*, volume 613 of *Contemp. Math.*, pages 111–176. Amer. Math. Soc., Providence, RI, 2014.

[Seg88] G. B. Segal. The definition of conformal field theory. In *Differential geometrical methods in theoretical physics (Como, 1987)*, volume 250 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pages 165–171. Kluwer Acad. Publ., Dordrecht, 1988.

[Sel11] P. Selinger. A survey of graphical languages for monoidal categories. In *New structures for physics*, volume 813 of *Lecture Notes in Phys.*, pages 289–355. Springer, Heidelberg, 2011. arXiv:0908.3347.

[ST09] Noah Snyder and Peter Tingley. The half-twist for $U_q(g)$ representations. *Algebra Number Theory*, 3(7):809–834, 2009. arXiv:0810.0084.

[ST11] Stephan Stolz and Peter Teichner. Supersymmetric field theories and generalized cohomology. In *Mathematical foundations of quantum field theory and perturbative string theory*, volume 83 of *Proc. Sympos. Pure Math.*, pages 279–340. Amer. Math. Soc., Providence, RI, 2011. arXiv:1108.0189.

[SW07] Adam S. Sikora and Bruce W. Westbury. Confluence theory for graphs. *Algebr. Geom. Topol.*, 7:439–478, 2007. arXiv:math/0609832.

[Swe69] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

[Tak02] Mitsuhiro Takeuchi. A short course on quantum matrices. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 383–435. Cambridge Univ. Press, Cambridge, 2002. Notes taken by Bernd Strüber.

[Tha21] Ying Hong Tham. On the Category of Boundary Values in the extended Crane-Yetter TQFT. 2021. PhD thesis, arXiv:2108.13467.

[Tin] Peter Tingley. A minus sign that used to annoy me but now I know why it is there. arXiv:1002.0555v2.

[Toë14] Bertrand Toën. Derived algebraic geometry. *EMS Surv. Math. Sci.*, 1(2):153–240, 2014. arXiv:1401.1044.

[Tur88] V. G. Turaev. The Conway and Kauffman modules of a solid torus. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 167:79–89, 190, 1988.

[Tur94] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[TV92] V. G. Turaev and O. Ya. Viro. State sum invariants of 3-manifolds and quantum $6j$-symbols. *Topology*, 31(4):865–902, 1992.

[Wala] Kevin Walker. A universal state sum. arXiv:2104.02101.

[Walb] Kevin Walker. Topological Quantum Field Theories. 2006, `http://canyon23.net/math/tc.pdf`.

[Wit89] Edward Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.