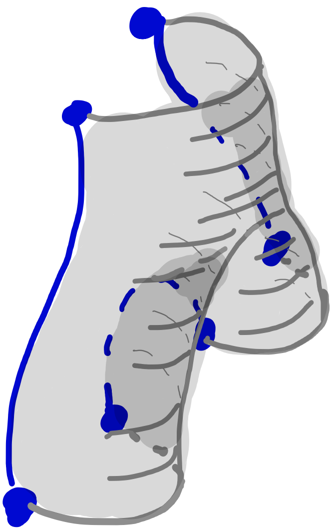
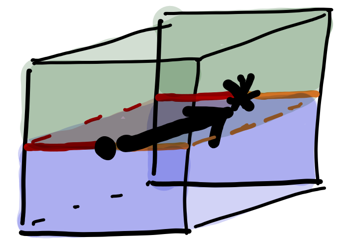
Une approche aux invariants quantiques non-semisimples via l'algèbre supérieure

Une thèse présentée par



Benjamin Haïoun



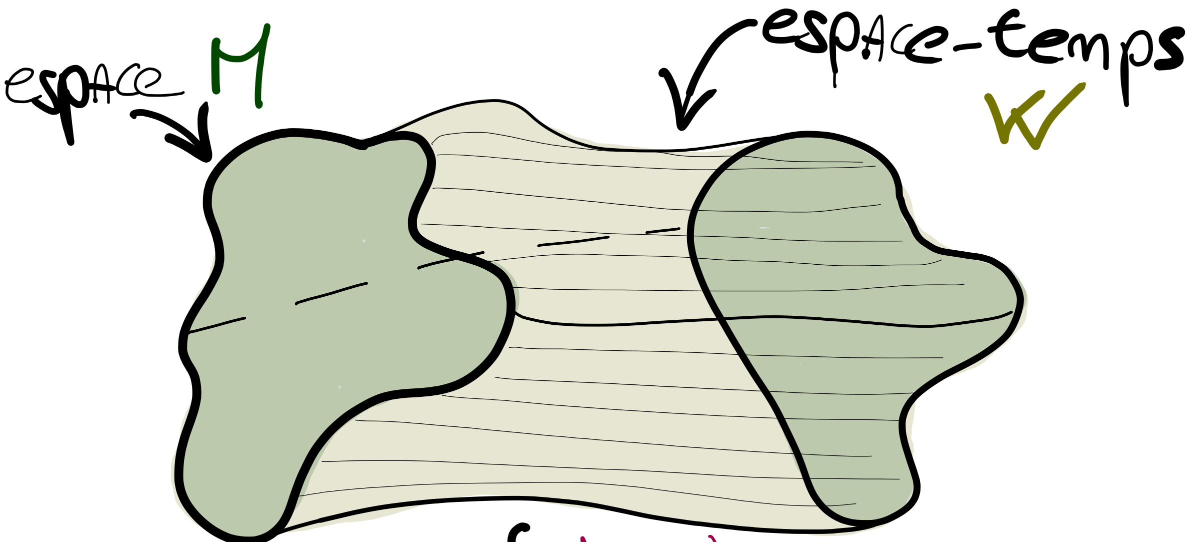
soutenue à

l'Université Toulouse 3 Paul Sabatier

effectuée sous la direction de

Francesco Costantino & David Jordan

Qu'est-ce qu'une théorie quantique des champs topologique ?



théorie physique

$$Z(M) \xrightarrow{Z(W)} Z(M')$$

états physiques sur M

opérateur d'évolution du temps

→ Idées de physique

⇝ Applications en mathématiques

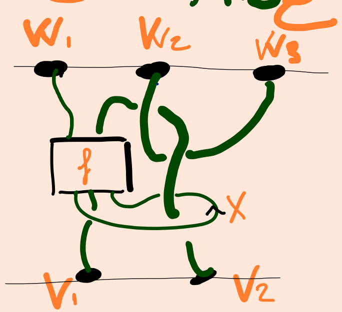
The birth of TQFTs

- 88' Atiyah, Segal
- 89' Witten predicts 3-TQFTs
- 91' Reshetikhin-Turaev

Def: An n-TQFT is a symmetric monoidal functor

$$Z : \mathrm{Cob}^{\sqcup}_{n-1,n} \to \mathrm{Vect}^{\otimes}$$

$$RT_{\mathcal{C}} : \mathrm{Rib}_{\mathcal{C}} \xrightarrow{\mathcal{C}\ \text{ribbon}} \mathcal{C}$$

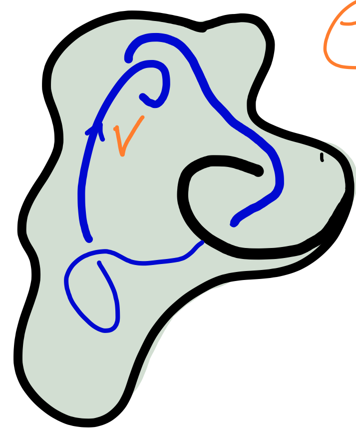


$$\longmapsto \quad \begin{array}{c} W_1 \otimes W_2 \otimes W_3 \\ T \uparrow \\ V_1 \otimes V_2 \end{array}$$

Thm: $\mathcal{C}$ $\begin{cases} \text{ribbon} \\ \text{finite} \\ \text{semisimple} \\ \text{modular} \end{cases}$

$$\exists\, Z_{RT} : \widetilde{\mathrm{Cob}}^{\sqcup}_{2,3} \to \mathrm{Vect}^{\otimes}$$

TQFTs coming from skein theory:



$\mathcal{C}$-decorated **skeins** in a 3-manifold

- 89' Witten
- 91' Reshetikhin – Turaev
- 92' Turaev–Viro
- 93' Crane – Yetter
  4-TQFT

"state sum"

- 95' Lyubashenko
- 96' Hennings
- ⋮
- 16' Blanchet – Costantino – Geer – Patureau-Mirand
- 22' De Renzi – Gainutdinov – Geer – Patureau-M – Runkel (DGGPR)
- ⋮

"Non semisimple" generalizations

⚠ Non-compact

Does the list ever end?

Cobordism Hypothesis:

(framed, fully extended) TQFTs are classified by "fully dualizable" objects

n	-TQFT	$\xleftrightarrow{C\text{-}H}$	dualizable object in ambient category
3	Turaev-Viro	←--→	fusion categories [DSS20'] in Tens [SS17']
4	Crane-Yetter	←--→	braided fusion categories [BJSS21'] in BrTens [SS]
4	?	←--→	non-semisimple modular tensor categories [BJSS21'] in BrTens
3	Witten-Reshetikhin-Turaev* (+ non-semisimple variants)	←--→	?

5

Goals:

This PhD

* construct (topologically) the predicted non-semisimple Crane-Yetter 4-TQFTs.

* provide the 3-dualizable objects describing (non-semisimple) WRT theories.

future work

* prove that the Cobordism Hypothesis does recover the expected topological constructions.

I. Missing

Non-semisimple

Crane-Yetter 4-TQFT

Non-semisimple Crane-Yetter 4-TQFTs

23' Costantino – Geer – H. – Patureau-Mirand

Thm Given a $\left\{\begin{array}{l}\text{finite}\\ \text{unimodular}\\ \text{ribbon}\\ \text{chromatic compact}\end{array}\right.$ category $\mathcal{C}$, we construct a 4-TQFT

modular ⇒ (finite, unimodular, ribbon, chromatic compact)

$$\mathcal{S}_{\mathcal{C}} : \mathrm{Cob}_{3+1} \longrightarrow \mathrm{Vect}$$

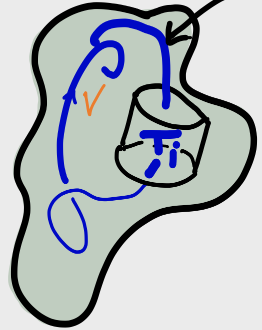
using skein theory.

The construction:

with projective colors in every c.c.

$$\mathcal{S}_e(M) := \Big\langle$$  $$\Big\rangle \Big/$$ Isotopy and

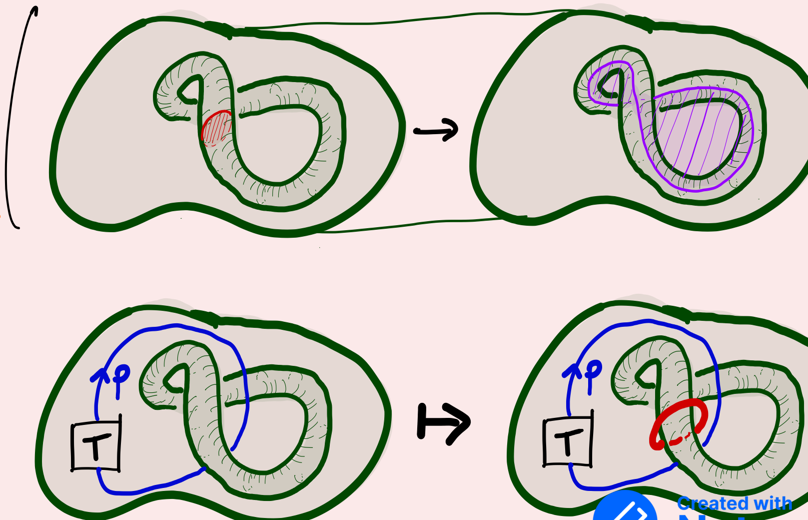
with projective outside of the cylinder

$$\sum \lambda_i$$  $$\sim 0 \text{ if } \sum \lambda_i \, RT_e(T_i) = 0$$

$\mathcal{S}_e$(4-dimensional handle attachment)

ii

explicit operations on skeins

Example: (the 2-handle) $\mathcal{S}_e\Big($  $\Big)$

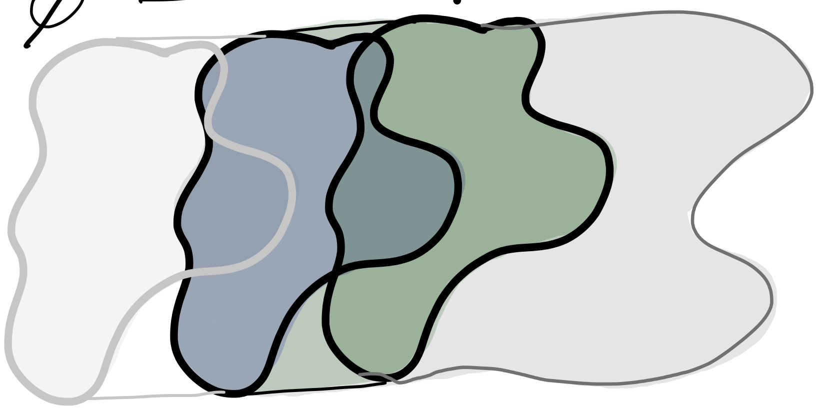
II. Missing (non-semisimple) WRT 3-TQFTs

Created with Notewise

WRT has an anomaly!

WRT

$\emptyset \xrightarrow{\text{boundary condition}} M \xrightarrow{\text{anomaly} = CY} \emptyset$



[Walker]

$\emptyset \xrightarrow{\text{boundary}^* \text{ condition}} M \xrightarrow{\text{anomaly} = \text{non-ss } CY} \emptyset$

DGGPR

[Jordan – Safronov]

Def (Johnson-Freyd–Scheimbauer (following F–T, S–T))

A boundary condition ∂ to an anomaly theory

$$\mathcal{S} : \mathrm{Bord}_4 \to \mathrm{BrTens}$$

is a symmetric monoidal functor

$$\partial : \mathrm{Bord}_3 \to \mathrm{BrTens}^{\rightarrow}$$

s.t. $s \circ \partial = \mathrm{Triv}$ and $t \circ \partial = \mathcal{S}$.

where

$\mathrm{BrTens}^{\rightarrow} =$
- obj: $X \xrightarrow{f} Y$
- 1-m: $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ h_X \downarrow & \overset{h^\#}{\nearrow\!\!\!\nearrow} & \downarrow h_Y \\ X' & \xrightarrow{f'} & Y' \end{array}$
- 2-m: some cubes
- ⋮

Dualizability of unit inclusion

3 WRT DGGPR

n-TQFT $\overset{C-H^*}{\longleftrightarrow}$ dualizable* object in ambient category

$\longleftrightarrow$ $\eta : 1 \to \mathcal{C}$ unit inclusion in $\overrightarrow{\mathrm{BrTens}}$ [SS]

Thm (H.) Let $\mathcal{C}$ be a (non-semisimple) modular tensor category and $\eta \in \overrightarrow{\mathrm{BrTens}}$ induced by the unit inclusion, then η is (non-compact) 3-dualizable

Rebuilding WRT

$\mathcal{C} \in \mathrm{BrTens}$ 4D [BSSS] $+$ $\eta \in \mathrm{BrTens}^{\rightarrow}$ nc3D [H]

orientation structure? (nc) C-H

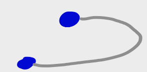
$\rho_e : \mathrm{Bord}_4 \to \mathrm{BrTens}$
$+$
$\partial : \mathrm{Bord}_3^{nc} \to \mathrm{BrTens}^{\rightarrow}$

"compose"

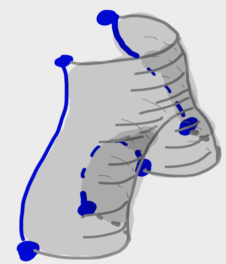
(non-ss) WRT

Def: The anomalous theory induced by ρ_e, ∂ is

$$\mathcal{A}_{\rho_e,\partial} : \underbrace{\mathrm{Bord}_{n-1}^{\text{filled},nc}}_{\overset{!!}{=}\ \mathrm{Bord}_n^{\to\emptyset,nc}} \longrightarrow \underbrace{\Omega\,\mathrm{BrTens}}_{\overset{!!}{=}\ \mathrm{End}_{\mathrm{BrTens}}(\mathbb{1})}$$

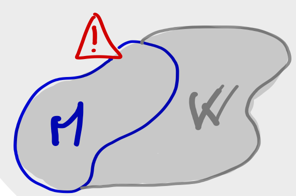


$$\longmapsto \begin{tikzcd} \mathbb{1} \arrow[r, "\partial(\cdot\,\cdot)"] \arrow[rr, bend right, "\mathcal{A}"'] & \rho_e(\cdot\,\cdot) \arrow[r, "\rho_e(\supset)"] & \mathbb{1} \end{tikzcd}$$



$$\longmapsto \begin{tikzcd} \mathbb{1} \arrow[r, "\partial(\cdot\,\cdot)"] \arrow[d, equal] & \rho_e(\cdot\,\cdot) \arrow[r, "\rho_e(\supset)"] \arrow[d, "\rho_e(\cdot)"] & \mathbb{1} \arrow[d, equal] \\ \mathbb{1} \arrow[r, "\partial(\cdot\cdot\cdot)"'] \arrow[ur, phantom, "\partial(\cdot)"] & \rho_e(\cdot\cdot\cdot) \arrow[r, "\rho_e(\supset)"'] \arrow[ur, phantom, "\rho_e(\cdot)"] & \mathbb{1} \end{tikzcd}$$

$\vdots$



$$\longmapsto \begin{tikzcd} \mathbb{1} \arrow[r, "\partial(M)"] \arrow[rr, bend right, "\mathcal{A}"'] & \rho_e(M) \arrow[r, "\rho_e(W)"] & \mathbb{1} \end{tikzcd}$$

Rebuilding WRT

bicategory of filled cobordisms of dimensions 1, 2 and 3

nice categories with all colimits

Conjecture:

$$
\begin{tikzcd}
h_2 \Omega \mathrm{Bord}_3^{\text{filled}, nc} \arrow[rr, "\mathcal{A}_{\mathcal{S}_e, \partial}"] \arrow[d, "\text{forgetful}"'] & & \Omega\mathrm{Tens} = \mathrm{PrCat} \\
\widetilde{\mathrm{Cob}}_{321}^{nc} \arrow[rr, "\mathrm{WRT}", "\mathrm{DGGPR}"'] \arrow[urr, phantom, "\circlearrowleft"] & & \widehat{\mathrm{Cat}} \arrow[u, "\text{Free}"']
\end{tikzcd}
$$

taking $\mathrm{Fun}(-^{op}, \mathrm{Vect})$

usual source for WRT:
surfaces + Lagrangian in H_1
3-manifolds + integer

Cauchy-complete linear categories

Expected description

dim :	0 1 2	3	4
$\mathcal{S}_e$:	skein categories ‖ [Cooke, Brown-H.] [Sch] factorization homology	Admissible skein module functors	CGHP's formulas for handle attachments
∂ :	empty object	empty skein*	

Conjecture check : $WRT(\Sigma) \cong SK(H, \emptyset)$ $\quad$ $WRT(M, \sigma(W)) = \mathcal{S}_e(W)(\emptyset)$

Thank you!