Non-semisimple WRT at the boundary of Crane–Yetter (DRAFT)

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Abstract

We prove the slogan, promoted by Walker twenty years ago, that

"The Witten–Reshetikhin–Turaev TQFT is a boundary condition for Crane–Yetter"

and generalize it to the non-semisimple case. To achieve this, we prove that the Crane–Yetter 4-TQFT and its non-semisimple version [CGHP23] are once-extended TQFTs, using the main result of [Haï24].

We define a boundary condition, partially defined in the non-semisimple case, for this theory. When the ribbon category used is modular, possibly non-semisimple, we check that the composition of this boundary condition with the values of the 4-TQFT on bounding manifolds, in a sense that we precise, reconstructs the Witten–Reshetikhin–Turaev 3-TQFT and its non-semisimple version [DGG+22].

Contents

1 Introduction

Walker explained in his unfinished notes [Wal06] how to obtain Witten–Reshetikhin–Turaev 3-TQFTs [Wit89, RT91, Tur94] at the boundary of Crane–Yetter [CY93] 4-TQFTs. This reflects the fact that WRT theories have an anomaly, described by Crane–Yetter.

Non-semisimple versions of Witten–Reshetikhin–Turaev theories appeared in [DGG+22]. It was expected by Jordan–Safronov that a similar story should apply to describe them. A non-semisimple version of Crane–Yetter appeared in [CGHP23], which indeed seems to model the anomaly of [DGG+22].

The notion of boundary condition is very natural in physics but has taken some time to be formalized in mathematics [Lur09, ST11, FT14, JS17]. Indeed, it only makes sense to talk about a boundary condition to an (at least once) extended TQFT, which not only assigns scalars to codimension 0 manifolds and vector spaces to codimension 1 manifolds, but also linear categories to codimension 2 manifolds, and so on, in a functorial manner. The Crane–Yetter theories, as any state sum theory, have long been known to be extended, though this is only a folklore result. To our knowledge, Crane–Yetter has never been written down in the literature as a symmetric monoidal 2-functor, i.e. has never been formalized as an extended theory. Our first step will therefore be to prove this folklore result, and extend it to the non-semisimple case.

Having defined Crane–Yetter as a once-extended 4-TQFT, we can define a boundary condition to it. Walker and Freed described how one should reconstruct WRT from this boundary condition and Crane–Yetter. We define the anomalous theory obtained from this construction, and check that it indeed reconstructs WRT theories and their non-semisimple versions, hence formalizing the ideas of Walker and generalizing them to the non-semisimple case.

From this description, one expects to be able to give a fully extended description of WRT (resp. [DGG+22]) as a fully extended boundary condition to the fully extended Crane–Yetter (resp. [CGHP23]) theory. The Crane–Yetter theory is expected to be obtained by the Cobordism Hypothesis from the input modular category $\mathcal{C}$, which is a 4-dualizable, and actually invertible, object in the 4-category $\mathrm{Alg}_2(\mathrm{Pr})$ [JS17, BJS21, BJSS21]. The boundary condition is expected to be obtained from the regular bimodule ${}_{\mathrm{Vect}}\mathcal{C}_{\mathcal{C}}$ which is a (resp. almost) 3-dualizable object in the 3-category $\mathrm{Alg}_2(\mathrm{Pr})^{\rightarrow}$ [Fre, JS17, Haï23]. These expectations and the fact that this boundary condition together with its anomaly recovers WRT theories has been formalized as a conjecture in [Haï23].

In this paper, we prove a version of these conjectures in a non-fully-extended setting, which do not involve the cobordism hypothesis.

2 Background

2.1 Cobordism bicategories and once-extended TQFTs

We begin by recalling the definition of the cobordism bicategory. We adopt the definition of [Haï24], but see also [Sch09].

Definition 2.1. The **bicategory of** $(2+1+1)$**-cobordisms** $\mathbf{Cob}_{2+1+1}$ is the symmetric monoidal bicategory with

objects: Closed oriented smooth surfaces Σ

1-morphisms: 3-cobordisms $M : \Sigma_- \to \Sigma_+$ equipped with a collar of their boundary $\Sigma_\pm \times [\pm 1, \pm\frac{1}{2}) \hookrightarrow M$. Composition is given by gluing the collars, which inherits a natural smooth structure.

2-morphisms: 4-cobordisms with corners $W : M_- \to M_+$, equipped with a side collar of their side boundary $\Sigma_\pm \times [-1,1] \times [\pm 1, \pm\frac{1}{2}) \hookrightarrow W$ compatible with the collars of $M_\pm$, and considered up to diffeomorphisms preserving $M_\pm$ and preserving side collars up to a reparametrization of the $[-1,1]$-coordinate. Horizontal composition is gluing the collars. Vertical composition is gluing along M's, whose smooth structure is well-defined up to diffeomorphism.

It is symmetric monoidal with disjoint union.

The **bicategory of non-compact** $(2+1+1)$**-cobordisms** $\mathbf{Cob}^{nc}_{2+1+1}$ is the symmetric monoidal sub-bicategory of $\mathbf{Cob}_{n+1+1}$ with the same objects and 1-morphisms but only those 2-morphisms where the target diffeomorphism is surjective on connected components, i.e. every connected component of the 4-cobordisms have non-empty outgoing boundary.

The **bicategory of** $(2+1+\varepsilon)$**-cobordisms** $\mathbf{Cob}_{2+1+\varepsilon}$ has the same objects and 1-morphisms as $\mathbf{Cob}_{2+1+1}$, but 2-morphisms are isotopy classes of diffeomorphisms preserving the side collars. It comes with a symmetric monoidal strict 2-functor $\mathbf{Cob}_{2+1+\varepsilon} \to \mathbf{Cob}^{nc}_{2+1+1} \subseteq \mathbf{Cob}_{2+1+1}$ which is the identity on objects and 1-morphisms and maps a diffeomorphism to its mapping cylinder.

The **bicategory of non-compact** $(2+1+\varepsilon)$**-cobordisms** $\mathbf{Cob}^{nc}_{2+1+\varepsilon}$ is the locally full symmetric monoidal sub-bicategory of $\mathbf{Cob}_{n+1+\varepsilon}$ with the same objects but only those 1-morphisms where the target diffeomorphism is surjective on connected components, i.e. every connected component of the 3-cobordisms have non-empty outgoing boundary, and all 2-morphisms between these.

Definition 2.2. Let $\mathcal{C}$ be a symmetric monoidal bicategory.
A **once-extended** 4**-TQFT**, or a $(2+1+1)$**-TQFT**, with values in $\mathcal{C}$ is a symmetric monoidal 2-functor

$$\mathcal{Z} : \mathbf{Cob}_{2+1+1} \to \mathcal{C} \ .$$

A **non-compact once-extended** 4**-TQFT** is a symmetric monoidal 2-functor

$$\mathcal{Z} : \mathbf{Cob}^{nc}_{2+1+1} \to \mathcal{C} \ .$$

A **categorified** 3**-TQFT**, or a $(2+1+\varepsilon)$**-TQFT**, is a symmetric monoidal 2-functor

$$\mathcal{Z} : \mathbf{Cob}_{2+1+\varepsilon} \to \mathcal{C} \ .$$

A **boundary condition** to a once-extended 4-TQFT $\mathcal{Z}$ is a symmetric monoidal oplax natural transformation

$$\partial : \mathrm{Triv} \Rightarrow \mathcal{Z}^{\varepsilon}$$

where $\mathcal{Z}^{\varepsilon} : \mathbf{Cob}_{n+1+\varepsilon} \to \mathbf{Cob}_{n+1+1} \overset{\mathcal{Z}}{\to} \mathcal{C}$ is the restriction of $\mathcal{Z}$ to $\mathbf{Cob}_{n+1+\varepsilon}$ and $\mathrm{Triv} : \mathbf{Cob}_{2+1+\varepsilon} \to \mathcal{C}$ is constant equal to the monoidal unit.
A **non-compact boundary condition** to a once-extended 4-TQFT $\mathcal{Z}$ is a symmetric monoidal oplax natural transformation

$$\partial : \mathrm{Triv} \Rightarrow \mathcal{Z}^{\varepsilon,nc}$$

where $\mathcal{Z}^{\varepsilon,nc} : \mathbf{Cob}^{nc}_{n+1+\varepsilon} \to \mathbf{Cob}_{n+1+1} \overset{\mathcal{Z}}{\to} \mathcal{C}$ is the restriction of $\mathcal{Z}$ to $\mathbf{Cob}^{nc}_{n+1+\varepsilon}$.

2.2 Categorified linear algebra

Let us quickly recall the definition of the target bicategory we will consider for our once-extended TQFTs. Details can be found in [AR94, DS97, Kel05, BCJ15, BJS21, GJS23].

Definition 2.3. Let $\Bbbk$ be a field.
The (strict) bicategory $\mathrm{Cat}_{\Bbbk}$ has objects small $\Bbbk$-linear categories, 1-morphisms linear functors and 2-morphisms natural transformations. It is symmetric monoidal the tensor product $\otimes$ which is Cartesian product on objects and tensor product on spaces of morphisms.
The bicategory Bimod has objects small $\Bbbk$-linear categories, 1-morphisms $\mathcal{C} \to \mathcal{D}$ are profunctors, or bimodule functors $F : \mathcal{C} \otimes \mathcal{D}^{op} \to \mathrm{Vect}_{\Bbbk}$, and 2-morphisms natural transformations. Composition of 1-morphisms $F : \mathcal{C} \otimes \mathcal{D}^{op} \to \mathrm{Vect}_{\Bbbk}$ and $G : \mathcal{D} \otimes \mathcal{E}^{op} \to \mathrm{Vect}_{\Bbbk}$ is given by the coend

$$(G \circ F)(C, E) := \int^{D \in \mathcal{D}} F(C, D) \otimes G(D, E)$$

It is symmetric monoidal with the usual tensor product $\otimes$ on linear categories.
There is a symmetric monoidal 2-functor Cat $\to$ Bimod which is the identity on objects and post-composition with the Yoneda embedding $\mathcal{D} \to \widehat{\mathcal{D}}$ on morphisms.
The (strict) bicategory Pr has objects presentable $\Bbbk$-linear categories, 1-morphisms cocontinuous functors and 2-morphisms natural transformations. It is symmetric monoidal with Kelly-Deligne tensor product.
There is a symmetric monoidal fully faithful embedding $\widehat{(-)}$: Bimod $\to$ Pr which maps a category $\mathcal{C}$ to its free cocompletion, or presheaf category $\widehat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Vect}_{\Bbbk})$. A profunctor $\mathcal{C} \to \widehat{\mathcal{D}}$ extends essentially uniquely to a cocontinuous functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ by the co-Yoneda Lemma. The essential image of $\widehat{(-)}$ consists of the presentable categories with enough compact-projective objects.

3 Skein theory as a categorified 3-TQFT

Ribbon categories are a class of particularly well-behaved $\mathbb{E}_2^{or}$-algebra in $\mathrm{Cat}_{\Bbbk}$ which have a graphical calculus, called skein theory, that makes sense in any 3-manifold. Skein theory for ribbon categories has been formalized in [Tur94]. Skein categories and skein module functors associated to cobordisms have been introduced in [Wal06, Joh21]. It is a well-known folklore result that these constructions form a categorified TQFT, though it has never been written down as a symmetric monoidal 2-functor. This is the subject of this section.

In order to adapt to the non-semisimple setting, we will also need to consider a tensor ideal $\mathcal{I}$ in a ribbon category $\mathcal{A}$ as in [CGP23, BH24]. These can also be thought of as a class of $\mathbb{E}_2^{or}$-algebras, this time in the bicategory Pr, by setting $\mathcal{E} := \widehat{\mathcal{I}}$. These correspond to the following:

Definition 3.1. A cp-ribbon category $\mathcal{E} \in \mathrm{Pr}$ is an $\mathbb{E}_2^{or}$-algebra in Pr, i.e. a presentable braided balanced category, such that:

- $\mathcal{E}$ has enough compact-projectives, i.e. $\mathcal{E} \simeq \widehat{\mathcal{I}}$ where $\mathcal{I}$ is the subcategory of compact-projective objects of $\mathcal{E}$,
- every object of $\mathcal{I}$ is dualizable, and
- the rigid balanced category $\mathcal{A}$ of dualizable objects[1] of $\mathcal{E}$ is ribbon.

From a cp-ribbon category $\mathcal{E}$ we extract an inclusion $(\mathcal{I} \subseteq \mathcal{A})$ of a tensor ideal in a ribbon category, and $\mathcal{E}$ can be reconstructed as $\widehat{\mathcal{I}}$ with tensor product, braiding and balancing induced by those of $\mathcal{I}$. Note that $\mathcal{I}$ will not in general contain the monoidal unit, but the unit of $\widehat{\mathcal{I}}$ is unique up to isomorphism, or can be reconstructed as $\mathrm{Hom}_{\mathcal{A}}(-, \mathbb{1})$ using the inclusion into $\mathcal{A}$.

The examples coming from the setting of [Wal06] are precisely those where $\mathcal{I} = \mathcal{A}$, i.e. where the unit of $\mathcal{E}$ is compact projective.

Let us recall the basic definitions of skein theory, adapted to the non-semisimple setting. Details can be found in [BH24], see also [CGP14, Tur94, Coo23, GJS23].

Definition 3.2. An **$\mathcal{I}$-labeling** X in a closed surface Σ is a collection of $\mathcal{I}$-colored framed oriented points in Σ. It is called **admissible** if there is at least one point per connected component of Σ.

An **$\mathcal{I}$-colored ribbon graph** $\overline{T}$ compatible with two $\mathcal{I}$-labellings $X \subseteq \Sigma_-$ and $Y \subseteq \Sigma_+$ in a 3-cobordism $M : \Sigma_- \to \Sigma_+$ is the image of an embedding $\Gamma \hookrightarrow M$ of a finite oriented graph Γ equipped with a smooth framing, with edges colored by objects of $\mathcal{I}$, inner vertices colored by appropriate morphisms in $\mathcal{I}$ and boundary vertices matching the colored oriented framed points X and Y. It is called **admissible** if $\Gamma \hookrightarrow M$ is surjective on connected components.

In our setting where cobordisms are equipped with a collar $\Sigma_\pm \times [\pm 1, \pm\frac{1}{2}) \hookrightarrow M$, we also require that Γ is strictly vertical inside the collar, i.e. $\Gamma \cap (\Sigma_- \times [-1, -\frac{1}{2})) = X \times [-1, -\frac{1}{2})$ and $\Gamma \cap (\Sigma_+ \times (\frac{1}{2}, 1]) = Y \times (\frac{1}{2}, 1]$. This replaces the weaker transversality requirement of [BH24] but does not affect the skein module where these are considered up to isotopy, as any ribbon graph transverse to the boundary is isotopic in an essentially unique way to one that is vertical on the collars.

The **relative admissible skein module** $\mathrm{Sk}_{\mathcal{I}}(M; X, Y)$ is the vector space freely generated by isotopy classes of admissible $\mathcal{I}$-colored ribbon graphs in M compatible with X and Y quotiented by **admissible skein relation**, which are usual local skein relations happening in a cube $[0,1]^3 \hookrightarrow M$ where we require that the ribbon graphs intersect the boundary on the cube at least once.

A diffeomorphism $f : M \to M'$ preserving orientation and collars induces an isomorphisms of vector spaces

$$\begin{array}{rcl} f_* : \mathrm{Sk}_{\mathcal{I}}(M; X, Y) & \to & \mathrm{Sk}_{\mathcal{I}}(M'; X, Y) \\ T & \mapsto & f(T) \end{array}$$

which depends on f only up to isotopy.

[1] Note that more precisely, for smallness issues, $\mathcal{I}$ and $\mathcal{A}$ are small subcategories of all compact-projective and dualizable objects that contain every isomorphism classes.

Definition 3.3. Given composeable cobordisms $\Sigma_1 \xrightarrow{M_{12}} \Sigma_2 \xrightarrow{M_{23}} \Sigma_3$ and $\mathcal{I}$-labelings $X_1 \subseteq \Sigma_1,\ X_2 \subseteq \Sigma_2, X_3 \subseteq \Sigma_3$ the **gluing of skeins** is the linear map

$$\begin{array}{ccccc} \mathrm{Sk}_{\mathcal{I}}(M_{12}; X_1, X_2) & \otimes & \mathrm{Sk}_{\mathcal{I}}(M_{23}; X_2, X_3) & \to & \mathrm{Sk}_{\mathcal{I}}(M_{23} \circ M_{12}; X_1, X_3) \\ T & \otimes & T' & \mapsto & \overline{T} \cup \overline{T}' \end{array}$$

where $\overline{T},\ \overline{T}'$ are ribbon graph representatives of the skeins T, T', and $\overline{T} \cup \overline{T}' \subseteq M_{12} \underset{\Sigma_2 \times I}{\cup} M_{23} = M_{23} \circ M_{12}$ is a ribbon graph as $\overline{T}$ and $\overline{T}'$ are both vertical in the collars of Σ_2 and glue smoothly.

Definition 3.4. The **skein category** $\mathrm{SkCat}_{\mathcal{I}}(\Sigma)$ of a surface Σ has:

Objects: Admissible $\mathcal{I}$-labelings in Σ

Morphisms: The relative admissible skein module $\mathrm{Sk}_{\mathcal{I}}(\Sigma \times [-1, 1]; X, Y)$

Composition: Gluing of skeins, i.e.

$$\mathrm{Sk}_{\mathcal{I}}(\Sigma \times [-1, 1]; X_1, X_2) \otimes \mathrm{Sk}_{\mathcal{I}}(\Sigma \times [-1, 1]; X_2, X_3) \to \mathrm{Sk}_{\mathcal{I}}(\Sigma \times [-1, 1]; X_1, X_3)$$

using the unitor diffeomorphism $\Sigma \times [-1, 1] \circ \Sigma \times [-1, 1] \simeq \Sigma \times [-1, 1]$.

The **admissible skein bimodule functor** of $M : \Sigma_- \to \Sigma_+$ is the functor

$$\begin{aligned} \underline{\mathrm{Sk}}_{\mathcal{I}}(M) : \mathrm{SkCat}_{\mathcal{I}}(\Sigma_+) \otimes \mathrm{SkCat}_{\mathcal{I}}(\Sigma_-)^{op} &\to \mathrm{Vect} \\ (Y, X) &\mapsto \mathrm{Sk}_{\mathcal{I}}(M; X, Y) \end{aligned} \tag{3.1}$$

The action of morphisms in $\mathrm{SkCat}_{\mathcal{I}}(\Sigma)$ and $\mathrm{SkCat}_{\mathcal{I}}(\Sigma')$ is induced by gluing of skeins and unitor diffeomorphisms.

A diffeomorphism $f : M \to M'$ defines a natural isomorphism $f_* : \underline{\mathrm{Sk}}_{\mathcal{I}}(M) \Rightarrow \underline{\mathrm{Sk}}_{\mathcal{I}}(M')$ whose components has been defined above. It depends on f only up to isotopy.

Note that $\underline{\mathrm{Sk}}_{\mathcal{I}}(M)$ is a morphism from $\mathrm{SkCat}_{\mathcal{I}}(\Sigma_+)$ to $\mathrm{SkCat}_{\mathcal{I}}(\Sigma_-)$ in Bimod. This contravariance also appears in [Wal06], where skeins are treated as the dual theory to a TQFT. We emphasize that this is only a nuisance and not a deep issue, since $\mathrm{Cob} \simeq \mathrm{Cob}^{op}$ via orientation reversal. We will denote Bimod^{hop} opposite bicategory in the horizontal direction (i.e. for 1-morphisms).

The following result has been shown in [BH24, Thm. 2.21] in a 1-categorical setting, see also [Wal06, Thm. 4.4.2]. The definition of a symmetric monoidal 2-functor is recalled in [Sch09, Def. A.5 and 2.5].

Theorem 3.5. *Given a tensor ideal $\mathcal{I}$ in a ribbon category $\mathcal{A}$, there exists a categorified TQFT*

$$\underline{\mathrm{Sk}}_{\mathcal{I}} : \mathbf{Cob}_{2+1+\varepsilon} \to \mathrm{Bimod}^{hop}$$

with $\underline{\mathrm{Sk}}_{\mathcal{I}}(\Sigma) = \mathrm{SkCat}_{\mathcal{I}}(\Sigma),\ \underline{\mathrm{Sk}}_{\mathcal{I}}(M) = \underline{\mathrm{Sk}}_{\mathcal{I}}(M)$ *and* $\underline{\mathrm{Sk}}_{\mathcal{I}}(f) = f_*$.

Proof. We much exhibit for any composable pair $\Sigma_1 \xrightarrow{M_{12}} \Sigma_2 \xrightarrow{M_{23}} \Sigma_3$ an isomorphism

$$\underline{\mathrm{Sk}}(M_{23}) \circ \underline{\mathrm{Sk}}(M_{12}) \tilde{\to} \underline{\mathrm{Sk}}(M_{23} \circ M_{12})$$

compatible with 2-morphisms in M and M'.

This is the data for any objects X_1, X_3 of $\mathrm{SkCat}(\Sigma_1), \mathrm{SkCat}(\Sigma_3)$ of an isomorphism

$$\int^{X_2 \in \mathrm{SkCat}(\Sigma_2)} \mathrm{Sk}_{\mathcal{I}}(M_{12}; X_1, X_2) \otimes \mathrm{Sk}_{\mathcal{I}}(M_{23}; X_2, X_3) \tilde{\to} \mathrm{Sk}_{\mathcal{I}}(M_{12} \underset{\Sigma_2}{\cup} M_{23}; X_1, X_3)\ .$$

There is a natural such morphism which is given by gluing of skeins. It is shown to be an isomorphism in [BH24, Thm. 2.21]. The proof there happens in two steps: first the arguments of [Wal06, Thm. 4.4.2] shows that the gluing of skeins is an isomorphism from the LHS to the space of skeins in $M_{12} \underset{\Sigma_2}{\cup} M_{23}$ which intersect Σ_2 at least once on every connected components. Then [BH24, Lem. 2.11] proves that the inclusion of this space in the RHS is an isomorphism.

This isomorphism is natural with respect to diffeomorphisms in M and M' preserving collars as $(f \cup f')(\overline{T} \cup \overline{T}') = f(\overline{T}) \cup f'(\overline{T}')$.

By definition, we also have

$$\underline{\mathrm{Sk}}_{\mathcal{I}}(\mathrm{id}_\Sigma) = \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{I}}(\Sigma)}(-,-) = \mathrm{id}_{\mathrm{SkCat}_{\mathcal{I}}(\Sigma)}$$

and these isomorphisms are readily checked to be compatible with associators and unitors. This proves that $\underline{\mathrm{Sk}}_{\mathcal{I}}$ is a 2-functor.

We now turn to symmetric monoidality. We have an isomorphism of vector spaces

$$\mathrm{Sk}_{\mathcal{I}}(M; X, Y) \otimes \mathrm{Sk}_{\mathcal{I}}(M'; X', Y') \tilde{\to} \mathrm{Sk}_{\mathcal{I}}(M \sqcup M'; X \sqcup X', Y \sqcup Y')$$

simply given by disjoint union of ribbon graphs. It induces an isomorphism of categories

$$\mathrm{SkCat}_{\mathcal{I}}(\Sigma) \otimes \mathrm{SkCat}_{\mathcal{I}}(\Sigma') \tilde{\to} \mathrm{SkCat}_{\mathcal{I}}(\Sigma \sqcup \Sigma')$$

which is disjoint union on objects, and a natural isomorphism

$$\underline{\mathrm{Sk}}_{\mathcal{I}}(M) \otimes \underline{\mathrm{Sk}}_{\mathcal{I}}(M') \tilde{\Rightarrow} \underline{\mathrm{Sk}}_{\mathcal{I}}(M \sqcup M')$$

where we implicitly used the isomorphism of categories above to match the source and target.

The skein category of the empty surface has only one object, the empty collection of points, with endomorphisms scalars times the empty ribbon graph, its identity, which is indeed the monoidal unit in Bimod.

All the coherence modifications of [Sch09, Def. 2.5] are identities. □

4 Extended non-semisimple Crane–Yetter

In this section we will extend the categorified TQFT

$$\underline{\mathrm{Sk}}_{\mathcal{I}} : \mathbf{Cob}_{2+1+\varepsilon} \to \mathrm{Bimod}^{hop}$$

into a possibly non-compact once-extended 4-TQFT

$$\mathcal{Z} : \mathbf{Cob}_{2+1+1} \to \mathrm{Bimod}^{hop}$$

under some additional conditions on the category $\mathcal{A}$.

We will construct this TQFT by specifying its values on the standard attachments of 0–4-handles, check that they satisfy handle cancellation and ι-invariance and use the main result of [Haï24].

This TQFT is a once-extended version of [CGHP23], and the values on handle attachments are constructed there. We will recall the definitions for the readers convenience, but refer to [CGHP23] for details.

4.1 Hypothesis and structure on the input category

In this section, $\mathcal{A}$ is a **finite ribbon tensor category** over an algebraically closed field $\Bbbk$ in the sense of [EGNO15] and $\mathcal{I} \subseteq \mathcal{A}$ is the tensor ideal of projective objects. We denote $P_{\mathbb{1}} \in \mathcal{I}$ a projective cover of the unit, equipped with its projection $\varepsilon_{\mathbb{1}} : P_{\mathbb{1}} \to \mathbb{1}$. For any projective object P, let $s_P : P \to P \otimes P_{\mathbb{1}}$ be a section of $\mathrm{id}_P \otimes \varepsilon_{\mathbb{1}} : P \otimes P_{\mathbb{1}} \to P$. By definition this means that $\mathrm{id}_P = (\mathrm{id}_P \otimes \varepsilon_{\mathbb{1}}) \circ s_P$ which at the levels of skeins means that we can introduce a $P_{\mathbb{1}}$-colored strand ending with a $\varepsilon_{\mathbb{1}}$ whenever there is a projective-colored strand.

We further assume that $\mathcal{A}$ is **unimodular** which is equivalent to asking that $P_{\mathbb{1}}$ is self dual. This implies that $\mathcal{A}$ has a non-degenerate modified trace

$$\mathrm{t}_P : \mathrm{End}_{\mathcal{A}}(P) \to \Bbbk\ , \quad P \in \mathcal{I}$$

which is unique up to scalar by [GKP22, Cor. 5.6]. We assume that a choice of modified trace has been made. It fixes the choice of a morphism $\eta_{\mathbb{1}} : \mathbb{1} \to P_{\mathbb{1}}$ such that $\mathrm{t}_{P_{\mathbb{1}}}(\eta_{\mathbb{1}} \circ \varepsilon_{\mathbb{1}}) = 1$. The modified trace being non-degenerate implies that it induces a non-degenerate pairing

$$\mathrm{t}_P(- \circ -) : \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, P) \otimes \mathrm{Hom}_{\mathcal{A}}(P, \mathbb{1}) \to \Bbbk$$

We denote

$$\Omega = \sum_i x^i \otimes x_i \in \mathrm{Hom}_{\mathcal{A}}(P, \mathbb{1}) \otimes \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, P)$$

the associated copairing, i.e. $(x_i)_i$ and $(x^i)_i$ are dual basis with respect to t_P. We also denote

$$\Lambda_P := \sum_i x_i \circ x^i \in \mathrm{End}_{\mathcal{A}}(P) \ .$$

Let $G \in \mathcal{I}$ be a projective generator of $\mathcal{A}$, e.g. take G to be the direct sum of all irreducible projectives. By [CGPV], or [CGHP23, Thm. 1.10], there exist a chromatic morphism based at any $P \in \mathcal{I}$ [CGHP23, Sec. 1.3]

$$\mathsf{c}_P : G \otimes P \to G \otimes P$$

We further assume that $\mathcal{A}$ is **chromatic non-degenerate** in the sense that the morphism

$$\Delta^0_{P_{\mathbb{1}}} := TIKZ$$

is non-zero. This implies the existence of a gluing morphism [CGHP23, Def 1.5]

$$\mathsf{g} : P_{\mathbb{1}} \to P_{\mathbb{1}}$$

We will sometimes assume that $\mathcal{A}$ is even **chromatic compact** in the sense that $\varepsilon_{\mathbb{1}} \circ \mathsf{g}$ is non-zero. This implies that there exists a non-zero global dimension

$$\zeta \in \Bbbk^{\times}$$

such that $\zeta^{-1}\,\mathrm{id}_{P_{\mathbb{1}}}$ is a gluing morphism.

Example 4.1. If $\mathcal{A}$ is semisimple and $(S_i)_{i=1,\dots,n}$ are its simples, with $S_0 = \mathbb{1}$, then:
- $P_{\mathbb{1}} = \mathbb{1}$,
- any scalar times the usual categorical trace $\lambda\,\mathrm{tr}$ is a modified trace,
- the copairing is $\Omega_{S_i} = \lambda^{-1}\delta_{i,0}\,\mathrm{id} \otimes \mathrm{id}$,
- a projective generator is given by $G = \oplus_i S_i$,
- $\mathsf{c} := \lambda \cdot \oplus_i \,\mathrm{qdim}(S_i)\,\mathrm{id}_{S_i}$ is a chromatic morphism,
- $\Delta^0_{P_{\mathbb{1}}} = \lambda \sum_i \mathrm{qdim}(S_i)^2\,\mathrm{id}_{\mathbb{1}}$ which is non-zero when $\mathrm{char}\,\Bbbk = 0$ or when $\mathcal{A}$ is separable
- in this case, $\mathsf{g} = \frac{1}{\lambda^2 \sum_i \mathrm{qdim}(S_i)^2}\,\mathrm{id}_{\mathbb{1}}$ is a gluing morphism, and
- the global dimension is $\zeta = \lambda^2 \sum_i \mathrm{qdim}(S_i)^2$.

Note that there exists precisely two values of λ for which $\zeta = 1$.

4.2 The construction

We give an operation on admissible skein module associated to each 4-dimensional handle attachment, and then check that they define an extended TQFT.

The 4-handle We define a natural transformation

$$Z_4 : \underline{\mathrm{Sk}}_{\mathcal{I}}(S^3) \Rightarrow \underline{\mathrm{Sk}}_{\mathcal{I}}(\emptyset)$$

In this case, the incoming and outgoing boundary of S^3 are both the empty surface, and $\underline{\mathrm{Sk}}_{\mathcal{I}}(S^3) : \mathrm{SkCat}(\emptyset) \otimes \mathrm{SkCat}_{\mathcal{I}}(\emptyset)^{op} \to \mathrm{Vect}$ is just the data of one vector space $\mathrm{Sk}_{\mathcal{I}}(S^3) = \underline{\mathrm{Sk}}_{\mathcal{I}}(S^3, \emptyset, \emptyset)$. The natural transformation Z_4 is just the data of a linear map, which we will still denote Z_4 by abuse,

$$Z_4 : \mathrm{Sk}_{\mathcal{I}}(S^3) \to \Bbbk$$

We take this map to be the invariant of $\mathcal{I}$-colored admissible ribbon graphs induced by the modified trace [GPT09, GP18] as in [CGHP23]. If $T \in \mathrm{Sk}_{\mathcal{I}}(S^3)$ is the closure of a 1-1-tangle T_{cut} whose endpoints are both colored by a projective P, we set

$$Z_4(T) := \mathrm{t}_P(\mathrm{RT}(T_{cut}))$$

where RT denote the usual Reshetikhin–Turaev evaluation functor [Tur94]. This is well-defined by the work of Geer–Patureau-Mirand et al. and there is no naturality to check in this case.

The 3-handle We define a natural transformation

$$Z_3 : \underline{\mathrm{Sk}}_{\mathcal{I}}(S^2 \times \mathbb{D}^1) \Rightarrow \underline{\mathrm{Sk}}_{\mathcal{I}}(\mathbb{D}^3 \times S^0) \ .$$

Both 3-cobordisms $S^2 \times \mathbb{D}^1$ and $\mathbb{D}^3 \times S^0$ have incoming boundary $S^2 \times S^0$, and outgoing boundary $\emptyset$. Let $X \in \mathrm{SkCat}_{\mathcal{I}}(S^2 \times S^0)$ be an admissible $\mathcal{I}$-labelling, which we may write $X = X_+ \sqcup X_-$ as $S^2 \times S^0 = S^2 \times \{+\} \sqcup S^2 \times \{-\}$, and $T \in \mathrm{Sk}_{\mathcal{I}}(S^2 \times \mathbb{D}^1; X_+ \sqcup X_-)$ an $\mathcal{I}$-colored ribbon graph. We want to "cut" T in two pieces.

An object $P \in \mathcal{I}$ induces an $\mathcal{I}$-labeling in S^2 with a single point colored by P, which by abuse we will still call $P \in \mathrm{SkCat}_{\mathcal{I}}(S^2)$. As S^2 is connected, any object of $\mathrm{SkCat}_{\mathcal{I}}(S^2)$ is actually isomorphic to an object of this form. If X is any configuration of points, then P is the tensor product of all its colors, with duals for negatively oriented points. Similarly, a morphism $f \in \mathrm{Hom}_{\mathcal{A}}(P, P')$ induces a morphism in $\mathrm{SkCat}_{\mathcal{I}}(S^2)$, and morphisms of this form generate all morphisms.

As in the proof of Theorem 3.5, we have an equivalence

$$\mathrm{Sk}_{\mathcal{I}}(S^2 \times \mathbb{D}^1, X_+ \sqcup X_-) \simeq \int^{P \in \mathrm{SkCat}_{\mathcal{I}}(S^2)} \mathrm{Sk}_{\mathcal{I}}(S^2 \times [0,1]; X_+, P) \otimes \mathrm{Sk}_{\mathcal{I}}(S^2 \times [-1,0]; P \sqcup X_-)$$

Let us denote $T_+ \otimes T_-$ two skeins in the RHS that glue to T.

Given a morphism $f : P \to \mathbb{1}$, we get a skein which by abuse we will still denote $f \in \mathrm{Sk}_{\mathcal{I}}(\mathbb{D}^3, P)$ which has a single vertex colored by f at $0 \in \mathbb{D}^3$ linked by a straight line to $P \in \mathrm{SkCat}_{\mathcal{I}}(S^2)$. Similarly, for $\Omega \in \mathrm{Hom}_{\mathcal{A}}(P, \mathbb{1}) \otimes \mathrm{Hom}_{\mathcal{A}}(\mathbb{1}, P)$ we get a skein $\Omega \in \mathrm{Sk}_{\mathcal{I}}(\mathbb{D}^3 \times S^0)$.

We set

$$Z_3(T) := (T_+ \sqcup T_-) \cdot \Omega_P$$

where $(T_+ \sqcup T_-) \cdot -$ is the action of morphisms in $\mathrm{SkCat}_{\mathcal{I}}(S^2 \times S^0)$ on $\mathrm{Sk}_{\mathcal{I}}(D^3 \times S^0)$. It is well-defined, i.e. preserves the coend relation in the coend above, by naturality of Ω_P [CGHP23, Lem. 1.1]. It is natural as for any morphism S in $\mathrm{SkCat}_{\mathcal{I}}(S^2 \times S^0)$ we have $Z_3(S \cdot T) = S \cdot (T_+ \sqcup T_-) \cdot \Omega_P$.

The 2-handle We define a natural transformation

$$Z_2 : \underline{\mathrm{Sk}}_{\mathcal{I}}(S^1 \times \mathbb{D}^2) \Rightarrow \underline{\mathrm{Sk}}_{\mathcal{I}}(\mathbb{D}^2 \times S^1) \ .$$

Both 3-cobordisms $S^1 \times \mathbb{D}^2$ and $\mathbb{D}^2 \times S^1$ have incoming boundary $S^1 \times S^1$, and outgoing boundary $\emptyset$. Let $X \in \mathrm{SkCat}_{\mathcal{I}}(S^1 \times S^1)$ be an admissible $\mathcal{I}$-labelling and $T \in \mathrm{Sk}_{\mathcal{I}}(S^1 \times \mathbb{D}^2; X)$.

We may isotope T so that it does not intersect the core $S^1 \times \{0\}$.

We set

$$Z_2(T) := T \cup (\{0\} \times S^1)$$

where the cocore $\{0\} \times S^1$ is colored in red, which is interpreted as a skein via the red-to-blue operation of [CGHP23], i.e. using a chromatic morphism.

This operation does not depend on how we isotoped T to be disjoint from the core by [CGHP23, Lem. 2.4] nor on how we applied the red-to-blue operation or on the choice of a chromatic morphism by [CGHP23, Lem. 2.3]. Both of these operations can be chosen to leave T unchanged near the boundary, which shows naturality of Z_2.

The 1-handle We define a natural transformation

$$Z_1 : \underline{\mathrm{Sk}}_{\mathcal{I}}(S^0 \times \mathbb{D}^3) \Rightarrow \underline{\mathrm{Sk}}_{\mathcal{I}}(\mathbb{D}^1 \times S^2) \ .$$

Both 3-cobordisms $S^0 \times \mathbb{D}^3$ and $\mathbb{D}^1 \times S^2$ have incoming boundary $S^0 \times S^2$, and outgoing boundary $\emptyset$. Let $X = X_+ \sqcup X_- \in \mathrm{SkCat}_{\mathcal{I}}(S^0 \times S^2)$ be an admissible $\mathcal{I}$-labelling and $T = T_+ \sqcup T_- \in \mathrm{Sk}_{\mathcal{I}}(S^0 \times \mathbb{D}^3; X_+ \sqcup X_-)$.

As above, me may isotope T to be disjoint from $S^0 \times \{0\}$ and introduce two $P_{\mathbb{1}}$-strands in T_+ and T_- ending by a vertex $v_\pm$ which we may isotope to be at $0 \in \mathbb{D}^3$. Removing a neighborhood of $S^0 \times \{0\}$ we obtain a skein which we will denote $T_+ \smallsetminus v_+ \sqcup T_- \smallsetminus v_- \in \mathrm{Sk}_{\mathcal{I}}(S^0 \times S^2 \times [0,1]); X, P_{\mathbb{1}} \sqcup P_{\mathbb{1}})$. This exhibits T as

$$T = (T_+ \smallsetminus v_+ \sqcup T_- \smallsetminus v_-) \cdot (\varepsilon_{\mathbb{1}} \sqcup \varepsilon_{\mathbb{1}})$$

where we see $\varepsilon_{\mathbb{1}}$ as a skein in $\mathrm{Sk}_{\mathcal{I}}(\mathbb{D}^3; P_{\mathbb{1}})$ as above.

Using the isomorphism $P_{\mathbb{1}} \simeq P_{\mathbb{1}}^*$ mapping $\varepsilon_{\mathbb{1}}$ to $\eta_{\mathbb{1}}$, we may think of a gluing morphism $\mathsf{g} : P_{\mathbb{1}} \to P_{\mathbb{1}}$ as a skein $\mathsf{g} \in \mathrm{Sk}_{\mathcal{I}}(\mathbb{D}^1 \times S^2; P_{\mathbb{1}} \sqcup P_{\mathbb{1}})$.

We set

$$Z_1(T) = (T_+ \smallsetminus v_+ \sqcup T_- \smallsetminus v_-) \cdot \mathsf{g}$$

This is well-defined and does not depend on the choice of a gluing morphism or on how we introduced $P_{\mathbb{1}}$-strands by [CGHP23, Proposition 5.1]

The 0-handle In the case where $\mathcal{A}$ is chromatic compact, i.e. there exists $\zeta \in \mathbb{k}^\times$ the global dimension satisfying $\varepsilon_{\mathbb{1}} \circ \mathsf{g} = \zeta^{-1} \varepsilon_{\mathbb{1}}$, we set

$$Z_0 : \underline{\mathrm{Sk}}_{\mathcal{I}}(\emptyset) \Rightarrow \underline{\mathrm{Sk}}_{\mathcal{I}}(S^3)$$

by mapping $1 \in \mathbb{k}$ to $\zeta \Gamma_0 \in \mathrm{Sk}_{\mathcal{I}}(S^3)$ where Γ_0 is the admissible skein in S^3 with a single strand colored by $P_{\mathbb{1}}$ and two coupons $\varepsilon_{\mathbb{1}}$ and $\eta_{\mathbb{1}}$.

Theorem 4.2. *Let $\mathcal{A}$ be a chromatic non-degenerate category, $\mathcal{I} = \mathrm{Proj}(\mathcal{A})$ and t a modified trace. Then there exists a unique non-compact (2+1+1)-TQFT*

$$\mathcal{Z}_{\mathcal{A}} : \mathbf{Cob}^{nc}_{2+1+1} \to \mathrm{Bimod}^{hop}$$

extending $\underline{\mathrm{Sk}}_{\mathcal{I}}$ and given by $Z_4, \dots, Z_1$ on the standard handle attachments of index $4, \dots, 1$.

If $\mathcal{A}$ is moreover chromatic compact, then $\mathcal{Z}_{\mathcal{A}}$ extends in a unique way to a fully defined $(2+1+1)$-TQFT

$$\mathcal{Z}_{\mathcal{A}} : \mathbf{Cob}_{2+1+1} \to \mathrm{Bimod}^{hop}$$

given by Z_0 on the 0-handle.

Proof. We apply [Haï24]. The fact that our assignment on handles satisfies handle cancellations and invariance under reversal of the attaching spheres is checked in [CGHP23]. □

4.3 Properties

For the applications in Section 6 we will be interested in the special case when $\mathcal{A}$ is modular in the sense of [Lyu95, DGG+22]. In this case, the TQFT constructed above is very simple in dimension 4, and almost all of its data is located "at the boundary".

Proposition 4.3. *Suppose $\mathcal{A}$ is a possibly non-semisimple modular tensor category. Then:*

$\mathcal{A}$ is chromatic compact and the TQFT $\mathcal{Z}_{\mathcal{A}}$ constructed above is invertible for any choice of non-degenerate modified trace t.

The red-to-blue operation used in our construction agrees with the red-to-blue operation of [DGG+22].

There are exactly two choices of modified trace for which $\mathcal{Z}_{\mathcal{A}}(S^4) = 1$. If we use one of these, the natural transformations $\mathcal{Z}_{\mathcal{A}}(W) : \mathrm{Sk}_{\mathcal{I}}(M_-) \to \mathrm{Sk}_{\mathcal{I}}(M_+)$, for fixed M_-, M_+, depend only on the signature of W. Moreover, if M is a closed 3-manifold, $T \subseteq M$ is admissible and $W : M \to \emptyset$ is a bounding 4-manifold, we have

$$\mathcal{Z}_{\mathcal{A}}(W)(T) = \mathrm{DGGPR}_{\mathcal{A}}(M, T, \sigma(W)) \ .$$

Proof. The fact that modular implies chromatic compact is immediate from [CGHP23, Def. 1.7]. The invertibility statement is [CGHP23, Thm. 5.8] for the (3+1)-part, and follows from [SP18] for the whole theory. The fact that the red-to-blue operations coincide is explained in [CGHP23, Thm. 1.10, Eq. (8)].

Choose a modified trace t. By [CGHP23, Prop 5.7], the (3+1)-TQFT obtained from $\kappa\,\mathrm{t}$ is differs from the one obtained from t by an Euler characteristic term. As $\chi(S^4) = 2$, there are exactly two choices for κ such that $\mathcal{Z}_{\mathcal{A},\kappa\,\mathrm{t}}(S^4) = 1$, namely $\kappa = \pm\mathcal{D}^{-1}$ for $\mathcal{D}$ a square root of the global dimension $\zeta = \mathcal{Z}_{\mathcal{A},\mathrm{t}}(S^4)$.

We now assume we have chosen one of the two modified traces above, so $\zeta = \Delta_+\Delta_- = 1$. As every cobordism act by isomorphisms, we can pre-compose and side compose W by bounding manifolds without loosing information and it is equivalent to check the dependence on W for W closed. As the signature is additive when gluing on closed boundary components, the signature of the closed up manifold is the sum of signature of the initial manifold and the manifold we closed it up with.

As $\mathsf{g} = 1$, the action of a disjoint union of two 4-handles is the same as that of first using a 3-handle to connect the two balls and then using only one 4-handle. This allows us to further reduce to the case where W is connected.

As observed in [CGHP23, Def. 1.7], or [DGG+22, Lem. 4.4], as $\mathcal{A}$ is modular we have

$$\Delta_P^0 = \Lambda_P, \quad P \in \mathcal{I}$$

where the endomorphism Δ_P^0 is obtained by encircling a P strand by a red circle. Reading the handles backwards, and using Akbulut's dotted circle convention, this is saying that one may replace a dotted circle (the RHS) by a plain circle (the LHS). As W is connected we may assume it has only one 0-handle, and applying this observation to every 1-handle we may further reduce to simply connected W.

The scalar $\mathcal{Z}(W)$ is multiplicative under connected sum as $\mathcal{Z}(S^4) = 1$. As $\mathcal{Z}(\mathbb{C}P_2)\mathcal{Z}(\overline{\mathbb{C}P}_2) \neq 0$, it is stable under $\mathbb{C}P_2$-stabilization. Finally, two simply connected closed 4-manifolds are $\mathbb{C}P_2$-stably diffeomorphic if and only if they have same signature. And $\mathcal{Z}(W)$ depends only on the signature of W as claimed.

Finally, if M is closed and $T \subseteq M$ is admissible by [CGHP23, Thm. 4.4 and 5.9] (if W is obtained by 2- and 4-handles there, and for any W by the arguments above) we have

$$\mathcal{Z}(W)(T) = \mathcal{Z}(\mathbb{C}P_2)^{\sigma(W)}\,\mathrm{L}'(M,T) = \mathrm{DGGPR}_{\mathcal{A}}(M,T,\sigma(W))$$

where L' is the renormalized Lyubashenko invariant introduced in [DGG+22] and $\mathrm{DGGPR}_{\mathcal{A}}$ is the TQFT introduced there, which gives invariants of 3-manifolds equipped with an admissible ribbon graph and a "signature defect" integer. □

5 The regular boundary conditions to skein theory

In this section we will define a possibly non-compact boundary condition

$$\partial : \mathrm{Triv} \Rightarrow \underline{\mathrm{Sk}_{\mathcal{I}}}$$

given by the empty skein.

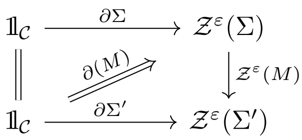
This boundary condition will be non-compact when $\mathcal{I} \neq \mathcal{A}$, which is quite different from the non-compact cases of Section 4. In the case of interest for Section 6, we will assume that $\mathcal{A}$ is modular, in which case $\mathcal{Z}$ is defined on all cobordisms, but ∂ will still be non-compact when $\mathcal{A}$ is non-semisimple.

Recall that $\mathrm{Sk}_{\mathcal{I}}$ is contravariant in the direction of 1-morphism. This is only an annoyance as one can always take opposite orientation on cobordisms to make it covariant, and when we say a boundary condition to $\mathrm{Sk}_{\mathcal{I}}$ we mean a boundary condition to this covariant functor. However, to avoid confusion or having to introduce a different notation, let us recall what explicit data a boundary condition to a contravariant functor represents.

Definition 5.1. A boundary condition to a categorified TQFT $\mathcal{Z}^\varepsilon : \mathbf{Cob}_{2+1+\varepsilon} \to \mathcal{C}$ contravariant in the direction of 1-morphism is the data of

For every object Σ: a 1-morphism $\partial(\Sigma) : \mathbb{1}_{\mathcal{C}} \to \mathcal{Z}^{\varepsilon}(\Sigma)$

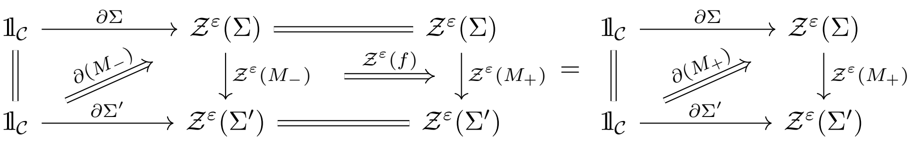
For every cobordism $M : \Sigma' \to \Sigma$: a 2-morphism



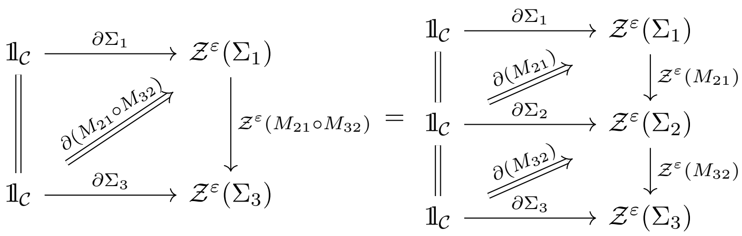
Symmetric monoidal structure: 2-isomorphisms $\partial(\Sigma) \otimes \partial(\Sigma') \Rightarrow \partial(\Sigma \sqcup \Sigma')$ and $\partial(\emptyset) \Rightarrow \mathbb{1}_{\mathcal{C}}$ with appropriate symmetric monoidal structure of $\mathcal{Z}^{\varepsilon}$ inserted to make the source and target of the 1-morphisms match.

Such that

For every diffeomorphism $f : M_- \to M_+$: the following equality holds:



For every composeable 1-morphisms $\Sigma_3 \overset{M_{32}}{\to} \Sigma_2 \overset{M_{21}}{\to} \Sigma_1$: the following equality holds:



Coherence of the symmetric monoidal structure: see [Sch09, Def. 2.7].

A non-compact boundary condition is one that only gives values to 3-cobordisms with non-empty incoming boundary Σ' in every connected component.

Let $\mathcal{I} \subseteq \mathcal{A}$ be a tensor ideal in a ribbon category.

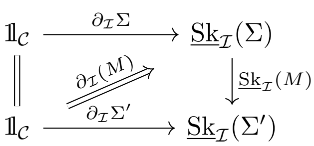
Definition 5.2. Let Σ be a surface. We define the 1-morphism $\partial_{\mathcal{I}}(\Sigma)$ from $\mathbb{1}$ to $\mathrm{SkCat}_{\mathcal{I}}(\Sigma)$ in Bimod by

$$\begin{array}{rcl} \partial_{\mathcal{I}}(\Sigma) : \mathrm{SkCat}_{\mathcal{I}}(\Sigma)^{op} & \to & \mathrm{Vect} \\ X & \mapsto & \mathrm{Sk}_{\mathcal{I}}(\Sigma \times [0,1]; X, \emptyset) \end{array}$$

called the distinguished object in [BH24].

If $\mathcal{I} = \mathcal{A}$, then this presheaf is actually representable, and represented by any number of points all colored by the monoidal unit, which we will call the empty collection of points.

Definition 5.3. Let $M : \Sigma' \to \Sigma$ be a cobordism with where $\pi_0(\Sigma') \to \pi_0(M)$ is surjective. We define a natural transformation



whose component on $X \in \mathrm{SkCat}_{\mathcal{I}}(\Sigma')$ is

$$\begin{array}{rcl} (\partial_{\mathcal{I}}(M))_X : \mathrm{Sk}_{\mathcal{I}}(\Sigma' \times [0,1]; X, \emptyset) & \to & \mathrm{Sk}_{\mathcal{I}}(M; X, \emptyset) \\ T & \mapsto & i_* T \end{array}$$

where $i : \Sigma' \times [0,1] \hookrightarrow M$ is the collar of the boundary, and we have used the equivalence

$$(\underline{\mathrm{Sk}}_{\mathcal{I}}(M) \circ \partial_{\mathcal{I}}(\Sigma))(X) := \int^{Y \in \mathrm{SkCat}_{\mathcal{I}}(\Sigma)} \mathrm{Sk}_{\mathcal{I}}(\Sigma \times [0,1]; Y, \emptyset) \otimes \mathrm{Sk}_{\mathcal{I}}(M; X, Y) \overset{glue}{\simeq} \mathrm{Sk}_{\mathcal{I}}(M; X, \emptyset)$$

as in Theorem 3.5. The transported skein $i_* T$ is indeed admissible as i is surjective on connected components. We think of this as "extending T by the empty skein in M".

Note that if $\mathcal{I} = \mathcal{A}$, then this is defined for every M.

Theorem 5.4. *There exists a (non compact if $\mathcal{I} \neq \mathcal{A}$) boundary condition*

$$\partial_{\mathcal{I}} : \mathrm{Triv} \Rightarrow \underline{\mathrm{Sk}}_{\mathcal{I}}$$

with $\partial_{\mathcal{I}}(\Sigma)$ and $\partial_{\mathcal{I}}(M)$ as defined above.

Proof. The symmetric monoidal structure is the identity, which simplifies a lot the verification. We simply have to check naturality with respect to diffeomorphisms and composition.

Let $f : M \to M'$ be a diffeomorphism preserving the collars, then indeed $f_*(i_* T) = i_* T$ as $i_* T$ is concentrated near the collars.

Let $\Sigma_3 \overset{M_{32}}{\to} \Sigma_2 \overset{M_{21}}{\to} \Sigma_1$ be composeable, and take $T \in \mathrm{Sk}_{\mathcal{I}}(\Sigma_3 \times [0,1]; X, \emptyset)$. Then $\partial(M_{32})(T)$ is $(i_3)_* T$, which we may isotope to meet $\Sigma_2 \times I \subseteq M_{23}$ and write as $T' \cup T''$ for some $T' \in \mathrm{Sk}_{\mathcal{I}}(\Sigma_2 \times [0,1]; Y, \emptyset), T'' \in \mathrm{Sk}_{\mathcal{I}}(M; X, Y)$. Then

$$(\partial_{\mathcal{I}}(M_{21}) \circ_h \mathrm{id}_{M_{32}}) \circ \partial_{\mathcal{I}}(M_{32})(T) = (i_2)_* T' \cup T'' \in \mathrm{Sk}_{\mathcal{I}}(M_{21} \circ M_{32}; X, \emptyset)$$

is equal to $i_* \circ (i_3)_*(T)$, where $i : M_{32} \hookrightarrow M_{21} \circ M_{32}$ is the canonical inclusion, and hence to $\partial_{\mathcal{I}}(M_{21} \circ M_{32})(T)$. ☐

6 Non-semisimple WRT at the boundary of Crane–Yetter

In this section we will explain how to obtain Witten–Reshetikhin–Turaev 3-TQFTs, and their non-semisimple generalizations [DGG+22], from the once-extended 4-TQFT $\mathcal{Z}_{\mathcal{I}}$ defined in Section 4 and its boundary condition $\partial_{\mathcal{I}}$ defined in Section 5.

We assume in this section that $\mathcal{A}$ is a modular category in the sense of [Lyu95, DGG+22] (which include semisimple modular categories in the sense of [Tur94] but allow non-semisimple examples), and that $\mathcal{I} \subseteq \mathcal{A}$ is the ideal of projective objects, so $\mathcal{I} = \mathcal{A}$ if and only if $\mathcal{A}$ is semisimple.

We begin by explaining how to obtain an anomalous theory out of the data $\mathcal{Z}, \partial$. This is a fairly general and known construction.

Definition 6.1. The **category of filled $(n+1)$-cobordisms** $\mathrm{Cob}_{n+1}^{filled}$ has:

Objects: Closed n-manifolds Σ equipped with a bounding $(n+1)$-manifold $H : \emptyset \to \Sigma$

Morphisms: Cobordisms $M : \Sigma \to \Sigma'$ equipped with a bounding $(n+2)$-manifold $\begin{array}{ccc} \emptyset & \xrightarrow{H} & \Sigma \\ \| & \overset{W}{\swarrow\!\!\swarrow} & \downarrow M \\ \emptyset & \xrightarrow{H'} & \Sigma' \end{array}$

Definition 6.2. Let $\mathcal{Z} : \mathbf{Cob}_{2+1+1} \to \mathcal{C}$ be a once-extended TQFT contravariant in the direction of 1-morphisms and $\partial : \mathrm{Triv} \to \mathcal{Z}^{\varepsilon}$ a (resp. non-compact) boundary condition to $\mathcal{Z}$.

The **anomalous theory**[2] $\mathscr{A}_{\mathcal{Z},\partial}$ associated to $\mathcal{Z}, \partial$ is the symmetric monoidal functor

$$\begin{array}{ccc} \mathscr{A}_{\mathcal{Z},\partial} : \mathrm{Cob}_{n+1}^{filled} & \to & \Omega\mathcal{C} := \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \\ \left[\emptyset \overset{H}{\to} \Sigma\right] & \mapsto & \left[\mathbb{1}_{\mathcal{C}} \overset{\partial(\Sigma)}{\longrightarrow} \mathcal{Z}(\Sigma) \overset{\mathcal{Z}(H)}{\longrightarrow} \mathbb{1}_{\mathcal{C}}\right] \\ \left[\begin{array}{ccc} \emptyset & \xrightarrow{H} & \Sigma \\ \| & \overset{W}{\swarrow\!\!\swarrow} & \downarrow M \\ \emptyset & \xrightarrow{H'} & \Sigma' \end{array}\right] & \mapsto & \left[\begin{array}{ccccc} \mathbb{1}_{\mathcal{C}} & \xrightarrow{\partial(\Sigma)} & \mathcal{Z}(\Sigma) & \xrightarrow{\mathcal{Z}(H)} & \emptyset \\ \| & \overset{\partial(M)}{\searrow\!\!\searrow} & \uparrow \mathcal{Z}(M) & \overset{\mathcal{Z}(W)}{\searrow\!\!\searrow} & \| \\ \mathbb{1}_{\mathcal{C}} & \xrightarrow{\partial(\Sigma')} & \mathcal{Z}(\Sigma') & \xrightarrow{\mathcal{Z}(H')} & \emptyset \end{array}\right] \end{array}$$

[2]This terminology is maybe only appropriate when $\mathcal{Z}$ is an invertible theory, which will be the case in our example.

If ∂ is non-compact, then $\mathscr{A}_{\mathcal{Z},\partial}$ is also non-compact, i.e. it is only defined on the category $\mathrm{Cob}_{2+1}^{filled,nc}$ of cobordisms with non-empty incoming boundary in every connected component.

The category of filled cobordisms has been suggested by Walker as the right source category for WRT theories. It is more standard to consider a smaller category where we have forgotten part of the data of the filling.

Definition 6.3. The category $\widetilde{\mathrm{Cob}}_{2+1}$ has objects surfaces equipped with a Lagrangian $L \subseteq H_1(\Sigma)$ and morphisms 3-cobordisms equipped with an integer $n \in \mathbb{Z}$. Composition is given by composing the cobordisms, adding the integers and adding a Maslov index of the three Lagrangians involved as defined in [Wal91, Section 2]. The category $\widetilde{\mathrm{Cob}}_{2+1}^{nc}$ is the subcategory where 3-cobordisms must have incoming boundary in every connected component.

The projection

$$\pi : \mathrm{Cob}_{2+1}^{filled} \to \widetilde{\mathrm{Cob}}_{2+1}$$

takes a pair (Σ, H) to $(\Sigma, \mathrm{Ker}(i_* : H_1(\Sigma \to H_1(H)))$ and a pair (M, W) to $(M, \sigma(W))$. Composition in $\widetilde{\mathrm{Cob}}_{2+1}$ is defined precisely to make this assignment preserve composition, using Wall's non-additivity theorem.

The TQFT

$$\mathrm{WRT}_{\mathcal{A}} : \widetilde{\mathrm{Cob}}_{2+1} \to \mathrm{Vect}$$

is defined in [Tur94, Wal91] for any semisimple modular category $\mathcal{A}$ with a chosen square root $\mathcal{D}$ of its global dimension. The TQFT

$$\mathrm{DGGPR}_{\mathcal{A}} : \widetilde{\mathrm{Cob}}_{2+1}^{nc} \to \mathrm{Vect}$$

is defined in [DGG+22] for any non-semisimple modular category $\mathcal{A}$ with a chosen modified trace t and square root $\mathcal{D}$ of its global dimension (called modularity parameter in [DGG+22]). In this paper, we will not consider decorations by ribbon graphs living in the 3-cobordisms.

We have seen in Section 4 that given such a modular category $\mathcal{A}$, we have a scalar choice of ways to extend $\underline{\mathrm{Sk}}_{\mathcal{I}}$, with $\mathcal{I} = \mathrm{Proj}(\mathcal{A})$, into a once-extended theory $\mathcal{Z}_{\mathcal{I}}$, corresponding to a choice of modified trace. If a choice of modified trace and square root of the global dimension has been made as above, we define $\mathcal{Z}_{\mathcal{I}}$ to be the theory obtained by using the modified trace $\mathcal{D}^{-1}\,\mathsf{t}$. We denote $\mathscr{A}_{\mathcal{I}} : \mathrm{Cob}_{2+1}^{filled,nc} \to \mathrm{Vect} \simeq \Omega\,\mathrm{Bimod}$ the anomalous theory associated to $\mathcal{Z}_{\mathcal{I}}, \partial_{\mathcal{I}}$, which is non-compact when $\mathcal{I} \neq \mathcal{A}$.

Theorem 6.4. *Let $\mathcal{A}$ be a semisimple modular tensor category and $\mathcal{I} = \mathcal{A}$, then*

$$\begin{array}{ccc} \mathrm{Cob}_{2+1}^{filled} & \xrightarrow{\ \mathscr{A}_{\mathcal{A}}\ } & \mathrm{Vect} \\ & {\scriptstyle \pi}\searrow \quad \nearrow{\scriptstyle \mathrm{WRT}_{\mathcal{A}}} & \\ & \widetilde{\mathrm{Cob}}_{2+1} & \end{array}$$

commutes up to symmetric monoidal natural isomorphism.

Theorem 6.5. *Let $\mathcal{A}$ be a non-semisimple modular tensor category and $\mathcal{I} = \mathrm{Proj}(\mathcal{A})$, then*

$$\begin{array}{ccc} \mathrm{Cob}_{2+1}^{filled,nc} & \xrightarrow{\ \mathscr{A}_{\mathcal{I}}\ } & \mathrm{Vect} \\ & {\scriptstyle \pi}\searrow \quad \nearrow{\scriptstyle \mathrm{DGGPR}_{\mathcal{A}}} & \\ & \widetilde{\mathrm{Cob}}_{2+1}^{nc} & \end{array}$$

commutes up to symmetric monoidal natural isomorphism.

Proof. Given $\emptyset \xrightarrow{H} \Sigma$, we need to give a natural isomorphism

$$\eta_{\Sigma,H} : \mathscr{A}_{\mathcal{I}}(\Sigma) \tilde{\to} \mathrm{DGGPR}_{\mathcal{A}}(\Sigma) \ .$$

On the one hand, we have

$$\mathscr{A}_{\mathcal{I}}(\Sigma) := \int^{X \in \mathrm{SkCat}_{\mathcal{I}}(\Sigma)} \mathrm{Sk}_{\mathcal{I}}(\Sigma \times [0,1]; X, \emptyset) \otimes \mathrm{Sk}_{\mathcal{I}}(H; X) \overset{glue}{\simeq} \mathrm{Sk}_{\mathcal{I}}(H; \emptyset)$$

is the admissible skein module of H with empty boundary points.

On the other hand, the state spaces of DGGPR are defined via the universal construction of [BHMV95], i.e. as a quotient of the vector space generated by all 3-manifolds N bounding Σ equipped with an admissible $\mathcal{I}$-colored ribbon graph T. Let us denote $[N,T] \in \mathrm{DGGPR}(\Sigma)$ the induced vector. The quotient asks the relation $\sum_i [N_i, T_i] = 0$ if for every $N', T_{N'}$ of boundary $-\Sigma$, the invariant of closed 3-manifold $\sum_i \mathrm{DGGPR}(N_i \underset{\Sigma}{\cup} N', T_i \sqcup T', n_i) = 0$, where n_i is a Maslov index computed in the composition of N' and N_i.

The map $\eta_{\Sigma,H}$ is the canonical map to the quotient. It is shown to be well-defined in [DGG+22, Prop. 4.11] (and is called π there), and surjective when H is connected. We defer to Lemma 6.6 the proof that it is an isomorphism.

The symmetric monoidal structure of DGGPR is given by taking disjoint union on these generators [DGG+22, Prop. 4.8], hence $\eta_{\Sigma,H}$ is symmetric monoidal.

We are left with the core of the proof: checking that $\eta_{\Sigma,H}$ is natural. Let $M : \Sigma \to \Sigma'$ be a 3-cobordism between filled surfaces equipped with a bounding 4-manifold $W : H \underset{\Sigma}{\cup} M \Rightarrow H'$.

The action of M, W on a vector $[H,T] \in \mathrm{DGGPR}(\Sigma)$ is given by

$$\mathrm{DGGPR}(M, \sigma(W))([H,T]) = \mathcal{Z}(\mathbb{C}P_2)^{\sigma(W)}[H \underset{\Sigma}{\cup} M, T]\ .$$

This is now a skein living in the gluing of M and H, and not in H' as we would hope. We would like to relate it to a element of the form $[H', T']$. This is asking: what is the skein $T' \subseteq H'$ so that the invariants

$$\mathcal{Z}(\mathbb{C}P_2)^{\sigma(W)+n-n'} \mathrm{DGGPR}(H \underset{\Sigma}{\cup} M \underset{\Sigma'}{\cup} N', T \cup T_{N'}) \overset{?}{=} \mathrm{DGGPR}(H' \underset{\Sigma'}{\cup} N', T' \cup T_{N'})$$

match for any $N', T_{N'}$, where n and n' are Maslov indices.

By Proposition 4.3, each of these scalars is given by bounding the 3-manifold by a 4-manifold, and evaluation it under the TQFT $\mathcal{Z}$. Let $W' : H' \underset{\Sigma'}{\cup} N' \to \emptyset$ be a bounding 4-manifold. Then $W' \circ (W \circ_h \mathrm{id}_{N'})$ is a bounding manifold for $H \underset{\Sigma}{\cup} M \underset{\Sigma'}{\cup} N'$. By construction, the skein $\mathcal{Z}(W)(T) \subseteq H'$ satisfies $\mathcal{Z}(W')(\mathcal{Z}(W)(T) \cup T_{N'}) = \mathcal{Z}(W' \circ (W \circ_h \mathrm{id}_{N'}))(T \cup T_{N'})$, i.e. precisely

$$\mathcal{Z}(\mathbb{C}P_2)^{\sigma(W)}[H \underset{\Sigma}{\cup} M, T] = [H', \mathcal{Z}(W)(T)]$$

as the integer $\sigma(W) + n - n'$ computes the difference of signature between W' and $W' \circ (W \circ_h \mathrm{id}_{N'})$. This concludes naturality. □

Lemma 6.6. *The natural transformation η is a natural isomorphism.*

Let us first describe explicitly the admissible skein modules of handle bodies.

Lemma 6.7. *Let $\mathcal{I} \subseteq \mathcal{A}$ be a tensor ideal in a ribbon category and H a genus-g handlebody. Then there is a vector space isomorphism*

$$\mathrm{Sk}_{\mathcal{I}}(H) \simeq \Big(\bigoplus_{(P_i)_i \in \mathcal{I}^g} \mathrm{Hom}_{\mathcal{A}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1}) \Big) / \langle (f, \mathrm{id}) \sim (\mathrm{id}, f^*), f : P_i \to P_i' \rangle$$

where for $f : P_i \to P_i'$ and $\psi : P_1 \otimes P_1^* \otimes \cdots \otimes P_i' \otimes P_i^* \otimes \cdots \otimes P_g \otimes P_g^* \to \mathbb{1}$, the relation $(f, \mathrm{id}) \sim (\mathrm{id}, f^*)$ denotes the usual "coend" relation

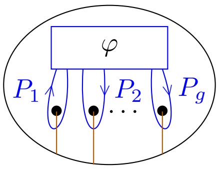
$$\psi \circ (\mathrm{id} \otimes \cdots \otimes f \otimes \mathrm{id}_{P_i^*} \otimes \cdots \otimes \mathrm{id}) \sim \psi \circ (\mathrm{id} \otimes \cdots \otimes \mathrm{id}_{P_i} \otimes f^* \otimes \cdots \otimes \mathrm{id})\ .$$

Proof. The handlebody H_g is obtained form a ball B^3 by gluing g pairs of disks.

By Theorem 3.5, the admissible skein module $\mathrm{Sk}_{\mathcal{I}}(H_g)$ is obtained as the coend

$$\mathrm{Sk}_{\mathcal{I}}(H_g) \simeq \int^{P_1,\dots,P_g\in\mathcal{I}} \mathrm{Sk}_{\mathcal{I}}(B^3; P_1, P_1^*, \dots, P_g, P_g^*)$$

which, as $\mathrm{Sk}_{\mathcal{I}}(B^3; X) \simeq \mathrm{Hom}_{\mathcal{A}}(X, \mathbb{1})$, is indeed the RHS above.

A morphism φ in the RHS maps to the skein 

□

Proof of Lemma 6.6. Let us first suppose that Σ is connected and $H : \emptyset \to \Sigma$ is a handle body. It is shown in [DGG+22, Prop. 4.11] that $\eta_{\Sigma,H}$ is surjective. We will prove that it is an isomorphism by showing that $\mathrm{Sk}_{\mathcal{I}}(H)$ and $\mathrm{DGGPR}_{\mathcal{I}}(\Sigma)$ have the same dimension.

The description of Lemma 6.7 can be related to the coend description of state spaces of [DGG+22, Sec. 4.1]. Remember that the coend $\mathcal{L}$ is defined as the colimit

$$\mathcal{L} = \int^{X\in\mathcal{A}} X\otimes X^* = \big(\oplus_{X\in\mathcal{A}} X\otimes X^*\big)/\langle (f,\mathrm{id}) \sim (\mathrm{id}, f^*), f : X \to Y\rangle$$

We only consider projectives in our case, but by [KL01, Proposition 5.1.7] this does not change this colimit and $\mathcal{L} \simeq \int^{P\in\mathcal{I}} P\otimes P^*$. Note that by [KL01, Corollary 5.1.8], the infinite nature of this colimit is unnecessary, and we could allow only $P = G$ the projective generator. We will still denote it $\int^{P\in\mathcal{I}}$, but it will be useful to remember that everything is finite.
It is shown in [DGG+22, Proposition 4.17 and Lemma 4.1 at $V = \mathbb{1}$] that

$$\mathrm{DGGPR}_{\mathcal{I}}(\Sigma) \simeq \big(\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}^{\otimes g}, \mathbb{1})\big)^*$$

where g is the genus of Σ. Using the definition of the colimit, the vector space $\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}^{\otimes g}, \mathbb{1})$ is obtained as a limit: the subspace of the product $\Pi\, \mathrm{Hom}_{\mathcal{A}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1})$ of the collections that satisfy the $(f,\mathrm{id}) \sim (\mathrm{id}, f^*)$ relations. The dual of this limit is then (using the fact everything is finite) the colimit

$$\mathrm{DGGPR}_{\mathcal{I}}(\Sigma) \simeq \Big(\bigoplus_{(P_i)_i\in\mathcal{I}^g} \mathrm{Hom}_{\mathcal{A}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1})^*\Big)/\langle (f,\mathrm{id}) \sim (\mathrm{id}, f^*), f : P_i \to P_i'\rangle$$

This is almost the same as the formula we gave for $\mathrm{Sk}_{\mathcal{I}}(H)$, though there are duals. We have an isomorphism $\mathrm{Hom}_{\mathcal{A}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1})^* \simeq \mathrm{Hom}_{\mathcal{A}}(P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*, \mathbb{1})$ given by the modified trace paring, and noticing that by design $P_1 \otimes P_1^* \otimes \cdots \otimes P_g \otimes P_g^*$ is self-dual up to isomorphism. These isomorphisms preserve the $(f,\mathrm{id}) \sim (\mathrm{id}, f^*)$ relations, and induce an isomorphism on the quotient. Hence $\mathrm{Sk}_{\mathcal{I}}(H)$ and $\mathrm{DGGPR}_{\mathcal{I}}(\Sigma)$ have the same dimension, and $\eta_{\Sigma,H}$ is an isomorphism when H is a handle body.

If H is a disjoint union of handle bodies, then $\eta_{\Sigma,H}$ is still an isomorphism by monoidality.

Now, consider a general bounding 3-manifold $M : \emptyset \to \Sigma$. Denote $H : \emptyset \to \Sigma$ a disjoint union of handle bodies. Any two 3-manifold with same boundary are related by a 4-cobordisms $W : M \Rightarrow H$. It can be thought of as a morphism (id_Σ, W) in $\mathrm{Cob}_{2+1}^{filled}$ where the 3-cobordism part is the identity. It induces a map $\mathcal{Z}(W) \circ_h \mathrm{id}_{\partial(\Sigma)} : \mathrm{Sk}_{\mathcal{I}}(M, \emptyset) \to \mathrm{Sk}_{\mathcal{I}}(H, \emptyset)$ which is an isomorphism because $\mathcal{Z}$ is invertible, and an isomorphism $\mathcal{Z}(\mathbb{C}P_2)^{\sigma(W)}\mathrm{id} : \mathrm{DGGPR}_{\mathcal{I}}(\Sigma) \to \mathrm{DGGPR}_{\mathcal{I}}(\Sigma)$. Naturality of η implies that $\eta_{\Sigma,M}$ and $\eta_{\Sigma,H}$ are related by these isomorphisms, hence $\eta_{\Sigma,M}$ is an isomorphism. □

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