STOAT 2 Syllabus

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Contents

Background

Students should do some background reading (if they are not already familiar) on the following topics, the first three of which will be reviewed on the first day.

- Khovanov homology [BN02].
- Bar-Natan Cob category [BN05].
- Lee homology and Rasmussen s invariant [Ras10].
- Basics of 4-manifolds X including handle decompositions (cores and cocores of handles), surfaces (including the intersection form on $H_2(X)$), and the long exact sequence in homology for a pair X and link $L \subset \partial X$.

1 Day 1: Link homology review

1.1 Khovanov homology review

- Bar-Natan category of cobordisms [BN05]
- Khovanov homology definition, quantum grading, link cobordism maps ; see[BN02] and [BN05] for Reidemeister maps
- Functoriality in $\mathbb{R}^3$ up to sign (movie moves) [Jac04, Theorem 2] no need for details on the proof, but indicate what needed to be done and the sign discrepancy

1.2 $\mathfrak{gl}_2$ homology review

Good general reviews are found in [QR16] and [ETW18]; see also [Bla10].

- Foam category for $\mathfrak{gl}_2$
- $\mathfrak{gl}_2$ homology definition and relationship with Khovanov homology (including cobordism maps) [RW24, Section 2.3]
- Functoriality in $\mathbb{R}^3$, no sign ambiguity (summarize only; see [Bla10, Section 5] and/or [ETW18, Section 4] for some ideas on what happened)

1.3 Lee homology review

- Lee homology definition, quantum filtration [Ras10]
- Lee generators and cobordism maps [Ras10]
- Rasmussen invariant, genus bounds, and easy formula for positive links [Ras10]
- The $\mathfrak{gl}_2$ versions of all of these can be summarized as in [RW24, Section 2.3]

2 Day 2: S^3 functoriality and skein lasagna invariants

2.1 The sweeparound move

- Functoriality in S^3 via proving the sweeparound move [MWW22, Section 3]
- Well-definedness of a cobordism map when two or more 4-balls are contained in a single 4-ball (again via sweeparound move; see end of proof of [MWW22, Theorem 5.2])

2.2 Skein lasagna invariants

- Category of skeins, lasagna fillings [RW24, Section 2.1]
- Defining the skein lasagna modules for $\mathfrak{gl}_2$ Khovanov and Lee homology [RW24, Section 2.1], see also [MWW22, Sections 4.2, 5.1, and 5.2]
- Homology class grading [RW24, beginning of Section 2.2]

2.3 Formal properties

- Disjoint unions, connected sums, boundary connected sums [MN22, Theorem 1.4, Corollary 7.3]
- General gluing formula [RW24, Proposition 2.3] and cobordism maps [RW24, Equations 7 and 15]
- Recovery of original theory for B^4 [RW24, Proposition 2.4], see also [MWW22, Example 5.6]

3 Day 3: 2-handles and vanishing results

3.1 Handle attachments

- 4-handles and 3-handles [MN22, Proposition 2.1]
- 2-handle formula with some proof details [RW24, Proposition 2.6] via [MN22, Proposition 3.8]; see also [SZ24, Figure 1, Section 4.2]

3.2 Vanishing result

- Combinatorial definition of TB and max TB from link diagrams ([Ng05, Section 2] will do)
- Ng's bound presented as [RW24, Lemma 3.1] via [Ng05, Theorem 1, Proposition 5] (emphasize purely combinatorial proof of Proposition 5, no geometry involved)
- Vanishing lasagna skein modules [RW24, Theorem 1.4]

3.3 Corollaries of vanishing result

- Unknot traces are embedded spheres; $S^2 \times S^2$ [RW24, Example 3.2] and the necessity for detecting exotica (Wall's stabilization theorem [Wal64])
- Trace embedding lemma and corresponding lasagna statement [RW24, Corollary 1.5]
- Rank inequality for Kh vs Lee and corresponding statement about filtration levels for "vanishing" lasagna [RW24, Theorem 1.12, Corollaries 5.1,5.2]

4 Day 4: Lee skein modules and Rasmussen invariants

4.1 Lee structure theorem

- Double skeins and canonical Lee lasagna fillings [RW24, Section 4.1]
- Lee lasagna modules [RW24, Theorem 4.1, Lemmas 4.2 and 4.3] (it's okay to focus on having no link in the boundary and skip the quantum filtration information)
- The lasagna Rasmussen invariant [RW24, Definition 4.6] (it's okay to ignore the double-class version)

4.2 Properties of lasagna Rasmussen invariants

- Basic properties [RW24, Theorem 4.7 (0)-(4)], without proof if necessary
- Gluing [RW24, Theorem 4.5] with proof sketch (okay to ignore the coefficients if necessary) and genus bounds [RW24, Theorem 4.7 (6), Proposition 1.11, and Corollary 4.13]
- Knot concordance and 2-handlebodies [RW24, Proposition 1.13] (see [RW24, Proposition 5.3], focusing on the case $L = L' = A = \varnothing$)

4.3 Lasagna Rasmussen invariants for 2-handles

- Lee homology of an n-cable as a S_n-representation [Ren23, Section 4, particularly Proposition 4.2], with certain filtration levels matching [Ren23, Section 4.3] (no proof necessary)
- Computing the lasagna Rasmussen invariant via usual Rasmussen invariants of cables [RW24, Proposition 5.4] with proof sketch

5 Day 5: Nonvanishing skein modules

5.1 Diagrammatic non-vanishing result

The non-vanishing result [RW24, Theorem 1.14]. This is [RW24, Theorem 5.5] for a single 2-handle so that $\alpha, P, W, N \in \mathbb{Z}$ (and $m = |K| = 1$). We will not need the full generality of Theorem 5.5. A proof for Theorem 1.14 can be sketched via the proof of Theorem 5.5 in the single 2-handle case with $\alpha = 0$ or $\alpha = 1$; see the list below.

- No need to present Lemmas 5.8 in full, but do indicate the last claim about the dotted annular creation map being injective
- Perhaps mention Lemma 5.9 as a simple way to bound extremal homological gradings
- The first portion of Step 1 of the proof (page 34) can be condensed into the claim (without proof) that, given the assumptions, the maximal homological grading can be computed as indicated and, in that maximal homological grading, the rank of Khovanov homology of the given cable is bounded above by the given sum.
- The second portion of Step 1 (top of page 35) should be sketched
- Step 2 can be skipped if necessary
- Step 3 should be sketched

If time allows, can discuss shake genus [RW24, Theorem 1.9(1)] as an immediate corollary.

5.2 Exotica and $\overline{\mathbb{CP}^2}$ (some basic knowledge about $\overline{\mathbb{CP}^2}$ might be helpful for these last two lectures)

- Akbulut's example [RW24, Theorem 1.1]. Can say just a few words about X_1 and X_2 being homeomorphic; see also [Akb91b, Akb91a]
- More exotica via connected sums [RW24, Corollary 1.2, Section 6.3], okay to ignore $E8$
- Lasagna Rasmussen invariants for $\overline{\mathbb{CP}^2}$ [RW24, Lemma 6.12 and Proposition 1.15]; no need to provide proof of Lemma 6.12.
- Embeddings in $k\overline{\mathbb{CP}^2}$ [RW24, Corollary 1.16]

5.3 Further examples and new exotica (some basic knowledge about $\overline{\mathbb{CP}^2}$ might be helpful for these last two lectures)

- The adjunction inequality for s [RW24, Equation 31] (see [Ren23, Corollary 1.5], but no need to prove) and the corresponding lasagna statement [RW24, Proposition 6.8]
- Basic examples [RW24, Section 6.1]; these can be skipped if time is short
- New exotica from Yasui's sattelites [RW24, Theorem 1.3] and negative torus knots [RW24, Lemma 6.10]; see in general [RW24, Section 6.6]

References

[Akb91a] Selman Akbulut. An exotic 4-manifold. *J. Differential Geom.*, 33(2):357–361, 1991.

[Akb91b] Selman Akbulut. A fake compact contractible 4-manifold. *J. Differential Geom.*, 33(2):335–356, 1991.

[Bla10] Christian Blanchet. An oriented model for Khovanov homology. *J. Knot Theory Ramifications*, 19(2):291–312, 2010.

[BN02] Dror Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Algebr. Geom. Topol.*, 2:337–370, 2002.

[BN05] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499, 2005.

[ETW18] Michael Ehrig, Daniel Tubbenhauer, and Paul Wedrich. Functoriality of colored link homologies. *Proc. Lond. Math. Soc. (3)*, 117(5):996–1040, 2018.

[Jac04] Magnus Jacobsson. An invariant of link cobordisms from khovanov homology. *Algebraic & Geometric Topology*, 4(2):1211–1251, 2004.

[MN22] Ciprian Manolescu and Ikshu Neithalath. Skein lasagna modules for 2-handlebodies. *J. Reine Angew. Math.*, 788:37–76, 2022.

[MWW22] Scott Morrison, Kevin Walker, and Paul Wedrich. Invariants of 4-manifolds from Khovanov-Rozansky link homology. *Geom. Topol.*, 26(8):3367–3420, 2022.

[Ng05] Lenhard Ng. A legendrian thurston–bennequin bound from khovanov homology. *Algebr. Geom. Topol.*, 5(4):1637–1653, 2005.

[QR16] Hoel Queffelec and David EV Rose. The sln foam 2-category: a combinatorial formulation of khovanov–rozansky homology via categorical skew howe duality. *Advances in Mathematics*, 302:1251–1339, 2016.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[Ren23] Qiuyu Ren. Lee filtration structure of torus links. *arXiv preprint arXiv:2305.16089*, 2023.

[RW24] Qiuyu Ren and Michael Willis. Khovanov homology and exotic 4-manifolds. *arXiv preprint arXiv:2402.10452*, 2024.

[SZ24] Ian A Sullivan and Melissa Zhang. Kirby belts, categorified projectors, and the skein lasagna module of $s^2 \times s^2$. *arXiv preprint arXiv:2402.01081*, 2024.

[Wal64] C Terence C Wall. On simply-connected 4-manifolds. *Journal of the London Mathematical Society*, 1(1):141–149, 1964.