Graph Homotopy and Graham Homotopy

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Abstract

Simple-homotopy for simplicial and CW complexes is a special kind of topological homotopy constructed by elementary collapses and expansions. In this paper we introduce graph homotopy for graphs and Graham homotopy for hypergraphs, and study the relation between these homotopies and the simple-homotopy for simplicial complexes. The graph homotopy is useful to describe topological properties of discretized geometric figures, while the Graham homotopy is essential to characterize acyclic hypergraphs and acyclic relational database schemes.

1 Introduction

Contractible transformations on graphs were introduced in [4] and [5] to study molecular spaces. The simplest model of a so called molecular space is a family of unit cubes in a Euclidean space. These models are useful in digital topology for image processing and computer graphics. Therefore the study of combinatorial, topological, and geometric properties of molecular spaces should be useful to understand

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properties of molecular objects. It has been illustrated that for a continuos surface the induced intersection graph from the molecular space usually preserves essential topological properties of the original surface. For example, given two topologically equivalent surfaces Σ_1 and Σ_2 of $\mathbf{R}^3$, if $\mathbf{R}^3$ is divided into a set of congruent cubes by coordinate hyperplanes, we have one family L_1 of cubes intersecting Σ_1 and another family L_2 of cubes intersecting Σ_2. If the division is refined enough, the intersection graphs $G(L_1)$ and $G(L_2)$ will be graph homotopy equivalent, that is, $G(L_1)$ can be transformed into $G(L_2)$ by a series of contractible graph transformations (see below). This means that some topological properties of the continuous surfaces have been encoded into the molecular spaces. Thus it should be interesting and important to find out what kinds of topological properties are preserved by the induced molecular spaces. In a series of papers, Ivashchenko and Yeh studied some of these preserved properties such as the Euler characteristic and homology groups, see [4-8] and [16]. They showed that graph transformations do not change the Euler characteristic and the homology group of graphs for some special cases. This leads us to ask whether the graph transformations are actually topological homotopy equivalence. The first half of this paper is to show that a contractible graph transformation can be decomposed into a series of compositions of elementary CW expansions and collapses of certain associated simplicial complexes, whereby graph homotopy is reduced to a special kind of simple-homotopy. Since simple-homotopy equivalence preserves "homotopy groups," and of course preserves "homology groups," all results of [4] and [5] are consequences of the present paper.

The graph homotopy and simple-homotopy are closely related to the Graham reduction for hypergraphs, which was origionally introduced to define acyclic hypergraphs and acyclic database schemes, see [12]. The importance of acyclic database schemes lies in the existence of their information-lossless decomposition, see [9] and [10]. Recall that the second operation in the Graham reduction for hypergraphs is to remove the hyperedges contained in another hyperedge. For this reason we associate a simplicial complex to each hypergraph by including all non-empty subsets of any hyperedge. With this association the first operation in the Graham reduction corresponds to a special kind of homotopy on the associated simplicial complexes, and we call this Graham homotopy. The purpose of the second half of the present paper is to give a topological interpretation for the Graham reduction and to derive a formula for counting the number of cycles in terms of the associated simplicial complexes.

The acyclic hypergraphs and relational database schemes were introduced easily by the Graham reduction. However, the concepts of cycles and independent cycles for hypergraphs and relational database schemes are still missing. We suspect that for some combinatorial optimization problem, it is the number of cycles of certain associated hypergraphs that determine the computational complexity. For instance, the satifiability problem may be transformed into a problem on an induced hypergraph and its computational complexity will be reduced to the cycle structures of the induced hypergraph. The detailed exposition of cycle structures for hypergraphs will be given elsewhere.

2 Graph Homotopy and Simple-Homotopy

Let $G = (V, E)$ be a simple graph, that is, a graph without loops and multiple edges, where V is the vertex set and E is the edge set; we always assume that V is finite. For each vertex v of G, let $N(v, G)$ denote the set of vertices of G that are adjacent to v; the *graph link* of v in G, denoted $L(v, G)$, is the subgraph of G induced by the vertex set $N(v, G)$. For a subgraph H of G, the *joint graph link* of H in G, denoted $L(H, G)$, is the subgraph of G induced by the intersection vertex set

$$N(H, G) = \bigcap_{v \in V(H)} N(v, G).$$

Notice that the graph link we defined for graphs is different from the topological link defined for simplicial complexes in combinatorial topology. We now define contractible graphs inductively by gluing and deleting vertices and edges as follows: (1) the graph of a single vertex is *contractible*; (2) a graph is said to be *contractible* if it can be obtained from a contractible graph by a sequence of the following graph operations.

(GO1) Deleting a vertex: A vertex v of a graph G can be deleted if its graph link $L(v, G)$ is contractible;

(GO2) Gluing a vertex: If H is a contractible subgraph of G, then a vertex v not in G can be glued to G to produce a new graph G' so that the graph link $L(v, G')$ is H;

(GO3) Deleting an edge: An edge uv of G can be deleted if the joint graph link $L(uv, G) = L(u, G) \cap L(v, G)$ of u and v is contractible;

(GO4) Gluing an edge: For two non-adjacent vertices u and v of G, the edge uv can be glued to G if the joint graph link $L(uv, G)$ of u and v is contractible.

By this definition all complete graphs are contractible; it is shown that chordal graphs are contractible, see [16]. Two graphs G_1 and G_2 are said to be *graph homotopy equivalent* if G_1 can be obtained from G_2 by a series of graph operations (GO1-GO4). It is clear that the graph homotopy defines an equivalence relation on the class of graphs so that graphs are classified into graph homotopy classes. This classification of graphs may be related to the classification of topological spaces by certain topological transformations. To see this relationship we associate with each graph a simplicial complex.

Let us recall that an abstract *simplicial complex* over a finite set V is a collection K of non-empty subsets of V, called (*open*) *cells*, such that $V = \bigcup_{\sigma \in K} \sigma$, and for each $\sigma \in K$, if $\rho \subset \sigma$ and $\rho \neq \emptyset$, then $\rho \in K$. An open cell σ is called a *face* of an open cell τ if $\sigma \subset \tau$, denoted $\sigma \leq \tau$ or $\tau \geq \sigma$; σ is called a *facet* of τ if $\tau = \sigma \cup \{v\}$ for some $v \notin \sigma$. The geometric realization $|K|$ is the metric space of non-negative real-valued functions f on V such that there is a simplex σ so that $\sum_{v \in \sigma} f(v) = 1$ and $f(v) = 0$

for all $v \notin \sigma$; and the metric is given by

$$d(f,g) = \left[\sum_{v\in V}(f(v) - g(v))^2\right]^{\frac{1}{2}},$$

see [14]. A *simplicial map* from a simplicial complex K_1 over V_1 to a simplicial complex K_2 over V_2 is a map $f : V_1 \longrightarrow V_2$ such that if σ is a simplex of K_1, then $f(\sigma)$ is a simplex of K_2. A *graph homomorphism* from a graph G_1 to a graph G_2 is map $f : V(G_1) \longrightarrow V(G_2)$ such that for each edge uv of G_1, $f(u)f(v)$ is an edge of G_2. Given a graph G, let $\Delta(G)$ denote the collection of all complete subgraphs of G; it is clear that $\Delta(G)$ is a simplicial complex, called the *clique complex* of G, for complete subgraphs are also called *cliques* in graph theory.

Let us denote by $\mathcal{G}$ the category of graphs with graph homomorphisms and by $\mathcal{K}$ the category of simplicial complexes with simplicial maps. If $f : G_1 \longrightarrow G_2$ is a graph homomorphism, then for any complete subgraph K_i of i vertices in G_1, its image $f(K_i)$ is a complete subgraph in G_2. Then we have an induced simplicial map $\Delta_f : \Delta(G_1) \longrightarrow \Delta(G_2)$, given by $\Delta_f(K_i) = f(K_i)$.

Conversely, given a simplicial complex K. Let $\mathrm{sk}^i(K)$ denote the i-dimensional skeleton of K, i.e., $\mathrm{sk}^i(K) = \{\sigma \in K : \sigma \text{ has at most } i+1 \text{ elements}\}$, $i \geq 0$. The 0-skeleton together with the 1-skeleton give rise to a graph $\mathrm{sk}(K) = (\mathrm{sk}^0(K), \mathrm{sk}^1(K) - \mathrm{sk}^0(K))$, where the vertex set is the 0-skeleton and the edge set is the pure 1-skeleton. Then sk defines a functor $\mathrm{sk} : \mathcal{K} \longrightarrow \mathcal{G}$. For each graph G, it is clear that $\mathrm{sk}\Delta(G) = G$. Notice that in general we do not have $\Delta\mathrm{sk}(K) = K$ for every simplicial complex K. For instance, if K is the boundary of a tetrahedron, i.e., K consists of all non-empty subsets of V except V itself, then $\mathrm{sk}(K)$ is the complete graph K_4, so $\Delta(K_4)$ represents a solid tetrahedron, including the cell V. Of course, $\Delta\mathrm{sk}K = \Delta(K_4)$ is different from K. However, if we take the first barycentric subdivision $\mathrm{sd}K$ of K, it is easy to see that $\Delta\mathrm{sk}(\mathrm{sd}K) = \mathrm{sd}K$. We state this as the following proposition.

Proposition 2.1 *The map Δ is a functor from the category $\mathcal{G}$ of graphs to the category $\mathcal{K}$ of simplicial complexes, and the map* sk *is a functor from $\mathcal{K}$ to $\mathcal{G}$. Moreover, for any graph G and any simplicial complex K,* $\mathrm{sk}\Delta(G) = G$ *and* $\Delta\mathrm{sk}(\mathrm{sd}K) = \mathrm{sd}K$.

□

In order to give topological interpretation for graph homotopy, we need the concept of simple-homotopy for simplicial complexes and CW complexes. Let K be a simplicial complex. A *face pair* is an ordered pair (σ, τ) of open cells such that σ is a facet of τ. A face pair (σ, τ) is said to be *free* in K if both σ and τ are open cells of K and σ is not a face of any open cells of K other than τ. If L can be obtained from K by a series of the following elementary simplicial collapses and elementary simplicial expansions, we say that K is *simple-homotopy equivalent to* a simplicial complex L.

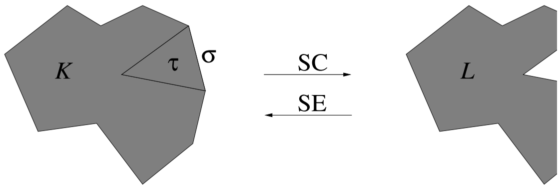

Figure 1: K collapses to L and L expands to K

(SC) Elementary simplicial collapse: A free face pair (σ, τ) of K can be deleted;

(SE) Elementary simplicial expansion: A face pair (σ, τ) can be added to K if σ and τ are not open cells of K, while all the faces of τ other than σ are open cells of K.

We say that K *collapses simplicially to* L or L *expands simplicially* to K, written $K \overset{SC}{\to} L$ or $L \overset{SE}{\to} K$, if L can be obtained from K by a sequence of only elementary simplicial collapses, see Figure 2. It is easy to see that every simplicial cone collapses to a point.

As pointed out by Whitehead [2], it is technically difficult to treat simple-homotopy in the context of simplicial complexes. To the author's understanding, in the category of simplical complexes one lacks flexibility to construct new cells without introducing new vertices. Thus Whitehead introduced CW complexes. A *CW complex* K is a Hausdorff space, which is divided into a collection of disjoint open cells $\{e_\alpha\}$ of various dimensions, such that the following conditions are satisfied.

(CW1) Each closed cell $\bar{e}_\alpha$ is a disjoint union of finitely many open cells e_β;

(CW2) For each open cell e_α, there is a continuous map $\phi_\alpha : Q^n \longrightarrow K$, where Q^n is homeomorphic to the standard closed unit ball B^n of dimension $n = \dim e_\alpha$, such that ϕ is a homeomorphism from $Q^n - \partial Q^n$ onto e_α and $\phi(\partial Q^n) \subset \bar{e}_\alpha - e_\alpha$, where ∂Q^n is homeomorphic to the standard unit sphere S^{n-1};

(CW3) A subset $A \subset K$ is closed if and only if $A \cap \bar{e}_\alpha$ is closed in $\bar{e}_\alpha$ for all e_α.

The maps ϕ_α are called *characteristic maps* of the cells e_α; the set $\mathrm{sk}^n(K) = \{e_\alpha : \dim e_\alpha \leq n\}$ is called the *n-skeleton* of K; and the geometric realization $|\mathrm{sk}^n(K)| = \bigcup_{\dim e_\alpha \leq n} e_\alpha$ is a closed subspace of K. We always assume that the number of cells in a CW complex is finite and the highest dimension of cells is called the *dimension* of the CW complex.

An ordered pair (e^{n-1}, e^n) of cells in a CW complex K is said to be *free* if $e^{n-1} \subset \partial e^n$ and there exists a ball pair (Q^{n-1}, Q^n) homeomorphic to (B^{n-1}, B^n), where $Q^{n-1} \subset \partial Q^n$, such that the restriction $\phi_e|_{Q^{n-1}}$ of the characteristic map $\phi_e : Q^n \longrightarrow K$ of e^n is the characteristic map of e^{n-1}. We say that K *collapses to a CW complex L by* *an elementary CW collapse* if K has a free pair (e^{n-1}, e^n) of cells and $L = K - e^{n-1} \cup e^n$. In this circumstance we also say that *L expands to K by an elementary CW expansion*. Two CW complexes are said to be *simple-homotopy equivalent* if one can be obtained from another by a series of elementary CW collapses and elementary CW expansions.

It is clear that simple-homotopy for simplicial complexes and CW complexes is a special kind of topological homotopy. In particular, elementary simplicial and CW collapses and expansions preserve topological invariants. In the following section we need regular CW complexes, regular CW expansions and regular CW collapses. A CW complex is called *regular* if every closed cell is homeomorphic to a closed simplex and its face ordering is isomorphic to the face ordering of the simplex. Expansions and collapses in regular CW complexes are called *regular CW expansions and collapses.*

3 Graph Homotopy Reduction

We shall show in this section that the graph homotopy can be reduced to simple-homotopy for regular CW complexes. To this end, we first show that the edge deletion and gluing are redundant in the following lemmas. This fact follows from a private communication with G. Chang [1].

Lemma 3.1 *For any graph G and a vertex v not in G, the* cone graph *$G * v$, where $V(G * v) = V(G) \cup \{v\}$ and $E(G * v) = E(G) \cup \{uv : u \in V(G)\}$, is contractible.*

Proof We show this by induction on the number of vertices of G. It is obviously true when G has only one vertex. Suppose it is true for any graph with k or less vertices. Now consider an arbitrary graph G with $k+1$ vertices. Let u be a vertex of G; the cone graphs $L(u, G) * v$ and $(G - u) * v$ are contractible by induction hypothesis. Notice that $L(u, G) * v$ is contained in $(G - u) * v$. Then $G * v$ can be obtained by gluing u to $L(u, G) * v$ in $(G - u) * v$. Thus $G * v$ is contractible by definition of vertex gluing. $\square$

Lemma 3.2 *The edge deletion (gluing) can be realized by the composition of a vertex deletion (gluing) and a vertex gluing (deletion).*

Proof Let G be a graph with vertices u and v adjacent. If the joint graph link $L(uv, G)$ is contractible, we need to find some vertex deletion and gluing to have the edge uv removed from G.

Let w be a vertex not in G. Since $(L(v, G) - u) * v$ is a contractible subgraph of G, we can glue w to $(L(v, G) - u) * v$ in G. Notice that both $L(uv, G)$ and $(L(v, G) - u) * w$ are contractible and $L(uv, G)$ is contained in $(L(v, G) - u) * w$. Then $L(uv, G) * u \bigcup (L(v, G) - u) * w$ is contractible, because it can be obtained by gluing u to $L(uv, G)$ in $(L(v, G) - u) * w$. It is clear that the graph link of v in $G \bigcup ((L(v, G) - u) * v) * w$ is $L(uv, G) * u \bigcup (L(v, G) - u) * w$, which has been shown to be contractible. Thus v can be removed by a vertex deletion. Rename the vertex

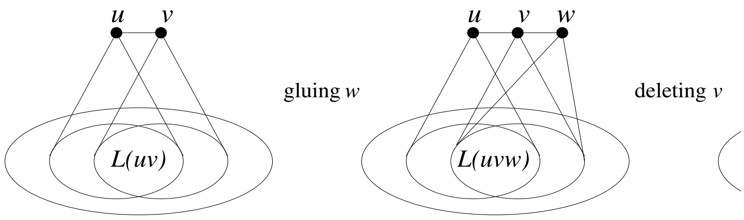

Figure 2: Edge deletion is realized by vertex gluing and vertex deletion

w as v. We have removed the only edge uv by a vertex gluing and a vertex deletion, see Figure 3.

The edge gluing is similar by reversing the procedure. $\square$

Theorem 3.3 *Let G be a graph and let H be a subgraph of G. If $\Delta(H)$ is regularly CW homotopy equivalent to a point, then, for a vertex v not in G, the simplicial complex $\Delta(G)$ is regularly CW homotopy equivalent to $\Delta(G \cup H * v)$.*

Proof To start with, let us assume that $\Delta(H)$ is constructed from a point $\{v_0\}$ of H by a series of regular CW expansions and collapses with free pairs (σ_i, τ_i) of cells, $1 \leq i \leq m$. We write

$$\{p\}(\sigma_1, \tau_1)^{t_1}(\sigma_2, \tau_2)^{t_2} \cdots (\sigma_m, \tau_m)^{t_m} = \Delta(H), \tag{1}$$

where $t_i = \pm 1$. For $(\sigma_i, \tau_i)^{\pm 1}$, the positive one $+1$ means that (σ_i, τ_i) is an expansion and the negative one -1 means that (σ_i, τ_i) is a collapse. The following algorithm tells us how to construct a series of regular CW expansions and collapses by which $\Delta(G \cup H * v)$ can be constructed from $\Delta(G)$. Let us run the expansions and collapses that correspond to the free pairs (σ_i, τ_i) of cells in getting $\Delta(H)$ from $\{v_0\}$ in (1).

The Reduction Algorithm

Step 1 For $\{v_0\}$, do expansion $(v, v_0 * v)$ to $\Delta(G)$.

Step 2 When applying the pair (σ_i, τ_i) in (1) with $t_i = +1$, if σ_i and τ_i are cells of $\Delta(G)$, do expansion $(\sigma_i * v, \tau_i * v)$; if σ_i and τ_i are not cells of $\Delta(G)$, do expansion (σ_i, τ_i) first and expansion $(\sigma_i * v, \tau_i * v)$ next.

Step 3 When applying the pair (σ_i, τ_i) in (1) with $t_i = -1$, if σ_i and τ_i are cells of $\Delta(G)$, do collapse $(\sigma_i * v, \tau_i * v)$; if σ_i and τ_i are not cells of $\Delta(G)$, do collapse $(\sigma_i * v, \tau_i * v)$ first and collapse (σ_i, τ_i) next.

For $i = 1, \cdots, m$, define

$$r_i = \begin{cases} +1 & \text{if } t_i = +1, \sigma_i \text{ and } \tau_i \text{ are not cells of } \Delta(G) \\ 0 & \text{otherwise;} \end{cases}$$

$$s_i = \begin{cases} -1 & \text{if } t_i = -1, \sigma_i \text{ and } \tau_i \text{ are not cells of } \Delta(G) \\ 0 & \text{otherwise.} \end{cases}$$

For a free pair (σ, τ) of cells, we understand that $(\sigma, \tau)^0 = 1$, meaning that (σ, τ) is neither an expansion nor a collapse, and so nothing will be changed when $(\sigma, \tau)^0$ is applied. The final CW complex constructed by the algorithm can be written as

$$\begin{aligned}\{v_0\}(v, v_0 * v)^{+1}(\sigma_1, \tau_1)^{r_1}(\sigma_1 * v, \tau_1 * v)^{t_1}(\sigma_1, \tau_1)^{s_1} \\ \cdots (\sigma_m, \tau_m)^{r_m}(\sigma_m * v, \tau_m * v)^{t_m}(\sigma_m, \tau_m)^{s_m}.\end{aligned} \tag{2}$$

We need to show that the algorithm actually works, that is, at each step, the expansions and collapses can be executed properly and the final CW complex (2) is the same as $\Delta(G \cup H * v)$.

In Step 1, since v is not a vertex of G, then $(v, v_0 * v)$ is a free pair of cells for $\Delta(G)$. Thus the expansion $(v, v_0 * v)^{+1}$ can be executed properly.

In Step 2, before we have applied expansion $(\sigma_i, \tau_i)^{+1}$ to

$$\{v_0\}(\sigma_1, \tau_1)^{t_1} \cdots (\sigma_{i-1}, \tau_{i-1})^{t_{i-1}} \tag{3}$$

in (1), by definition of free pair of cells all the facets ρ_i of τ_i other than σ_i must have been contained in (3), but σ_i and τ_i are not contained in (3). This means that in (3) the last operation related to ρ_i is an expansion, while the last operations related to either σ_i or τ_i, if exist, are collapses. It follows clearly that $\rho_i * v$ is contained in

$$\begin{aligned}\{v_0\}(v, v_0 * v)^{+1}(\sigma_1, \tau_1)^{r_1}(\sigma_1 * v, \tau_1 * v)^{t_1}(\sigma_1, \tau_1)^{s_1} \\ \cdots (\sigma_{i-1}, \tau_{i-1})^{r_{i-1}}(\sigma_{i-1} * v, \tau_{i-1} * v)^{t_{i-1}}(\sigma_{i-1}, \tau_{i-1})^{s_{i-1}},\end{aligned} \tag{4}$$

and $(\sigma_i * v, \tau_i * v)$ are not contained in (4). If σ_i and τ_i are cells of $\Delta(G)$, then $(\sigma_i * v, \tau_i * v)$ is already a free pair of cells to (4). If σ_i and τ_i are not cells of $\Delta(G)$, then (σ_i, τ_i) is a free pair of cells to (4). Thus we can do expansion (σ_i, τ_i) to (4) to have

$$\begin{aligned}\{v_0\}(v, v_0 * v)^{+1}(\sigma_1, \tau_1)^{r_1}(\sigma_1 * v, \tau_1 * v)^{t_1}(\sigma_1, \tau_1)^{s_1} \\ \cdots (\sigma_{i-1}, \tau_{i-1})^{r_{i-1}}(\sigma_{i-1} * v, \tau_{i-1} * v)^{t_{i-1}}(\sigma_{i-1}, \tau_{i-1})^{s_{i-1}}(\sigma_i, \tau_i)^{r_i}.\end{aligned} \tag{5}$$

Therefore $(\sigma_i * v, \tau_i * v)$ is a free pair of cells to (5).

The argument in Step 3 is similar. When (1) is running over, ending at $\Delta(H)$, the complexes $\Delta(G)$ and $\Delta(H * v)$ are contained in (2), and the cells not in $\Delta(G)$, if expanded at some time, must have been collapsed finally. So (2) is the same as $\Delta(G \cup H * v)$. $\square$

Corollary 3.4 *If G is a contractible graph, then $\Delta(G)$ is regularly CW homotopy equivalent to a point.*

Proof Let $T_i^{t_i}$ $(1 \le i \le n)$ denote the vertex gluing or vertex deletion such that

$$\{v\}T_1^{t_1} \cdots T_n^{t_n} = G,$$

where v is a vertex (may or may not be in G), $t_i = \pm 1$, $T_i^{t_i}$ is a vertex gluing for $t_i = +1$ and a vertex deletion for $t_i = -1$. Obviously, the single point $\Delta(\{v\})$ is simple-homotopy equivalent to a point. By Theorem 3.3, $\Delta(\{v\})$ is regularly CW homotopy equivalent to $\Delta(\{v\}T_1^{t_1})$; again, $\Delta(\{v\}T^{t_1})$ is regularly CW homotopy equivalent to $\Delta(\{v\}T_1^{t_1}T_2^{t_2})$; and so on. By transitivity, $\Delta(\{v\}T_1^{t_1}\cdots T_n^{t_n})$ is regularly CW homotopy equivalent to a point. □

The following theorem follows immediately from Theorem 3.3 and Corollary 3.4.

Theorem 3.5 *Let G be a graph and H a contractible subgraph. Then for a vertex v not in G, the simplicial complex $\Delta(G)$ is regularly CW homotopy equivalent to $\Delta(G \cup H * v)$.*

We have shown that homotopy groups, as well as homology groups, are invariant under graph transformations. Of course, the Euler characteristic is unchanged under graph transformations. Especially, the main results in [4-8] and [16] are consequences of Theorem 3.5.

4 Graham Homotopy

Hypergraphs are useful structures to study relational databases, see [12]. Acyclic hypergraphs, which are the extension of trees in graph theory, correspond to acyclic database schemes. The Graham reduction for defining acyclic hypergraphs can be viewed as a new type of combinatorial homotopy, which is much stronger than simplicial homotopy, CW homotopy, and graph homotopy. The acyclic database schemes came from the work of many people, see the bibliography and comments of [12], pp. 482-284. The database scheme problems were first formulated by Namibar [13] in terms of hypergraphs. The algorithm (Graham reduction) to test acyclicity for hypergraphs was first introduced by M. H. Graham [3]; Yu and Ozsoyoglu [17-18] also independently formulated the algorithm in terms of "join graphs."

A *hypergraph* $H = (V, \mathcal{E})$ consists of a finite set V, whose elements are called *vertices*, and a collection $\mathcal{E} = \{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ of non-empty subsets of V, called *hyperedges* or *cells*, such that $V = \bigcup_{i=1}^n \sigma_i$; H is said to be *reduced* if there are no hyperedges σ_i and σ_j such that $\sigma_i \subset \sigma_j$ and $\sigma_i \neq \sigma_j$. A vertex v of H is said to be *isolated* if $v \in \sigma_i$ for one i and $v \notin \sigma_j$ for all $j \neq i$. The Graham reduction for hypergraphs are the following operations.

(GR1) Deleting an isolated vertex;

(GR2) Deleting σ_i if $\sigma_i \subset \sigma_j$ for some $j \neq i$.

A hypergraph is said to be *acyclic* if it can be reduced to have no hyperedges by the Graham reduction. Otherwise, it is said to be *cyclic*.

For any hypergraph $H = (V, \mathcal{E})$ we associate a simplicial complex

$$\Delta(H) = \{\sigma : \emptyset \neq \sigma \subset \sigma_i \text{ for some } \sigma_i \in \mathcal{E}\}.$$

It is clear that the reduced hypergraphs over V are in one-to-one correspondence with the simplicial complexes over V. We shall give a topological interpretation of the Graham reduction on simplicial complexes. Let us first combine (GR1) and (GR2) together as one operation.

(GR) Delete a vertex v from V and from all hyperedges σ_i if v belongs to exactly one maximal hyperedge.

If a vertex v belongs to exactly one maximal hyperedge, say σ_1, then $\Delta(H)$ can be obtained by gluing v to the closed simplex $\Delta(\sigma_1 - v)$. Obviously, this gluing can be obtained by a series of elementary simplicial expansions. Therefore we have the following theorem.

Theorem 4.1 *The application of a Graham reduction* (GR) *to a hypergraph H corresponds to a simple collapse on the associated simplicial complex $\Delta(H)$. Moreover, if $\Delta(H) = \Delta(\mathrm{sk}(\Delta(H)))$, then* (GR) *corresponds to a graph homotopy on* $\mathrm{sk}(\Delta(H))$.

□

For each cell $\sigma \in \Delta(H)$, the *link* of σ in $\Delta(H)$ is the simplicial complex

$$\mathrm{lk}(\sigma, \Delta(H)) = \{\tau \in \Delta(H) : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta(H)\}.$$

Notice that for a maxiaml cell σ the link $\mathrm{lk}(\sigma, \Delta(H))$ is an empty set. Let $c_0(H)$ denote the number of connected components of $\Delta(H)$ and $c(\sigma, H)$ the number of connected components of $\mathrm{lk}(\sigma, \Delta(H))$ for each $\sigma \in \Delta(H)$. The cycle rank $r(H)$ was introduced by Lee in [9-11] to generalize the ordinary cycle rank in graph theory.

Definition 4.2 *The* cycle rank *of a hypergraph H is the integer*

$$r(H) = c_0(H) + \sum_{\sigma \in \Delta(H)} [c(\sigma, H) - 1].$$

Theorem 4.3 *The cycle rank of a hypergraph is an integer invariant under Graham reduction* (GR).

Proof Let v be a vertex of a maximal cell, say τ, such that v is not a vertex of any other maximal cellss. Let H' be the hypergraph on $V - \{v\}$ obtained from H by having v removed. For convenience we write $\tau - v$ instead of $\tau - \{v\}$. For any cell σ in $\Delta(H)$, we have the following four cases.

Case 1: $v \notin \sigma$ and $\sigma \neq \tau - v$. Then $c(\sigma, H') = c(\sigma, H)$;
Case 2: $v \notin \sigma$ and $\sigma = \tau - v$. Then $c(\tau - v, H') = c(\tau - v, H) - 1$;
Case 3: $v \in \sigma$ and $\sigma \neq \tau$. Then $c(\sigma, H) = 1$;
Case 4: $v \in \sigma$ and $\sigma = \tau$. Then $c(\tau, H) = 0$.

Notice that the cell σ in Case 3 and Case 4 will vanish from $\Delta(H')$ when v is removed from H. Thus we have

$$\begin{aligned} r(H) &= c_0(H) + [c(\tau - v, H) - 1] + [c(\tau, H) - 1] \\ &+ \sum_{\substack{v \notin \sigma \\ \sigma \neq \tau - v}} [c(\sigma, H) - 1] + \sum_{\substack{v \in \sigma \\ \sigma \neq \tau}} [c(\sigma, H) - 1] \end{aligned}$$

$$
\begin{aligned}
&= c_0(H) + [c(\tau - v, H') - 1] + \sum_{\substack{v \notin \sigma \\ \sigma \neq \tau - v}} [c(\sigma, H') - 1] \\
&= c_0(H') + \sum_{\sigma \in \Delta(H')} [c(\sigma, H') - 1] \\
&= r(H').
\end{aligned}
$$

□

Theorem 4.4 *A hypergraph H is acyclic if and only if $r(H) = 0$. Moreover, the cycle rank is always non-negative.*

Proof It follows from Theorem 4.3 that if H is acyclic, then $r(H) = 0$. Suppose H is cyclic. Apply (GR) to reduce the hypergraph H to a hypergraph H' so that it cannot be further reduced by (GR). Then the maximal hyperedges $\tau_1, \cdots, \tau_m$ of H' must satisfy the following properties:

$$
\tau_i \subset \bigcup_{j \neq i} \tau_j; \quad \tau_i \not\subset \tau_j \text{ for } i \neq j \text{ and } (\tau_i - \tau_j) \cap \tau_k \neq \emptyset \text{ for some } k \neq i. \tag{6}
$$

Notice that $c(\sigma, H') \geq 1$ for any $\sigma \in \Delta(H')$, except $\tau_1, \cdots, \tau_m$, and $c(\tau_i, H') = 0$, $1 \leq i \leq m$. Denote $I(H') = \{\tau_i \cap \tau_j \neq \emptyset : 1 \leq i \leq m, 1 \leq j \leq m, i \neq j\}$. Since $c_0(H') \geq 1$, it suffices to show that

$$
\sum_{\sigma \in I(H')} [c(\sigma, H') - 1] \geq m.
$$

For each maximal cell of $\Delta(H')$, say τ_1, there exists a maximal cell, say τ_2, such that $\tau_1 \cap \tau_2 \neq \emptyset$. Since (6), $\tau_1 - \tau_2 \neq \emptyset$. Again by (6), $\tau_1 - \tau_2$ must intersect another maximal cell, say τ_3. Thus $(\tau_1 - \tau_2) \cap \tau_3 \neq \emptyset$. Of course $\tau_1 \cap \tau_2 \neq \tau_1 \cap \tau_3$. This shows that τ_1 contributes a component $\tau_1 - \tau_2$ in $\mathrm{lk}(\tau_1 \cap \tau_2, \Delta(H')$ and a component $\tau_1 - \tau_3$ in $\mathrm{lk}(\tau_1 \cap \tau_3, \Delta(H))$. This means that for each cell $\tau_i \cap \tau_j \neq \emptyset$ with $i \neq j$, $\mathrm{lk}(\tau_i \cap \tau_j, \Delta(H'))$ contains at least two components $\tau_i - \tau_j$ and $\tau_j - \tau_i$. Consider the bipartite graph with the vertex set $I(H') \cup \{\tau_1, \cdots, \tau_m\}$ and edges $(\tau_i \cap \tau_j, \tau_i)$ and $(\tau_i \cap \tau_j, \tau_j)$, then $\sum_{\sigma \in I(H')} c(\sigma, H')$ should be the number of edges of the bipartite graph. Thus

$$
\begin{aligned}
\sum_{\sigma \in I(H')} c(\sigma, H') &= \frac{1}{2} \left(\sum_{i=1}^{m} \deg(\tau_i) + \sum_{\sigma \in I(H')} \deg(\sigma) \right) \\
&\geq \frac{1}{2} (2m + 2|I(H')|) \\
&= m + |I(H')|.
\end{aligned}
$$

So we have a contradiction $r(H) = r(H') \geq c_0(H') \geq 1$. The non-negativity of the cycle rank follows from the same argument. □

The non-negativity of the cycle rank $r(H)$ for a hypergraph H automatically gives rise to an inequality about the number of components of $\Delta(H)$ and the number of components of the links $\mathrm{lk}(\sigma, \Delta(H))$ at cells σ.

Corollary 4.5 *Let Δ be a simplicial complex with $\#(\Delta)$ cells. Let $c_0(\Delta)$ denote the number of connected components of $|\Delta|$ and $c(\sigma)$ the number of components of* $\mathrm{lk}(\sigma, \Delta))$. *Then*

$$\sum_{\sigma \in \Delta} c(\sigma) \geq \#(\Delta) - c_0(\Delta).$$

□

The present proof of Theorem 4.3 has a clear topological interpretation, that is, (GR2) is unnecessary from topological point of view; this is why we can ignore (GR2) and modify (GR1) to (GR). It should be pointed out that Theorem 4.3 was first proved in [15] with respect to ear removal, which can be viewed as composition of a series of consecutive operations of (GR1) and (GR2). Another proof of Theorem 4.3 with respect to (GR1) and (GR2) is given in [11].

Let H_1 and H_2 be hypergraphs. If H_2 can be reduced from H_1 by one (GR), the operation from H_1 to H_2 is called an *elementary Graham collapse*; the reverse operation from H_2 to H_1 is called an *elementary Graham expansion.* Two hypergraphs are said to be *Graham homotopy equivalent* if one can be obtained from another by a series of elementary Graham collapses and expansions. There are examples of hypergraphs that are simple-homotopy equivalent to a point, but not Graham homotopy equivalent to a point, namely, cyclic.

Example 4.6 The hypergraph S consisting of the following hyperedges

$$\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$$

is not Graham homotopy equivalent to a point. Since the reduced hypergraph consists of the hyperedges $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 3, 4\}$, it is easy to check that $r(S) = 1$. The geometric realization of S is the boundary of a tetrahedron with one open face removed. Obviously, S is simple-homotopy equivalent to a point. This example also shows that the converse of Theorem 4.7 is not true, because $\mathrm{sk}(S)$ is chordal.

Theorem 4.7 *If H is an acyclic hypergraph, then $G = \mathrm{sk}\Delta(H)$ is chordal.*

Proof Suppose G is not chordal for some acyclic hypergraph H. Then there is a cycle $v_1 v_2 \cdots v_n v_1$ in G without chords, $n \geq 4$. If $v_{i-1}v_i \in \tau_i$ and $v_i v_{i+1} \in \tau_{i+1}$ for some maximal hyperedges τ_i and τ_{i+1}, then $v_{i-1} \notin \tau_{i+1}$ and $v_{i+1} \notin \tau_i$, for in other case $v_{i-1}v_{i+1}$ will be a chord. Hence $\tau_i \neq \tau_{i+1}$ and each v_i belongs to two maximal hyperedges. Thus no vertex v_i can be removed by (CR). Notice that when some vertices other than those v_i are removed by (GR) in the reduction process, the non-removable property of v_i still holds. By definition of acyclicity, H is cyclic, a contradiction. □

Corollary 4.8 *If a hypergraph H is Graham homotopy equivalent to a point, then H can be reduced to a point only by elementary Graham collapses.*

Proof By Theorem 4.3 the cycle rank $r(H)$ is invariant under Graham reduction (GR). Since the cycle rank of a point is zero, then $r(H) = 0$. By Theorem 4.4, H is acyclic. Thus H can be contracted to a point only by elementary Graham collapses. □

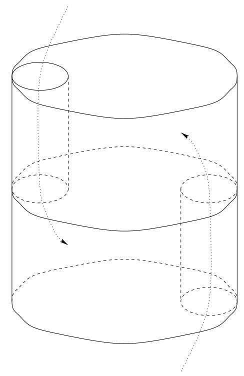

Figure 3: Bing's house

5 Bing's House

In this section we use Bing's house to describe the significant difference of simple-homotopy and graph homotopy from Graham homotopy. Bing's house is a closed topological surface (not a 2-manifold), see Figure 3. It is well-known that Bing's house is simple-homotopy equivalent to a point, but can *not* be contracted to a point only by elementary collapses, see [2], because there is no free pair of cells. Bing's house can be realized by a graph G_b, whose vertices and edges are shown in Figure **??**. Like the situation of simple-homotopy, we shall see that G_b is graph homotopy equivalent to a point, but can not be contracted to a point only by vertex deletion. The latter can be checked directly by computing the graph links of vertices of G_b, all are not contractible, as follows:

$L(u_1)$ is the union of two cycles $u_2u_7v_7v_1u_2$ and $u_4u_5v_5v_1u_4$;
$L(u_2)$ is the cycle $u_1u_7u_6u_3v_3v_2v_1u_1$;
$L(u_3)$ is the cycle $u_2u_6u_4v_3u_2$;
$L(u_5)$ is the cycle $u_1u_4u_6v_5u_1$;
$L(u_6)$ is the cycle $u_2u_3u_4u_5v_5v_6v_7u_7u_2$;
$L(v_1)$ is the union of three cycles $u_1u_2v_2v_7u_1$, $u_1u_4v_4v_5u_1$, and $u_1v_5w_1v_7u_1$;
$L(v_2)$ is the cycle $u_2v_1v_7v_6w_2v_3u_2$;
$L(v_3)$ is the cycle $u_2u_3u_4v_4w_4w_3w_2v_2u_2$;
$L(v_5)$ is the union of two cycles $u_1u_5u_6v_6v_4v_1u_1$ and $v_1v_4v_6w_6w_5w_1v_1$.
By symmetry, $L(u_4)$ and $L(u_7)$ are isomorphic to $L(u_2)$ and $L(u_5)$ respectively; $L(v_4)$, $L(v_6)$, and $L(v_7)$ are isomorphic to $L(v_2)$, $L(v_1)$, and $L(v_5)$ respectively; $L(w_i)$ are isomorphic to $L(u_i)$ for $i = 2, 3, 4, 5, 7$; $L(w_1)$ and $L(w_6)$ are isomorphic to $L(u_6)$ and $L(u_1)$ respectively.

To see that G_b is graph homotopy equivalent to a point, notice that we can glue the edges u_2v_7, u_4v_5, u_2v_6, u_4v_6, u_3v_6, v_3v_6, v_6w_3, and w_3w_6 consecutively to G_b to fill up the "lower room" of $\Delta(G_b)$, because the joint graph links of these pair of vertices, listed below, are all contractible.

$L(u_2v_7)$ is the path $u_6u_7u_1v_1v_2$;
$L(u_4v_5)$ is the path $u_6u_5u_1v_1v_4$, isomorphic to $L(u_2v_7)$;

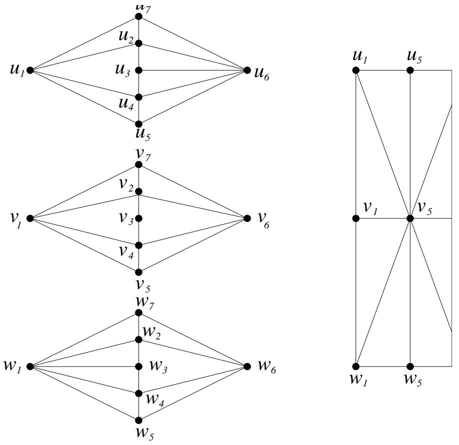

Figure 4: The graph G_b of Bing's house

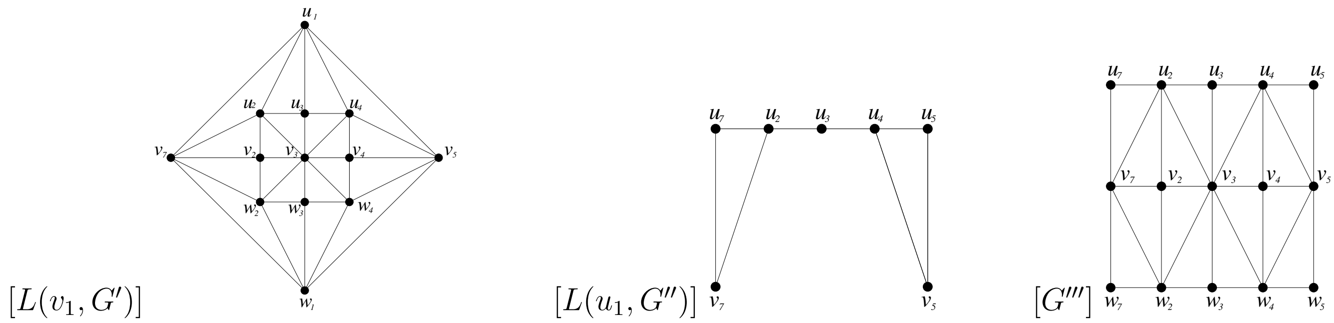

Figure 5: G_b is graph homotopy equivalent to G' and G' is contractible

$L(u_2v_6)$ is the path $u_6v_7v_2$;
$L(u_4v_6)$ is the path $u_6v_5v_4$, isomorphic to $L(u_2v_6)$;
$L(u_3v_6)$ is the path $u_2u_6u_4$;
$L(v_3v_6)$ is the path $w_2v_2u_2u_3u_4v_4w_4$;
$L(v_6w_3)$ is the path $w_2v_3w_4$;
$L(w_3w_6)$ is the path $w_2v_6w_4$.

Similarly, we can glue the edges v_7w_2, v_5w_4, v_1w_2, v_1w_4, v_1w_3, v_1v_3, u_3v_1, and u_1u_3 consecutively to fill up the "upper room" of $\Delta(G_b)$, because the joint graph links of these pair of vertices are isomorphic to those joint graph links of the edges for filling up the "lower room." By gluing all those edges for filling up the two "rooms," G_b is expanded to a graph G'; $\Delta(G')$ is a triangulated solid cube. Notice that $L(v_1, G')$ and $L(v_6, G')$ are isomorphic and contractible, see Part (a) of Figure 5. Then v_1 and v_6 can be deleted from G' to obtain a graph G''. Now the graph links of u_1, u_6, w_1, and w_6 in G'' are all isomorphic and contractible, see Part (b) of Figure 5. Remove the vertices u_1, u_6, w_1, w_6 from G'', we obtain a contractible graph G''', see Part (c) of Figure 5. We thus have proved that G_b is graph homotopy equivalent to a point.

The above example shows that for graph homotopy, as well as for simple-homotopy, there is no straightforward algorithm to test whether a simplicial complex is contractible. However, for Graham homotopy, the situation is quite different. If a simplicial complex is Graham homotopy equivalent to a point, then it can always be reduced to a point only by elementary Graham collapses. This special property of Graham homotopy makes it useful to the theory of algorithms of theoretical computer science. This hints that, if we compare elementary collapses to the forward steps in a computer algorithm and elementary expansions to backtracks, then an algorithm with backtracks is essentially different from one without backtracks (possibly the difference is between polynomial and exponential). Though we could not formulate our ideas on the comparison of algorithms and homotopies in a precise way, yet we believe that our point of view is still useful and is worth to be explored. Graham homotopy also suggests that some stronger homotopies need be studied even for contractible spaces. The numerical characterizations for simple-homotopy and graph homotopy of simplicial complexes are particularly wanted.

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