Foldings in graphs and relations with simplicial complexes and posets

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Abstract

We study dismantlability in graphs. In order to compare this notion to similar operations in posets (partially ordered sets) or in simplicial complexes, we prove that a graph G dismants on a subgraph H if and only if H is a strong deformation retract of G. Then, by looking at a triangle relating graphs, posets and simplicial complexes, we get a precise correspondence of the various notions of dismantlability in each framework. As an application, we study the link between the graph of morphisms from a graph G to a graph H and the polyhedral complex $\mathrm{Hom}(G, H)$; this gives a more precise statement about well known results concerning the polyhedral complex $\mathrm{Hom}(G, H)$ and its relation with foldings in G or H.

Keywords : dismantlability; foldings; Hom complex; posets; simplicial complexes; strong deformation retract

1 Introduction

A vertex g of a graph G is said *dismantlable* if there is another vertex a in G such that $N_G(x) \subset N_G(a)$ where $N_G(x) := \{y \in V(G), y \sim x\}$ is the open neighborhood of x. This will be denoted $x \overset{d}{\vdash} a$ and we will also say that a dominates x. The passage from G to $G - x$ by deleting a dismantlable vertex x is called a *folding* and denoted $G \searrow^d G - x$; the resultant graph $G - x$ is called a *fold* of G. A succession of foldings will be called a *dismantling*. If there is a dismantling from a graph G to a subgraph H, we say that G *is dismantlable on* and write $G \searrow^d H$; this means that there is a *dismantling sequence* $x_1, \ldots, x_k$ from G to H, i.e. $V(G) = V(H) \cup \{x_1, \ldots, x_k\}$ with x_i dismantlable in the subgraph induced by $V(H) \cup \{x_i, x_{i+1}, \ldots, \ldots, x_k\}$ for $i = 1, 2, \ldots, k$; this will be also denoted $H \,{}^d\!\nearrow G$. A reflexive graph G is said *dismantlable* if it is dismantlable on a looped vertex. Following [HN04], a graph whose every vertex is non dismantlable is called *stiff*.

It seems that the the first papers which focused on vertices whose open neighborhood is included in the open neighborhood of another vertex[3] are [Qui83] and [NW83]where it was proved independently that a reflexive graph is cop win if, and only if, it is dismantlable. The reflexive bridged and connected graphs (a graph is bridged if it contains no isometric cycles of length greater than three; in particular, the chordal graphs are bridged) are examples of dismantlable graphs ([AF88]). In this paper, the objective is to give a precise description of the relation between dismantlability in graphs and similar operations in partially ordered sets (posets) or in simplicial complexes. In section 2, we give a characterization of foldings and dismantlings by the way of morphisms and homotopies. The key result (Proposition 2.2) is that a graph G dismants on a subgraph H if, and only if, H is a strong deformation retract of G. As a useful corollary, we get that if G' and G'' are two subgraphs of a graphs G such that G'' is a subgraph of G', $G \searrow^d G'$ and $G \searrow^d G''$ then we can conclude that $G' \searrow^d G''$ (Corollary 2.1).

In the framework of posets, there is also a very well known notion of dismantlability (most frequently named *irreducibility*; see Section 3 for a brief discussion). From the seminal paper [Sto66], we know that the dismantlings in posets allow to describe the homotopy type of a poset (its *real* homotopy type, i.e. the homotopy type of the poset considered as a topological space and not the homotopy type of its order complex). Dismantlability in posets has been studied in various articles, in particular in relation with the fixed point property ([BB79],[Riv76],[Sch03],[Wal84]). It is known ([BCF94],[Gin94]) that the dismantlability of a poset P is equivalent to the dismantlability of its *comparibility graph* (which will be called $Comp(P)$). In [Gin94], it was also proved that the dismantlability of a graph G is equivalent to the dismantlability of the *poset of complete subgraphs* of G (which will be called $C(G)$). In section 3, we will give a generalization of

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[3] It is important to note that several papers (as [AF88],[BFJ08],[Gin94],[LPVF08],[Qui83]) take another definition of dismantlability : a vertex x is dismantlable if there is another vertex a such that $N_G[x] \subset N_G[a]$ where $N_G[x] := N_G(x) \cup \{x\}$ is the closed neighborhood of x. Of course, the two definitions are the same when the graphs are reflexive.

these results. More recently ([BM09]), J. Barmak and A. Minian have introduced a notion of dismantlability in the category $\mathscr{K}$ of finite simplicial complexes. We show in section 4 that it « corresponds » to the dismantlability in graphs under natural functors relating $\mathscr{G}$ and $\mathscr{K}$.

So, this gives a « good » behaviour of a triangle $(\mathscr{G}^\circ, \mathscr{P}, \mathscr{K})$ in relation to the various notions of dismantlability in $\mathscr{G}^\circ$, $\mathscr{P}$ or $\mathscr{K}$ and, consequently, with the equivalences classes (named *homotopy classes*) defined by the operation of dismantlability (section 5). A motivation for this question is given by the polyhedral complex $\mathsf{Hom}(G, H)$ associated to two graphs G and H. This construction is due to Lovasz after its pioneering work ([Lov78]) where he solved the Kneser conjecture by using the simplicial complex $\mathcal{N}(G)$, the *neighborhood complex* of G. Since the article [BK06] (where the authors proved in particular that $\mathsf{Hom}(K_2, G)$ and $\mathcal{N}(G)$ have the same homotopy type), the Hom complex has became an important tool for determining lower bounds to the chromatic number of certain graphs (see [Koz08] for a complete exposition and more references). For obtaining topological information about the polyhedral complex $\mathsf{Hom}(G, H)$ (which is not in general a simplicial complex), it is usual to look at its face poset $\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G, H))$ or at the order complex of its face poset, i.e. its barycentric subdivision $Bd(\mathsf{Hom}(G, H)) = \Delta_{\mathscr{P}}(\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G, H)))$ (which is a simplicial complex). On the other hand, the set of morphisms from G to H is the vertex set of a graph (called $\mathsf{hom}_{\mathscr{G}}(G, H)$) and is also the vertex set of $\mathsf{Hom}(G, H)$; we will study the relation between the graph $\mathsf{hom}_{\mathscr{G}}(G, H)$ and the polyhedral complex $\mathsf{Hom}(G, H)$ by using the triangle $(\mathscr{G}^\circ, \mathscr{P}, \mathscr{K})$ and regarding them in $\mathscr{P}$ (Proposition 6.1). In particular, this gives another proof of a result describing the dismantlings on $\mathsf{Hom}(G, H)$ induced by foldings on G or H. However, this result which is usually formulated in terms of simplicial complexes is formulated here in terms of graphs.

Notations In this paper, the graphs will be finite, undirected and without parallel edges. The vertex set of a graph G is denoted $V(G)$. The set of these graphs will be denoted $\mathscr{G}$ and eventually considered as a category where a morphism $f : G \to G'$ from a graph G to a graph G' is an application from $V(G)$ to $V(G')$ which preserves adjacency ($x \sim y \Longrightarrow f(x) \sim f(y)$); $\mathscr{G}^\circ$ will denote the subcategory obtained by retricting to reflexive graphs (i.e., graphs G such that $x \sim x$ for all x in $V(G)$).

Let $G \in \mathscr{G}$. If X is a subset of $V(G)$, the notation $G - X$ will indicate the subgraph of G induced by the set of vertices $V(G) \setminus X$. In particular, if $x \in V(G)$, $G - x$ will be an abbreviated form of $G - \{x\}$ and $i_x : G - x \to G$ will denote the inclusion morphism. If $x \overset{d}{\vdash} a$, the folding $G \to G - x$ which sends x to a (and is the identity on $G - x$) will be denoted $r_{x,a}$.

The notation $G \overset{d}{\nearrow} G + y$ means that we have a added a vertex y to G in such a way that y is dismantlable in the new graph.

2 Morphisms

In this section, we characterize foldings and dismantlings in terms of morphisms. Let $G, G' \in \mathscr{G}$. The set of morphisms from G to G' is the vertex set of a graph, denoted $\mathsf{hom}_{\mathscr{G}}(G, G')$, where $f \sim f'$ in $\mathsf{hom}_{\mathscr{G}}(G, G')$ if and only if $x \sim y$ in G implies $f(x) \sim f'(y)$ in G' ([HHMNL88],[BCF94]); this graph is reflexive because $f \sim f$ means precisely that f is a morphism of graph. By an abuse of notation, « $f \in \mathsf{hom}_{\mathscr{G}}(G, G')$ » will mean that f is a morphism from G to G' (in place of $f \in V(\mathsf{hom}_{\mathscr{G}}(G, G'))$).

Remark 2.1 *Let $G, G', G'' \in \mathscr{G}$, $f, f' \in \mathsf{hom}_{\mathscr{G}}(G, G')$ and $h, h' \in \mathsf{hom}_{\mathscr{G}}(G', G'')$. If $f \sim f'$ and $h \sim h'$, then $h \circ f \sim h' \circ f'$ because $x \sim y$ in G implies $f(x) \sim f'(y)$ in G' (by $f \sim f'$) which implies $h \circ f(x) \sim h' \circ f'(y)$ in G' (by $h \sim h'$).*

2.1 Foldings and retraction

An important class of morphisms is given by retractions. A retraction of a graph G to a subgraph H of G is a morphism $r : G \to H$ such that $r(x) = x$ for all x in $V(H)$. So, a morphism $r : G \to G$ such that $r \circ r = r$ is a retraction of G to $r(G)$. The results of this paragraph are based on the following remarks:

Remark 2.2 *a. Let $f \in \mathsf{hom}_{\mathscr{G}}(G, G)$. If $f \sim 1_G$ (where 1_G is the identity morphism on G), then every vertex x of G verifies either $f(x) = x$, or $x \overset{d}{\vdash} f(x)$.*

b. In particular, if $f : G \to G$ is a retraction such that $f \sim 1_G$, then $f \overset{d}{\searrow} f(G)$.

We note that Remark 2.2.a. implies that 1_G is an isolated vertex in $\mathsf{hom}_{\mathscr{G}}(G, G)$ when G is a stiff graph (this is a classical result used in [BCF94], [Doc09]). By definition, a folding is a retraction $G \to G - x$ which sends x to a vertex a which dominates x. However, a general retraction $G \to G - x$ is not necessarily a folding (see Figure 1)

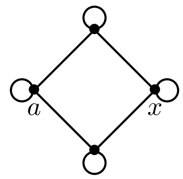


Figure 1: The retraction $G \to G - x$ (which sends x to a) is not a folding

From Remark 2.2.a, we get the following characterization of foldings:

Lemma 2.1 *Let $G \in \mathscr{G}$, $x \in V(G)$ and $f : G \to G - x$ a retraction; the following assertions are equivalent:*

1. *f is a folding (i.e., $x \overset{d}{\vdash} f(x)$)*
2. *$i_x \circ f \sim 1_G$ (where i_x is the inclusion $G - x \hookrightarrow G$)*

We conclude also from Remark 2.2.b that foldings on graphs induce dismantlability in graphs of morphisms:

Proposition 2.1 *1. If x is dismantlable in G, then $\mathsf{hom}_{\mathscr{G}}(G, H) \overset{d}{\searrow} \mathsf{hom}_{\mathscr{G}}(G - x, H)$ (by identifying $\mathsf{hom}_{\mathscr{G}}(G - x, H)$ with an induced subgraph of $\mathsf{hom}_{\mathscr{G}}(G, H)$).*

2. If x is dismantlable in H, then $\mathsf{hom}_{\mathscr{G}}(G, H) \overset{d}{\searrow} \mathsf{hom}_{\mathscr{G}}(G, H - x)$ (by identifying $\mathsf{hom}_{\mathscr{G}}(G, H - x)$ with an induced subgraph of $\mathsf{hom}_{\mathscr{G}}(G, H)$).

Proof : 1. Let x dismantlable in G with $x \overset{d}{\vdash} a$. Then, the map $\Psi_{x,a} : \mathsf{hom}_{\mathscr{G}}(G-x, H) \to \mathsf{hom}_{\mathscr{G}}(G, H)$ defined by $\Psi_{x,a}(f) = f \circ r_{x,a}$ is an injective morphism of graphs and we identify $\mathsf{hom}_{\mathscr{G}}(G - x, H)$ with the subgraph $\Psi_{x,a}(\mathsf{hom}_{\mathscr{G}}(G - x, H))$ of $\mathsf{hom}_{\mathscr{G}}(G, H)$. Let us denote $\Phi_x : \mathsf{hom}_{\mathscr{G}}(G, H) \to \mathsf{hom}_{\mathscr{G}}(G - x, H)$ the restriction morphism defined by $\Phi_x(f) = f \circ i_x \equiv f_{|G-x}$. If $f \in \mathsf{hom}_{\mathscr{G}}(G - x, H)$, then $(\Phi_x \circ \Psi_{x,a})(f) = (f \circ r_{x,a}) \circ i_x = f \circ (r_{x,a} \circ i_x) = f$; so $\Phi_x \circ \Psi_{x,a} = 1_{\mathsf{hom}_{\mathscr{G}}(G-x,H)}$ and this means that $\Psi_{x,a} \circ \Phi_x : \mathsf{hom}_{\mathscr{G}}(G, H) \to \mathsf{hom}_{\mathscr{G}}(G, H)$ is a retraction to $\mathsf{hom}_{\mathscr{G}}(G - x, H)$ identified with $\Psi_{x,a}(\mathsf{hom}_{\mathscr{G}}(G - x, H))$. If $f \in \mathsf{hom}_{\mathscr{G}}(G, H)$, $\Psi_{x,a} \circ \Phi_x(f)$ takes the same value as f on vertices distinct from x and takes the value $f(a)$ on x. Let $f, f' \in \mathsf{hom}_{\mathscr{G}}(G, H)$ with $f \sim f'$. As $i_x \circ r_{x,a} \sim 1_G$ (Lemma 2.1), we have $f \circ i_x \circ r_{x,a} \sim f'$ by Remark 2.1 ; this proves that $\Psi_{x,a} \circ \Phi_x \sim 1_{\mathsf{hom}_{\mathscr{G}}(G,H)}$ and we conclude $\mathsf{hom}_{\mathscr{G}}(G, H) \overset{d}{\searrow} \mathsf{hom}_{\mathscr{G}}(G - x, H)$ by Remark 2.2.b.

2. Similarly, if x is dismantlable in H with $x \overset{d}{\vdash} b$, we denote $\Phi_{x,b} : \mathsf{hom}_{\mathscr{G}}(G, H) \to \mathsf{hom}_{\mathscr{G}}(G, H - x)$ the morphism of graphs defined by $\Phi_{x,b}(f) = r_{x,b} \circ f$ and we identify $\mathsf{hom}_{\mathscr{G}}(G, H - x)$ with the induced subgraph of $\mathsf{hom}_{\mathscr{G}}(G, H)$ given by its image under the injection $\Psi_x : \mathsf{hom}_{\mathscr{G}}(G, H - x) \to \mathsf{hom}_{\mathscr{G}}(G, H)$ defined by $\Psi_x(f) = i_x \circ f$. Then $\Phi_{x,b} \circ \Psi_x = 1_{\mathsf{hom}_{\mathscr{G}}(G,H-x)}$ and $\Psi_x \circ \Phi_{x,b} : \mathsf{hom}_{\mathscr{G}}(G, H) \to \mathsf{hom}_{\mathscr{G}}(G, H)$ is a retraction to $\mathsf{hom}_{\mathscr{G}}(G, H - x)$ identified with $\Psi_x(\mathsf{hom}_{\mathscr{G}}(G, H - x))$. If $f \in \mathsf{hom}_{\mathscr{G}}(G, H)$, $\Psi_x \circ \Phi_{x,b}(f)$ takes at a vertex z the same value as f when $f(z) \neq x$ and the value b when $f(z) = x$. It is easy to verify that $\Psi_x \circ \Phi_{x,b} \sim 1_{\mathsf{hom}_{\mathscr{G}}(G,H)}$ and this proves $\mathsf{hom}_{\mathscr{G}}(G, H) \overset{d}{\searrow} \mathsf{hom}_{\mathscr{G}}(G, H - x)$. $\square$

2.2 Dismantlings and homotopy

Morphisms give rise to a notion of homotopy and it was noticed in [Qui83] that a graph is dismantlable if and only if the identity morphism is homotopic to a constant morphism. Following [Doc09], for $N \in \mathbb{N}^*$, I_N is the reflexive graph with looped vertices $0, 1, 2, \ldots, N$ and adjacencies $0 \sim 1 \sim 2 \sim 3 \sim \ldots \sim N - 1 \sim N$.

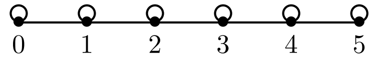


Figure 2: The reflexive path I_5

If $f, f' \in \mathsf{hom}_{\mathscr{G}}(G, G')$, a homotopy from f to f' is a morphism of graphs $\mathcal{H} : I_N \to \mathsf{hom}_{\mathscr{G}}(G, G')$, $i \mapsto \mathcal{H}_i$ such that $\mathcal{H}_0 = f$ and $\mathcal{H}_N = f'$; this will be denoted $f \simeq f'$ and this means that f and f' are in the same connected component of $\mathsf{hom}_{\mathscr{G}}(G, G')$. A subgraph G' of G is a *strong deformation retract* if there is a homotopy $\mathcal{H} : I_N \to \mathsf{hom}_{\mathscr{G}}(G, G)$ such that $\mathcal{H}_0 = 1_G$, $\mathcal{H}_{i|G'} = 1_{G'}$ for all $i \in \{1, 2, \ldots, N\}$ and $\mathcal{H}_N : G \to G$ is actually a retraction to G'. The following results will be useful in the sequel:

Lemma 2.2 *Let $G'' \subset G' \subset G$ inclusions of graphs.*

1. If G'' is a strong deformation retract of G' and G' is a strong deformation retract of G, then G'' is a strong deformation retract of G.

2. If G'' is a strong deformation retract of G and G' a retract of G, then G'' is a strong deformation retract of G'.

Proof : 1. Straightforward. 2. Let $i_{G'} : G' \hookrightarrow G$ the inclusion and $r_{G'} : G \to G'$ a retraction of G to G'. If $\mathcal{H} : I_N \to \mathsf{hom}_{\mathscr{G}}(G, G)$ is a homotopy proving that G'' is a strong deformation retract of G, then $\mathcal{H}' : I_N \to \mathsf{hom}_{\mathscr{G}}(G', G')$ defined by $\mathcal{H}'_i = r_{G'} \circ \mathcal{H}_i \circ i_{G'}$ is a homotopy proving that G'' is a strong deformation retract of G'. □

By Lemma 2.1, a fold $G - x$ of G is a strong deformation retract of G; more generally, dismantlability is characterized by strong deformations:

Proposition 2.2 *Let H be a subgraph of a graph G. Then, $G \searrow^{d} H$ if, and only if, H is a strong deformation retract of G.*

Proof : Let us suppose that $G \searrow^{d} H$. This means that one can go from G to H by a composition of foldings ; each fold being a strong deformation retract, H is a strong deformation retract of G by Lemma 2.2 a. If we suppose now that H is a strong deformation retract of G, we can use an argument similar to that used in the proof of Théorème 4.4 in [BCF94]. Let $\mathcal{H} : I_N \to \mathsf{hom}_{\mathscr{G}}(G, G)$ be a homotopy proving that H is a strong deformation retract of G. If $H \neq G$, $\mathcal{H}_N \neq 1_G$, and we can suppose $\mathcal{H}_1 \neq 1_G$. If $a \in G$ is such that $\mathcal{H}_1(a) \neq a$, then $a \notin H$, $a \overset{d}{\vdash} \mathcal{H}_1(a)$ in G and $G \searrow^{d} G - a$. For $i \in \{0, 1, \ldots, N\}$, define $\mathcal{H}'_i : G - a \to G - a$ by $\mathcal{H}'_i(x) = (r_{a,\mathcal{H}_1(a)} \circ \mathcal{H}_i)(x)$ for $x \in G - a$. Clearly, these morphisms define a homotopy $\mathcal{H}' : I_N \to \mathsf{hom}_{\mathscr{G}}(G - a, G - a)$, $\mathcal{H}'_1 \sim 1_{G-a}$ and $\mathcal{H}'_N$ is a retraction from $G - a$ to H. Thus, H is a strong deformation retract of $G - a$. If $G - a \neq H$, we can iterate, and so $G \searrow^{d} H$. □

Corollary 2.1 *Let $G'' \subset G' \subset G$ inclusions of graphs such that $G \searrow^{d} G'$ and $G \searrow^{d} G''$. Then $G' \searrow^{d} G''$.*

Proof : Straightforward from Lemma 2.2 b. and Proposition 2.2. □

Let us recall that two graphs G and H are homotopically equivalent if there is $f \in \mathsf{hom}_{\mathscr{G}}(G, H)$ and $g \in \mathsf{hom}_{\mathscr{G}}(H, G)$ such that $g \circ f \simeq 1_G$ and $f \circ g \simeq 1_H$. In particular, if H is a strong deformation retract of G, H and G are homotopically equivalent. We mention the following well known result (two quite different proofs are given in [BCF94] and [HN04]):

Proposition 2.3 *Let $G \in \mathscr{G}$ and H, H' two stiff subgraphs such that $G \searrow^{d} H$ and $G \searrow^{d} H'$. Then H is isomorphic to H'.*

Proof : By Proposition 2.2, H and H' are strong deformation retracts of G. So, H and H' are homotopically equivalent. Let $f \in \mathsf{hom}_{\mathscr{G}}(H, H')$ and $g \in \mathsf{hom}_{\mathscr{G}}(H', H)$ such that $g \circ f \simeq 1_H$ and $f \circ g \simeq 1_{H'}$. As the graphs H and H' are stiff, the connected components of 1_G and 1_H are reduced, respectively, to $\{1_G\}$ and $\{1_H\}$. So, we conclude that $g \circ f = 1_H$ and $f \circ g = 1_{H'}$ and that H and H' are isomorphic. □

3 Foldings versus $(\mathscr{G}, \mathscr{P})$

Let $\mathscr{P}$ the category of finite posets. If $P, Q \in \mathscr{P}$, a morphism of posets $f : P \to Q$ is a map from P to Q which preserves the order (i.e. $x \leq y$ in P implies $f(x) \leq f(y)$ in Q). An element p of a poset P will be called *dismantlable* if either $P_{>p} := \{y \in P, y > p\}$ has a least element or $P_{<p} := \{y \in P, y < p\}$ has a greatest element. There are already various denominations for this notion; the most classical are: *irreducible* (in many papers, following [Riv76]), *linear* and *antilinear* (in [Sto66]), *upbeat points* and *downbeat points* ([May03],[BM09]); in this paper, we adopt the denomination « *dismantlable* » in order to emphasize the link with graphs.

Let p a dismantlable point in P. If $a = \sup P_{<p}$ or $a = \inf P_{>p}$, p will be said *dominated by* a. The deletion of the dismantlable element x, will be denoted $P \searrow^{d} P \setminus \{x\}$ and $P \searrow^{d} Q$ means that one can go from the poset P to a subposet Q by successive deletions of dismantlable elements.

Proposition 3.1 *Let $f : P \to P$ a morphism of posets map such that either $f \leq 1_P$ or $f \geq 1_P$. Then $P \searrow^{d} \mathrm{Fix}\,(f)$ where $\mathrm{Fix}\,(f) := \{p \in P, f(p) = p\}$.*

Proof : We suppose that $\mathrm{Fix}\,(f) \neq P$ (i.e., $f \neq 1_P$) and we consider the case $f \leq 1_P$. Let x minimal in $P\backslash \mathrm{Fix}(f)$. Let $y < x$ (for example, $y = f(x)$). Then $y = f(y)$ (by minimality of x in $P\backslash\mathrm{Fix}(f)$) and $y \leq f(x)$ (because $y < x \Rightarrow f(y) \leq f(x)$). Thus, $f(x)$ is the greatest element of $P_{<x}$ and x is dismantlable. So, we have $P \searrow^{d} P\backslash\{x\}$. Now, we define $\widetilde{f} : P\backslash\{x\} \to P\backslash\{x\}$ by $\widetilde{f}(y) = f(y)$ if $f(y) \neq x$ and $\widetilde{f}(y) = f(x) = f^2(y)$ if $f(y) = x$. Let us verify that $\widetilde{f}$ is a morphism of posets, i.e. $y \leq z \Longrightarrow \widetilde{f}(y) \leq \widetilde{f}(z)$, for all $y, z \in P \setminus \{x\}$. This is clear if either $(f(y) \neq x$ and $f(z) \neq x)$, or $(f(y) = f(z) = x)$. If $f(y) = x$ and $f(z) \neq x$, then $\widetilde{f}(y) = f^2(y) \leq f(y) \leq f(z) = \widetilde{f}(z)$ ($f^2(y) \leq f(y)$ follows from $f \leq 1_P$ and $f(y) \leq f(z)$ because f is a morphism of posets). Finally, if $f(y) \neq x$ and $f(z) = x$, we have $f(y) \leq f(z)$ (because f is a morphism of posets), so $f(y) < x$ (because $f(z) = x$ and $f(y) \neq x$). By minimality of x in $P \setminus \mathrm{Fix}\,(f)$, this means that $f(y) \in \mathrm{Fix}\,(f)$. So we get $f^2(y) = f(y)$ and $\widetilde{f}(y) = f(y) = f^2(y) \leq f^2(z) = f(x) = \widetilde{f}(z)$ (because f^2 is a morphism of posets). It is clear that $\widetilde{f} : P \setminus \{x\} \to P \setminus \{x\}$ satisfies $\widetilde{f} \leq 1_{P\setminus\{x\}}$ and that $\mathrm{Fix}\,(\widetilde{f}) = \mathrm{Fix}\,(f)$. So, we can iterate the procedure and finally we get $P \searrow^{d} P \setminus \mathrm{Fix}\,(f)$. The proof is similar if $f \geq 1_P$. $\square$

Remark 3.1 *This proof is essentially the proof given by Kozlov in the particular case $f^2 = f$ ([Koz06, Theorem 2.1] or [Koz08, Theorem 13.12], where the conclusion is given in terms of simplicial complexes).*

3.1 Dismantlability and functor $Comp : \mathscr{P} \to \mathscr{G}^{\circ}$

Let $P \in \mathscr{P}$. The comparability graph of P, denoted $Comp(P)$, is the graph whose vertex set is P with adjacencies $x \sim y$ if and only if x and y are comparable in P. In particular, for every poset P, $Comp(P)$ is a reflexive graph. We will say that a graph G is a cone with apex a if $y \sim a$ for all $y \in V(G)$ (this definition implies that the apex is a looped vertex). The following facts are easy :

- If x is a looped vertex of a graph G, then x is dismantlable if, and only if, $N_G(x) - x$ is a cone.
- $Comp(P) - x = Comp(P \setminus \{x\})$.
- $N_{Comp(P)}(x) - x = Comp(P_{>x} \cup P_{<x})$

Proposition 3.2 *Let $P, Q \in \mathscr{P}$. If $P \searrow^{d} Q$, then $Comp(P) \searrow^{d} Comp(Q)$.*

Proof : Clearly, if x is dominated by an element a in P, then x is dominated by the vertex a in $Comp(P)$. Consequently, $P \searrow^{d} P \setminus \{x\} \Longrightarrow Comp(P) \searrow^{d} Comp(P) - x = Comp(P \setminus \{x\})$ and the proposition follows by iteration. $\square$

Reciprocally, $Comp(P) \searrow^{d} Comp(Q)$ doesn't imply in general $P \searrow^{d} Q$ because a dismantlable vertex in $Comp(P)$ is not necessarily a dismantlable element in P (see, for example, the poset $P = \{a, b, c, d\}$ with $d < b, c < a$ given in Figure 3).

P $\qquad$ $Comp(P)$

Figure 3: a and d are non dismantlable in P and dismantlable in $Comp(P)$

A poset will be called a *double cone with apex a* if it admits an element a comparable with all elements of the poset. It is clear that P is a double cone if, and only if, $Comp(P)$ is a cone and this motivates the following definition.

Définition 3.1 *An element p of a poset P is said weakly dominated by a if $P_{>p} \cup P_{<p}$ is a double cone with apex a. In this case, p will be said weakly dismantlable. We note $P \searrow^{wd} P \setminus \{x\}$ the deletion of a weak dismantlable vertex and $P \searrow^{wd} Q$ means that one can go from P to a subposet Q by successive deletions of weak dismantlable vertices.*

In other terms, p is weakly dominated by a if p and a are comparable and if every element comparable with p is also comparable with a. Of course, if p is dominated by a, then p is weakly dominated by a but the reverse implication is false in general (in the poset P given in Figure 3, d is weakly dominated by a but is not dominated by a). Let p an element of a poset P; the following assertions are equivalents :

1) p is dominated by a in $Comp(P)$
2) $N_{Comp(P)}(p) - p$ is a cone with apex a
3) $Comp(P_{>p} \cup P_{<p})$ is a cone with apex a
4) $P_{>p} \cup P_{<p}$ is a double cone with apex a
5) p is weakly dominated by a in P

In particular, an element p of a poset P is weakly dismantlable in P if, and only if, p is dismantlable in $Comp(P)$ and the following equivalence is immediate :

Theorem 3.1 *Let $P, Q \in \mathscr{P}$. Then, $P \searrow^{wd} Q \iff Comp(P) \searrow^{d} Comp(Q)$*

3.2 Dismantlability and functor $C : \mathscr{G} \to \mathscr{P}$

Let $G \in \mathscr{G}$. We recall that a *complete subgraph* H of G is an induced subraph of G such that $x \sim y$ for any distinct vertices x and y of H; a complete subgraph of G will be identified with its set of vertices. The poset of complete subgraphs of G, denoted $C(G)$, is the poset given by the set of non empty complete subgraphs of G with the inclusion as order relation.

Theorem 3.2 *Let $G \in \mathscr{G}$ and H a subgraph of G such that $G \searrow^{d} H$. If all vertices in $V(G) \setminus V(H)$ are looped, then $C(G) \searrow^{d} C(H)$.*

In particular, if $G \in \mathscr{G}^{\circ}$ and H a subgraph of G, then $G \searrow^{d} H \Longrightarrow C(G) \searrow^{d} C(H)$.

Proof : Let $x \in V(G) \setminus V(H)$ be a dismantlable and looped vertex with a which dominates x. We define $f_1 : C(G) \to C(G)$ by $f_1(c) = c \cup \{a\}$ if $x \in c$ and $f_1(c) = c$ if $x \notin c$; note that f_1 is well defined because x is looped. Then $f_1 \geq 1_{C(G)}$ and, by Proposition 3.1, $C(G) \searrow^{d} \text{Im}(f_1)$. Now, let $f_2 : \text{Im}(f_1) \to \text{Im}(f_1)$ defined by $f_2(c) = c \setminus \{x\}$ if $x \in c$ and $f_2(c) = c$ if $x \notin c$. Then $f_2 \leq 1_{\text{Im}(f_1)}$ and, by Proposition 3.1, $\text{Im}(f_1) \searrow^{d} \text{Im}(f_2) = C(G - x)$. So, $C(G) \searrow^{d} C(G - x)$ and the proposition follows by iterating the process. □

Now, before studying the reciprocal of Theorem 3.2, we recall that an element p of a poset P is an atom if $P_{<p} = \emptyset$; the set of atoms of a poset P will be denoted $\mathcal{A}(P)$. We introduce the applications

$$RUB : \mathscr{P} \to \mathscr{G}^{\circ} \qquad \text{and} \qquad m : \mathscr{P} \to \mathscr{G}^{\circ}$$

- $RUB(P)$ is the *reflexive upper bound graph* of P : $V(RUB(P)) = P$ et and $p \sim q$ in $RUB(P)$ if there is a $z \in P$ such that $z \geq p$ and $z \geq q$ (in other words, $p \sim q \Longleftrightarrow P_{\geq p,q} := P_{\geq p} \cap P_{\geq q} \neq \emptyset$).
- $m(P)$ is the subgraph of $RUB(P)$ induced by $\mathcal{A}(P)$, the set of atoms of P (i.e., $V(m(P)) = \mathcal{A}(P)$ and $a \sim b \in m(P)$ if there is a $p \in P$ such that $p \geq a$ and $p \geq b$).

Proposition 3.3 *For all $P \in \mathscr{P}$, $RUB(P) \searrow^{d} m(P)$.*

Proof : As $m(P)$ is the subgraph of $RUB(P)$ induced by the set of atoms $\mathcal{A}(P)$, it suffices to prove that every vertex in $V(RUB(P)) \setminus V(m(P))$ (i.e., every element of P which is not an atom) is dominated by a vertex of $m(P)$. Let $q \in V(RUB(P)) \setminus V(m(P)) = P \setminus \mathcal{A}(P)$. It is immediate that $q \overset{d}{\vdash} x$ for every vertex $x \in V(m(P)) = \mathcal{A}(P)$ such that $x < q$ (because $z \sim q \Longleftrightarrow P_{\geq z,q} \neq \emptyset \Longrightarrow P_{\geq z,x} \neq \emptyset \Longrightarrow z \sim x$). □

Proposition 3.4 *Let P in $\mathscr{P}$ and x dismantlable in P. Then, $RUB(P) \searrow^{d} RUB(P \setminus \{x\})$.*

As a consequence, $P \searrow^{d} Q \Longrightarrow RUB(P) \searrow^{d} RUB(Q)$.

Proof : Let us suppose that x is dominated by a in P. First, we verify that x is dominated by a in $RUB(P)$. So, let $y \in P$ such that $y \sim x$ in $RUB(P)$. If $y = x$, then $x \sim a$ in $RUB(P)$ because $P_{\geq y,a} = P_{\geq x,a} \neq \emptyset$. If $y \neq x$ and $z \in P_{\geq y,x}$, then $z \in P_{\geq y,a}$ (because $z \geq x$ and x is dominated by a) ; so, $y \sim a$ and x is also dominated by a in $RUB(P)$. Hence we have $RUB(P) \searrow^{d} RUB(P) - x$. Now, we compare the graphs $RUB(P) - x$ and $RUB(P \setminus \{x\})$. They have the same vertex sets and clearly $RUB(P \setminus \{x\})$ is a subgraph of $RUB(P) - x$ (if $P_{\geq y,z} \neq \emptyset$ in $P \setminus \{x\}$, we have also $P_{\geq y,z} \neq \emptyset$ in P).

Now, let us suppose that $y \sim z$ in $RUB(P) - x$; this means that $P_{\geq y,z} \neq \emptyset$. If x is in $P_{\geq y,z}$, then a is also in $P_{\geq y,z}$; so, $P_{\geq y,z} \cap (P \setminus \{x\}) \neq \emptyset$ and this proves that $y \sim z$ in $RUB(P \setminus \{x\})$. In conclusion, $RUB(P) - x = RUB(P \setminus \{x\})$ and $RUB(P) \searrow^{d} RUB(P \setminus \{x\})$. □

Let us denote by G° the reflexive graph obtained from a graph G by adding loops to its non looped vertices. We note that, by identifying $\mathcal{A}(C(G))$ with $V(G)$, we get $m(C(G)) = G^{\circ}$ for every $G \in \mathscr{G}$.

Theorem 3.3 *Let $G \in \mathscr{G}$ and H a subgraph of G such that $C(G) \searrow^{d} C(H)$. Then $G^{\circ} \searrow^{d} H^{\circ}$.*
In particular, if $G \in \mathscr{G}^{\circ}$ and H is a subgraph of G, then $C(G) \searrow^{d} C(H) \Longrightarrow G \searrow^{d} H$.

Proof : By Proposition 3.3, we have two dismantling $f_G : RUB(C(G)) \searrow^{d} m(C(G)) = G^{\circ}$ and $f_H : RUB(C(H)) \searrow^{d} m(C(H)) = H^{\circ}$. There is also a dismantling $\varphi : RUB(C(G)) \searrow^{d} RUB(C(H))$ from $C(G) \searrow^{d} C(H)$ and Proposition 3.4. So, we have the following diagram:

$$\begin{array}{ccc} RUB(C(G)) & \xrightarrow{f_G} & m(C(G)) = G^{\circ} \\ \varphi \downarrow & & \downarrow \\ RUB(C(H)) & \xrightarrow{f_H} & m(C(H)) = H^{\circ} \end{array}$$

The conclusion $G^{\circ} \searrow^{d} H^{\circ}$ follows from Corollary 2.1 applied to $(G, G', G'') = (RUB(C(G)), G^{\circ}, H^{\circ})$. □

4 Foldings versus $(\mathscr{G}, \mathscr{K})$

4.1 Dismantlability in $\mathscr{K}$

Let K be the category of finite simplicial complexes (cf. [Koz08] for a reference textbook) and let $K \in \mathscr{K}$. If σ is a simplex of K, we write $\sigma \in K$. A simplicial complex K is a *simplicial cone* if there is a subcomplex L and a vertex a of $K \setminus L$ such that the set of simplices of K is $\{\{a\}, \sigma, \{a\} \cup \sigma, \sigma \in L\}$; in this case, K is denoted aL. Let us recall the following definitions for a vertex x of K:

- $\mathrm{star}^{o}_{K}(x) := \{\sigma \in K,\ x \in \sigma\}$
- $\mathrm{lk}_{K}(x) := \{\sigma \in K,\ \{x\} \cup \sigma \in K \text{ and } x \notin \sigma\}$
- $\mathrm{star}_{K}(x) := \{\sigma \in K,\ \{x\} \cup \sigma \in K\} = \mathrm{star}^{o}_{K}(x) \cup \mathrm{lk}_{K}(x)$
- $K - x := \{\sigma \in K,\ x \notin \sigma\}$.

We note that a simplicial complex K is a simplicial cone if, and only if, one can write $K = xL$ with $L = K - x$ for some vertex x. In [BM09], a notion of dismantlability is defined in the framework of simplicial complexes. A vertex x of a simplicial complex K is said *dominated* by the vertex a of K if $\mathrm{lk}_K(x)$ is a simplicial cone aK' for some subcomplex K' of K; in this case, the deletion of the vertex x in K is called *an elementary strong collapse* and denoted $K \searrow\!\!\searrow K - x$. A strong collapse, denoted $K \searrow\!\!\searrow L$, is the succession of elementary strong collapses. In this paper, by analogy with the situation in graphs and posets, a dominated vertex in a simplicial complex K will be said *dismantlable* in K.

Remark 4.1 *In [CY07], the authors introduce the notion of linear coloring on simplicial complexes. The Theorem 6.2 of [CY07] shows that the notion of LC-reduction in [CY07, §6] and the notion of strong reduction defined in [BM09] are equivalent.*

4.2 Dismantlability and functor $\Delta_{\mathscr{G}} : \mathscr{G} \to \mathscr{K}$

Let $G \in \mathscr{G}$. We recall that $\Delta_{\mathscr{G}}(G)$ (sometimes called *the clique complex* of G) is the simplicial complex whose simplices are given by sets of vertices of complete subgraphs of G. The following facts are easy :

- If G is a reflexive graph, then G is a cone if, and only if, $\Delta_{\mathscr{G}}(G)$ is a simplicial cone.
- For every vertex x of a graph G, $\Delta_{\mathscr{G}}(N_G(x) - x) = \mathrm{lk}_{\Delta_{\mathscr{G}}(G)}(x)$.

Lemma 4.1 *Let $G \in \mathscr{G}$, $a, x \in V(G)$ such that $a \neq x$ and x looped. Then, x is dominated by a in G if, and only if, x is dominated by a in $\Delta_{\mathscr{G}}(G)$.*

Proof : Let x a looped vertex. If $x \overset{d}{\vdash} a$, then $N_G(x) - x$ is a cone with apex a and $\mathrm{lk}_{\Delta_{\mathscr{G}}(G)}(x) = \Delta_{\mathscr{G}}(N_G(x) - x) = aL$ with $L = \big(\Delta_{\mathscr{G}}(N_G(x) - x)\big) - a$, i.e. x is dominated by a in $\Delta_{\mathscr{G}}(G)$. Conversely, if $\mathrm{lk}_{\Delta_{\mathscr{G}}(G)}(x) = \Delta_{\mathscr{G}}(N_G(x) - x)$ is a simplicial cone aL, then necessarily $y \sim a$ for all $y \in N_G(x) - x$ and $x \sim a$; in other terms, $N_G(x) \subset N_G(a)$, i.e. $x \overset{d}{\vdash} a$. □

Theorem 4.1 *Let $G, H \in \mathscr{G}^\circ$. Then, $G \searrow^d H \iff \Delta_{\mathscr{G}}(G) \searrow\searrow \Delta_{\mathscr{G}}(H)$.*

Proof : Follows by iteration of Lemma 4.1. □

4.3 Dismantlability and functor $\mathcal{F}_{\mathscr{G}} : \mathscr{K} \to \mathscr{G}^\circ$

Let K a simplicial complex. The *face graph* ([BFJ08]) $\mathcal{F}_{\mathscr{G}}(K)$ of K is the reflexive graph whose vertices are the non empty simplices of K with an edge between two simplices if one contains the other. If x is a vertex of K, $\{x\}$ will denote the same vertex as a 0-simplex of K or as a vertex of $\mathcal{F}_{\mathscr{G}}(K)$. More generally, if σ is a simplex of K, we also denote σ the corresponding vertex of $\mathcal{F}_{\mathscr{G}}(K)$.

Theorem 4.2 *Let $K, L \in \mathscr{K}$. Then, $K \searrow\searrow L \implies \mathcal{F}_{\mathscr{G}}(K) \searrow^d \mathcal{F}_{\mathscr{G}}(L)$.*

Proof : It is sufficient to prove $K \searrow\searrow K - x \implies \mathcal{F}_{\mathscr{G}}(K) \searrow^d \mathcal{F}_{\mathscr{G}}(K - x)$. As $V(\mathcal{F}_{\mathscr{G}}(K)) \setminus V(\mathcal{F}_{\mathscr{G}}(K - x)) = \mathrm{star}^o_K(x)$, we have to verify that one can dismant, one by one, all the elements of $\mathrm{star}^o_K(x)$ when $\mathrm{lk}_K(x)$ is a cone. So, let x a dismantlable vertex in K and a a vertex which dominates x in K; we have $\mathrm{star}^o_K(x) = \Gamma_x \cup \Gamma_{x,a}$ with $\Gamma_x := \{\sigma \in K, x \in \sigma \text{ and } a \notin \sigma\}$ and $\Gamma_{x,a} := \{\sigma \in K, x \in \sigma \text{ and } a \in \sigma\}$. As the neighborhood in $\mathcal{F}_{\mathscr{G}}(K)$ of a maximal simplex σ of Γ_x is $\{\{a\} \cup \sigma\} \cup \{\tau, \tau \subset \sigma\}$, we have $\sigma \overset{d}{\vdash} \{a\} \cup \sigma$ in $\mathcal{F}_{\mathscr{G}}(K)$. So, all maximal simplices of Γ_x are dismantlable and, when they have been deleted, the maximal simplices of the resulting subset of Γ_x are also dismantlable by the same argument and the iteration of this procedure showes that all vertices of Γ_x are dismantlable (the procedure ends when the 0-simplex $\{x\}$ is dominated by the 1-simplex $\{a, x\}$). Next, it remains to prove that one can dismant all vertices of $\Gamma_{x,a}$. This follows from the existence of a similar procedure to the precedent, in the reverse order. First, the vertex $\{x, a\}$ is dominated by a. Next, after the removing of $\{x, a\}$, vertices of type $\{a, x, y\}$ are dominated by $\{a, y\}$ and after the removing of these vertices, vertices of type $\{a, x, y, z\}$ are dominated by $\{a, y, z\}$ and so on, until all vertices of $\Gamma_{x,a}$ have been deleted. □

Remark 4.2 *There is an obvious morphism $f \circ g : \mathcal{F}_{\mathscr{G}}(K) \to \mathcal{F}_{\mathscr{G}}(K - x)$ where $g : \mathcal{F}_{\mathscr{G}}(K) \to \mathcal{F}_{\mathscr{G}}(K) - \Gamma_x$ is defined by $g(\sigma) = \{a\} \cup \sigma$ on Γ_x and $g(\sigma) = \sigma$ otherwise and $f : \mathcal{F}_{\mathscr{G}}(K) - \Gamma_x \to \big(\mathcal{F}_{\mathscr{G}}(K) - \Gamma_x\big) - \Gamma_{x,a} = \mathcal{F}_{\mathscr{G}}(K - x)$ is defined by $f(\sigma) = \sigma \setminus \{x\}$ on $\Gamma_{x,a}$ and $f(\sigma) = \sigma$ otherwise. Nevertheless, in general, we don't have $g \sim 1_{\mathcal{F}_{\mathscr{G}}(K)}$, nor $f \sim 1_{\mathcal{F}_{\mathscr{G}}(K) - \Gamma_x}$ and the preceding proof shows the necessity of deleting the vertices of $\mathrm{star}^o_K(x)$ in a certain order.*

To establish the reciprocal statement of Theorem 4.2, we need two lemmas.

Lemma 4.2 *Let $K \in \mathscr{K}$ and L a subcomplex of K such that $\mathcal{F}_{\mathscr{G}}(K) \searrow^d \mathcal{F}_{\mathscr{G}}(L)$. If σ is a maximal simplex of K which appears in a dismantling sequence from $\mathcal{F}_{\mathscr{G}}(K)$ to $\mathcal{F}_{\mathscr{G}}(L)$, then there is a 0-simplex $\{x\}$ with $x \in \sigma$ which appears before σ in the same dismantling sequence.*

Proof : Let us suppose that σ is a maximal simplex of K which appears in a dismantling sequence from $\mathcal{F}_{\mathscr{G}}(K)$ to $\mathcal{F}_{\mathscr{G}}(L)$. This means that after having removed some vertices, we get a subgraph $\mathcal{F}'$ of $\mathcal{F}_{\mathscr{G}}(K)$ and there is a simplex σ' which dominates σ in $\mathcal{F}'$. As σ is a maximal simplex and $\sigma \sim \sigma'$, we must have $\sigma' \subsetneq \sigma$. Now, let $x \in \sigma$; $\sigma \overset{d}{\vdash} \sigma'$ implies $\{x\} \sim \sigma'$, i.e. $x \in \sigma'$. In particular, if no vertex of σ has been dismantled, then $\sigma \subset \sigma'$. But this contradict $\sigma' \subsetneq \sigma$. So, there must be at least one vertex of σ which has been dismantled before σ. □

Lemma 4.3 *Let $K \in \mathscr{K}$ and L a subcomplex of K such that $\mathcal{F}_{\mathscr{G}}(K) \searrow^d \mathcal{F}_{\mathscr{G}}(L)$. If $\{x\}$ is the first 0-simplex dismantled in a dismantling sequence from $\mathcal{F}_{\mathscr{G}}(K)$ to $\mathcal{F}_{\mathscr{G}}(L)$, then x is dismantlable in K.*

Proof : Let $\{x\}$ be the first 0-simplex dismantled in a dismantling sequence from $\mathcal{F}_{\mathscr{G}}(K)$ to $\mathcal{F}_{\mathscr{G}}(L)$ and σ a simplex such that $\{x\} \overset{d}{\vdash} \sigma$ in the dismantling process. We will show that every element of σ dominates x in K. So, let us take $a \in \sigma$, $a \neq x$ and $\tau \in \mathrm{lk}_K(x)$. We have to prove that $\tau \cup \{a\}$ is a simplex of $\mathrm{lk}_K(x)$. Let $\tau_{\max}$ be a maximal simplex of K containing $\tau \cup \{x\}$; by Lemma 4.2, we know that $\tau_{\max}$ has not been dismantled before x. As $x \in \tau_{\max}$ and $\{x\} \overset{d}{\vdash} \sigma$, we conclude that σ is adjacent to $\tau_{\max}$, i.e. $\sigma \subset \tau_{\max}$ (because $\tau_{\max}$ is maximal). Consequently, $a \in \tau_{\max}$ and $\tau \cup \{a, x\} \subset \tau_{\max}$; this shows that $\tau \cup \{a\}$ is a simplex of $\mathrm{lk}_K(x)$. □

Theorem 4.3 *Let $K \in \mathscr{K}$ and L a subcomplex of K such that $\mathcal{F}_{\mathscr{G}}(K) \searrow^{d} \mathcal{F}_{\mathscr{G}}(L)$. Then $K \searrow\!\!\searrow L$.*

Proof : By Lemma 4.3, we know that there exists a vertex x of $K - L$ such that $K \searrow\!\!\searrow K - x$. Now, from Theorem 4.2, we get a dismantling $f_x : \mathcal{F}_{\mathscr{G}}(K) \searrow^{d} \mathcal{F}_{\mathscr{G}}(K - x)$. So, with the hypothesis of a dismantling $\varphi : \mathcal{F}_{\mathscr{G}}(K) \to \mathcal{F}_{\mathscr{G}}(L)$, we have the following triangle:

$$\begin{array}{ccc} \mathcal{F}_{\mathscr{G}}(K) & \xrightarrow{f_x} & \mathcal{F}_{\mathscr{G}}(K - x) \\ & \overset{\varphi}{\searrow} & \downarrow \\ & & \mathcal{F}_{\mathscr{G}}(L) \end{array}$$

which allows to conclude $\mathcal{F}_{\mathscr{G}}(K-x) \searrow^{d} \mathcal{F}_{\mathscr{G}}(L)$ from Corollary 2.1 because $\mathcal{F}_{\mathscr{G}}(L)$ is a subgraph of $\mathcal{F}_{\mathscr{G}}(K-x)$. Now, we iterate the argument with $\mathcal{F}_{\mathscr{G}}(K-x)$. The iteration ends when all 0-simplices which are not vertices of $\mathcal{F}_{\mathscr{G}}(L)$ have been dismantled and this proves that $K \searrow\!\!\searrow L$. □

Remark 4.3 *We also deduce from the proof of Theorem 4.3 that a dismantling sequence from K to L is obtained by keeping the 0-simplices (or vertices of K) in a dismantling sequence from $\mathcal{F}_{\mathscr{G}}(K)$ to $\mathcal{F}_{\mathscr{G}}(L)$.*

5 Homotopy classes and the triangle $(\mathscr{G}^\circ, \mathscr{P}, \mathscr{K})$

5.1 Posets, simplicial complexes and dismantlability

The order complex of a poset $P \in \mathscr{P}$ is the simplicial complex $\Delta_{\mathscr{P}}(P)$ whose simplices are given by the chains of P. First, we note the elementary facts:

- $\Delta_{\mathscr{P}}(P_{>x} \cup P_{<x}) = \mathrm{lk}_{\Delta_{\mathscr{P}}(P)}(x)$.
- A poset P is a double cone with apex a (i.e., $P = P_{>a} \cup P_{<a} \cup \{a\}$) if, and only if, $\Delta_{\mathscr{P}}(P)$ is a simplicial cone with apex a.

As a consequence of these facts, an element x of P is weakly dismantlable if, and only if, x is dismantlable in $\Delta_{\mathscr{P}}(P)$ and:

Theorem 5.1 *Let $P, Q \in \mathscr{P}$. Then, $P \searrow^{wd} Q \iff \Delta_{\mathscr{P}}(P) \searrow\!\!\searrow \Delta_{\mathscr{P}}(Q)$.*

Remark 5.1 *We know from [BM09, Theorem 4.14.a] that $P \searrow^{d} Q$ implies $\Delta_{\mathscr{P}}(P) \searrow\!\!\searrow \Delta_{\mathscr{P}}(Q)$; the example of the poset P given in Figure 3 (d is dominated by a in $\Delta_{\mathscr{P}}(P)$ but not dominated in P) shows that the reciprocal statement is not true in general.*

Let K a simplicial complex. The *face poset* $\mathcal{F}_{\mathscr{P}}(K)$ of K is the poset given by the set of non empty simplices of K with the inclusion as order relation. From [BM09, Theorem 4.14.b], we know that $K \searrow\!\!\searrow L \implies \mathcal{F}_{\mathscr{P}}(K) \searrow^{d} \mathcal{F}_{\mathscr{P}}(L)$; the reciprocal statement is true:

Theorem 5.2 *Let $K, L \in \mathscr{K}$. If $\mathcal{F}_{\mathscr{P}}(K) \searrow^{d} \mathcal{F}_{\mathscr{P}}(L)$, then $K \searrow\!\!\searrow L$.*

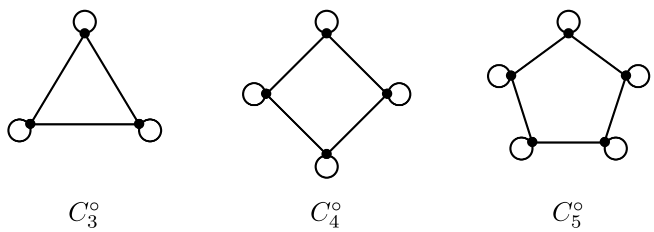
Proof : Let us suppose that $\mathcal{F}_{\mathscr{P}}(K) \searrow^{d} \mathcal{F}_{\mathscr{P}}(L)$ in $\mathscr{P}$. By Proposition 3.2 and identity $Comp \circ \mathcal{F}_{\mathscr{P}} = \mathcal{F}_{\mathscr{G}}$, $\mathcal{F}_{\mathscr{G}}(K) \searrow^{d} \mathcal{F}_{\mathscr{G}}(L)$ in $\mathscr{G}^\circ$ and, by Theorem 4.3, $K \searrow\!\!\searrow L$. □

5.2 Homotopy classes

Addition or deletion of dismantlable vertices define an equivalence relation in $\mathscr{G}$: $[G]_d = [H]_d$ if there is in $\mathscr{G}$ a sequence $G = J_0, J_1, J_2, \ldots, J_{n-1}, J_n = H$ such that $J_i \searrow^d J_{i+1}$ or $J_i \,{}^d\!\!\nearrow J_{i+1}$ or $J_i \cong J_{i+1}$ (J_i and J_{i+1} are isomorphic graphs) for $i = 0, 1, \ldots, N-1$. The equivalence class $[G]_d$ of G will be called the *d-homotopy type* of G.

The term *homotopy* is given here by analogy with the equivalence class $[P]_d$ of a poset P in $\mathscr{P}$ which is defined in a similar way by dismantlings in $\mathscr{P}$. It is well known ([Sto66]) that $[P]_d$ is actually the homotopy class of the poset P considered as a topological space (with $\{P_{\leq x}, x \in P\}$ as a base of neighborhoods). In a similar way and following [BM09, Definition 2.1], two simplicial complexes K and L have the same *strong homotopy type* if one can go from K to L by a succession of strong collapses or strong expansions.

Remark 5.2 *The d-homotopy type is quite rigid. The example (see Figure 4) of the reflexive cycles C_n° shows the important gap with the s-homotopy (two graphs G and H have the same s-homotopy type if, and only if, the simplicial complexes $\Delta_{\mathscr{G}}(G)$ and $\Delta_{\mathscr{G}}(H)$ have the same simple homotopy type, cf. [BFJ08]).*



$$[C_n^\circ]_d \neq [C_m^\circ]_d, \ \forall\, n, m \geq 3, \ n \neq m$$

$$[C_n^\circ]_s = [C_m^\circ]_s \neq [C_3^\circ]_s, \ \forall\, n, m \geq 4.$$

Figure 4: d-homotopy classes and s-homotopy classes

Proposition 5.1 *Let $P \in \mathscr{P}$ and x a weak dismantlable element in P. Then $[P]_d = [P \setminus \{x\}]_d$.*

Proof : As x is a weak dismantlable element in P, it exists an element a comparable with all elements of $P_{>x} \cup P_{<x}$. Let us suppose that $a > p$ and let $y \in \max[x, a[$ where $[x, a[= P_{\geq x} \cap P_{<a}$. It is easy to see that $y \overset{d}{\vdash} a$ (indeed, if $z > y$, then $z > x$, so z is comparable with a; but $z < a$ would contradict $y \in \max[x, a[$, so $z \geq a$). So, one can remove by dismantlability all maximal elements of $[x, a[$ and the iteration of this reasoning until x is removed proves that $P \searrow^d Q$ and $P \setminus \{x\} \searrow^d Q$ with $Q = P \setminus [x, a[= (P \setminus \{x\}) \setminus]x, a[$. If we suppose that $a < p$, we get $P \searrow^d Q'$ and $P \setminus \{x\} \searrow^d Q'$ with $Q' = P \setminus]a, x] = (P \setminus \{x\}) \setminus]a, x[$. $\square$

Remark 5.3 *As a useful consequence of Proposition 5.1, if two posets P and Q are such that $P \searrow^{wd} Q$, then $[P]_d = [Q]_d$. In other terms, the weak dismantlability preserves the homotopy type in $\mathscr{P}$.*

5.3 The triangle $(\mathscr{G}^\circ, \mathscr{P}, \mathscr{K})$

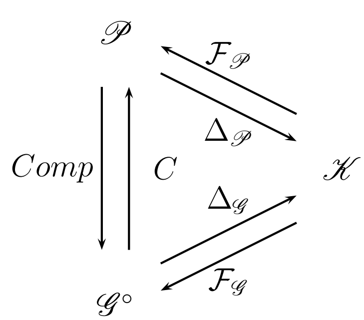


Figure 5: The triangle $(\mathscr{G}^\circ, \mathscr{P}, \mathscr{K})$

The functors in the triangle $(\mathscr{G}^\circ, \mathscr{P}, \mathscr{K})$ are compatible with the various homotopy classifications (d-homotopy type in $\mathscr{G}$, homotopy type in $\mathscr{P}$ and strong homotopy type in $\mathscr{K}$):

Theorem 5.3
1. Let $G, H \in \mathscr{G}^{\circ}$.

a. G and H have the same d-homotopy type if, and only if, $C(G)$ and $C(H)$ have the same homotopy type.

b. G and H have the same d-homotopy type if, and only if, $\Delta_{\mathscr{G}}(G)$ and $\Delta_{\mathscr{G}}(H)$ have the same strong homotopy type.

2. Let $P, Q \in \mathscr{P}$.

a. P and Q have the same homotopy type if, and only if, $Comp(P)$ and $Comp(Q)$ have the same d-homotopy type.

b. P and Q have the same homotopy type if, and only if, $\Delta_{\mathscr{P}}(P)$ and $\Delta_{\mathscr{P}}(Q)$ have the same strong homotopy type.

3. Let $K, L \in \mathscr{K}$.

a. K and L have the same strong homotopy type if, and only if, $\mathcal{F}_{\mathscr{G}}(K)$ and $\mathcal{F}_{\mathscr{G}}(L)$ have the same d-homotopy type.

b. K and L have the same strong homotopy type if, and only if, $\mathcal{F}_{\mathscr{P}}(K)$ and $\mathcal{F}_{\mathscr{P}}(L)$ have the same homotopy type.

Proof : All these equivalences are immediate corollaries of previous results: Theorems 3.2 and 3.3 (1.a), Theorem 4.1(1.b), Theorem 3.1 and Proposition 5.1 (2.a), Theorem 5.1 and Proposition 5.1 (2.b), Theorems 4.2 and 4.3 (3.a), [BM09, Theorem 4.14.b] and Theorem 5.2 (3.b). □

We recall that there is an operation of barycentric subdivision either for graphs, for posets, or for simplicial complexes ([BFJ08]) verifying $Bd = C \circ Comp = \mathcal{F}_{\mathscr{G}} \circ \Delta_{\mathscr{G}}$ in $\mathscr{G}^{\circ}$, $Bd = Comp \circ C = \mathcal{F}_{\mathscr{P}} \circ \Delta_{\mathscr{P}}$ in $\mathscr{P}$ and $Bd = \Delta_{\mathscr{G}} \circ FG = \Delta_{\mathscr{P}} \circ \mathcal{F}_{\mathscr{P}}$ in $\mathscr{K}$.

Proposition 5.2
1. Let $G, H \in \mathscr{G}$; then, $G \searrow^{d} H \Longleftrightarrow Bd(G) \searrow^{d} Bd(H)$

2. Let $K, L \in \mathscr{K}$; then, $K \searrow\searrow L \Longleftrightarrow Bd(K) \searrow\searrow Bd(L)$

3. Let $P, Q \in \mathscr{P}$; then, $P \searrow^{wd} Q \Longleftrightarrow Bd(P) \searrow^{d} Bd(Q)$

Proof : The assertions 1 and 2 are corollaries of Theorems 4.1, 4.2 and 4.3 by using, respectively, $\mathcal{F}_{\mathscr{G}} \circ \Delta_{\mathscr{G}} = Bd$ (in $\mathscr{G}$) and $\Delta_{\mathscr{G}} \circ \mathcal{F}_{\mathscr{G}} = Bd$ (in $\mathscr{K}$). The assertion 3 is a consequence of Theorems 3.1, 3.2, 3.3 and equality $C \circ Comp = Bd$ (in $\mathscr{P}$). □

Remark 5.4 *If L is reduced to a vertex of K, the assertion 2 of Proposition 5.2 is [BM09, Theorem 4.15].*

6 Remarks about the Hom complex

Let $G, H \in \mathscr{G}$. The set of morphisms from G to H is the vertex set of the reflexive graph $\mathsf{hom}_{\mathscr{G}}(G, H)$ and is also the set of vertices (or 0-dimensional cells) of the polyhedral complex $\mathsf{Hom}(G, H)$ ([BK06],[Koz08]) whose cells are indexed by functions (which will be called *indexing functions*) $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that if $(x, y) \in E(G)$, then $\eta(x) \times \eta(y) \subset E(H)$.

Example 6.1 *We will illustrate the results of this section with the example given by the path $G = P_3$ (i.e., $V(G) = \{0, 1, 2\}$ and $0 \sim 1 \sim 2$) and the complete graph $H = K_3$ (i.e., $V(K) = \{a, b, c\}$ and $a \sim b \sim c \sim a$). The notation $\left| \begin{array}{l} r \\ s \\ t \end{array} \right.$ will indicate a morphism from P_3 to K_3 which sends 0 to r, 1 to s and 2 to t. There are 12 morphisms from P_3 to K_3:*

u	v	w	x	y	z	f	g	h	j	k	l
a	b	b	c	c	a	a	b	b	c	a	c
c	c	a	a	b	b	c	c	a	a	b	b
a	b	b	c	c	a	b	a	c	b	c	a

The graph $\mathsf{hom}_{\mathscr{G}}(P_3, K_3)$ and the polyhedral complex $\mathsf{Hom}(K_3, P_3)$ are represented in Figure 6.

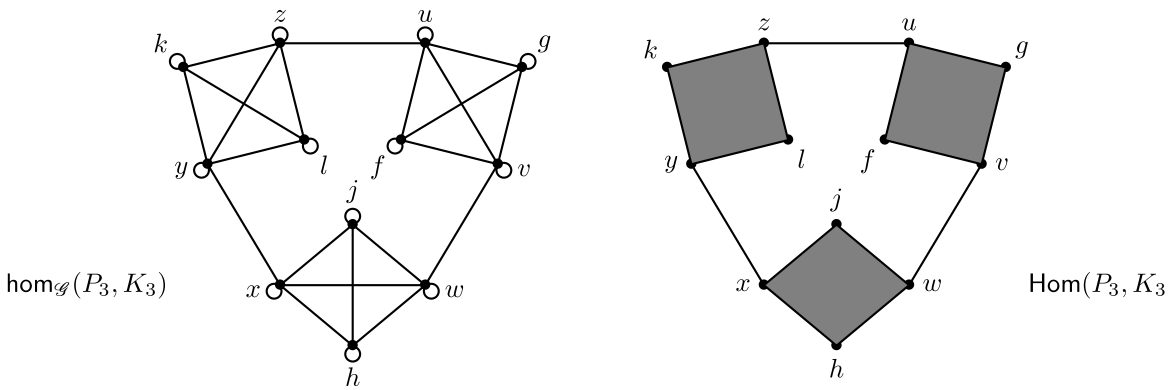


Figure 6: The graph $\mathsf{hom}_{\mathscr{G}}(P3, K3)$ and the polyhedral complex $\mathsf{Hom}(P_3, K_3)$

6.1 Hom$(-,-)$ and hom$_{\mathscr{G}}(-,-)$

For studying the polyhedral complex $\mathsf{Hom}(G,H)$, it is usual to consider its face poset $\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H))$ whose elements are all indexing functions with order given by $\eta \leq \eta'$ if and only if $\eta(x) \subset \eta'(x)$ for all x in $V(G)$.

Actually, there is a natural identification of $\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H))$ with a subposet of $C(\mathsf{hom}_{\mathscr{G}}(G,H))$, the poset of complete subgraphs of $\mathsf{hom}_{\mathscr{G}}(G,H)$. Indeed, let $\eta \in \mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H))$ and for every vertex x of G, let us choose an element $y_x \in \eta(x)$. Then the application $f : V(G) \to V(H), x \mapsto y_x$ is actually a morphism from G to H ; such an application will be called an *associated morphism to* η. The set of all morphisms associated to η will be called $\Psi(\eta)$. By definition of indexing functions, $\Psi(\eta)$ induces a complete subgraph of $\mathsf{hom}_{\mathscr{G}}(G,H)$ and we get an injective poset map

$$\begin{array}{rcl} \Psi \ : \ \mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H)) & \longrightarrow & C(\mathsf{hom}_{\mathscr{G}}(G,H)) \\ \eta & \mapsto & \Psi(\eta) \end{array}$$

which identifies $\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H))$ with a subposet of $C(\mathsf{hom}_{\mathscr{G}}(G,H))$.

Now let $[f_1, f_2, \ldots, f_k] \in C(\mathsf{hom}_{\mathscr{G}}(G,H))$ (i.e., the set $\{f_1, f_2, \ldots, f_k\}$ of morphisms from G to H induces a complete subgraph of $\mathsf{hom}_{\mathscr{G}}(G,H)$). We define the indexing function $\Phi([f_1, f_2, \ldots, f_k]) : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ by $\Phi([f_1, f_2, \ldots, f_k])(x) = \{f_1(x), f_2(x), \ldots, f_k(x)\}$ for all $x \in V(G)$. This gives a morphism of posets:

$$\begin{array}{rcl} \Phi \ : \ C(\mathsf{hom}_{\mathscr{G}}(G,H)) & \hookrightarrow & \mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H)) \\ [f_1, f_2, \ldots, f_k] & \mapsto & \Phi([f_1, f_2, \ldots, f_k]) \end{array}$$

Proposition 6.1 *Let $G, H \in \mathscr{G}$. By identifying $\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H))$ with a subposet of $C(\mathsf{hom}_{\mathscr{G}}(G,H))$, we have:*

$$C(\mathsf{hom}_{\mathscr{G}}(G,H)) \searrow^{d} \mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H))$$

Proof : First, we note that $\Phi \circ \Psi = 1_{\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H))}$. This implies that $(\Psi \circ \Phi)^2 = \Psi \circ \Phi$, i.e. $\Psi \circ \Phi : C(\mathsf{Hom}(G,H)) \to C(\mathsf{Hom}(G,H))$ is a retraction on $\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H))$ (identified with $\Psi(\mathsf{Hom}(G,H))$). Next, for all $[f_1, f_2, \ldots, f_k] \in C(\mathsf{hom}_{\mathscr{G}}(G,H))$, $[f_1, f_2, \ldots, f_k] \subset \Psi \circ \Phi([f_1, f_2, \ldots, f_k])$, i.e. $1_{C(\mathsf{hom}_{\mathscr{G}}(G,H))} \leq \Psi \circ \Phi$. The conclusion follows from Proposition 3.1. $\square$

Example 6.2 *The posets obtained when $G = P_3$ and $K = K_3$ are drawed in Figures 7 and 8. The dismantling sequence: $fuv, guv, fgu, fgv, fg, uv, hwx, jwx, hjw, hjx, wx, hj, kyz, lyz, kly, klz, yz, kl$ illustrates the Proposition 6.1.*

The *face graph* $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G,H)$ of the polyhedral complex $\mathsf{Hom}(G,H)$ is the graph whose vertices are the indexing functions of $\mathsf{Hom}(G,H)$ with edges $\eta \sim \eta'$ if and only if either $\eta(x) \subset \eta'(x)$ for all x in $V(G)$, or $\eta'(x) \subset \eta(x)$ for all x in $V(G)$. In other words, $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G,H)) = Comp(\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(G,H)))$.

Corollary 6.1 *Let $G, H \in \mathscr{G}$. we have the following dismantling in $\mathscr{G}$:*

$$Bd(\mathsf{hom}_{\mathscr{G}}(G,H)) \searrow^{d} \mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G,H))$$

Proof : Follows from Proposition 6.1 by using $Comp \circ C = Bd$, $Comp \circ \mathcal{F}_{\mathscr{P}} = \mathcal{F}_{\mathscr{G}}$ and Proposition 3.2. $\square$

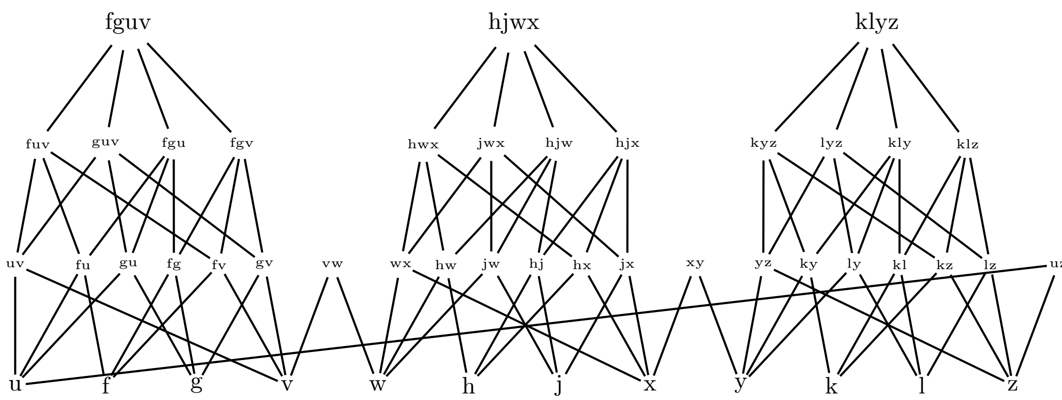


Figure 7: $C(\mathsf{hom}_{\mathscr{G}}(P3, K3))$

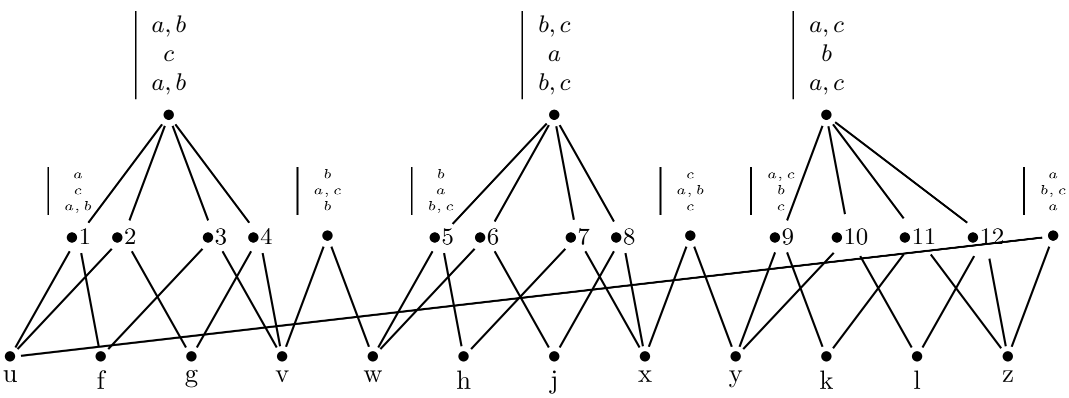


Figure 8: $\mathcal{F}_{\mathscr{P}}(\mathsf{Hom}(P3, K3))$

6.2 Hom(G, H) and foldings in G or in H

Theorem 6.1 *Let $G, H \in \mathscr{G}$.*

1. *If a is dismantlable in G, then $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G, H)) \searrow^{d} \mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G - a, H))$ (by identifying $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G - a, H))$ with a subgraph of $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G, H))$).*

2. *If u is dismantlable in H, then $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G, H)) \searrow^{d} \mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G, H - u))$ (by identifying $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G, H - u))$ with a subgraph of $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G, H))$).*

Proof : 1. We have the following diagram where the morphisms A and A' are dismantlings given by Corollary 6.1 and the morphism B is a dismantling given by Propositions 2.1 and 5.2.1:

$$\begin{array}{ccc} Bd(\mathsf{hom}_{\mathscr{G}}(G, H)) & \xrightarrow[\mathbf{A}]{} & \mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G, H)) \\ \mathbf{B}\downarrow & & \vdots \\ Bd(\mathsf{hom}_{\mathscr{G}}(G - a, H)) & \xrightarrow{\mathbf{A}'} & \mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G - a, H)) \end{array}$$

By considering $(G, G', G'') = (Bd(\mathsf{hom}_{\mathscr{G}}(G, H)), \mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G, H)), \mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(G - a, H)))$, the conclusion follows from Corollary 2.1.

2. The proof is similar. □

Example 6.3 *Returning to the case $G = P_3$ and $H = K_3$, we have $2 \vdash^{d} 0$ in P_3 and $P_3 \searrow^{d} P_3 - 2 = K_2$. The deletion in $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(P_3, K_3))$ of the twelve numbered vertices in the order indicated in Figure 8 followed by the deletion of the vertices f, g, h, j, k and l is a dismantling sequence from $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(P_3, K_3))$ to*

$\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(K_2, K_3))$ *(identified with a subgraph of* $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(P_3, K_3))$*). We note that* $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(K_2, K_3))$ *is a stiff graph.*

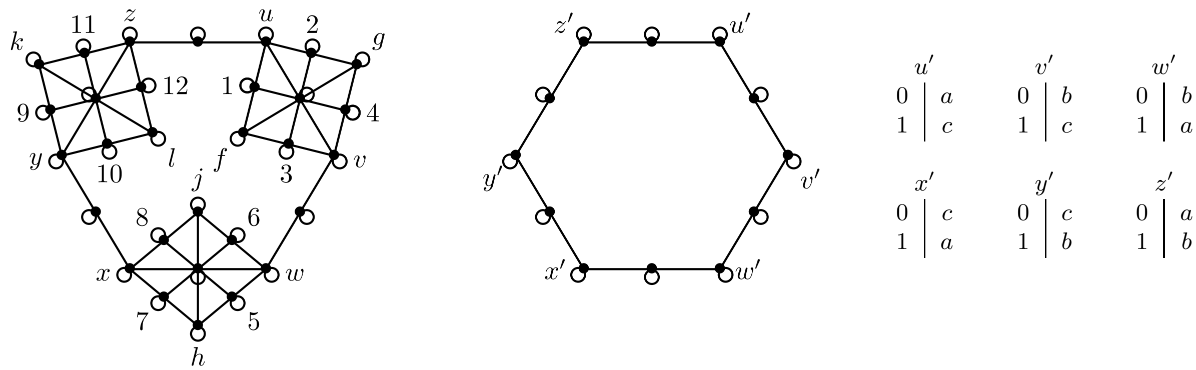


Figure 9: The graphs $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(P3, K3))$ and $\mathcal{F}_{\mathscr{G}}(\mathsf{Hom}(K2, K3))$

Using $\Delta_{\mathscr{G}} \circ \mathcal{F}_{\mathscr{G}} = Bd$ (in $\mathscr{K}$) and Theorem 4.1, we get the following corollary:

Corollary 6.2 *Let* $G, H \in \mathscr{G}$*.*

1. *If* a *is dismantlable in* G*, then* $Bd(\mathsf{Hom}(G, H)) \searrow\searrow Bd(\mathsf{Hom}(G - a, H))$

   In particular, $Bd(\mathsf{Hom}(G, H))$ *and* $Bd(\mathsf{Hom}(G - a, H))$ *have the same strong homotopy type.*

2. *If* u *is dismantlable in* H*, then* $Bd(\mathsf{Hom}(G, H)) \searrow\searrow Bd(\mathsf{Hom}(G, H - u))$

   In particular, $Bd(\mathsf{Hom}(G, H)$ *and* $Bd(\mathsf{Hom}(G, H - u))$ *have the same strong homotopy type.*

Remark 6.1 *The result of the corollary implies the well known results about the behaviour of* $\mathsf{Hom}(G, H)$ *in relation to foldings in* G *or* H *([BK06], [Koz06],[Cso08]). It is actually more precise because the strong homotopy type is very much stronger than the simple homotopy type (cf. Remark 5.2) and the equality of the two strong homotopy types follows from a strong collapse. Moreover, this result in the framework of simplicial complexes is itself a consequence of Theorem 6.1 in the framework of graphs.*

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