Structure theorem for tournaments excluding U_5

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Abstract

Let U_5 be the tournament with vertices $v_1, \ldots, v_5$ such that $v_2 \to v_1$, and $v_i \to v_j$ if $j - i \equiv 1, 2 \pmod 5$ and $\{i, j\} \neq \{1, 2\}$. In this paper we describe the tournaments which do not have U_5 as a subtournament. Specifically, we show that if a tournament G is "prime"—that is, if there is no subset $X \subseteq V(G)$, $1 < |X| < |V(G)|$, such that for all $v \in V(G) \setminus X$, either $v \to x$ for all $x \in X$ or $x \to v$ for all $x \in X$—then G excludes U_5 if and only if either G is a specific tournament T_n or $V(G)$ can be partitioned into sets X, Y, Z such that $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive. From the prime tournaments that exclude U_5 we can construct all the tournaments which exclude U_5.

1 Introduction

A *tournament* G is a loopless directed graph such that for any two distinct vertices $u, v \in V(G)$, there is exactly one edge with both ends in $\{u, v\}$. In this paper, all tournaments are finite. A *subtournament* of a tournament G is a tournament induced on a subset of $V(G)$. For $X \subseteq V(G)$, let $G[X]$ denote the subtournament of G induced on X. Given two tournaments G and H, we say that G *excludes* H if G has no subtournament isomorphic to H; otherwise, G *contains* H. Given a tournament G and a vertex $v \in V(G)$, let $B_G(v) = \{u \in V(G) : u \to v\}$ be the set of *predecessors* of v in G, and let $A_G(v) = \{u \in V(G) : v \to u\}$ be the set of *successors* of v in G. For two disjoint sets X, $Y \subseteq V(G)$, we write $X \Rightarrow Y$ if $x \to y$ for all $x \in X$, $y \in Y$. We use $v \Rightarrow X$ and $X \Rightarrow v$ to mean $\{v\} \Rightarrow X$ and $X \Rightarrow \{v\}$, respectively. The *dual* of a tournament G is the tournament obtained by reversing all edges of G. A *cyclic triangle* in a tournament G is a set $\{v_1, v_2, v_3\} \subseteq V(G)$ of three distinct vertices such that $v_1 \to v_2 \to v_3 \to v_1$. A *transitive* tournament is a tournament with no cyclic triangle; a tournament G is transitive if and only if its vertices can be ordered $v_1, \ldots, v_{|V(G)|}$ such that $v_i \to v_j$ if $i < j$. Let I_n denote the transitive tournament with n vertices. If the subtournament induced on a subset $X \subseteq V(G)$ is transitive, we say that X is transitive.

Given a tournament G, a *homogeneous set* of G is a subset $X \subseteq V(G)$ such that for all vertices $v \in V(G) \setminus X$, either $v \Rightarrow X$ or $X \Rightarrow v$. A homogeneous set $X \subseteq V(G)$ is *nontrivial* if $1 < |X| < |V(G)|$; otherwise it is *trivial*. A tournament is *prime* if all of its homogeneous sets are trivial. Given a tournament G and a nonempty homogeneous set X of G, let G/X denote the tournament isomorphic to $G\left[(V(G) \setminus X) \cup \{v\}\right]$, where v is any vertex in X. (Note that G/H is well-defined up to isomorphism.) Thus, if G has a nontrivial homogeneous set X, we can express it as the combination of two tournaments G/X and $G[X]$ each of which has less vertices than G. In addition, note that if H is a prime tournament, then G excludes H if and only if G/X and $G[X]$ exclude H.

Define U_5 to be the tournament with vertices $v_1, \ldots, v_5$ such that $v_2 \to v_1$, and $v_i \to v_j$ if $j - i \equiv 1, 2 \pmod 5$ and $\{i, j\} \neq \{1, 2\}$. (Alternatively, U_5 is the tournament with vertices $u_1,\ldots,u_5$ such that for any $1 \leq i < j \leq 5$, we have $u_i \to u_j$ if i, j are not both odd, and $u_j \to u_i$ otherwise.) The tournament U_5 is prime. In this paper, we characterize the tournaments which exclude U_5. To do this, it suffices to characterize the prime tournaments which exclude U_5, because any tournament G with a nontrivial homogeneous set X excludes U_5 if and only if the strictly smaller tournaments G/X and $G[X]$ exclude U_5.

To state the main theorem, we define T_n for odd $n \geq 1$ to be the tournament with vertices $v_1,\ldots,v_n$ such that $v_i \to v_j$ if $j - i \equiv 1, 2, \ldots, (n-1)/2 \pmod n$. The theorem is as follows.

Theorem 1.1. *Let G be a prime tournament. Then G excludes U_5 if and only if G is T_n for some odd $n \geq 1$ or $V(G)$ can be partitioned into sets X, Y, Z such that $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive.*

The paper is organized as follows. In Section 2, we review some results on prime tournaments and introduce the "critical" tournaments. In Section 3, we give some examples of prime tournaments that exclude U_5 and verify that they satisfy Theorem 1.1. In Section 4, we prove several preliminary facts that will be used in the proof of the main theorem. Section 5 is the proof of the main theorem.

2 Prime tournaments

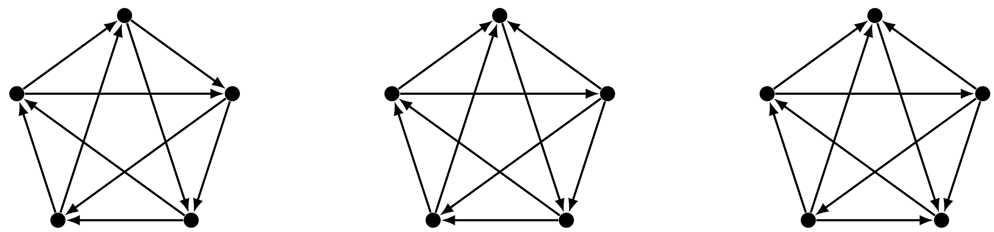
We list some properties of prime tournaments. First, note that each strong component of a tournament is a homogeneous set. Thus, if a tournament is prime, either it is strongly connected or all of its strong components have exactly one vertex. In the latter case, the tournament is transitive, and a transitive tournament is prime if and only if it has at most two vertices. Thus, every prime tournament with at least three vertices is strongly connected.

All tournaments with at most two vertices are prime. The only prime tournament with three vertices is the tournament whose vertex set is a cyclic triangle. There are no prime tournaments with four vertices. For five vertices, there are exactly three prime tournaments T_5, U_5, and W_5, drawn in Figure 1. Every prime tournament with at least five vertices contains at least one of T_5, U_5, or W_5 (Ehrenfeucht and Rozenberg [2]).

The tournaments T_5, U_5, and W_5 have the following generalizations to any odd number of vertices.

Definition 2.1. *Let $k \geq 0$ and $n = 2k + 1$. The tournaments T_n, U_n, and W_n are defined as follows.*

- *T_n is the tournament with vertices $v_1, \ldots, v_n$ such that $v_i \to v_j$ if $j - i \equiv 1, 2, \ldots, k \pmod n$.*
- *U_n is the tournament obtained from T_n by reversing all edges which have both ends in $\{v_1, \ldots, v_k\}$.*



(a) T_5 (b) U_5 (c) W_5

Figure 1: The three five-vertex prime tournaments

- *W_n is the tournament with vertices v, $w_1, \ldots, w_{n-1}$ such that $w_i \to w_j$ if $i < j$, and $\{w_i : i \text{ even}\} \Rightarrow v \Rightarrow \{w_i : i \text{ odd}\}$.*

The tournaments T_n, U_n, and W_n are prime for all odd $n \geq 1$, and their only prime subtournaments with at least three vertices are T_m, U_m, and W_m, respectively, for odd $3 \leq m < n$. These tournaments are all isomorphic to their duals. Finally, note that T_1, U_1, and W_1 are all the one-vertex tournament, and T_3, U_3, and W_3 are all the cyclic triangle tournament.

The tournaments T_n, U_n, and W_n are known as the "critical tournaments" due to the following theorem by Schmerl and Trotter.

Theorem 2.2 (Schmerl and Trotter [5]). *If G is a prime tournament with $|V(G)| \geq 6$, and G is not T_n, U_n, or W_n for any odd $n \geq 1$, then G has a prime subtournament with $|V(G)| - 1$ vertices.*

In [4], the author proved the following strengthening of Theorem 2.2.

Theorem 2.3 ([4]). *Let G be a prime tournament which is not T_n, U_n, or W_n for any odd $n \geq 1$, and let H be a prime subtournament of G with $5 \leq |V(H)| < |V(G)|$. Then there exists a prime subtournament of G with $|V(H)| + 1$ vertices that has a subtournament isomorphic to H.*

This theorem can be used to prove the following result, which appears in Belkhechine and Boudabbous [1].

Theorem 2.4 (Belkhechine and Boudabbous [1]). *Let G be a prime tournament which contains T_5. Then either G is T_n for some odd $n \geq 1$ or G contains U_5 and W_5.*

Proof. Suppose G is not T_n for any n. In particular, since G contains T_5, $|V(G)| \geq 6$. Since G contains T_5, it is not U_n or W_n for any n. So by Theorem 2.3, there exists a prime subtournament H of G, $|V(H)| = 6$, that contains T_5. Let $V(H) = \{u, v_1, \ldots, v_5\}$, where $v_i \to v_j$ if $j - i \equiv 1$ or $2 \pmod 5$.

Let $B = B_H(u)$ and $A = A_H(u)$. By taking the dual if necessary (we can do this because T_5, U_5, and W_5 are isomorphic to their duals), we can assume $|B| \leq 2$. If $|B| = 0$, then $u \Rightarrow V(H) \setminus \{u\}$, contradicting the primeness of H. Suppose $|B| = 1$. Without loss of generality, assume $v_1 \in B$ and $v_2, \ldots, v_5 \in A$. Then the tournament induced on $\{v_2, u, v_4, v_5, v_1\}$ is isomorphic to U_5, and the tournament induced on $\{v_5, v_1, u, v_2, v_3\}$ is isomorphic to W_5, as desired.

Finally, suppose $|B| = 2$. We either have v_i, $v_{i+1} \in B$ for some i or v_i, $v_{i+2} \in B$ for some i, where the indices are taken modulo 5. Suppose we have the former case; without loss of generality, assume $i = 1$. Then $\{u, v_3\}$ is a homogeneous set in H, which contradicts the primeness of H. So we must have v_i, $v_{i+2} \in B$ for some i; without loss of generality, assume $i = 1$. Then the tournament induced on $\{v_2, v_1, v_3, u, v_5\}$ is isomorphic to U_5, and the tournament induced on $\{v_2, v_3, u, v_4, v_5\}$ is isomorphic to W_5. □

Theorem 2.4 implies that to prove Theorem 1.1, it suffices to prove the following.

Theorem 2.5. *Let G be a prime tournament. Then G excludes T_5 and U_5 if and only if $V(G)$ can be partitioned into sets X, Y, Z such that $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive.*

To conclude this section, we state a subtournament exclusion result different from ours. In [3], Latka characterized all the tournaments which exclude W_5. To state the theorem, we define Q_7 to be the Paley tournament on 7 vertices; that is, the tournament with vertices $v_1, \ldots, v_7$ such that $v_i \to v_j$ if $j - i$ is a quadratic residue modulo 7. Let $Q_7 - v$ be the tournament obtained from Q_7 by deleting a vertex.

Theorem 2.6 (Latka [3])**.** *A prime tournament excludes W_5 if and only if it is isomorphic to one of I_2, $Q_7 - v$, Q_7, T_n, or U_n for some some odd $n \geq 1$.*

3 Examples

We now give some examples of families of prime tournaments that exclude U_5. First, we have that T_n excludes U_5 for all odd $n \geq 1$; this obviously agrees with Theorem 1.1.

The tournament W_n excludes U_n for for all odd $n \geq 1$. To see that this agrees with Theorem 1.1, let $V(W_n) = \{v, w_1, \ldots, w_{n-1}\}$ as in Definition 2.1, and let $X = \{v\}$, $Y = \{w_i : i \text{ odd}\}$, and $Z = \{w_i : i \text{ even}\}$. Then X, Y, and Z partition $V(W_n)$, and $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive, as desired.

Finally, let P_n be the tournament with vertices $v_1, \ldots, v_n$ such that $v_i \to v_j$ if $j - i \geq 2$, and $v_{i+1} \to v_i$ for all $1 \leq i < n$. Then P_n is prime for all $n \neq 4$, and P_n excludes U_5 for all n. Let $X = \{v_i : i \equiv 0 \pmod 3\}$, $Y = \{v_i : i \equiv 1 \pmod 3\}$, and $Z = \{v_i : i \equiv 2 \pmod 3\}$. Then X, Y, and Z partition $V(P_n)$, and $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive.

4 Preliminaries

Before proving the main theorem, we establish some facts that will aid us in the proof.

Proposition 4.1. *Let G be a tournament, and let X, $Y \subseteq V(G)$ be disjoint transitive sets. Let $X = \{x_1, \ldots, x_\ell\}$ and $Y = \{y_1, \ldots, y_m\}$, where $x_i \to x_j$ if $i < j$ and $y_i \to y_j$ if $i < j$. Suppose that for each $x_i \in X$, there is an integer $1 \leq s_i \leq m + 1$ such that $\{y_j : j < s_i\} \Rightarrow x_i \Rightarrow \{y_j : j \geq s_i\}$. Then $X \cup Y$ is transitive if and only if $s_1 \leq s_2 \leq \ldots \leq s_\ell$.*

Proposition 4.1 is clear; we omit the proof.

Proposition 4.2. *Let G be a strongly connected tournament, $|V(G)| \geq 3$, such that $V(G)$ can be partitioned into sets X, Y, Z such that $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive. Let $X = \{x_1, \dots, x_\ell\}$, $Y = \{y_1, \dots, y_m\}$, and $Z = \{z_1, \dots, z_n\}$, where $x_i \to x_j$ if $i < j$, and similarly for Y and Z. Then there exists a sequence C_1, C_2, ..., $C_{|V(G)|-2}$ of cyclic triangles such that $C_1 = \{x_1, y_1, z_1\}$ and if $C_r = \{x_i, y_j, z_k\}$, $1 \leq r < |V(G)| - 2$, then $C_{r+1} = \{x_{i+1}, y_j, z_k\}$, $\{x_i, y_{j+1}, z_k\}$, or $\{x_i, y_j, z_{k+1}\}$.*

Note that if $C_1, \dots, C_{|V(G)|-2}$ is such a sequence, then $V(G) = C_1 \cup \dots \cup C_{|V(G)|-2}$.

Proof. First, note that if any of X, Y, or Z are empty, then G is transitive, which contradicts the fact that G is strongly connected since $|V(G)| \geq 3$. So X, Y, and Z are all nonempty.

It suffices to show that $\{x_1, y_1, z_1\}$ is a cyclic triangle, and if $\{x_i, y_j, z_k\}$ is a cyclic triangle and $i + j + k < |V(G)|$, then one of $\{x_{i+1}, y_j, z_k\}$, $\{x_i, y_{j+1}, z_k\}$, or $\{x_i, y_j, z_{k+1}\}$ exists (e.g., $\{x_{i+1}, y_j, z_k\}$ exists if and only if $i < \ell$) and is a cyclic triangle. First, suppose that $\{x_1, y_1, z_1\}$ is not a cyclic triangle. Then one of the vertices in $\{x_1, y_1, z_1\}$ is a predecessor of the other two; without loss of generality, assume $x_1 \Rightarrow \{y_1, z_1\}$. Then since $X \cup Y$ and $X \cup Z$ are transitive, we have $x_1 \Rightarrow V(G) \setminus \{x\}$. This contradicts the strong connectivity of G. So $\{x_1, y_1, z_1\}$ is a cyclic triangle.

Finally, suppose that $\{x_i, y_j, z_k\}$ is a cyclic triangle and $i + j + k < |V(G)|$. Without loss of generality, assume $x_i \to y_j \to z_k \to x_i$. Suppose that none of $\{x_{i+1}, y_j, z_k\}$, $\{x_i, y_{j+1}, z_k\}$, and $\{x_i, y_j, z_{k+1}\}$ are cyclic triangles (if they exist). If $i < \ell$, then since $z_k \to x_i$ and $Z \cup X$ is transitive, we have $z_k \to x_{i+1}$. Then since $\{x_{i+1}, y_j, z_k\}$ is not a cyclic triangle and $y_j \to z_k$, we have $y_j \to x_{i+1}$ as well, so $\{x_i, y_j, z_k\} \Rightarrow \{x_{i+1}\}$. Since $X \cup Y$ and $X \cup Z$ are transitive, we thus have

$$\{x_1, \dots, x_i\} \cup \{y_1, \dots, y_j\} \cup \{z_1, \dots, z_k\} \Rightarrow \{x_{i'} : i' > i\}.$$

The above statement also holds if $i = \ell$, since in that case the right side is empty.

Now, we similarly have $\{x_1, \dots, x_i\} \cup \{y_1, \dots, y_j\} \cup \{z_1, \dots, z_k\} \Rightarrow \{y_{j'} : j' > j\}$ and $\{x_1, \dots, x_i\} \cup \{y_1, \dots, y_j\} \cup \{z_1, \dots, z_k\} \Rightarrow \{z_{k'} : k' > k\}$. Thus,

$$\{x_1, \dots, x_i\} \cup \{y_1, \dots, y_j\} \cup \{z_1, \dots, z_k\} \Rightarrow \{x_{i'} : i' > i\} \cup \{y_{j'} : j' > j\} \cup \{z_{k'} : k' > k\}.$$

Since $i + j + k < |V(G)|$, the right side of the above statement is nonempty. This contradicts the strong connectivity of G. Thus, one of $\{x_{i+1}, y_j, z_k\}$, $\{x_i, y_{j+1}, z_k\}$, or $\{x_i, y_j, z_{k+1}\}$ is a cyclic triangle, completing the proof. □

Proposition 4.3. *Let G be a tournament which excludes T_5 and U_5, and let $v \in V(G)$. Let $B = B_G(v)$ and $A = A_G(v)$. Suppose that $C = \{x, y, z\} \subseteq V(G) \setminus \{v\}$ is a cyclic triangle. Then the following hold.*

(a) If $|C \cap A| = 1$ and there is $u \in V(G) \setminus \{v\}$ such that $u \Rightarrow C$, then $u \in B$.

(b) If $|C \cap A| = 2$ and there is $u \in V(G) \setminus \{v\}$ such that $C \Rightarrow u$, then $u \in A$.

If in addition we have $x' \in V(G) \setminus \{v\}$ such that $C' = \{x', y, z\}$ is a cyclic triangle and $x \to x'$, then the following hold.

(c) If $|C \cap A| > 0$, then $|C' \cap A| > 0$.

(d) If $|C \cap A| = 3$, then $|C' \cap A| = 3$.

(e) If $y \to z$ and $y \in B$, $z \in A$, then x and x' are either both in B or both in A.

Proof. Assume without loss of generality that $x \to y \to z \to x$. For part (a), assume without loss of generality that $x \in A$ and $y, z \in B$. Let $u \in V(G) \setminus \{v\}$ such that $u \Rightarrow C$. If $u \in A$, then the tournament induced on $\{x, u, y, z, v\}$ is U_5, a contradiction. So $u \in B$. Part (b) follows from (a) by taking the dual.

For parts (c) through (e), we have $x' \to y \to z \to x'$. For part (c), suppose that $|C \cap A| > 0$ and $|C' \cap A| = 0$. Then we must have $x \in A$ and x', y, $z \in B$. But then the tournament induced on $\{v, z, x, x', y\}$ is U_5, a contradiction. This proves (c). Part (d) follows from the contrapositive of (c) by taking the dual.

Finally, for part (e), suppose $y \in B$ and $z \in A$. First, suppose that $x \in B$ and $x' \in A$. Then the tournament induced on $\{x', x, y, v, z\}$ is U_5, a contradiction. Now suppose that $x \in A$ and $x' \in B$. Then the tournament induced on $\{x, x', y, v, z\}$ is T_5, a contradiction. So x and x' are either both in A or both in B, as desired. □

In particular, given a sequence $C_1, \ldots, C_{|V(G)|-2}$ of cyclic triangles as in Proposition 4.2, we can apply Proposition 4.3(c)-(e) to consecutive cyclic triangles C_r, C_{r+1}.

5 Proof of theorem

We now prove Theorem 1.1. Recall that by Theorem 2.4, it suffices to prove the following.

Theorem 2.5. *Let G be a prime tournament. Then G excludes T_5 and U_5 if and only if $V(G)$ can be partitioned into sets X, Y, Z such that $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive.*

Proof. We organize the proof into three steps.

Step 1: "If" direction. Suppose $V(G)$ can be partitioned into sets X, Y, Z such that $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive. Then for any five-vertex subtournament H of G, at least one of $|V(H) \cap X|$, $|V(H) \cap Y|$, $|V(H) \cap Z|$ is at most 1, so at least one of $|V(H) \cap (X \cup Y)|$, $|V(H) \cap (Y \cup Z)|$, $|V(H) \cap (Z \cup X)|$ is at least 4. Hence, H has a four-vertex transitive subtournament, and thus is not T_5 or U_5. So G excludes T_5 and U_5, proving one direction of the theorem.

Step 2: Setting up the induction. We now prove the other direction. Suppose G excludes T_5 and U_5. We wish to prove that $V(G)$ can be partitioned into sets X, Y, Z such that $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive. We proceed by induction on $|V(G)|$. If $|V(G)| \leq 5$, then since G is prime and excludes T_5 and U_5, G is either I_2 or W_n for $n = 1$, 3, or 5. In the former case, let $X = V(G)$ and Y, $Z = \varnothing$; in the latter case, let $X = \{v\}$, $Y = \{w_i : i \text{ odd}\}$, and $Z = \{w_i : i \text{ even}\}$, where v, $w_1, \ldots, w_n$ are as in Definition 2.1. In either case, $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive, as desired.

Now, assume $|V(G)| \geq 6$, and that the theorem holds for all prime tournaments G' with $|V(G')| < |V(G)|$. If G is W_n for some n, then we are done by setting $X = \{v\}$, $Y = \{w_i : i \text{ odd}\}$, and $Z = \{w_i : i \text{ even}\}$ as before. Assume G is not W_n for any n. Since G excludes T_5 and U_5, it is not T_n or U_n for any n. Thus, by Theorem 2.2, there is a prime subtournament G' of G with $|V(G')| = |V(G)| - 1$. Now, G' excludes T_5 and U_5, so by the inductive hypothesis we can partition $V(G')$ into sets X, Y, and Z such that $X \cup Y$, $Y \cup Z$, and $Z \cup X$ are transitive. Let $X = \{x_1, \ldots, x_\ell\}$, $Y = \{y_1, \ldots, y_m\}$, and $Z = \{z_1, \ldots, z_n\}$ such that $x_i \to x_j$ if $i < j$, and similarly for Y and Z. Since G' is strongly connected (because G' is prime and $|V(G')| \geq 5$), there exists a sequence $C_1, C_2, \ldots, C_{|V(G')-2}$ of cyclic triangles in G' as in Theorem 4.2.

Now, G is prime and $|V(G')| \geq 5$, so neither $v \Rightarrow V(G')$ nor $V(G') \Rightarrow v$ hold. In addition, we have $V(G') = C_1 \cup \cdots \cup C_{|V(G')|-2}$ and $C_r \cap C_{r+1} \neq \varnothing$ for all $1 \leq r < |V(G')| - 2$. Thus, there is some $1 \leq r \leq |V(G')| - 2$ such that $|C_r \cap A| \neq 0, 3$; i.e., there is some r such that $|C_r \cap A| = 1$ or 2. By taking the dual if necessary, we may assume there is some r such that $|C_r \cap A| = 2$. Choose r_0 to be the minimum r such that $|C_r \cap A| = 2$. Let $C_{r_0} = \{x_{i_0}, y_{j_0}, z_{k_0}\}$, and without loss of generality, assume $x_{i_0} \to y_{j_0} \to z_{k_0} \to x_{i_0}$ and $x_{i_0} \in A$, $y_{j_0} \in A$, and $z_{k_0} \in B$.

We set some final notation before proceeding. Let s be the smallest integer such that $y_s \in A$ (thus, $s \leq j_0$), and let t be the largest integer such that $z_{t-1} \in B$ (thus, $t \geq k_0 + 1$). Since $X \cup Y$ and $X \cup Z$ are transitive, for each $x_i \in X$ let $1 \leq s_i \leq m + 1$ be the integer such that $\{y_j : j < s_i\} \Rightarrow x_i \Rightarrow \{y_j : j \geq s_i\}$, and let $1 \leq t_i \leq n + 1$ be the integer such that $\{z_k : k < t_i\} \Rightarrow x_i \Rightarrow \{z_k : k \geq t_i\}$. Note that $s_1 \leq \cdots \leq s_\ell$ and $t_1 \leq \cdots \leq t_\ell$.

Step 3: Completing the induction. We claim that for the partition $\{X \cup \{v\}, Y, Z\}$ of $V(G)$, the sets $X \cup \{v\} \cup Y$, $X \cup \{v\} \cup Z$, and $Y \cup Z$ are transitive, which will prove the theorem. To prove this claim, it suffices (by Proposition 4.1) to show the following.

(1) $\{x_i : i < i_0\} \Rightarrow v \Rightarrow \{x_i : i \geq i_0\}$.

(2) $\{y_j : j < s\} \Rightarrow v \Rightarrow \{y_j : j \geq s\}$.

(3) $\{z_k : k < t\} \Rightarrow v \Rightarrow \{z_k : k \geq t\}$.

(4) $s_{i_0-1} \leq s \leq s_{i_0}$ (if $i_0 = 1$, only the right-hand inequality needs to be proved).

(5) $t_{i_0-1} \leq t \leq t_{i_0}$ (if $i_0 = 1$, only the right-hand inequality needs to be proved).

We prove these statements through a series of claims.

Claim 1. *For all $1 \leq r \leq |V(G')| - 2$, if $C_r = \{x_i, y_j, z_k\}$, then $x_i \to y_j \to z_k \to x_i$.*

This follows from the assumption $x_{i_0} \to y_{j_0} \to z_{k_0} \to x_{i_0}$ and the properties of the sequence $C_1, \ldots, C_{|V(G')|-2}$.

Claim 2. *Suppose $C_r = \{x_i, y_j, z_k\}$ for some r and $x_i \in B$, $y_j \in A$, and $z_k \in B$. Then $s_i \leq s$ and $t_i \leq t$.*

Suppose $s_i > s$. Then by the definition of s_i, we have $y_s \to x_i$. Since $x_i \to y_j$ and $X \cup Y$ is transitive, we also have $y_s \to y_j$. Since $y_j \to z_k$ and $Y \cup Z$ is transitive, we then have $y_s \to z_k$. So $y_s \Rightarrow C_r$. But $|C_r \cap A| = 1$, and by the definition of s, we have $y_s \in A$. This contradicts Proposition 4.3(a). So we must have $s_i \leq s$.

Now suppose that $t_i > t$. In particular, this means $t \leq t_i - 1 \leq n$, so the vertex z_t exists. By the definition of t_i, we have $z_t \to x_i$. Also, since $z_k \in B$, by the definition of t

we have $k < t$, so $z_k \to z_t$. Finally, since $y_j \to z_k$ and $Y \cup Z$ is transitive, we have $y_j \to z_t$. Thus, $\{z_k, x_i, y_j\}$ and $\{z_t, x_i, y_j\}$ are cyclic triangles with $z_k \to z_t$. However, $x_i \in B$, $y_j \in A$, $z_k \in B$, and by the definition of t, $z_t \in A$. This contradicts Proposition 4.3(e). So $t_i \leq t$, as desired.

Claim 3. *Suppose $C_r = \{x_i, y_j, z_k\}$ for some r and $x_i \in A$, $y_j \in A$, and $z_k \in B$. Then $s_i \geq s$ and $t_i \geq t$.*

Suppose $s_i < s$. In particular, we have $s - 1 \geq s_i \geq 1$, so the vertex y_{s-1} exists. By the definition of s_i, $x_i \to y_{s-1}$. Also, since $y_j \in A$, by the definition of s we have $s - 1 < j$, so $y_{s-1} \to y_j$. Finally, since $y_j \to z_k$ and $Y \cup Z$ is transitive, we have $y_{s-1} \to z_k$. Thus, $\{y_{s-1}, z_k, x_i\}$ and $\{y_j, z_k, x_i\}$ are cyclic triangles with $y_{s-1} \to y_j$. However, $z_k \in B$, $x_i \in A$, $y_j \in A$, and by the definition of s, $y_{s-1} \in B$. This contradicts Proposition 4.3(e), so $s_i \geq s$.

Now suppose $t_i < t$. By the definition of t_i, we have $x_i \to z_{t-1}$. Since $z_k \to x_i$ and $Z \cup X$ is transitive, we also have $z_k \to z_{t-1}$. Since $y_j \to z_k$ and $Y \cup Z$ is transitive, we then have $y_j \to z_{t-1}$. So $C_r \Rightarrow z_{t-1}$. But $|C_r \cap A| = 2$, and by the definition of t, we have $z_{t-1} \in B$. This contradicts Proposition 4.3(b). So $t_i \geq t$, as desired.

Claim 4. *Suppose $r_0 > 1$. Then $C_{r_0-1} = \{x_{i_0-1}, y_{j_0}, z_{k_0}\}$, and $x_{i_0-1} \in B$, $y_{j_0} \in A$, and $z_{k_0} \in B$.*

Assume $r_0 > 1$. Recall that by assumption, $x_{i_0} \in A$, $y_{j_0} \in A$, and $z_{k_0} \in B$. By the defintion of r_0, we must have $|C_{r_0-1}| \neq 2$. So one of the following holds.

- $C_{r_0-1} = \{x_{i_0-1}, y_{j_0}, z_{k_0}\}$ and $x_{i_0} \in B$.
- $C_{r_0-1} = \{x_{i_0}, y_{j_0-1}, z_{k_0}\}$ and $y_{j_0-1} \in B$.
- $C_{r_0-1} = \{x_{i_0}, y_{j_0}, z_{k_0-1}\}$ and $z_{j_0-1} \in A$.

If we have the first case then we are done. The second case contradicts Proposition 4.3(e). The third case contradicts Proposition 4.3(d). This proves the claim.

Claim 5. *For all $1 \leq r < r_0$, if $C_r = \{x_i, y_j, z_k\}$, then $x_i \in B$, $z_k \in B$, and $y_j \in A$ if $j \geq s$ and $y_j \in B$ otherwise.*

If $r_0 = 1$, there is nothing to prove. So assume $r_0 > 1$. We prove the claim by downward induction on r. If $r = r_0 - 1$, then the conclusion follows by Claim 4 and the fact that $s \leq j_0$. Now suppose the conclusion holds for some $2 \leq r < r_0$. Let $C_r = \{x_i, y_j, z_k\}$. First suppose that $j < s$. Then $|C_r \cap A| = 0$ by the inductive hypothesis. Thus, by Proposition 4.3(c) we have $|C_{r-1} \cap A| = 0$ as well, as desired.

Now suppose that $j \geq s$. By the inductive hypothesis, $x_i \in B$, $y_j \in A$, and $z_k \in B$. We have three cases.

- $C_{r-1} = \{x_{i-1}, y_j, z_k\}$.
- $C_{r-1} = \{x_i, y_{j-1}, z_k\}$.
- $C_{r-1} = \{x_i, y_j, z_{k-1}\}$.

For the first case, since $|C_{r-1} \cap A| \neq 2$ by the definition of r_0, we must have $x_{i-1} \in B$, as desired. Similarly, in the third case we must have $z_{k-1} \in B$, as desired.

Finally, suppose we have the second case. We wish to prove that $y_{j-1} \in B$ if $j = s$, and $y_{j-1} \in A$ if $j > s$. If $j = s$, then by the definition of s we must have $y_{j-1} \in B$, as desired. Now suppose $j > s$. Suppose $y_{j-1} \in B$. Then $|C_{r-1} \cap A| = 0$, so by Proposition 4.3(c) and downward induction, we have $|C_{r'} \cap A| = 0$ for all $r' \leq r - 1$. However, then $y_{j'} \in B$ for all $j' \leq j - 1$, and in particular $y_s \in B$ since $j > s$. This contradicts the fact that $y_s \in A$ by the definition of s. Thus, $y_{j-1} \in A$, as desired.

Claim 6. *For all $r_0 \leq r \leq |V(G')| - 2$, if $C_r = \{x_i, y_j, z_k\}$, then $x_i \in A$, $y_j \in A$, and $z_k \in B$ if $k < t$ and $z_k \in A$ otherwise.*

We induct upwards on r. If $r = r_0$, the conclusion holds by assumption and the fact that $t \geq k_0 + 1$. Now suppose the conclusion holds for some $r_0 \leq r \leq |V(G')| - 3$. Let $C_r = \{x_i, y_j, z_k\}$. First suppose $k \geq t$. Then $|C_r \cap A| = 3$ by the inductive hypothesis, so by Proposition 4.3(d), we have $|C_{r+1} \cap A| = 3$, as desired.

Now suppose that $k < t$. By the inductive hypothesis, $x_i \in A$, $y_j \in A$, and $z_k \in B$. We have three cases.

- $C_{r+1} = \{x_{i+1}, y_j, z_k\}$.
- $C_{r+1} = \{x_i, y_{j+1}, z_k\}$.
- $C_{r+1} = \{x_i, y_j, z_{k+1}\}$.

For the second case, by Proposition 4.3(e) we must have $y_{j+1} \in A$, as desired. Now suppose we have the third case. We wish to prove that $z_{k+1} \in A$ if $k = t - 1$, and $z_{k+1} \in B$ if $k < t - 1$. If $k = t - 1$, then by the definition of t we have $z_{k+1} \in A$, as desired. Suppose $k < t - 1$, and suppose that $z_{k+1} \in A$. Then $|C_{r+1} \cap A| = 3$, so by Proposition 4.3(d) and upward induction, we have $|C_{r'} \cap A| = 3$ for all $r' \geq r + 1$. Then $z_{k'} \in A$ for all $k' \geq k + 1$, so in particular $z_{t-1} \in A$ since $k < t - 1$. This contradicts the fact that $z_{t-1} \in B$ by the definition of t. Thus, $z_{k+1} \in B$, as desired.

Finally, suppose we have the first case. We wish to prove $x_{i+1} \in A$. Suppose that $x_{i+1} \in B$. Applying Claim 2 to C_{r+1}, we have $s_{i+1} \leq s$ and $t_{i+1} \leq t$. However, applying Claim 3 to C_r, we have $s_i \geq s$ and $t_i \geq t$. Since $s_i \leq s_{i+1}$ and $t_i \leq t_{i+1}$ by Proposition 4.1, we must therefore have $s_i = s_{i+1}$ and $t_i = t_{i+1}$. It follows that $\{x_i, x_{i+1}\}$ is a homogeneous set in G'. This contradicts the fact that G' is prime. So we must have $x_{i+1} \in A$, as desired.

Claim 7. *(1) through (5) hold.*

(1), (2), and (3) follow by Claims 4, 5 and 6. For $i_0 > 1$, the lower bounds of (4) and (5) follow by Claims 4 and 2. The upper bounds of (4) and (5) follow by Claim 3. This completes the proof of the theorem.

□

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