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PARTIALLY CRITICAL TOURNAMENTS AND PARTIALLY CRITICAL SUPPORTS

MOHAMED Y. SAYAR

Abstract. Given a tournament $T = (V, A)$, with each subset X of V is associated the subtournament $T[X] = (X, A \cap (X \times X))$ of T induced by X. A subset I of V is an interval of T provided that for any $a, b \in I$ and $x \in V \setminus I$, $(a, x) \in A$ if and only if $(b, x) \in A$. For example, $\varnothing$, $\{x\}$, where $x \in V$, and V are intervals of T called trivial. A tournament is indecomposable if all its intervals are trivial; otherwise, it is decomposable. Let $T = (V, A)$ be an indecomposable tournament. The tournament T is critical if for every $x \in V$, $T[V \setminus \{x\}]$ is decomposable. It is partially critical if there exists a proper subset X of V such that $|X| \geq 3$, $T[X]$ is indecomposable and for every $x \in V \setminus X$, $T[V \setminus \{x\}]$ is decomposable. The partially critical tournaments are characterized. Lastly, given an indecomposable tournament $T = (V, A)$, consider a proper subset X of V such that $|X| \geq 3$ and $T[X]$ is indecomposable. The partially critical support of T according to $T[X]$ is the family of $x \in V \setminus X$ such that $T[V \setminus \{x\}]$ is indecomposable and $T[V \setminus \{x, y\}]$ is decomposable for every $y \in (V \setminus X) \setminus \{x\}$. It is shown that the partially critical support contains at most three vertices. The indecomposable tournaments whose partially critical supports contain at least two vertices are characterized.

1. Introduction

A *digraph* $D = (V, A)$ consists of a finite and nonempty *vertex* set V and of an *arc* set A where an arc is an ordered pair of distinct vertices. With each nonempty subset X of V associate the *subdigraph* $D[X] = (X, A \cap (X \times X))$ of D induced by X. For convenience, given $X \subsetneq V$, $D[V \setminus X]$ is also denoted by $D - X$ and by $D - x$ when $X = \{x\}$. Given a digraph $D = (V, A)$, we define an equivalence relation $\mathcal{S}$ on V in the following way. For any $x \neq y \in V$, $x\mathcal{S}y$ if $x = y$ or $x \neq y$ and there are vertices $x = x_0, \ldots, x_m = y$ and $y = y_0, \ldots, y_n = x$ such that $(x_i, x_{i+1}) \in A$ for $0 \leq i \leq m-1$ and $(y_j, y_{j+1}) \in A$ for $0 \leq j \leq n-1$. The equivalence classes of $\mathcal{S}$ are called the *strongly connected* components of D.

A digraph $D = (V, A)$ is a *tournament* provided that for any $x \neq y \in V$, $(x, y) \in A$ if and only if $(y, x) \notin A$. Let $T = (V, A)$ be a tournament. Given

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$x \neq y \in V$, $x \longrightarrow y$ means $(x, y) \in A$. Given $Y \subseteq V$ and $x \in V \setminus Y$, $x \longrightarrow Y$ means $x \longrightarrow y$ for every $y \in Y$. We similarly define $Y \longrightarrow x$. Given $X, Y \in V$, $X \longrightarrow Y$ means $x \longrightarrow Y$ for every $x \in X$. With each tournament $T = (V, A)$ associate its *dual* $T^\star = (V, A^\star)$ defined as follows: $(x, y) \in A^\star$ if $(y, x) \in A$ for any $x, y \in V$. A tournament $T = (V, A)$ is a *total order* provided that for any $x, y, z \in V$, if $x \longrightarrow y$ and $y \longrightarrow z$, then $x \longrightarrow z$. Given a total order $T = (V, A)$ and $x, y \in V$, $x \longrightarrow y$ means $x < y$ modulo T.

A *graph* $G = (V, E)$ consists of a finite and nonempty *vertex* set V and of an *edge* set E where an edge is an unordered pair of distinct vertices. With each nonempty subset X of V associate the *subgraph* $G[X] = (X, E \cap \binom{X}{2})$ of G induced by X. For instance, given a set V, $(V, \varnothing)$ is the *empty* graph on V whereas $(V, \binom{V}{2})$ is the *complete* graph. With each graph $G = (V, E)$ associate its *complement* $\overline{G} = (V, \overline{E})$ defined as follows: given $x \neq y \in V$, $\{x, y\} \in \overline{E}$ if $\{x, y\} \notin E$. Given a graph $G = (V, E)$, consider a partition p of V. The graph G is *multipartite* by p if for every $M \in p$, $G[M]$ is empty. It is *bipartite* when $|p| = 2$. Given a graph $G = (V, E)$, we define an equivalence relation $\mathcal{C}$ on V in the following way. For any $x \neq y \in V$, $x\mathcal{C}y$ if $x = y$ or $x \neq y$ and there are vertices $x = x_0, \ldots, x_n = y$ such that $\{x_i, x_{i+1}\} \in E$ for $i \in \{0, \ldots, n-1\}$. The equivalence classes of $\mathcal{C}$ are called the *connected* components of G. The number of connected components of G is denoted by $c(G)$. The graph G is *connected* if $c(G) = 1$, otherwise it is *disconnected*. A vertex x of G is *isolated* if $\{x\}$ constitutes a connected component of G. Given a vertex x of G, the *neighbourhood* $N_G(x)$ of x in G is the set of $y \in V$ such that $\{x, y\} \in E$. The *degree* $d_G(x)$ of x in G is the cardinality of $N_G(x)$.

1.1. **Indecomposable digraphs.** Given a digraph $D = (V, A)$, a subset I of V is an *interval* [1, 7, 6, 10] (or a *module* [11] or a *clan* [5, 4]) of D provided that for any $a, b \in I$ and $x \in V \setminus I$, $(a, x) \in A$ if and only if $(b, x) \in A$, and $(x, a) \in A$ if and only if $(x, b) \in A$. For example, $\varnothing$, $\{x\}$, where $x \in V$, and V are intervals of D called *trivial*. For tournaments, the notion of interval generalizes the usual notion of interval of a total order. Furthermore, the intervals of a digraph share the same properties as those of a total order.

Proposition 1. *Let $D = (V, A)$ be a digraph.*

(i) *Given a nonempty subset W of V, if I is an interval of D, then $I \cap W$ is an interval of $D[W]$.*
(ii) *Given an interval I of D, if J is an interval of $D[I]$, then J is an interval of D.*
(iii) *If I and J are intervals of D, then $I \cap J$ is an interval of D.*
(iv) *If I and J are intervals of D such that $I \cap J \neq \varnothing$, then $I \cup J$ is an interval of D.*
(v) *If I and J are intervals of D such that $I \setminus J \neq \varnothing$, then $J \setminus I$ is an interval of D.*

A digraph is *indecomposable* [1, 7, 10] (or *prime* [11] or *primitive* [5, 4]) if all its intervals are trivial, otherwise it is *decomposable*. For instance, the 3-cycle T_3 defined on $\{0,1,2\}$ by $0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 0$ is indecomposable. On the other hand, a total order on at least three vertices is decomposable. Notice that a tournament with one or two vertices is indecomposable.

Remark 1. *Given a tournament $T = (V, A)$, each strongly connected component C of T is an interval of T. Indeed consider $a, b \in C$ and $c \in V$ such that $a \longrightarrow c \longrightarrow b$. Since $a, b \in C$, there exist $b = b_0, \dots, b_n = a \in C$ such that $b_i \longrightarrow b_{i+1}$ for $0 \leq i \leq n-1$. By considering the sequences $b = b_0, \dots, b_n, b_{n+1} = c$ and $c = c_0, c_1 = b$, we obtain $c \in C$. Therefore, for every $x \in V \setminus C$, we have either $x \longrightarrow C$ or $C \longrightarrow x$. Given distinct strongly connected components C and D of T, it follows that $C \longrightarrow D$ or $D \longrightarrow C$. Consequently T induces a total order on its strongly connected components.*

We review some of the relevant properties of the indecomposable subdigraphs of an indecomposable digraph. We begin with the existence of an indecomposable subdigraph with three or four vertices.

Proposition 2 (Sumner [12])**.** *Given a digraph $D = (V, A)$, with $|V| \geq 3$, if D is indecomposable, then there is a subset X of V such that $|X| = 3$ or 4 and $D[X]$ is indecomposable.*

To construct indecomposable subdigraphs of a larger size, we use the following. Let $D = (V, A)$ be a digraph. Given a proper subset X of V such that $|X| \geq 3$ and $D[X]$ is indecomposable, consider the following subsets of $V \setminus X$:

- $\mathrm{Ext}(X)$ is the family of the elements x of $V \setminus X$ such that $D[X \cup \{x\}]$ is indecomposable;
- $\langle X \rangle$ is the family of the elements x of $V \setminus X$ such that X is an interval of $D[X \cup \{x\}]$;
- For each $u \in X$, $X(u)$ is the family of the elements x of $V \setminus X$ such that $\{u, x\}$ is an interval of $D[X \cup \{x\}]$.

The family $\{\mathrm{Ext}(X), \langle X \rangle\} \cup \{X(u); u \in X\}$ is denoted by $p_{D[X]}$. Using Proposition 1, it is simply verified that $p_{D[X]}$ is a partition of $V \setminus X$. Moreover, given $M \in p_{D[X]} \setminus \{\mathrm{Ext}(X)\}$, $D[X \cup \{x, y\}]$ is decomposable for $x, y \in M$. Indeed, if $x, y \in \langle X \rangle$, then X is an interval of $D[X \cup \{x, y\}]$. Given $u \in X$, if $x, y \in X(u)$, then $\{u, x, y\}$ is an interval of $D[X \cup \{x, y\}]$. When $x \in M$ and $y \in (V \setminus X) \setminus M$, where $M \in p_{D[X]} \setminus \{\mathrm{Ext}(X)\}$, or when $x \neq y \in \mathrm{Ext}(X)$, we have the following.

Lemma 1 (Ehrenfeucht and Rozenberg [5])**.** *Given a digraph $D = (V, A)$, consider $X \subsetneq V$ such that $|X| \geq 3$ and $D[X]$ is indecomposable.*

(i) *For $x \in \langle X \rangle$ and $y \in V \setminus (X \cup \langle X \rangle)$, if $D[X \cup \{x, y\}]$ is decomposable, then $X \cup \{y\}$ is an interval of $D[X \cup \{x, y\}]$.*
(ii) *Given $u \in X$, for $x \in X(u)$ and $y \in V \setminus (X \cup X(u))$, if $D[X \cup \{x, y\}]$ is decomposable, then $\{x, u\}$ is an interval of $D[X \cup \{x, y\}]$.*

(iii) *For* $x \neq y \in \mathrm{Ext}(X)$*, if* $D[X \cup \{x, y\}]$ *is decomposable, then* $\{x, y\}$ *is an interval of* $D[X \cup \{x, y\}]$*.*

The next proposition results from the preceding lemma.

Proposition 3 (Ehrenfeucht and Rozenberg [5])**.** *Given a digraph* $D = (V, A)$*, consider* $X \subseteq V$ *such that* $|X| \geq 3$ *and* $D[X]$ *is indecomposable. If* D *is indecomposable and* $|V \setminus X| \geq 2$*, then there exist* $a \neq b \in V \setminus X$ *such that* $D[X \cup \{a, b\}]$ *is indecomposable.*

Given a digraph $D = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$, $|V \setminus X| \geq 2$ and $D[X]$ is indecomposable. Proposition 3 leads us to associate with $D[X]$ the *outside* graph $G_{D[X]} = (V \setminus X, E_{D[X]})$ defined in the following manner. For any $x \neq y \in V \setminus X$, $\{x, y\} \in E_{D[X]}$ if $D[X \cup \{x, y\}]$ is indecomposable.

Remark 2. *Given a digraph* $D = (V, A)$*, consider* $X \subsetneq V$ *such that* $|X| \geq 3$ *and* $D[X]$ *is indecomposable. Before Lemma 1, we observed that* $G_{D[X]}[M]$ *is empty for each* $M \in p_{D[X]} \setminus \{\mathrm{Ext}(X)\}$*. Consequently, if* $\mathrm{Ext}(X) = \varnothing$*, then* $G_{D[X]}$ *is multipartite by* $p_{D[X]}$*.*

The following result is a consequence of Propositions 2 and 3.

Corollary 1 (Ehrenfeucht and Rozenberg [5])**.** *Given an indecomposable digraph* $D = (V, A)$ *such that* $|V| \geq 5$*. There exist* $x, y \in V$ *such that* $D - \{x, y\}$ *is indecomposable.*

1.2. **Critical and partially critical digraphs.** In Corollary 1, we may have $x = y$. Whence the following definition. A vertex x of an indecomposable digraph $D = (V, A)$ is *critical* when $D - x$ is decomposable. Given a subset X of V, D is X*-critical* if all the elements of X are critical. An indecomposable digraph $D = (V, A)$ is *critical* [10] if it is V-critical. An indecomposable digraph $D = (V, A)$ is *partially* critical [1] if there exists a proper subset X of V such that $|X| \geq 3$, $D[X]$ is indecomposable and D is $(V \setminus X)$-critical. The following is an easy consequence of Proposition 3. For its proof, we refer to [1, Lemma 4.1 and Proposition 4.2].

Lemma 2. *Given an indecomposable digraph* $D = (V, A)$*, consider* $X \subsetneq V$ *such that* $|X| \geq 3$ *and* $G[X]$ *is indecomposable. Assume that* G *is* $(V \setminus X)$*-critical. For every* $Y \subsetneq V$ *such that* $Y \supseteq X$*, if* $G[Y]$ *is indecomposable, then* $|V \setminus Y|$ *is even. The three assertions below follow*

(i) $|V \setminus X|$ *is even;*
(ii) $G[X \cup Z]$ *is decomposable for every subset* Z *of* $V \setminus X$ *such that* $|Z| = 1$ *or* 3*;*
(iii) *for every* $Y \subsetneq V$ *such that* $Y \supsetneq X$*, if* $G[Y]$ *is indecomposable, then* $G[Y]$ *is* $(Y \setminus X)$*-critical.*

The notions of interval, of indecomposable digraph, of critical digraph and of partially critical digraph are also introduced for graphs by identifying a graph $G = (V, E)$ with the digraph $D = (V, A)$ defined as follows. For any

$x \neq y \in V$, $(x, y) \in A$ if $\{x, y\} \in E$. The analogue of Remark 1 for graphs is simple.

Remark 3. *Let C be a connected component of a graph $G = (V, E)$. For $c \in C$ and $x \in V \setminus C$, we have $\{c, x\} \notin E$. Thus C is an interval of G. Furthermore G induces an empty graph on its connected components.*

To state the characterization of the critical graphs attributed to Schmerl and Trotter [10], we introduce for $n \geq 1$ the graph $G_{2n} = (\{0, \ldots, 2n-1\}, E_{2n})$ defined as follows (see Figure 1). For any $x \neq y \in \{0, \ldots, 2n-1\}$, $\{x, y\} \in E_{2n}$ if there exist $i \leq j \in \{0, \ldots, n-1\}$ such that $\{x, y\} = \{2i, 2j+1\}$.

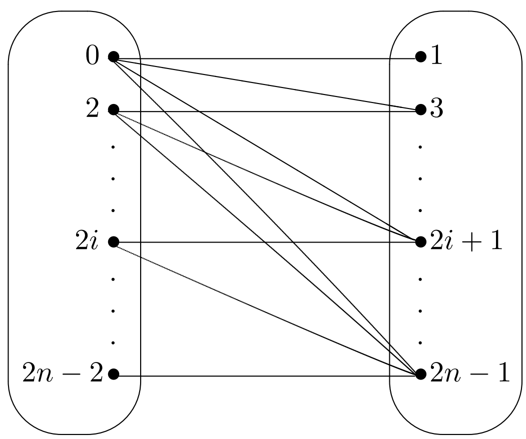

FIGURE 1. G_{2n}.

Theorem 1 (Schmerl and Trotter [10])**.** *Given an indecomposable graph $G = (V, E)$ such that $|V| \geq 4$, G is critical if and only if G is isomorphic to G_{2n} or to $\overline{G_{2n}}$ where $n \geq 2$.*

Breiner, Deogun and Ille [1] characterized the partially critical graphs. We adopt the same approach to study the partially critical tournaments. We use similar preliminary results and we omit their proofs when they are easily adaptable from the analogues for graphs. Our first characterization of the partially critical tournaments (see Theorems 3 and 4) was published in French without a proof [8]. The second one (see Theorem 5) provides a simple proof of the main result of [7] for tournaments. Using Theorems 3 and 4, we conclude with the study of the partial critical support of a tournament. The notion of partially critical support comes from that of critical support introduced in [9] as follows.

Given an indecomposable digraph $D = (V, A)$, the family of its non-critical vertices is called its *support* and is denoted by $\sigma(D)$. The *critical support* of D is the family $\sigma^c(D)$ of $x \in \sigma(D)$ such that $D - x$ is critical. The next two results are obtained in [9].

Proposition 4. *For every indecomposable tournament $T = (V, A)$, with $|V| \geq 7$, $|\sigma^c(T)| \leq 1$.*

Theorem 2. *For every indecomposable digraph $D = (V, A)$, with $|V| \geq 7$, $|\sigma^c(D)| \leq 2$.*

In terms of partial criticality, we obtain the following notion. Given an indecomposable digraph $D = (V, A)$, consider $X \subsetneq V$ such that $|X| \geq 3$ and $D[X]$ is indecomposable. The *partially critical* support of D according to $D[X]$ is the family $\sigma^p_{D[X]}(D)$ of $x \in V \setminus X$ such that $D - x$ is indecomposable and $((V \setminus \{x\}) \setminus X)$-critical. In the last section, we show by using Theorem 5 that the partially critical support of an indecomposable tournament contains at most three vertices (see Lemma 10). Lastly, we characterize the indecomposable tournaments whose partially critical supports contain at least two vertices (see Theorem 6 and Corollary 4).

2. Preliminaries

We use the following easily verified properties of G_{2n}.

Observation 1. *Consider an integer $n \geq 1$.*

- (i) *For $i \in \{0, \ldots, n-1\}$, $d_{G_{2n}}(2i) = n - i$ and $d_{G_{2n}}(2i+1) = i + 1$.*
- (ii) *G_{2n} is bipartite by $B(G_{2n}) = \{\{0, 2, \ldots, 2n-2\}, \{1, 3, \ldots, 2n-1\}\}$.*
- (iii) *The permutation ψ_{2n} of $\{0, \ldots, 2n-1\}$, which interchanges i and $(2n-1) - i$ for $i \in \{0, \ldots, 2n-1\}$, and $\mathrm{Id}_{\{0,\ldots,2n-1\}}$ are the only automorphisms of G_{2n}.*
- (iv) *Assume that $n \geq 2$. For every $j \in \{0, \ldots, 2n-1\}$, $G_{2n} - j$ admits a single non-trivial interval I_j determined by $I_0 = \{2, \ldots, 2n-1\}$, $I_{2n-1} = \{0, \ldots, 2n-3\}$ and $I_j = \{j-1, j+1\}$ for $1 \leq j \leq 2n-2$.*

In the case of tournaments, we refine the partition $p_{D[X]}$ as follows. Given a tournament $T = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$, $|V \setminus X| \geq 2$ and $T[X]$ is indecomposable. The element $\langle X \rangle$ of $p_{T[X]}$ is divided into $X^- = \{x \in \langle X \rangle : x \longrightarrow X\}$ and $X^+ = \{x \in \langle X \rangle : X \longrightarrow x\}$. Similarly, for each $u \in X$, $X(u)$ is divided into $X^-(u) = \{x \in X(u) : x \longrightarrow u\}$ and $X^+(u) = \{x \in X(u) : u \longrightarrow x\}$. We introduce the three families $q_{T[X]} = \{\mathrm{Ext}(X), X^-, X^+\} \cup \{X^-(u), X^+(u)\}_{u \in X}$, $q^-_{T[X]} = \{X^-\} \cup \{X^-(u)\}_{u \in X}$ and $q^+_{T[X]} = \{X^+\} \cup \{X^+(u)\}_{u \in X}$.

Remark 4. *Given a tournament $T = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$, $|V \setminus X| \geq 2$ and $T[X]$ is indecomposable. Assume that $\mathrm{Ext}(X) = \varnothing$. For any $x \neq y \in V \setminus X$, we have $\{x, y\}$ is an interval of $T[X \cup \{x, y\}]$ if and only if x and y belong to the same element of $q_{T[X]}$.*

Now, we examine the outside graph $G_{T[X]}$ associated with an indecomposable subtournament $T[X]$ of an indecomposable tournament T. We omit the proof of the next two results. Their corresponding results are [1, Lemma 2.6] and [1, Lemma 2.7] respectively.

Lemma 3. *Given a tournament $T = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable. If $\mathrm{Ext}(X) = \varnothing$, then the following two assertions hold.*

(i) *If I is an interval of T such that $I \cap X = \varnothing$, then I is an interval of $G_{T[X]}$ and there exists $N \in q_{T[X]}$ such that $I \subseteq N$.*

(ii) *Let $M \in p_{T[X]}$ and $N \in q_{T[X]}$ such that $N \subseteq M$. If I is an interval of $G_{T[X]}$ such that $I \subseteq N$ and if I is an interval of $T[M]$, then I is an interval of T.*

Lemma 4. *Given a tournament $T = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable.*

(i) *If I is an interval of T such that $X \subseteq I$, then the elements of $V \setminus I$ are isolated vertices of $G_{T[X]}$.*

(ii) *Given $u \in X$, if I is an interval of T such that $I \cap X = \{u\}$, then the elements of $I \setminus \{u\}$ are isolated vertices of $G_{T[X]}$.*

Consequently, if T admits a non-trivial interval I such that $I \cap X \neq \varnothing$, then $G_{T[X]}$ possesses isolated vertices.

Given an indecomposable tournament $T = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable. By Lemma 2, if T is $(V \setminus X)$-critical, then $T[X \cup \{a, b, c\}]$ is decomposable for distinct elements a, b, c of $V \setminus X$. We consider the following two situations. Once again, we refer to the corresponding results [1, Lemma 4.3] and [1, Lemma 4.4] respectively.

Lemma 5. *Given a tournament $T = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable. Given distinct elements a, b, c of $V \setminus X$ such that $\{a, b\}, \{a, c\} \in E_{T[X]}$, if $T[X \cup \{a, b, c\}]$ is decomposable, then $\{b, c\}$ is an interval of $T[X \cup \{a, b, c\}]$ and hence there exists $N \in q_{T[X]}$ such that $b, c \in N$.*

Lemma 6. *Given a tournament $T = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable. Given $M, N \in p_{T[X]}$, consider $a \in M$ and $b \neq c \in N$ such that $\{a, b\} \in E_{T[X]}$ and $\{a, c\} \notin E_{T[X]}$. If $T[X \cup \{a, b, c\}]$ is decomposable, then the following assertions are satisfied*

(1) *If $N = \langle X \rangle$, then $X \cup \{a, b\}$ is an interval of $T[X \cup \{a, b, c\}]$;*

(2) *If $N = X(u)$, where $u \in X$, then $\{u, c\}$ is an interval of $T[X \cup \{a, b, c\}]$.*

3. Example

The following example illustrates the main steps of the characterization of the partially critical tournaments based on the outside graph (see Theorems 3 and 4 below). Consider the tournament $T = (V, A)$ defined on $V = X \cup \{x_0, \ldots, x_{2m-1}\} \cup \{y_0, \ldots, y_{2n-1}\}$ as follows, where $X = \{0, 1, 2\}$, $m \geq 2$ and $n \geq 2$ (see Figure 2).

- $T[X] = T_3$.
- $\{x_0, x_2, \ldots, x_{2m-2}\} \longrightarrow X \longrightarrow \{y_0, y_2, \ldots, y_{2n-2}\}$.

- $2 \longrightarrow \{x_1, x_3, \ldots, x_{2m-1}\} \longrightarrow \{0, 1\}$.
- $\{0, 2\} \longrightarrow \{y_1, y_3, \ldots, y_{2n-1}\} \longrightarrow 1$.
- $\{x_0, x_1, \ldots, x_{2m-1}\} \longrightarrow \{y_0, y_1, \ldots, y_{2n-1}\}$.
- $T[\{x_0, x_2, \ldots, x_{2m-2}\}] = x_{2m-2} < x_{2m-4} < \cdots < x_0$ and $T[\{x_1, x_3, \ldots, x_{2m-1}\}] = x_{2m-1} < x_{2m-3} < \cdots < x_1$.
- $T[\{y_0, y_2, \ldots, y_{2n-2}\}] = y_0 < y_2 < \cdots < y_{2n-2}$ and $T[\{y_1, y_3, \ldots, y_{2n-1}\}] = y_1 < y_3 < \cdots < y_{2n-1}$.
- For $i, j \in \{0, \ldots, m-1\}$, $x_{2i+1} \longrightarrow x_{2j}$ if and only if $j \leq i$ (in Figure 2, only the arcs $x_{2i+1} \longrightarrow x_{2j}$ are represented).
- For $i, j \in \{0, \ldots, n-1\}$, $y_{2i} \longrightarrow y_{2j+1}$ if and only if $i \leq j$ (in Figure 2, only the arcs $y_{2i} \longrightarrow y_{2j+1}$ are represented).

Clearly $X^- = \{x_0, x_2, \ldots, x_{2m-2}\}$, $X^+ = \{y_0, y_2, \ldots, y_{2n-2}\}$, $X^-(0) = \{x_1, x_3, \ldots, x_{2m-1}\}$ and $X^+(0) = \{y_1, y_3, \ldots, y_{2n-1}\}$. Therefore $\mathrm{Ext}(X) = \varnothing$, $p_{T[X]} = \{\langle X \rangle, X(0)\}$ and $q_{T[X]} = \{X^-, X^+, X^-(0), X^+(0)\}$. The first claim follows from Remark 2 and from Lemma 1.

Claim 1.

(1) *The function* $\{0, \ldots, 2m-1\} \longrightarrow \{x_0, \ldots, x_{2m-1}\}$*, defined by* $i \mapsto x_i$ *for* $i \in \{0, \ldots, 2m-1\}$*, realizes an isomorphism from* G_{2m} *onto* $G_{T[X]}[\{x_0, \ldots, x_{2m-1}\}]$*;*
(2) *The function* $\{0, \ldots, 2n-1\} \longrightarrow \{y_0, \ldots, y_{2n-1}\}$*, defined by* $j \mapsto y_j$ *for* $j \in \{0, \ldots, 2n-1\}$*, realizes an isomorphism from* G_{2n} *onto* $G_{T[X]}[\{y_0, \ldots, y_{2n-1}\}]$*;*
(3) *For* $x \in \{x_0, x_1, \ldots, x_{2m-1}\}$ *and* $y \in \{y_0, y_1, \ldots, y_{2n-1}\}$*,* $\{x, y\} \notin E_{T[X]}$*.*

Claim 2. *The tournament* T *is indecomposable.*

Proof. Consider an interval I of T such that $|I| \geq 2$. We must show that $I = V$. For a contradiction, suppose that $I \cap X = \varnothing$. As $\mathrm{Ext}(X) = \varnothing$, it follows from Lemma 3 that I is an interval of $G_{T[X]}$ contained in an element N of $q_{T[X]}$. Thus there exists $Y = \{x_0, \ldots, x_{2m-1}\}$ or $\{y_0, \ldots, y_{2n-1}\}$ such that $N \subseteq Y$. By Proposition 1, I is an interval of $G_{T[X]}[Y]$. Since $|I| \geq 2$ and since $N \subsetneq Y$, I would be a non-trivial interval of $G_{T[X]}[Y]$. But $G_{T[X]}[Y]$ is indecomposable by Theorem 1 and Claim 1. Therefore $I \cap X \neq \varnothing$. It follows also from Claim 1 that $\{x_0, \ldots, x_{2m-1}\}$ and $\{y_0, \ldots, y_{2n-1}\}$ are the connected components of $G_{T[X]}$. In particular $G_{T[X]}$ has no isolated vertices. It follows from Lemma 4 that I is a trivial interval of T and hence $I = V$. □

Claim 3. *The tournament* T *is* $(V \setminus X)$*-critical.*

Proof. It suffices to verify the following

- $\{x_1, 0\}$ is an interval of $T - x_0$;
- $V \setminus \{x_{2m-1}, x_{2m-2}\}$ is an interval of $T - x_{2m-1}$;
- for $i \in \{1, \ldots, 2m-2\}$, $\{x_{i-1}, x_{i+1}\}$ is an interval of $T - x_i$;
- $\{y_1, 0\}$ is an interval of $T - y_0$;

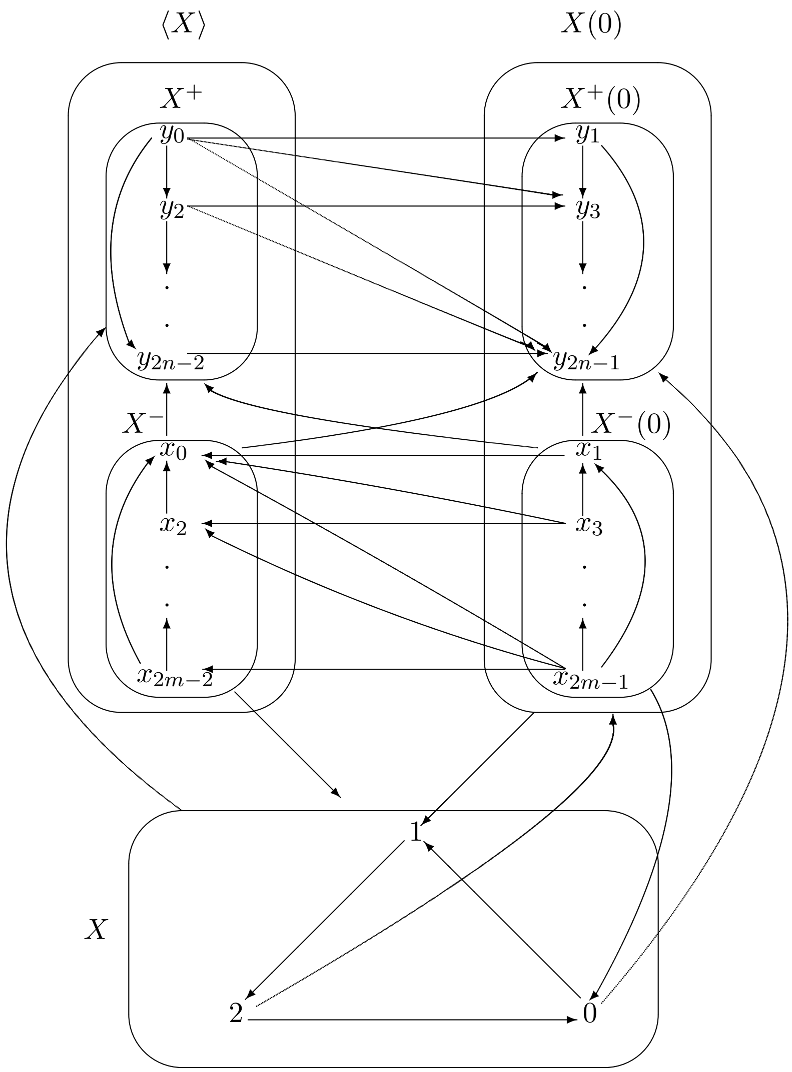

FIGURE 2. A $(V \setminus X)$-critical tournament.

- $V \setminus \{y_{2n-1}, y_{2n-2}\}$ is an interval of $T - y_{2n-1}$;
- for every $j \in \{1, \dots, 2n-2\}$, $\{y_{j-1}, y_{j+1}\}$ is an interval of $T - y_j$.

□

4. The first results

We consider an indecomposable tournament $T = (V, A)$ and a proper subset X of V such that $|X| \geq 3$ and $T[X]$ is indecomposable. We do not assume that T is $(V \setminus X)$-critical. Following Lemma 2, we only assume that $T[X \cup Y]$ is decomposable for every subset Y of $V \setminus X$ such that $|Y| = 1$

or 3. In particular note that $\mathrm{Ext}(X) = \varnothing$. The proof of the first lemma is identical to that of [1, Corollary 4.5].

Lemma 7. *The outside graph $G_{T[X]}$ has no isolated vertices.*

Lemma 8. *If $X^- \neq \varnothing$ and $X^+ \neq \varnothing$, then $X^- \longrightarrow X^+$. Similarly, for every $u \in X$, if $X^-(u) \neq \varnothing$ and $X^+(u) \neq \varnothing$, then $X^-(u) \longrightarrow X^+(u)$.*

Proof. First, consider $a \in X^-$ and $b \in X^+$. By Lemma 7, there exists $a' \in V \setminus X$ such that $\{a, a'\} \in E_{T[X]}$. Since a and b are not in the same element of $q_{T[X]}$, it follows from Lemma 5 that $\{a', b\} \notin E_{T[X]}$. By Lemma 6, $X \cup \{a, a'\}$ is an interval of $T[X \cup \{a, a', b\}]$. Thus $a \longrightarrow b$ because $X \longrightarrow b$.

Second, given $u \in X$, consider $a \in X^-(u)$ and $b \in X^+(u)$. By Lemma 7, there exists $a' \in V \setminus X$ such that $\{a, a'\} \in E_{T[X]}$. By Lemma 5 $\{a', b\} \notin E_{T[X]}$. It follows from Lemma 6 that $\{u, b\}$ is an interval of $T[X \cup \{a, a', b\}]$. Thus $a \longrightarrow b$ because $a \longrightarrow u$. □

Lemma 9. *For every $M \in q_{T[X]}$, $T[M]$ is a total order.*

Proof. Let $M \in q_{T[X]}$. By interchanging T and $T^\star$, assume that $M \in q^-_{T[X]}$. By Remark 1, it suffices to establish that each strongly connected component of $T[M]$ is reduced to a singleton. Since T is indecomposable, we verify that each strongly connected component of $T[M]$ is an interval of T. Let S be a strongly connected component of $T[M]$. For a contradiction, suppose that there exists $x \in V \setminus S$ such that S is not an interval of $T[S \cup \{x\}]$. Thus $S^- = \{s \in S : s \longrightarrow x\}$ and $S^+ = \{s \in S : x \longrightarrow s\}$ are nonempty. It follows from Remark 4 that S is an interval of $T[X \cup S]$. Moreover S is an interval of $T[M]$ by Remark 1. Therefore $x \notin M \cup X$. We show that $S^- \longrightarrow S^+$ or $S^+ \longrightarrow S^-$ so that $T[S]$ would not be strongly connected.

First, assume that $M = X^-$. By Lemma 8, $X^- \longrightarrow X^+$ when $X^+ \neq \varnothing$. Therefore $x \notin X \cup \langle X \rangle$. Let $s^- \in S^-$ and $s^+ \in S^+$. Since $x \longrightarrow s^+ \longrightarrow X$ and $s^- \longrightarrow X \cup \{x\}$, it follows from Lemma 1 that $\{s^+, x\} \in E_{T[X]}$ and $\{s^-, x\} \notin E_{T[X]}$. By Lemma 6, $X \cup \{s^+\}$ is an interval of $T[X \cup \{s^-, s^+, x\}]$. Thus $s^- \longrightarrow s^+$ because $s^- \longrightarrow X$. Consequently $S^- \longrightarrow S^+$.

Second, assume that $M = X^-(u)$ where $u \in X$. By Lemma 8, $X^-(u) \longrightarrow X^+(u)$ when $X^+(u) \neq \varnothing$. Thus $x \notin X \cup X(u)$. For example, assume that $u \longrightarrow x$ and consider any $s^- \in S^-$ and $s^+ \in S^+$. Since $u \longrightarrow x \longrightarrow s^+$ and $\{u, s^-\} \longrightarrow x$, it follows from Lemma 1 that $\{s^+, x\} \in E_{T[X]}$ and $\{s^-, x\} \notin E_{T[X]}$. By Lemma 6, $\{u, s^-\}$ is an interval of $T[X \cup \{s^-, s^+, x\}]$. Thus $s^+ \longrightarrow s^-$ because $s^+ \longrightarrow u$. Consequently $S^+ \longrightarrow S^-$ when $u \longrightarrow x$. □

Proposition 5. *For every connected component C of $G_{T[X]}$, $G_{T[X]}[C]$ is bipartite by two elements of $q_{T[X]}$ and $T[X \cup C]$ is indecomposable.*

Proof. We begin with the following observation. For every $M \in q_{T[X]}$, if $M \cap C \neq \varnothing$, then $M \subseteq C$. Suppose for a contradiction that there is $M \in q_{T[X]}$ such that $M \cap C \neq \varnothing$ and $M \setminus C \neq \varnothing$. Let $y \in M \cap C$ and $z \in M \setminus C$. By Lemma 7, there are $y' \in C$ and $z' \in V \setminus C$ such that $\{y, y'\} \in E_{T[X]}$ and

$\{z, z'\} \in E_{T[X]}$. Clearly $\{y, z\}, \{y', z\}, \{y, z'\}, \{y', z'\} \notin E_{T[X]}$. Assume that $M \subseteq \langle X \rangle$. By interchanging T and $T^\star$, assume that $M = X^-$. It follows from Lemma 6 applied to $T[X \cup \{y, y', z\}]$ that $X \cup \{y, y'\}$ is an interval of $T[X \cup \{y, y', z\}]$. Since $z \longrightarrow X$, we get $z \longrightarrow y$. But, by the same lemma applied to $T[X \cup \{y, z, z'\}]$, we obtain that $X \cup \{z, z'\}$ is an interval of $T[X \cup \{y, z, z'\}]$ and $y \longrightarrow z$. A similar approach provides a contradiction when $M \subseteq X(u)$ where $u \in X$.

Now, we prove that $G_{T[X]}[C]$ is bipartite. By Lemma 7, there exist $a \neq a' \in C$ such that $\{a, a'\} \in E_{T[X]}$. Consider $M_C, N_C \in q_{T[X]}$ such that $a \in M_C$ and $a' \in N_C$. By Remark 2, $M_C \neq N_C$. Furthermore $M_C \cup N_C \subseteq C$ by the above observation. We show that $C \subseteq M_C \cup N_C$. Let $b \in C$. There exists a sequence $a = a_0, \ldots, a_n = b \in C$ such that $\{a_i, a_{i+1}\} \in E_{T[X]}$ for every $i \in \{0, \ldots, n-1\}$. We verify that $a_1 \in N_C$. This is obvious when $a_1 = a'$. When $a_1 \neq a'$, a_0, a_1, a' are distinct element of $V \setminus X$ such that $\{a_0, a_1\}, \{a_0, a'\} \in E_{T[X]}$. By Lemma 5 applied to $T[X \cup \{a_0, a_1, a'\}]$, $a_1 \in N_C$. Assume that $n \geq 2$ and consider $i \in \{1, \ldots, n-1\}$. Since $\{a_i, a_{i-1}\}, \{a_i, a_{i+1}\} \in E_{T[X]}$, it follows from Lemma 5 that a_{i-1} and a_{i+1} belong to the same element of $q_{T[X]}$. Therefore $a_0, a_2, \ldots \in M_C$ and $a_1, a_3, \ldots \in N_C$. In particular $b = a_n \in M_C \cup N_C$. Consequently $C = M_C \cup N_C$. As $G_{T[X]}[M_C]$ and $G_{T[X]}[N_C]$ are empty by Remark 2, $G_{T[X]}[C]$ is bipartite by $\{M_C, N_C\}$.

Lastly, we establish that $T[X \cup C]$ is indecomposable. More precisely, we prove that if $T[X \cup C]$ admits a non-trivial interval I, then $I \cap X = \varnothing$ and I would be a non-trivial interval of T as well. Suppose for a contradiction that $I \cap X \neq \varnothing$. By Lemma 4 applied to $T[X \cup C]$, $G_{T[X]}[C]$ admits isolated vertices. Thus C would be a singleton which contradicts Lemma 7. Therefore $I \cap X = \varnothing$. As $\text{Ext}(X) = \varnothing$, it follows from the first assertion of Lemma 3 applied to $T[X \cup C]$ that I is an interval of $G_{T[X]}[C]$ and $I \subseteq M \cap C$ where $M \in q_{T[X]}$. Since $C = M_C \cup N_C$, $I \subseteq M_C$ or $I \subseteq N_C$. For instance, assume that $I \subseteq M_C$. We apply the second assertion of Lemma 3 to show that I is an interval of T. By denoting by M' the element of $p_{T[X]}$ containing M_C, we must verify that I is an interval of $G_{T[X]}$ and of $T[M']$. By Remark 3, C is an interval of $G_{T[X]}$. As I is an interval of $G_{T[X]}[C]$, it follows from Proposition 1(ii) that I is an interval of $G_{T[X]}$. It remains to verify that I is an interval of $T[M']$. Since I is an interval of $T[X \cup C]$ with $I \subseteq M_C$, I is an interval of $T[M_C]$ by Proposition 1(i). By Lemma 8, M_C is an interval of $T[M']$. Thus I is an interval of $T[M']$ by Proposition 1(ii). □

5. Characterization of partially critical tournaments

Proposition 5 and Lemma 2 suggest an hereditary property of the partial criticality by considering the connected components of the corresponding outside graph. The first theorem follows.

Theorem 3. *Given a tournament $T = (V, A)$, consider a proper subset X of V such that $|X| \geq 3$ and $T[X]$ is indecomposable. The tournament T is* *indecomposable and* $(V\setminus X)$*-critical if and only if the three assertions below hold.*

(H1) *If* $X^- \neq \varnothing$ *and* $X^+ \neq \varnothing$*, then* $X^- \longrightarrow X^+$*. Similarly, for every* $u \in X$*, if* $X^-(u) \neq \varnothing$ *and* $X^+(u) \neq \varnothing$*, then* $X^-(u) \longrightarrow X^+(u)$*;*

(H2) *For every connected component* C *of* $G_{T[X]}$*, there are distinct elements* M_C *and* N_C *of* $q_{T[X]}$ *such that* $G_{T[X]}[C]$ *is bipartite by* $\{M_C, N_C\}$*;*

(H3) *For each connected component* C *of* $G_{T[X]}$*,* $T[X \cup C]$ *is indecomposable and* C*-critical.*

Proof. To begin, assume that T is indecomposable and $(V\setminus X)$-critical. Assertion H1 is Lemma 8. Assertion H2 follows from Proposition 5 and Assertion H3 follows from Proposition 5 and Lemma 2.

Conversely, assume that Assertions H1, H2 and H3 are satisfied. We begin with three remarks. First, consider a connected component C of $G_{T[X]}$. By Assertion H3, $T[X \cup C]$ is indecomposable and C-critical. It follows that $|C| > 1$, that is, $G_{T[X]}$ does not have isolated vertices.

Second, given $x \in V \setminus X$, denote by C the connected component of $G_{T[X]}$ which contains x. By Assertion H3, $T[X \cup C]$ is indecomposable and C-critical. By Lemma 2 applied to $T[X \cup C]$, $T[X \cup \{x\}]$ is decomposable. Consequently $\mathrm{Ext}(X) = \varnothing$.

Third, consider a connected component C of $G_{T[X]}$. By the first remark, $|C| > 1$. By Assertion H2, there are distinct elements M_C and N_C of $q_{T[X]}$ such that $G_{T[X]}[C]$ is bipartite by $\{M_C, N_C\}$. Denote by M and N the elements of $p_{T[X]}$ such that $M_C \subseteq M$ and $N_C \subseteq N$. Since $|C| > 1$, there are $c \neq c' \in C$ such that $\{c, c'\} \in E_{T[X]}$. By Remark 2, $G_{T[X]}[M]$ and $G_{T[X]}[N]$ are empty and hence $M \neq N$. So we may also assume in Assertion H2 that M_C and N_C are not included in the same element of $p_{T[X]}$.

Now, we show that T is indecomposable. Otherwise consider a non-trivial interval I of T. Since $\mathrm{Ext}(X) = \varnothing$ and since $G_{T[X]}$ does not have isolated vertices, it follows from Lemma 4 that $I \cap X = \varnothing$. By lemma 3, there is $M \in q_{T[X]}$ such that $I \subseteq M$. Let C be a connected component of $G_{T[X]}$ such that $M \cap C \neq \varnothing$. It follows from Assertion H2 that $M \subseteq C$. Thus I would be a non-trivial interval of $T[X \cup C]$ which contradicts Assertion H3.

Lastly, we prove that T is $(V\setminus X)$-critical. Consider an element x of $V \setminus X$ and denote by C the connected component of $G_{T[X]}$ which contains x. By Assertion H3, $T[X \cup C] - x$ admits a non-trivial interval J. As $T[X]$ is indecomposable, $X \subseteq J$, $|J \cap X| = 1$ or $J \cap X = \varnothing$. We distinguish the following three cases to obtain a non-trivial interval of $T - x$.

First, assume that $X \subseteq J$. We prove that $V\setminus(C\setminus J)$ is an interval of $T-x$. Clearly $(C\setminus J)\setminus\{x\} \neq \varnothing$ and $(C\setminus J)\setminus\{x\} \subseteq \langle X\rangle$. For instance, assume that $C \cap X^- \neq \varnothing$. By Assertion H2, $X^- \subseteq C$. Furthermore it follows from our third remark that there exists $M \in q_{T[X]} \setminus \{X^-, X^+\}$ such that $G_{T[X]}[C]$ is bipartite by $\{X^-, M\}$. Thus $(C \setminus J) \setminus \{x\} \subseteq X^-$. Given $y \in (C \setminus J) \setminus \{x\}$, it is sufficient to verify that $X \cup \{z\}$ is an interval of $T[X \cup \{y, z\}]$ for every

$z \in V \setminus (C \setminus J)$. If $z \in X \cup C$, then $z \in J$ and $J \cap (X \cup \{y, z\}) = X \cup \{z\}$ is an interval of $T[X \cup \{y, z\}]$. Assume that $z \in V \setminus (X \cup C)$. Since C is a connected component of $G_{T[X]}$, we have $\{y, z\} \notin E_{T[X]}$. By Lemma 1, if $z \notin \langle X\rangle$, then $X \cup \{z\}$ is an interval of $T[X \cup \{y, z\}]$. If $z \in \langle X\rangle$, then $z \in X^+$ because $X^- \subseteq C$. By Assertion H1, $X^- \longrightarrow X^+$ and hence $y \longrightarrow z$. As $y \in X^-$, we obtain $y \longrightarrow X \cup \{z\}$.

Second, assume that there exists $u \in X$ such that $J \cap X = \{u\}$. We prove that J is an interval of $T - x$. We have $J \setminus \{u\} \neq \varnothing$ and $J \setminus \{u\} \subseteq X(u) \cap C$. For instance, assume that $C \cap X^-(u) \neq \varnothing$. By Assertion H2, $X^-(u) \subseteq C$. It follows from our third remark that there exists $M \in q_{T[X]} \setminus \{X^-(u), X^+(u)\}$ such that $G_{T[X]}[C]$ is bipartite by $\{X^-(u), M\}$. Thus $J \setminus \{u\} \subseteq X^-(u)$. Given $y \in J \setminus \{u\}$, it is sufficient to verify that $\{u, y\}$ is an interval of $T[X \cup \{y, z\}]$ for every $z \in (V \setminus J) \setminus \{x\}$. If $z \in X \cup C$, then $z \notin J$ and $J \cap (X \cup \{y, z\}) = \{u, y\}$ is an interval of $T[X \cup \{y, z\}]$. Assume that $z \in V \setminus (X \cup C)$. As C is a connected component of $G_{T[X]}$, we have $\{y, z\} \notin E_{T[X]}$. Lemma 1 permits us to conclude when $z \notin X(u)$. When $z \in X(u)$, $z \in X^+(u)$ because $X^-(u) \subseteq C$. By Assertion H1, $X^-(u) \longrightarrow X^+(u)$. In particular $y \longrightarrow z$ and thus $\{u, y\} \longrightarrow z$ because $z \in X^+(u)$. Since $y \in X(u)$, $\{u, y\}$ is an interval of $T[X \cup \{y\}]$. Therefore $\{u, y\}$ is an interval of $T[X \cup \{y, z\}]$.

Third, assume that $J \cap X = \varnothing$. We prove that J is an interval of $T - x$. By Assertion H2, there exist $M_C \neq N_C \in q_{T[X]}$ such that $G_{T[X]}[C]$ is bipartite by $\{M_C, N_C\}$. Since $\mathrm{Ext}(X) = \varnothing$, it follows from the first assertion of Lemma 3 applied to $T[X \cup C] - x$ that J is an interval of $G_{T[X]}[C] - x$ and $J \subseteq N \setminus \{x\}$ where $N = M_C$ or N_C. Therefore J is an interval of $T[N \setminus \{x\}]$. To show that J is an interval of $T - x$, we apply the second assertion of Lemma 3 to $T - x$. Denote by M the element of $p_{T[X]}$ which contains N. By Assertion H1, N is an interval of $T[M]$ and hence $N \setminus \{x\}$ is an interval of $T[M \setminus \{x\}]$. By Proposition 1(ii), J is an interval of $T[M \setminus \{x\}]$. It remains to verify that J is an interval of $G_{T[X]} - x$. As already observed, J is an interval of $G_{T[X]}[C] - x$. It follows from Remark 3 that $C \setminus \{x\}$ is an interval of $G_{T[X]} - x$. By Proposition 1(ii), J is an interval of $G_{T[X]} - x$. □

Given a tournament $T = (V, A)$, consider a proper subset X of V such that $|X| \geq 3$ and $T[X]$ is indecomposable. Now, we characterize the tournament T when it is indecomposable and $(V \setminus X)$-critical and when the outside graph $G_{T[X]}$ is connected. When $|V \setminus X| \leq 2$, we have: T is indecomposable and $(V \setminus X)$-critical if and only if $\mathrm{Ext}(X) = \varnothing$. Consequently, we assume that $|V \setminus X| \geq 3$. Proposition 5 is completed by the next proposition.

Proposition 6. *Given an indecomposable tournament $T = (V, A)$, consider a proper subset X of V such that $|X| \geq 3$, $|V \setminus X| \geq 3$ and $T[X]$ is indecomposable. Assume that $T[X \cup Y]$ is decomposable for every $Y \subseteq V \setminus X$ such that $|Y| = 1$ or 3. If $G_{T[X]}$ is connected, then there exists an isomorphism* *f from G_{2n} onto $G_{T[X]}[C]$ such that*

$$p_{T[X]} = q_{T[X]} = \{f(\{0,\dots,2n-2\}), f(\{1,\dots,2n-1\})\}$$

and satisfying

(K1) *if $\{f(0), f(2),\dots, f(2n-2)\} = X^-$ or $X^+(u)$, where $u \in X$, then $T[\{f(0),\, f(2),\dots, f(2n-2)\})] = f(2n-2) < f(2n-4) < \dots < f(0)$;*
(K2) *if $\{f(0), f(2),\dots, f(2n-2)\} = X^+$ or $X^-(u)$, where $u \in X$, then $T[\{f(0),\, f(2),\dots, f(2n-2)\}] = f(0) < f(2) < \dots < f(2n-2)$;*
(K3) *if $\{f(1), f(3),\dots, f(2n-1)\} = X^-$ or $X^+(u)$, where $u \in X$, then $T[\{f(1),\, f(3),\dots, f(2n-1)\}] = f(1) < f(3) < \dots < f(2n-1)$;*
(K4) *if $\{f(1), f(3),\dots, f(2n-1)\} = X^+$ or $X^-(u)$, where $u \in X$, then $T[\{f(1),\, f(3),\dots, f(2n-1)\}] = f(2n-1) < f(2n-3) < \dots < f(1)$.*

Proof. To begin, we prove that $G_{T[X]}$ is indecomposable. We establish that if I is a non-trivial interval of $G_{T[X]}$ which is maximal under inclusion among the non-trivial intervals of $G_{T[X]}$, then I would be a non-trivial interval of T. By Proposition 5, there exist $M \neq N \in q_{T[X]}$ such that $G_{T[X]}$ is bipartite by $\{M, N\}$. We have $p_{T[X]} = q_{T[X]} = \{M, N\}$. Since $G_{T[X]}$ is a connected and bipartite graph, $I \subseteq M$ or $I \subseteq N$. For instance, assume that $I \subseteq M$. Since $\mathrm{Ext}(X) = \varnothing$, we apply the second assertion of Lemma 3 to prove that I is an interval of T. It suffices to verify that I is an interval of $T[M]$. Let $x \in M \setminus I$. As I is a maximal non-trivial interval of $G_{T[X]}$, $I \cup \{x\}$ is not an interval of $G_{T[X]}$. There is $\alpha \in (M \cup N) \setminus (I \cup \{x\})$ such that $I \cup \{x\}$ is not an interval of $G_{T[X]}[I \cup \{x, \alpha\}]$. Since $G_{T[X]}[M]$ is empty by Remark 2, $\alpha \in N$. As I is an interval of $G_{T[X]}$, we have either $\{\alpha, x\} \in E_{T[X]}$ and $\{\alpha, i\} \notin E_{T[X]}$ for $i \in I$ or $\{\alpha, x\} \notin E_{T[X]}$ and $\{\alpha, i\} \in E_{T[X]}$ for $i \in I$. We apply Lemma 6 as follows. First, assume that $M = \langle X\rangle$. By interchanging T and $T^\star$, assume that $M = X^-$. If $\{\alpha, x\} \in E_{T[X]}$ and $\{\alpha, i\} \notin E_{T[X]}$ for $i \in I$, then $X \cup \{\alpha, x\}$ is an interval of $T[X \cup \{\alpha, i, x\}]$ for $i \in I$ so that $I \longrightarrow x$. If $\{\alpha, x\} \notin E_{T[X]}$ and $\{\alpha, i\} \in E_{T[X]}$ for $i \in I$, then $X \cup \{\alpha, i\}$ is an interval of $T[X \cup \{\alpha, i, x\}]$ for $i \in I$ so that $x \longrightarrow I$. Second, assume that there exists $u \in X$ such that $M = X(u)$. By interchanging T and $T^\star$, assume that $M = X^-(u)$. If $\{\alpha, x\} \in E_{T[X]}$ and $\{\alpha, i\} \notin E_{T[X]}$ for $i \in I$, then $\{i, u\}$ is an interval of $T(X \cup \{\alpha, i, x\})$ for $i \in I$ so that $x \longrightarrow I$. If $\{\alpha, x\} \notin E_{T[X]}$ and $\{\alpha, i\} \in E_{T[X]}$ for $i \in I$, then $\{x, u\}$ is an interval of $T[X \cup \{\alpha, i, x\}]$ for $i \in I$ so that $I \longrightarrow x$. It follows that I is an interval of $T[M]$.

Now, we establish that there exists an isomorphism f from G_{2n} onto $G_{T[X]}$. First, we verify that $\{N_{G_{T[X]}}(x); x \in M\}$ is totally ordered by inclusion. Otherwise there are $x \neq y \in M$ and $\alpha, \beta \in N$ such that $\{x, \alpha\} \in E_{T[X]}$, $\{y, \alpha\} \notin E_{T[X]}$, $\{x, \beta\} \notin E_{T[X]}$ and $\{y, \beta\} \in E_{T[X]}$. For instance, assume that $M = X^-$. By Lemma 6 applied to $T([X \cup \{x, y, \alpha\}]$, $X \cup \{x, \alpha\}$ is an interval of $T([X \cup \{x, y, \alpha\}]$ so that $y \longrightarrow x$. On the other hand, by Lemma 6 applied to $T([X \cup \{x, y, \beta\}]$, we obtain $x \longrightarrow y$ as well. We get a similar contradiction when $M = X^+, X^-(u)$ or $X^+(u)$ where $u \in X$.

Therefore we may set $M = \{x_0, \dots, x_{n-1}\}$ with $N_{G_{T[X]}}(x_i) \supseteq N_{G_{T[X]}}(x_{i+1})$ for $0 \leq i \leq n-2$. As $G_{T[X]}$ is connected, $G_{T[X]}$ does not admit isolated vertices. Thus $N_{G_{T[X]}}(x_0) = N$ and $N_{G_{T[X]}}(x_{n-1}) \neq \varnothing$. Clearly $N_{G_{T[X]}}(x_{n-1})$ is an interval of $G_{T[X]}$. Since $G_{T[X]}$ is indecomposable, $|N_{G_{T[X]}}(x_{n-1})| = 1$. Let $i \in \{0, \dots, n-2\}$. We have $N_{G_{T[X]}}(x_i) \setminus N_{G_{T[X]}}(x_{i+1})$ is an interval of $G_{T[X]}$. Furthermore $\{x \in M : N_{G_{T[X]}}(x) = N_{G_{T[X]}}(x_i)\}$ is an interval of $G_{T[X]}$. It follows that $|N_{G_{T[X]}}(x_i) \setminus N_{G_{T[X]}}(x_{i+1})| = 1$ for $0 \leq i \leq n-2$. Consequently we may denote the elements of N by $\alpha_0, \dots, \alpha_{n-1}$ in such a way that $N_{G_{T[X]}}(x_i) = \{\alpha_0, \dots, \alpha_{n-i-1}\}$ for $0 \leq i \leq n-1$. Therefore $f : \{0, \dots, 2n-1\} \longrightarrow V \setminus X$, defined by $2i \mapsto x_i$ and $2i+1 \mapsto \alpha_i$ for $0 \leq i \leq n-1$, realizes an isomorphism from G_{2n} onto $G_{T[X]}$. Clearly $p_{T[X]} = q_{T[X]} = \{f(\{0, \dots, 2n-2\}), f(\{1, \dots, 2n-1\})\}$.

Finally, we verify that Assertions K1, …, K4 are satisfied. For Assertion K1, assume that $\{f(0), f(2), \dots, f(2n-2)\} = X^-$ or $X^+(u)$ where $u \in X$. Given $i < j \in \{0, \dots, n-1\}$, we have $\{f(2i), f(2i+1\} \in E_{T[X]}$ and $\{f(2i+1), f(2j)\} \notin E_{T[X]}$. We apply Lemma 6 to $T[X \cup \{f(2i), f(2i+1), f(2j)\}]$ as follows. If $\{f(0), f(2), \dots, f(2n-2)\} = X^-$, then $X \cup \{f(2i), f(2i+1)\}$ is an interval of $T[X \cup \{f(2i), f(2i+1), f(2j)\}]$. If $\{f(0), f(2), \dots, f(2n-2)\} = X^+(u)$, then $\{u, f(2j)\}$ is an interval of $T[X \cup \{f(2i), f(2i+1), f(2j)\}]$. In both cases, we obtain $f(2j) \longrightarrow f(2i)$. Therefore

$$T[\{f(0), f(2), \dots, f(2n-2)\}] = f(2n-2) < f(2n-4) < \cdots < f(0).$$

Assertion K2 is deduced from Assertion K1 by considering $T^\star$ instead of T. Indeed $T^\star$ is also indecomposable and $(V \setminus X)$-critical. Moreover $G_{T^\star[X]} = G_{T[X]}$. Thus f is also an isomorphism from G_{2n} onto $G_{T^\star[X]}$. Finally, Assertions K3 and K4 are deduced from the two first by considering the isomorphism $f \circ \psi_{2n}$ (see Observation 1(iii)) from G_{2n} onto $G_{T[X]}$ instead of f. Indeed $f(\{1, 3, \dots, 2n-1\}) = (f \circ \psi_{2n})(\{0, 2, \dots, 2n-2\})$. For instance, concerning Assertion K3, assume that $\{f(1), f(3), \dots, f(2n-1)\} = X^-$ or $X^+(u)$ where $u \in X$. It follows from Assertion K1 applied to $f \circ \psi_{2n}$ that

$$T[\{(f \circ \psi_{2n})(0), (f \circ \psi_{2n})(2), \dots, (f \circ \psi_{2n})(2n)\}] \\ = (f \circ \psi_{2n})(2n-2) < (f \circ \psi_{2n})(2n-4) < \cdots < (f \circ \psi_{2n})(0),$$

that is,

$$T[\{f(1), f(3), \dots, f(2n-1)\}] = f(1) < f(3) < \cdots < f(2n-1).$$

□

Using Assertions K1,...,K4, we obtain the following characterization.

Theorem 4. *Given a tournament $T = (V, A)$, consider a proper subset X of V such that $|X| \geq 3$, $|V \setminus X| \geq 3$ and $T[X]$ is indecomposable. Assume that $G_{T[X]}$ is connected. The tournament T is indecomposable and $(V \setminus X)$-critical if and only if $\mathrm{Ext}(X) = \varnothing$ and there exists an isomorphism f from G_{2n} onto*

$G_{T[X]}$ such that $p_{T[X]} = q_{T[X]} = \{f(\{0,\dots,2n-2\}), f(\{1,\dots,2n-1\})\}$ and satisfying Assertions K1,...,K4.

Proof. To begin, assume that T is indecomposable and $(V\setminus X)$-critical. It follows from Lemma 2 and Proposition 6 that $\mathrm{Ext}(X) = \varnothing$ and such an isomorphism f from G_{2n} onto $G_{T[X]}$ exists.

Conversely, suppose for a contradiction that T admits a non-trivial interval I. Since $G_{T[X]}$ does not admit isolated vertices, it follows from Lemma 4 that $I \cap X = \varnothing$. As $\mathrm{Ext}(X) = \varnothing$, it follows from Lemma 3 that I is an interval of $G_{T[X]}$ and either $I \subseteq f(\{0,\dots,2n-2\})$ or $I \subseteq f(\{1,\dots,2n-1\})$. Thus I would be a non-trivial interval of $G_{T[X]}$ which contradicts Theorem 1. Finally, we prove that T is $(V\setminus X)$-critical. Given $x \in V\setminus X$, we show that $T-x$ admits a non-trivial interval. By considering $f \circ \psi_{2n}$ (see Observation 1(iii)) instead of f, assume that $x \in \{f(0), f(2), \dots, f(2n-2))\}$. By Observation 1(iv), either $G_{T[X]} - x$ admits a single isolated vertex or $G_{T[X]} - x$ admits an unordered pair as a non-trivial interval. Since $\mathrm{Ext}(X) = \varnothing$, we distinguish the three following cases.

First, assume that $G_{T[X]} - x$ admits an isolated vertex y belonging to $\langle X\rangle$. We verify that $V \setminus \{x,y\}$ is an interval of $T - x$. It follows from Observation 1(i) that $x = f(0)$ and $y = f(1)$. By interchanging T and $T^\star$, assume that $\{f(1), f(3), \dots, f(2n-1)\} = X^-$. By Assertion K3, $f(1) \longrightarrow \{f(3), \dots, f(2n-1)\}$. Moreover, for $i \in \{1,\dots,n-1\}$, we have $\{f(1), f(2i)\} \notin E_{T[X]}$. Since $\{f(0), f(2), \dots, f(2n-2)\} \neq \langle X\rangle$, it follows from Lemma 1 that $X \cup \{f(2i)\}$ is an interval of $T[X \cup \{f(1), f(2i)\}]$. Thus $f(1) \longrightarrow f(2i)$ because $f(1) \in X^-$. Therefore $f(1) \longrightarrow X \cup \{f(2), \dots, f(2n-2)\} \cup \{f(3), \dots, f(2n-1)\}$, that is, $y \longrightarrow V \setminus \{x,y\}$. Consequently $V \setminus \{x,y\}$ is an interval of $T - x$.

Second, assume that $G_{T[X]} - x$ admits an isolated vertex y belonging to $X(u)$ where $u \in X$. We verify that $\{u,y\}$ is an interval of $T - x$. As previously, $x = f(0)$ and $y = f(1)$. It follows from Assertions K3 and K4 that $T[\{f(1), f(3), \dots, f(2n-1)\} \cup \{u\}] = u < f(1) < f(3) < \cdots < f(2n-1)$ or $f(2n-1) < f(2n-3) < \cdots < f(1) < u$. In both instances, $\{u, f(1)\}$ is an interval of $T[\{f(1), f(3), \dots, f(2n-1)\} \cup \{u\}]$. Moreover, for $i \in \{1,\dots,n-1\}$, we have $\{f(1), f(2i)\} \notin E_{T[X]}$. Since $f(2i) \notin X(u)$, it follows from Lemma 1 that $\{u, f(1)\}$ is an interval of $T[X \cup \{f(1), f(2i)\}]$. Thus $\{u, f(1)\}$ is an interval of $T[X \cup \{f(2), \dots, f(2n-2)\}]$. It follows that $\{u,y\} = \{u, f(1)\}$ is an interval of $T - x$.

Third, assume that $G_{T[X]} - x$ admits no isolated vertices. By Observation 1(iv), there exists $i \in \{1,\dots,n-1\}$ such that $x = f(2i)$ and $\{f(2i-1), f(2i+1)\}$ is an interval of $G_{T[X]} - x$. It follows from Assertions K3 and K4 that $\{f(2i-1), f(2i+1)\}$ is an interval of $T[\{f(1), f(3), \dots, f(2n-1)\}]$. Since $\mathrm{Ext}(X) = \varnothing$, it follows from Lemma 3 applied to $T - x$ that $\{f(2i-1), f(2i+1)\}$ is an interval of $T - x$. □

Corollary 2. *Given an indecomposable tournament $T=(V,A)$, consider $X \subsetneq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable. If T is $(V \setminus X)$-critical, then for every $x \in V \setminus X$, $G_{T[X]} - x$ admits at most one isolated vertex and $T - x$ admits a unique non-trivial interval I_x. More precisely*

1. *if $G_{T[X]} - x$ admits a single isolated vertex y and if $y \in \langle X \rangle$, then $I_x = V \setminus \{x, y\}$;*
2. *if $G_{T[X]} - x$ admits a single isolated vertex y and if $y \in X(u)$, where $u \in X$, then $I_x = \{u, y\}$;*
3. *if $G_{T[X]} - x$ admits no isolated vertices, then $|I_x| = 2$ and I_x is the unique interval of $G_{T[X]}[C] - x$, where C is the connected component of $G_{T[X]}$ containing x.*

Proof. Consider an element x of $V \setminus X$. Denote by C the connected component of $G_{T[X]}$ which contains x and by M_C the element of $q_{T[X]}$ which contains x. By Proposition 5, there is $N_C \in q_{T[X]} \setminus \{M_C\}$ such that $G_{T[X]}[C]$ is bipartite by $\{M_C, N_C\}$. Assume that $|C| \geq 3$. By Theorem 3, $T[X \cup C]$ is indecomposable and C-critical. By Theorem 4, there exists an isomorphism f_C from $G_{|C|}$ onto $G_{T[X]}[C]$ such that

$$\{M_C, N_C\} = \{f_C(\{0, 2, \ldots, |C|-2\}), f_C(\{1, 3, \ldots, |C|-1\})\}.$$

By considering $f_C \circ \psi_{|C|}$ (see Observation 1(iii)) instead of f_C, assume that $M_C = f_C(\{0, 2, \ldots, |C|-2\})$. When $|C| = 2$, f_C denotes the bijection from $\{0,1\}$ onto C such that $f_C(0) = x$ and we set $M_C = \{x\}$ and $N_C = \{f_C(1)\}$.

Denote by W_x the set of the isolated vertices of $G_{T[X]} - x$. By Lemma 7, $G_{T[X]}$ does not admit isolated vertices. Consequently W_x is the set of the isolated vertices of $G_{T[X]}[C] - x$ and $W_x \subseteq N_C$. Clearly $W_x = \{f_C(1)\}$ if $|C| = 2$. When $|C| \geq 3$, it follows from Observation 1(i) that $W_x \neq \varnothing$ if and only if $x = f_C(0)$. Moreover $W_x = \{f_C(1)\}$ if $x = f_C(0)$. Using Observation 1(iv), we associate with x subsets I_x and J_x of $V \setminus \{x\}$ by distinguishing the following three cases. First, if $W_x = \varnothing$, then $I_x = J_x$ is the unique interval of $G_{T[X]}[C] - x$. Second, if $W_x = \{y\}$ and $y \in X(u)$, where $u \in X$, then $I_x = J_x = \{u, y\}$. Third, if $W_x = \{y\}$ and $y \in \langle X \rangle$, then $J_x = (X \cup C) \setminus \{x, y\}$ and $I_x = V \setminus \{x, y\}$. By the discussion at the end of the proof of Theorem 4 applied to $T[X \cup C]$, J_x is an interval of $T[X \cup C] - x$. Then, by the discussion at the end of the proof of Theorem 3 applied to T, I_x is an interval of $T - x$.

Now, we verify that I_x is the unique non-trivial interval of $T - x$. Let L_x be a non-trivial interval of $T - x$.

First, assume that $X \subseteq L_x$. We have $(V \setminus \{x\}) \setminus L_x \neq \varnothing$ and $(V \setminus \{x\}) \setminus L_x \subseteq \langle X \rangle$. Moreover, it follows from the first assertion of Lemma 4 that $(V \setminus \{x\}) \setminus L_x \subseteq W_x$. As previously observed, $|W_x| \leq 1$ and hence there is $y \in (V \setminus X) \setminus \{x\}$ such that $W_x = \{y\} \subseteq \langle X \rangle$. It follows that $L_x = V \setminus \{x, y\}$ and $L_x = I_x$.

Second, assume that $L_x \cap X = \{u\}$. We have $L_x \setminus \{u\} \neq \varnothing$ and $L_x \setminus \{u\} \subseteq X(u)$. Furthermore $L_x \setminus \{u\} \subseteq W_x$ by the second assertion of Lemma 4. Since

$|W_x| \leq 1$, there is $y \in (V \setminus X) \setminus \{x\}$ such that $W_x = \{y\}$. Thus $L_x = \{u, y\}$ and $L_x = I_x$.

Finally, assume that $L_x \cap X = \varnothing$. As $\text{Ext}(X) = \varnothing$ by Lemma 2, it follows from Lemma 3 applied to $T - x$ that L_x is an interval of $G_{T[X]} - x$ and there is $N \in q_{T[X]}$ such that $L_x \subseteq N \setminus \{x\}$. By Proposition 5, there exists a connected component D of $G_{T[X]}$ such that $N \subseteq D$. Clearly L_x is a non-trivial interval of $T[X \cup D] - x$. As $T[X \cup D]$ is indecomposable by Theorem 3, $x \in D$ and hence $C = D$. Therefore $N \in \{M_C, N_C\}$ and L_x is a non-trivial interval of $G_{T[X]}[C] - x$ because $L_x \subseteq N \setminus \{x\}$. Necessarily $|C| \geq 4$ and it follows from Observation 1(iv) that $(f_C)^{-1}(x) \neq 0$. As above observed, $W_x \neq \varnothing$ if and only if $(f_C)^{-1}(x) = 0$. Thus $W_x = \varnothing$. Since I_x is the unique non-trivial interval of $G_{T[X]}[C] - x$ when $W_x = \varnothing$, $L_x = I_x$. □

Discussion. Let us explain how Theorems 3 and 4 allow us to generate partially critical tournaments. Consider an indecomposable tournament $\tau = (X, A_\tau)$ with $|X| \geq 3$. Let $G = (V, E)$ be a disconnected graph such that $V \cap X = \varnothing$ and $|X| \geq c(G) - 1$. Denote the connected components of G by $C_1, \ldots, C_{c(G)}$. Moreover, for each $1 \leq d \leq c(G)$, assume that $|C_d|$ is even and there exists an isomorphism f_d from $G_{|C_d|}$ onto $G[C_d]$. Set $M_d = f_d(\{0, 2, \ldots, |C_d| - 2\})$ and $N_d = f_d(\{1, 3, \ldots, |C_d| - 1\})$.

We construct a V-critical and indecomposable tournament $T = (X \cup V, A)$ such that $T[X] = \tau$ and $G_{T[X]} = G$. By Theorems 3 and 4, we must have $q_{T[X]} = \{M_d, N_d\}_{1 \leq d \leq c(G)}$ and N_d, M_d are not included in the same element of $p_{T[X]}$. By choosing $\langle X \rangle$ as an element of $p_{T[X]}$, we need $c(G) - 1$ distinct elements $u_1, \ldots, u_{c(G)-1}$ of X to obtain $p_{T[X]} = \{\langle X \rangle\} \cup \{X(u_d)\}_{1 \leq d \leq c(G)-1}$ and $q_{T[X]} = \{X^-, X^+\} \cup \{X^-(u_d), X^+(u_d)\}_{1 \leq d \leq c(G)-1}$. Given $1 \leq d \leq c(G)$, we have to associate with M_d and N_d two distinct elements of $q_{T[X]} = \{X^-, X^+\} \cup \{X^-(u_d), X^+(u_d)\}_{1 \leq d \leq c(G)-1}$ which are not included in the same element of $p_{T[X]} = \{\langle X \rangle\} \cup \{X(u_d)\}_{1 \leq d \leq c(G)-1}$. For example, set $M_1 = X^-$, $N_{c(G)} = X^+$ and for $1 \leq d \leq c(G) - 1$, $N_d = X^+(u_d)$ and $M_{d+1} = X^-(u_d)$. Now, the tournament T is entirely determined in the following way.

- Given $v \in V$ and $x \in X$, the arc of T between x and v comes from the definition of $q_{T[X]}$. For instance, we have $M_1 = X^- \longrightarrow X$.
- Let $v \neq w \in V$ such that v and w do not belong to the same element of $p_{T[X]}$. The arc of T between v and w is provided by Lemma 1 using the fact that $\{v, w\}$ is an edge of G or not. For instance, given $d \neq e \in \{1, \ldots, c(G) - 1\}$ such that $(u_d, u_e) \in A_\tau$, consider $v \in X(u_d)$ and $w \in X(u_e)$. We have $(v, w) \in A$ if and only if $\{v, w\} \notin E$.
- Let $v \neq w \in V$ such that v and w belong to the same element of $p_{T[X]}$ without belonging to the same element of $q_{T[X]}$. The arc of T between v and w is given by Assertion H1 of Theorem 3. For instance, we have $M_1 = X^- \longrightarrow N_{c(G)} = X^+$.
- Let $v \neq w \in V$ such that v and w belong to the same element of $q_{T[X]}$. The arc of T between v and w is given by Assertions K1,…,K4 of

Theorem 4. For instance, let $v = f_1(2i) \in X^-$ and $w = f_1(2j) \in X^-$ where $i \neq j \in \{0, \ldots, |C_1|/2\}$. We have $(v, w) \in A$ if and only if $j < i$.

If G is connected, we proceed in the same way by choosing $p_{T[X]} = \{\langle X \rangle, X(u)\}$ and $q_{T[X]} = \{X^-, X^+(u)\}$ where $u \in X$.

The example presented in Section 3 (see Figure 2) is obtained as above from $\tau = T_3$ and from a graph G admitting two connected components C_1 and C_2 by choosing $M_1 = X^-$, $N_1 = X^-(0)$, $M_2 = X^+$ and $N_2 = X^+(0)$.

6. Applications

6.1. **A new proof of the main result of** [7] **for tournaments.** We begin with a new characterization of partially critical tournaments when the considered tournaments are assumed to be indecomposable. This is naturally suggested by Propositions 5 and 6.

Theorem 5. *Given an indecomposable tournament $T = (V, A)$, consider a proper subset X of V such that $|X| \geq 3$ and $T[X]$ is indecomposable. The tournament T is $(V \setminus X)$-critical if and only if $T[X \cup Y]$ is decomposable for every $Y \subseteq V \setminus X$ such that $|Y| = 1$ or 3.*

Proof. If T is $(V \setminus X)$-critical, it suffices to apply Lemma 2. Conversely, assume that $T[X \cup Y]$ is decomposable for every $Y \subseteq V \setminus X$ such that $|Y| = 1$ or 3. We prove that the tournament T is $(V \setminus X)$-critical by using Theorem 3. Assertions H1 of Theorem 3 follows from Lemma 8. For Assertions H2 and H3, consider a connected component C of $G_{T[X]}$. By Proposition 5, there exist $M_C \neq N_C \in q_{T[X]}$ such that $G_{T[X]}[C]$ is bipartite by $\{M_C, N_C\}$. Lastly, we must show that $T[X \cup C]$ is indecomposable and C-critical. By Proposition 5, $T[X \cup C]$ is indecomposable. If $|C| \leq 2$, then $|C| = 2$ and $T[X \cup C]$ is C-critical because $\text{Ext}(X) = \varnothing$. Assume that $|C| \geq 3$. It follows from Proposition 6 applied to $T[X \cup C]$ that there exists an isomorphism f from G_{2n} onto $G_{T[X]}$ such that $\{M_C, N_C\} = \{f(\{0, \ldots, 2n-2\}), f(\{1, \ldots, 2n-1\})\}$ and satisfying Assertions K1,...,K4. As $\text{Ext}(X) = \varnothing$, it follows from Theorem 4 applied to $T[X \cup C]$ that $T[X \cup C]$ is C-critical. Consequently, Assertion H3 of Theorem 3 holds also and hence T is $(V \setminus X)$-critical. □

Theorem 5 provides a quick and simple proof of the main result of [7] for tournaments.

Corollary 3. *Given an indecomposable tournament $T = (V, A)$, consider $X \subseteq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable. If $|V \setminus X| \geq 4$, then there exist $x \neq y \in V \setminus X$ such that $T - \{x, y\}$ is indecomposable.*

Proof. We apply several times Proposition 3 from an indecomposable subtournament $T[Z]$ of T where $X \subseteq Z \subsetneq V$ and $|V \setminus Z|$ is even. If $|V \setminus X|$ is even, then choose $Z = X$. Assume that $|V \setminus X|$ is odd so that $|V \setminus X| \geq 5$. By Lemma 2, the tournament T is not $(V \setminus X)$-critical because $|V \setminus X|$ is odd. It follows from Theorem 5 that there exists a subset Y of $(V \setminus X)$ such that $|Y| = 1$ or 3 and $T[X \cup Y]$ is indecomposable. It suffices to choose $Z = X \cup Y$. □

For a digraph, we have to assume that $|V \setminus X| \geq 6$ (see [7]). The above improvement of this threshold for tournaments answers also a question of Dammak [3]. As noticed in [7, Remark 1], this improvement does not hold for digraphs. In fact, Theorem 5 is false, even for graphs, as shown in the next remark.

Remark 5. *Recall that a graph $G = (V, E)$ is identified with the digraph $D = (V, A)$ defined by: for $x \neq y \in V$, $(x, y) \in A$ if $\{x, y\} \in E$. Let $H = (X, E)$ be an indecomposable graph with $|X| \geq 4$. Given $v_0, \dots, v_{2n} \notin X$ (where $n \geq 2$) and $u \in X$, consider the graph H' defined on $X \cup \{v_0, \dots, v_{2n}\}$ by*

- *$H'[X] = H$;*
- *for $x \in X$ and $w \in \{v_0, \dots, v_{2n}\}$, $\{x, w\}$ is an edge of H' if and only if $\{x, u\}$ is an edge of H and there is $i \in \{0, \dots, n-1\}$ such that $w = v_{2i+1}$;*
- *for $w, w' \in \{v_0, \dots, v_{2n}\}$, $\{w, w'\}$ is an edge of H' if and only if there is $i \in \{0, \dots, 2n-1\}$ such that $\{w, w'\} = \{v_i, v_{i+1}\}$.*

We obtain $p_{H'[X]} = \{\langle X \rangle, X(u)\}$ where $\langle X \rangle = \{v_{2i}; 0 \leq i \leq n\}$ and $X(u) = \{v_{2i+1}; 0 \leq i \leq n-1\}$. Furthermore the outside graph $G_{H'[X]}$ is the path on $\{v_0, \dots, v_{2n}\}$ whose edges are $\{v_i, v_{i+1}\}$ for $0 \leq i \leq 2n-1$. It follows that H' is indecomposable and $H'[X \cup Y]$ is decomposable for each $Y \subseteq \{v_0, \dots, v_{2n}\}$ such that $|Y| = 1$ or 3. But H' is not $\{v_0, \dots, v_{2n}\}$-critical because $H' - v_0$ is indecomposable.

6.2. **The partially critical support of tournaments.** By adding one vertex to a partially critical tournament, it is easy to construct an indecomposable tournament whose partially critical support is a singleton. Consider a $(V \setminus X)$-critical and indecomposable tournament $T = (V, A)$ where $X \subsetneq V$ such that $|X| \geq 5$ and $T[X]$ is indecomposable. Given $\alpha \notin V$, consider any indecomposable tournament T' defined on $X \cup \{\alpha\}$ such that $T'[X] = T$. Then consider the unique tournament T_1 defined on $V \cup \{\alpha\}$ such that $T_1[X \cup \{\alpha\}] = T'$, $T_1[V] = T$ and $\{\alpha, v\} \notin E(G_{T_1[X]})$ for every $v \in V$. It is simply verified that T_1 is indecomposable and $\sigma^p_{T_1[X]}(T_1) = \{\alpha\}$.

To extend a partially critical tournament to an indecomposable tournament whose partially critical support contains at least two vertices, we must add at least three vertices and we cannot use an element of $\mathrm{Ext}(X)$. For instance, consider the tournament $T = (V, A)$ examined in Section 3. We verified that T is indecomposable and $(V \setminus X)$-critical where $X = \{0, 1, 2\}$. Given $\alpha, \beta, \gamma \notin V$, consider the tournament T_2 defined on $V \cup \{\alpha, \beta, \gamma\}$ by

- $T_2[V] = T$;
- $0 \longrightarrow \{\alpha, \beta\} \longrightarrow 2$ and $\alpha \longrightarrow 1 \longrightarrow \beta$;
- $\{1, 2\} \longrightarrow \gamma \longrightarrow 0$;

- $\gamma \longrightarrow \{\alpha, \beta\}$ and $\alpha \longrightarrow \beta$;
- $\gamma \longrightarrow \{x_1, x_3, \ldots, x_{2m-1}\} \cup \{y_1, y_3, \ldots, y_{2n-1}\} \longrightarrow \{\alpha, \beta\}$;
- $\{x_0, x_2, \ldots, x_{2m-2}\} \longrightarrow \{\alpha, \beta, \gamma\} \longrightarrow \{y_0, y_2, \ldots, y_{2n-2}\}$.

We have $X^-(1) = \{\alpha\}$, $X^+(1) = \{\beta\}$ and $X^+(2) = \{\gamma\}$ so that $\{\alpha\}, \{\beta\}, \{\gamma\} \in q_{T_2[X]}$. Since $\{\alpha, \gamma\}, \{\beta, \gamma\} \in E(G_{T_2[X]})$ and $\{\alpha, \beta, \gamma\}$ is a connected component of $G_{T_2[X]}$, we obtain $\alpha, \beta \in \sigma^p_{T_2[X]}(T_2)$. This construction is generalized in Theorem 6. For the sequel, it is important to notice the following.

Remark 6. *Given an indecomposable tournament $T = (V, A)$, consider $X \subsetneq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable. Let $\alpha \in \sigma^p_{T[X]}(T)$. It follows from Theorems 3 and 4 applied to $T - \alpha$ that for each $M \in q_{T[X]}$ such that $M \setminus \{\alpha\} \neq \varnothing$, there exists $N \in q_{T[X]} \setminus \{M\}$ such that $N \setminus \{\alpha\} \neq \varnothing$ satisfying*

- *$(M \setminus \{\alpha\}) \cup (N \setminus \{\alpha\})$ is a connected component of $G_{T[X]} - \alpha$;*
- *$G_{T[X]}[(M \setminus \{\alpha\}) \cup (N \setminus \{\alpha\})]$ is bipartite by $\{M \setminus \{\alpha\}, N \setminus \{\alpha\}\}$;*
- *$G_{T[X]}[(M \setminus \{\alpha\}) \cup (N \setminus \{\alpha\})]$ is isomorphic to $G_{2|M \setminus \{\alpha\}|}$.*

Thus $|M \setminus \{\alpha\}| = |N \setminus \{\alpha\}|$.

As another consequence of Theorem 5, we obtain that the partially critical support of an indecomposable tournament contains at most three vertices.

Lemma 10. *Let $T = (V, A)$ be an indecomposable tournament. For every $X \subsetneq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable, $|\sigma^p_{T[X]}(T)| \leq 3$.*

Proof. As $\sigma^p_{T[X]}(T) \subseteq \sigma(T) \cap (V \setminus X)$, assume that $\sigma(T) \cap (V \setminus X) \neq \varnothing$, that is, T is not $(V \setminus X)$-critical. By Theorem 5, there is $Y_0 \subseteq V \setminus X$ such that $T[X \cup Y_0]$ is indecomposable and $|Y_0| = 1$ or 3. Let $\alpha \in \sigma^p_{T[X]}(T)$. Since $T - \alpha$ is $((V \setminus \{\alpha\}) \setminus X)$-critical, it follows from Theorem 5 that $T[X \cup Y]$ is indecomposable for each $Y \subseteq (V \setminus \{\alpha\}) \setminus X$ with $|Y| = 1$ or 3. Thus $Y_0 \nsubseteq (V \setminus \{\alpha\}) \setminus X$, that is, $\alpha \in Y_0$. Consequently $\sigma^p_{T[X]}(T) \subseteq Y_0$. □

The next constitutes the main step in describing an indecomposable tournament whose partially critical support contains at least two vertices.

Proposition 7. *Given an indecomposable tournament $T = (V, A)$, consider $X \subsetneq V$ such that $|X| \geq 3$ and $T[X]$ is indecomposable. If $|\sigma^p_{T[X]}(T)| = 2$ or 3, then $\{\alpha\} \in q_{T[X]}$ for every $\alpha \in \sigma^p_{T[X]}(T)$.*

Proof. Consider an element α of $\sigma^p_{T[X]}(T)$ and denote by M the element of $q_{T[X]}$ containing α. Seeking a contradiction, suppose that $|M| \geq 2$. By Remark 6 there exists $N \in q_{T[X]} \setminus \{M\}$ such that $(M \setminus \{\alpha\}) \cup N$ is a connected component of $G_{T[X]} - \alpha$ and

$$|M| = |N| + 1. \tag{1}$$

Consider an element β of $\sigma^p_{T[X]}(T) \setminus \{\alpha\}$. By Remark 6, there exists $N' \in q_{T[X]} \setminus \{M\}$ such that $N' \setminus \{\beta\} \neq \varnothing$ satisfying $(M \setminus \{\beta\}) \cup (N' \setminus \{\beta\})$ is a connected component of $G_{T[X]} - \beta$ and

$$|M \setminus \{\beta\}| = |N' \setminus \{\beta\}|. \tag{2}$$

We prove that $\beta \in M$ and $N = N'$ by distinguishing the following two cases.

First, assume that there exist $u \in M \setminus \{\alpha\}$ and $v \in N$ such that $\{u, v\} \in E_{T[X]}$ and $\beta \notin \{u, v\}$. As $(M \setminus \{\beta\}) \cup (N' \setminus \{\beta\})$ is a connected component of $G_{T[X]} - \beta$, $\{u, v\} \subseteq (M \setminus \{\beta\}) \cup (N' \setminus \{\beta\})$. Thus $v \in N \cap N'$ and hence $N = N'$. It follows from (1) and (2) that $\beta \in M$.

Second, assume that for any $u \in M \setminus \{\alpha\}$ and $v \in N$, if $\{u, v\} \in E_{T[X]}$, then $\beta \in \{u, v\}$. By Remark 6, $G_{T[X]}[(M \setminus \{\alpha\}) \cup N]$ is bipartite by $\{M \setminus \{\alpha\}, N\}$ and is isomorphic to $G_{2|N|}$. Thus $|M \setminus \{\alpha\}| = |N| = 1$. By denoting by u the unique element of $M \setminus \{\alpha\}$ and by v this one of N, we have $\{u, v\} \in E_{T[X]}$ and $\beta \in \{u, v\}$. By Remark 2, $\{\alpha, u\} \notin E_{T[X]}$. Since $\{u, v\}$ is a connected component of $G_{T[X]} - \alpha$, u is an isolated vertex of $G_{T[X]} - v$. By Lemma 7, $G_{T[X]} - \beta$ does not admit isolated vertices so that $\beta = u$. Therefore v is an isolated vertex of $G_{T[X]} - \{\alpha, \beta\}$. As v is not an isolated vertex of $G_{T[X]} - \beta$, $\{\alpha, v\} \in E_{T[X]}$. Consequently $v \in N \cap N'$ and hence $N = N'$.

In both cases, we obtain $\beta \in M$ and $N = N'$. Since $(M \setminus \{\alpha\}) \cup N$ is a connected component of $G_{T[X]} - \alpha$ and $(M \setminus \{\beta\}) \cup N$ is a connected component of $G_{T[X]} - \beta$, $M \cup N$ is a connected component of $G_{T[X]}$.

To obtain a contradiction, we prove that $\{\alpha, \beta\}$ is an interval of T. Denote by L the element of $p_{T[X]}$ such that $M \subseteq L$. Using Lemma 3, we have to prove that $\mathrm{Ext(X)} = \varnothing$, $\{\alpha, \beta\}$ is an interval of $G_{T[X]}$ and $\{\alpha, \beta\}$ is an interval of $T[L]$.

As seen in the proof of Lemma 10, if there exists $a \in V \setminus X$ such that $a \in \mathrm{Ext}(X)$, that is, $T[X \cup \{a\}]$ is indecomposable, then $\sigma^p_{T[X]}(T) \subseteq \{a\}$. Thus $\mathrm{Ext}(X) = \varnothing$.

To show that $\{\alpha, \beta\}$ is an interval of $G_{T[X]}$, it suffices to prove that $\{\alpha, \beta\}$ is an interval of $G_{T[X]}[M \cup N]$ because $M \cup N$ is a connected component of $G_{T[X]}$. By Remark 6, there exists an isomorphism f from $G_{2|N|}$ onto $G_{T[X]}[(M \setminus \{\alpha\}) \cup N]$. Since $G_{T[X]}[(M \setminus \{\alpha\}) \cup N]$ is bipartite by $\{M \setminus \{\alpha\}, N\}$,

$$N = \{f(0), f(2), \ldots, f(2|N| - 2)\} \text{ or } \{f(1), f(3), \ldots, f(2|N| - 1)\}.$$

By considering $f \circ \psi_{2|N|}$ (see Observation 1(iii)) instead of f, assume that $N = \{f(1), f(3), \cdots, f(2|N| - 1)\}$. Similarly, there exists an isomorphism g from $G_{2|N|}$ onto $G_{T[X]}[(M \setminus \{\beta\}) \cup N]$ such that $N = \{g(1), g(3), \cdots, g(2|N| - 1)\}$. By Theorem 4, either

$$\begin{aligned} T[N] &= f(1) < f(3) < \cdots < f(2|N| - 1) \\ &= g(1) < g(3) < \cdots < g(2|N| - 1) \end{aligned}$$

or

$$\begin{aligned} T[N] &= f(2|N|-1) < f(2|N|-3) < \cdots < f(1) \\ &= g(2|N|-1) < g(2|N|-3) < \cdots < g(1). \end{aligned}$$

It follows that $f(2i+1) = g(2i+1)$ for $0 \leq i \leq |N|-1$. Thus

$$d_{G_{T[X]}[(M\setminus\{\alpha\})\cup N]}(\gamma) = d_{G_{T[X]}[(M\setminus\{\beta\})\cup N]}(\gamma)$$

for every $\gamma \in N$. Therefore, for every $\gamma \in N$, $\{\alpha, \gamma\} \in E_{T[X]}$ if and only if $\{\beta, \gamma\} \in E_{T[X]}$. As $G_{T[X]}[M]$ is empty by Remark 2, $\{\alpha, \beta\}$ is an interval of $G_{T[X]}[M \cup N]$.

Lastly, we prove that $\{\alpha, \beta\}$ is an interval of $T[L]$. Since $\{\alpha, \beta\}$ is an interval of $G_{T[X]}[M \cup N]$, the function $h : (M \setminus \{\alpha\}) \cup N \longrightarrow (M \setminus \{\beta\}) \cup N$, defined by $\beta \mapsto \alpha$ and $\gamma \mapsto \gamma$ for $\gamma \in (M \cup N) \setminus \{\alpha, \beta\}$, is an isomorphism from $G_{T[X]}[(M \setminus \{\alpha\}) \cup N]$ onto $G_{T[X]}[(M \setminus \{\beta\}) \cup N]$. Thus $g^{-1} \circ h \circ f$ is an automorphism of $G_{2|N|}$. As $f(2i+1) = g(2i+1)$ for $0 \leq i \leq |N|-1$, $(g^{-1} \circ h \circ f)(\{1, 3, \ldots, 2|N|-1\}) = \{1, 3, \ldots, 2|N|-1\}$. It follows from Observation 1(iii) that $g^{-1} \circ h \circ f = \mathrm{Id}_{\{0,\cdots,2|\mathrm{N}|-1\}}$. Therefore $f^{-1}(\beta) = g^{-1}(\alpha)$ and $f^{-1}(\gamma) = g^{-1}(\gamma)$ for every $\gamma \in (M \setminus \{\alpha, \beta\}) \cup N$. We obtain $f(m) = g(m)$ for every $m \in \{0, 2, \ldots, 2|N|-2\} \setminus \{f^{-1}(\beta)\}$. By Theorem 4, either

$$\begin{aligned} &T[M \setminus \{\alpha\}] = f(0) < f(2) < \cdots < f(2|N|-2) \text{ and} \\ &T[M \setminus \{\beta\}] = g(0) < g(2) < \cdots < g(2|N|-2) \end{aligned}$$

or

$$\begin{aligned} &T[M \setminus \{\alpha\}] = f(2|N|-2) < f(2|N|-4) < \cdots < f(0) \text{ and} \\ &T[M \setminus \{\beta\}] = g(2|N|-2) < g(2|N|-4) < \cdots < g(0). \end{aligned}$$

Thus $\{\alpha, \beta\}$ is an interval of $T[M]$. Furthermore, it follows from Lemma 8 that either $\{\alpha, \beta\} \longrightarrow L \setminus M$ or $L \setminus M \longrightarrow \{\alpha, \beta\}$. Consequently $\{\alpha, \beta\}$ is an interval of $T[L]$. ☐

Theorem 6. *Given a tournament $T = (V, A)$, consider $X \subsetneq V$ such that $|X| \geq 3$, $T[X]$ is indecomposable and* $\mathrm{Ext}(X) = \varnothing$. *Given $\alpha \neq \beta \in V \setminus X$, T is indecomposable and $\alpha, \beta \in \sigma^p_{T[X]}(T)$ if and only if there exists $\gamma \in (V \setminus X) \setminus \{\alpha, \beta\}$ satisfying*

- $\{\alpha\}, \{\beta\}, \{\gamma\} \in q_{T[X]}$;
- $\{\alpha, \beta, \gamma\}$ *is a connected component of* $G_{T[X]}$;
- $\{\alpha, \gamma\} \in E_{T[X]}$ *and* $\{\beta, \gamma\} \in E_{T[X]}$;
- $T - \{\alpha, \beta, \gamma\}$ *is indecomposable and* $((V \setminus \{\alpha, \beta, \gamma\}) \setminus X)$*-critical.*

Proof. To begin, assume that T is indecomposable and $\alpha, \beta \in \sigma^p_{T[X]}(T)$. It follows from Proposition 7 that $\{\alpha\}, \{\beta\} \in q_{T[X]}$. As $\alpha \in \sigma^p_{T[X]}(T)$ and $\{\beta\} \in q_{T[X]}$, it follows from Remark 6 that there is $N \in q_{T[X]} \setminus \{\{\beta\}\}$ such that $\{\beta\} \cup (N \setminus \{\alpha\})$ is a connected component of $G_{T[X]} - \alpha$ with

$|N \setminus \{\alpha\}| = 1$. Since $\{\alpha\} \in q_{T[X]}$, there is $\gamma \in (V \setminus X) \setminus \{\alpha, \beta\}$ such that $N = \{\gamma\}$. Thus γ is an isolated vertex of $G_{T[X]} - \{\alpha, \beta\}$. By Lemma 7, $G_{T[X]} - \beta$ does not admit isolated vertices and hence $\{\alpha, \gamma\} \in E_{T[X]}$. Since $\beta \in \sigma^p_{T[X]}(T)$ and since $\{\beta\}, \{\gamma\} \in q_{T[X]}$, with $\{\alpha, \gamma\} \in E_{T[X]}$, it follows from Remark 6 that $\{\alpha, \gamma\}$ is a connected component of $G_{T[X]} - \beta$. Furthermore, as $\{\beta\} \cup (N \setminus \{\alpha\}) = \{\beta, \gamma\}$ is a connected component of $G_{T[X]} - \alpha$, $\{\alpha, \beta, \gamma\}$ is a connected component of $G_{T[X]}$. Lastly, since $\alpha \in \sigma^p_{T[X]}(T)$ and $\{\beta, \gamma\}$ is a connected component of $G_{T[X]} - \alpha$, it follows from Theorems 3 and 4 that $(T - \alpha) - \{\beta, \gamma\}$ is indecomposable and $((V \setminus \{\alpha, \beta, \gamma\}) \setminus X)$-critical.

Conversely, assume that the four assertions above are satisfied. For a contradiction, suppose that T admits a non-trivial interval I. Since $T - \{\alpha, \beta, \gamma\}$ is indecomposable and $((V \setminus \{\alpha, \beta, \gamma\}) \setminus X)$-critical, $G_{T[X]} - \{\alpha, \beta, \gamma\}$ has no isolated vertex. As $\{\alpha, \gamma\}, \{\beta, \gamma\} \in E_{T[X]}$, $G_{T[X]}$ has no isolated vertex as well. It follows from Lemma 4 that $I \cap X = \varnothing$. By Lemma 3, there exists $M \in q_{T[X]}$ such that $I \subseteq M$. Since $\{\alpha\}, \{\beta\}, \{\gamma\} \in q_{T[X]}$, $M \subseteq (V \setminus X) \setminus \{\alpha, \beta, \gamma\}$ and hence I would be a non-trivial interval of $T - \{\alpha, \beta, \gamma\}$. It follows that T is indecomposable. Lastly, we verify that $\alpha \in \sigma^p_{T[X]}(T)$. As $\{\alpha, \beta, \gamma\}$ is a connected component of $G_{T[X]}$ and $\{\beta, \gamma\} \in E_{T[X]}$, $\{\beta, \gamma\}$ is a connected component of $G_{T[X]} - \alpha$. Thus the connected components of $G_{T[X]} - \alpha$ are $\{\beta, \gamma\}$ and those of $G_{T[X]} - \{\alpha, \beta, \gamma\}$. Since $T - \{\alpha, \beta, \gamma\}$ is indecomposable and $((V \setminus \{\alpha, \beta, \gamma\}) \setminus X)$-critical, it follows from Theorems 3 and 4 that $T - \alpha$ is indecomposable and $((V \setminus \{\alpha\}) \setminus X)$-critical, that is, $\alpha \in \sigma^p_{T[X]}(T)$. Similarly $\beta \in \sigma^p_{T[X]}(T)$. □

In the first part of the last proof, we can also observe that $\gamma \in \sigma^p_{T[X]}(T)$ if and only if $\{\alpha, \beta\} \in E_{T[X]}$. The following is then an immediate consequence of Theorem 6.

Corollary 4. *Given a tournament $T = (V, A)$, consider $X \subsetneq V$ such that $|X| \geq 3$, $T[X]$ is indecomposable and $\mathrm{Ext}(X) = \varnothing$. Given distinct elements α, β, γ of $V \setminus X$, T is indecomposable and $\sigma^p_{T[X]}(T) = \{\alpha, \beta, \gamma\}$ if and only if the following hold*

- *$\{\alpha\}, \{\beta\}, \{\gamma\} \in q_{T[X]}$;*
- *$\{\alpha, \beta, \gamma\}$ is a connected component of $G_{T[X]}$;*
- *$G_{T[X]}[\{\alpha, \beta, \gamma\}]$ is complete;*
- *$T - \{\alpha, \beta, \gamma\}$ is indecomposable and $((V \setminus \{\alpha, \beta, \gamma\}) \setminus X)$-critical.*

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Faculté des Sciences de Sfax, Département de Mathématiques,
BP 802, 3018 Sfax, Tunisie
E-mail address: sayar_mohamed@yahoo.fr