On motives for Deligne-Mumford stacks
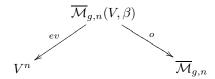
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Abstract

We define and compare two different definitions of Chow motives for Deligne-Mumford stacks, associated with two different definitions of Chow rings. The main result we prove is that both categories of motives are equivalent to the usual category of motives of algebraic varieties, but the motives of a given stack associated with both theories are not isomorphic. We will also give some examples of motives associated with some algebraic stacks.

Contents

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The construction of Gromov-Witten invariants in algebraic geometry is based on two fundamental objects. The first one is a diagram of algebraic stacks

$$\overline{\mathcal{M}}_{g,n}(V,\beta)$$

$$ev \swarrow \qquad \searrow o$$

$$V^n \qquad\qquad\qquad \overline{\mathcal{M}}_{g,n}$$

where $\overline{\mathcal{M}}_{g,n}(V,\beta)$ is the stack of stable maps $f : C \longrightarrow V$, with $f_*([C]) = \beta$, o the morphism which forgets f then stabilizes, and ev the evaluation morphism at the marked points on the curve C.

The second one is the virtual fundamental class, $\mathcal{I}_{g,n}(V,\beta) \in A_D(\overline{\mathcal{M}}_{g,n}(V,\beta))$ ([B-F]).

These two objects combine to give the Gromov-Witten correspondence ([B])

$$I_{g,n}(V,\beta) := (ev,o)_*(\mathcal{I}_{g,n}(V,\beta)) \in A_D(V^n \times \overline{\mathcal{M}}_{g,n}),$$

from which the Gromov-Witten invariants are defined. This shows the motivic nature of Gromov-Witten invariants, and raises the question of the construction of a good theory of motives for Deligne-Mumford stacks.

We possess now at least two different ways to define such a theory, corresponding to two different definitions of Chow cohomology for Deligne-Mumford stacks. There exist first the A^* theories, which are described in [E-G, G, Jo, K, V], and which all coincide with rational coefficients. They satisfy every expected properties of a Chow cohomology, except the Riemann-Roch theorem, and in particular the Riemann-Roch isomorphism $K_0(F)_{\mathbb{Q}} \simeq A^*(F)_{\mathbb{Q}}$. On the other hand, we have the enriched theory A_χ^*, which was used in [T1, T2] to prove the Grothendieck-Riemann-Roch formula, and for which $K_0(F)_{\mathbb{Q}} \simeq A_\chi^*(F)_{\mathbb{Q}}$.

In this paper we will show that the two categories of motives associated to these two previous theories are equivalent. This will be shown using the fact that both of them are equivalent to the usual category of Chow motives for algebraic varieties (answering a question of Y. Manin and K. Behrend [B-M, 8.2]). However this does not implies that the two associated motivic theories are equivalent. Indeed, we will show that the motive of a Deligne-Mumford stack associated with the theory A^* is only a direct factor of the one associated with the theory A_χ^*.

Basically, all the results proved in this paper can be seen as a motivic interpretation of the computation of the rationnal G-theory spectrum of a Deligne-Mumford stacks which appears in [T1, T2, T3].

In the first part of this work we will review briefly the two different definitions of Chow rings, as well as some results about the K-theory of stacks which explain them. In the second and third part we will define the associated categories of motives and prove that they are equivalent to the category of usual

Chow motives. Finally, we will give some examples of motives associated to stacks.

Notations: We will work over a perfect base field k, of any caracteristic. An algebraic variety will be a scheme, smooth and proper over $Spec\, k$. A DM-stack will be a Deligne-Mumford stack of finite type over $Spec\, k$, smooth and proper over $Spec\, k$ ([D-M]). As a convention we will work in the homotopy category of stacks. Thus a morphism of stacks will be for us the class of a 1-morphism up to 2-isomorphisms. The category of DM-stacks thus obtained will be denoted by $\mathcal{DM}$. The full sub-category of varieties will be denoted by $\mathcal{VAR}$.

1 Preliminaries on Chow rings of Deligne-Mumford stacks

We start with the first definition of the Chow rings of Deligne-Mumford stacks.

For every DM-stack F, we consider $\mathcal{K}_m$, the sheaf of the m-th K-groups on F_{et} associated with the abelian presheaf

$$\mathcal{K}_m : \quad \begin{array}{ccc} F_{et} & \longrightarrow & Ab \\ U & \mapsto & K_m(U) \end{array}$$

Definition 1.1 *([G]) The codimension m rationnal Chow group of F is defined to be*
$$A^m(F) := H^m(F_{et}, \mathcal{K}_m \otimes \mathbb{Q}).$$
We will note $A^*(F) := \bigoplus_m A^m(F)$.

As it is shown in [G], the theory $A^*(F)$ is a good Chow cohomology theory. Without recalling all the properties w recall three of them which will be usefull for us.

- (*Product*) For every DM-stack $A^*(F)$ has a structure of a graded commutative ring.

- (*Functoriality*) For every morphism of DM-stacks, $f : F \longrightarrow F'$, there exist an inverse image

$$f^* : A^*(F') \longrightarrow A^*(F)$$

  which is a morphism of graded rings.

  There is also a direct image

$$f_* : A^*(F) \longrightarrow A^*(F')$$

  which is a morphism of $\mathbb{Q}$-vector spaces. This morphism is moreover graded of degree $DimF' - DimF$ if F and F' are pure dimensional.

- (*Projection formula*) For every morphism $f : F \longrightarrow F'$ between two DM-stacks we have
$$f_*(x.f^*(y)) = f_*(x).y$$
  for every $x \in A^*(F)$ and $y \in A^*(F')$.

  In particular, if F and F' are connected and of the same dimension, we have
$$f_* f^* = \times m$$
  where m is the generic degree of f (in the stack sense [V]).

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- (*Compatibility*) For every variety X, $A^*(X) \simeq CH^*(X)_{\mathbb{Q}}$ is the usual Chow ring of X.

In order to introduce the second definition A_χ^* recall the main result of [T3]. It will not be used but explains the definition of A_χ^*.

We begin with a DM-stack F, and consider C_F^t, the classifying stack of cyclic subgroups of automorphisms in F. More precisely it is defined in the following way.

A S-group scheme $G \longrightarrow S$ is called cyclic (and finite), if locally for the etale topology on S it is isomorphic (as a S-group scheme) to $Spec \dfrac{\mathcal{O}_S[T]}{T^m - 1}$. In other words, G is a multiplicative type S-group scheme whose sheaf of characters has cyclic geometric fibres.

The stack C_F^t is now defined by the following.

- For any k-scheme X, the objects in $C_F^t(X)$ are pairs (s, c), where s is an object in $F(X)$, and c is a sub-group scheme of the X-group scheme of automorphisms of s, $Aut_X(s) \longrightarrow X$, such that c is a cyclic X-group scheme.

- An isomorphism between two objects in $C_F^t(X)$, (s, c) and (s', c'), is an isomorphism $u : s \simeq s'$ in $F(X)$, such that $u^{-1}.c'.u = c$.

The map which sends (s, c) to s gives a morphism $\pi_F : C_F^t \longrightarrow F$, which is oubviously representable. Futhermore, we have the following local description of π_F. Locally (for the etale topology) on its moduli space F is given by a quotient of a smooth scheme X by a finite group H. So, to obtain a local descrition of π_F it is enough to consider the case where $F = [X/H]$.

Let $c(H)$ be a set of representative of conjugacy classes of cyclic subgroups of H whose orders are prime to the caracteristic of k. For each $c \in c(H)$, let X^c the closed sub-scheme of fixed points of c in X, and N_c the normalisor of c in H. Note that N_c acts on X^c by restriction. Then we have a natural equivalence

$$C_F^t \simeq \coprod_{c \in c(H)} [X^c / N_c].$$

As every c is cyclic of order invertible on X, it is a diagonalisable group scheme, and so as X is smooth, X^c is also smooth.

From this local description we deduce that the stack C_F^t is smooth, and the natural morphism $\pi_F : C_F^t \longrightarrow F$ is representable, finite and unramified. In particular C_F^t is again smooth and proper.

On the stack C_F^t lives the universal cyclic group stack, $q : \mathcal{C}_F^t \longrightarrow C_F^t$. It classifies triplets (s, c, h), where (s, c) is an object of $C_F^t(X)$ and h a section of c over X. Thus, for any morphism $(s, c) : U \longrightarrow C_F^t$, the pull-back of $\mathcal{C}_F^t$ on U is isomorphic to the cyclic U-group scheme $c \longrightarrow U$.

Let χ_F be the sheaf of characters of $\mathcal{C}_F^t$ on C_F^t. It is defined by $\chi_F := \underline{Hom}_{Gp}(\mathcal{C}_F^t, \mathbb{G}_m)$. More explicitly, its restriction on the small etale site of C_F^t is given by

$$\chi_F : \qquad (C_F^t)_{et} \qquad \longrightarrow \qquad Ab$$
$$((s,c) \in C_F^t(U)) \quad \mapsto \quad Hom_{Gp}(c, \mathbb{G}_{m,U})$$

As $\mathcal{C}_F^t$ is a cyclic group stack, χ is a locally constant sheaf on $(C_F^t)_{et}$, locally isomorphic to a constant finite cyclic group sheaf.

Let us consider the sheaf of group-algebras associated to χ, $\mathbb{Q}[\chi_F]$. It is a locally constant sheaf of $\mathbb{Q}$-algebras on $(C_F^t)_{et}$, which is locally isomorphic to the constant sheaf with fibre $\dfrac{\mathbb{Q}[T]}{T^m - 1}$. As $\dfrac{\mathbb{Q}[T]}{T^m - 1}$ is a product of cyclotomic fields with only one of maximal degree, namely $\mathbb{Q}(\zeta_m)$, the kernels of the local quotients $\dfrac{\mathbb{Q}[T]}{T^m - 1} \longrightarrow \mathbb{Q}(\zeta_m)$ glue together to give a well defined ideal sheaf $\mathcal{I}_F \hookrightarrow \mathbb{Q}[\chi_F]$. We then define

$$\Lambda_F := \frac{\mathbb{Q}[\chi_F]}{\mathcal{I}_F}.$$

Note that this is a well defined sheaf of $\mathbb{Q}$-algebras on C_F^t, locally isomorphic to the constant sheaf associated with a cyclotomic field.

We can now state the main result of [T3]. For a sketch of proof the reader can consult [T3], or [T1, 3.15] for a particular case.

Theorem 1.2 *There exist a functorial ring isomorphism*

$$\phi_F : K_*(F) \otimes \mathbb{Q} \simeq H^{-*}((C_F^t)_{et}, \underline{K} \otimes \Lambda_F).$$

Remark: Here $K_*(F)$ is the ring of K-theory of perfect complexes on F, and $\underline{K}$ is the presheaf of K-theory spectrum on $(C_F^t)_{et}$.

The theorem justifies the following definition.

Definition 1.3 *For any DM-stack F, the codimension m rationnal Chow group with coefficients in the characters of F is defined by*

$$A_\chi^m(F) := H^m((C_F^t)_{et}, \mathcal{K}_m \otimes \Lambda_F).$$

We will note $A_\chi^*(F) := \bigoplus_m A_\chi^m(F)$.

There is a natural decomposition $C_F^t \simeq F \coprod C_{F,+}^t$ coming from the section $F \longrightarrow C_F^t$ maping an object s to $(s, \{e\})$. As the sheaf χ_F restricts to the constant sheaf $\mathbb{Q}$ on F_{et}, this induces a group decomposition

$$A_\chi^*(F) \simeq A^*(F) \times A_{\chi \neq 1}^*(F).$$

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Proposition 1.4 *1. (Product) For every DM-stack, there is structure of graded commutative $\mathbb{Q}$-algebra on $A_\chi^*(F)$. Furthermore, the decomposition $A_\chi^*(F) \simeq A^*(F) \times A_{\chi \neq 1}^*(F)$ becomes a $\mathbb{Q}$-algebra decomposition.*

 2. (Functoriality) For every morphism of DM-stacks $f : F \longrightarrow F'$, there is an inverse image

$$f^* : A_\chi^*(F') \longrightarrow A_\chi^*(F)$$

 which makes A_χ^* into a functor $\mathcal{DM}^o \longrightarrow (graded \,\, \mathbb{Q}-algebras)$. Furthermore the decomposition $A_\chi^* \simeq A^* \times A_{\chi \neq 1}^*$ is compatible with these inverse images.

 There exist a direct image

$$f_* : A_\chi^*(F) \longrightarrow A_\chi^*(F')$$

 which makes A_χ^* into a functor $\mathcal{DM} \longrightarrow \mathbb{Q}-Vect$.

 3. (Projection formula) For every morphism of DM-stacks $f : F \longrightarrow F'$, we have

$$f_*(x.f^*(y)) = f_*(x).y$$

 for every $x \in A_\chi^*(F)$ and $y \in A_\chi^*(F')$.

 4. (Compatibility) For every variety X, $A_\chi^*(X) \simeq CH^*(X)_\mathbb{Q}$ is the usual Chow ring of X.

 Proof: (1) The product in K-theory gives morphisms of sheaves on $(C_F^t)_{et}$

$$\mathcal{K}_p \otimes \mathcal{K}_m \longrightarrow \mathcal{K}_{p+m},$$

defining a graded ring structure on $\mathcal{K}_* := \bigoplus_m \mathcal{K}_m$. By tensoring with the sheaf of algebras Λ_F we obtain a sheaf of graded $\mathbb{Q}$-algebras $\mathcal{K}_* \otimes \Lambda_F$. It is then a general fact that the cohomology

$$A_\chi^*(F) \simeq H^*((C_F^t)_{et}, \mathcal{K}_* \otimes \Lambda_F)$$

is naturally a graded $\mathbb{Q}$-algebra.

 (2) Every morphism between two DM-stacks $f : F \longrightarrow F'$ induces a morphism $Cf : C_F^t \longrightarrow C_{F'}^t$. It sends an object $(s, c) \in C_F^t(X)$ to $(f(s), f(c)) \in C_{F'}^t$. Furthermore there is a morphism of sheaves of groups $Cf^{-1}(\chi_{F'}) \longrightarrow \chi_F$ given by restrictions of characters, giving a morphism of sheaves of algebras

$$Res_f : Cf^{-1}(\Lambda_{F'}) \longrightarrow \Lambda_F.$$

On the other hand we have inverse image in K-theory, which gives a morphism of sheaves on graded algebras

$$Cf^* : Cf^{-1}(\mathcal{K}_*) \longrightarrow \mathcal{K}_*.$$

By tensorisation this gives

$$Cf^{-1}(\mathcal{K}_* \otimes \Lambda_{F'}) \longrightarrow \mathcal{K}_* \otimes \Lambda_F$$

which allows to define inverse images

$$f^* : H^*((C_{F'}^t)_{et}, \mathcal{K}_* \otimes \Lambda_{F'}) \longrightarrow H^*((C_F^t)_{et}, Cf^{-1}(\mathcal{K}_* \otimes \Lambda_{F'})) \longrightarrow H^*((C_F^t)_{et}, \mathcal{K}_* \otimes \Lambda_F).$$

To define the direct images we use the induction morphism of characters

$$Ind_f : \chi_F \longrightarrow Cf^{-1}(\chi_{F'}).$$

This induces a morphism of sheaves of $\mathbb{Q}$-vector spaces

$$Ind_f : \Lambda_F \longrightarrow Cf^{-1}(\Lambda_{F'}).$$

For every m, we use the Gersten resolution of the sheaf $\mathcal{K}_m$ ([G2, 7])

$$\mathcal{K}_m \longrightarrow \mathcal{R}_m^m \longrightarrow \mathcal{R}_m^{m-1} \longrightarrow \cdots \longrightarrow \mathcal{R}_m^0.$$

Let $\mathcal{R}_*^\bullet := \bigoplus_m \mathcal{R}_m^\bullet$. Thinking of $\mathcal{R}_m^0$ in cohomological degree 0, we have

$$A_\chi^*(F) \simeq H^0((C_F^t)_{et}, \mathcal{R}_*^\bullet \otimes \Lambda_F).$$

The direct image is a morphism of complexes of sheaves on $(C_{F'}^t)_{et}$ ([G2, 7])

$$Cf_*(\mathcal{R}_*^\bullet) \longrightarrow \mathcal{R}_*^\bullet.$$

Tensoring with $\Lambda_{F'}$ gives

$$Cf_*(\mathcal{R}_*^\bullet \otimes Cf^{-1}(\Lambda_{F'})) \simeq Cf_*(\mathcal{R}_*^\bullet) \otimes \Lambda_{F'} \longrightarrow \mathcal{R}_*^\bullet \otimes \Lambda_{F'}.$$

We then compose with Ind_f and take the cohomology to obtain

$$f_* : A_\chi^*(F) \longrightarrow A_\chi^*(F').$$

(3) Using the two previous explicits definitions of f_* and f^* the proof is exactly the same as for the case of scheme ([G2, 7]).

(4) If X is a variety, then $C_X^t \simeq X$ and $\Lambda_X \simeq \mathbb{Q}$, so the isomorphism $A_\chi^*(X) \simeq CH^*(X)_{\mathbb{Q}}$ is given by the Bloch's formula ([G2, 7])

$$CH^p(X)_{\mathbb{Q}} \simeq H^p(X_{zar}, \mathcal{K}_p) \otimes \mathbb{Q} \simeq H^p(X_{et}, \mathcal{K}_p \otimes \mathbb{Q}).$$

$\square$

Remark: The Riemann-Roch formula of [T1, 4.11] extends to a formula with values in A_χ^*. Indeed, by using the construction of chern classes in [G2] and the theorem 1.2 one can define a Chern character

$$Ch^\chi : K_0(F) \longrightarrow A_\chi^*(F).$$

The Todd class $Td(F)$ defined in [T1, 4.8] can also be defined as $Td^\chi(F) \in A^*_\chi(F)$ in a very similar manner.

To prove the Riemann-Roch formula for $Ch^\chi(-).Td^\chi$, we first use the projection formula to do galois descent and reduce the problem to the case where k is algebraically closed. We choose an embending $\mu_\infty(k) \hookrightarrow \mathbb{C}^*$. Then the formula follows from [T2, 3.36] and the fact that (see 3.6 for a proof of this)

$$A^*_\chi(F) \otimes \mathbb{Q}(\mu_\infty(k)) \simeq A^*_{rep}(F).$$

It is also true that the Chern character

$$Ch^\chi : K_0(F)_\mathbb{Q} \longrightarrow A^*_\chi(F)$$

is a ring isomorphism.

2 First construction

The construction of the category of Chow motives for DM-stacks using the theory A^* was done in [B-M, 8]. We will denote it by $\mathcal{M}^{DM}$, and call its objects the DMC-motives, as suggested in [B-M]. We start by recalling briefly its construction.

For $F, F' \in \mathcal{DM}$, we define the vector space of correspondences of degree m between F and F'

$$S^m(F, F') := \{x \in A^*(F \times F')/(p_2)_*(x) \in A^m(F')\}$$

where $p_2 : F \times F' \longrightarrow F'$ is the second projection.

We have the usual composition

$$\circ : S^m(F, F') \otimes S^n(F', F'') \longrightarrow S^{p+m}(F, F'')$$

given by the formula

$$x \circ y := (p_{13})_*(p_{12}^*(x).p_{23}^*(y)),$$

where the p_{ij} are the natural projections of $F \times F' \times F''$ on two of the three factors.

Objects of $\mathcal{M}^{DM}$ are triplets (F, p, m), with $F \in \mathcal{DM}$, p an idempotent in the ring of correspondences $S^0(F, F)$, and $m \in \mathbb{Z}$. The morphisms between (F, p, m) and (F', q, n) are given by

$$Hom_{\mathcal{M}^{DM}}((F, p, m), (F', q, n)) := q \circ S^{n-m}(F, F') \circ p \subset S^{n-m}(F, F').$$

Recall also that for any morphism in $\mathcal{DM}$, $f : F \longrightarrow F'$, we have its graph

$$\Gamma_f = f \times Id : F \longrightarrow F' \times F,$$

and so a well defined element

$$[f^*] := (\Gamma_f)_*(1) \in S^0(F', F).$$

We can also consider its transposed

$$[f_*] := [f^*]^t \in S^*(F, F').$$

This allows us to define a functor

$$
\begin{array}{cccc}
h: & \mathcal{DM}^o & \longrightarrow & \mathcal{M}^{DM} \\
 & F & \mapsto & (F, Id, 0) \\
 & f & \mapsto & [f^*]
\end{array}
$$

As for the case of varieties, the category $\mathcal{M}^{DM}$ is $\mathbb{Q}$-linear karoubian category ([B-M, 8.1]). In particular, this implies that if a morphism in $\mathcal{M}^{DM}$ possesses a left inverse then it is a direct factor.

It is also symetric monoidal for the tensor product defined by

$$(F, p, m) \otimes (F', q, n) := (F \times F', p \otimes q, n + m).$$

As usual we shall write $\mathbb{L}^m = (Spec\, k, Id, m)$ for the m-th power of the Lefschetz motive. Note that for every DMC-motive M, we have $M \simeq (F, p, 0) \otimes \mathbb{L}^m$. As $(F, p, 0)$ is a direct factor in $h(F)$, this shows that M is a direct factor in some $h(F) \otimes \mathbb{L}^m$.

Finally, there is a natural fully faithfull tensorial functor

$$\mathcal{M} \longrightarrow \mathcal{M}^{DM}$$

from the usual category of Chow motives of varieties to the category of DMC-motives. This functor fits into a commutative diagramm

$$
\begin{array}{ccc}
\mathcal{VAR}^o & \longrightarrow & \mathcal{DM}^o \\
\Big\downarrow h & & \Big\downarrow h \\
\mathcal{M} & \longrightarrow & \mathcal{M}^{DM}
\end{array}
$$

The following theorem is a positive answer to the question [B-M, 8.2].

Theorem 2.1 *The previous functor*

$$\mathcal{M} \longrightarrow \mathcal{M}^{DM}$$

is an equivalence of $\mathbb{Q}$-tensorial categories.

Proof: By noticing that the essential image is closed by direct factors (because any direct factor of (X, p, m) in $\mathcal{M}^{DM}$ is of the form $(X, p \circ q, m)$), we only have to check that for each connected $F \in \mathcal{DM}$, $h(F)$ is a direct factor of

some $h(X)$ for $X \in \mathcal{VAR}$.

Let $F \in \mathcal{DM}$, and by [L-M, 16.6] choose an integral scheme X and a finite and surjective morphism $X \longrightarrow F$. Using [J] we can find $Y \longrightarrow X$ which is generically finite, with Y a variety. We know consider the composed morphism $f : Y \longrightarrow F$, as well as

$$[f_*] : h(Y) \longrightarrow h(F)$$

$$[f^*] : h(F) \longrightarrow h(Y).$$

The indentity principle [B-M, 8.2] and the projection formula implies that $\frac{1}{m}.[f_*]$ is a left inverse to $[f^*]$. This implies that $[f^*]$ is a direct factor. More explicitely we have $h(F) \simeq (X, \frac{1}{m}[f_*] \circ [f^*], 0)$. $\square$

Inverting the equivalence $\mathcal{M} \longrightarrow \mathcal{M}^{DM}$ gives a functor

$$h : \mathcal{DM}^o \longrightarrow \mathcal{M}.$$

As an inverse of a monoidal functor has a natural monoidal structure, h is naturally a monoidal functor. We obtain this way natural isomorphisms

$$h(F \times F') \simeq h(F) \otimes h(F')$$

wich are associatives, commutatives and unitaries. In particular, the diagonal of a DM-stack F gives a commutative algebra structure on the motive $h(F)$. This can be used for example to show that every good cohomology theory for varieties extends to DM-stacks.

3 Second construction

Definition 3.1 *For two DM-stacks F and F', we define the vector space of χ-correspondences of degree m between F ans F' by*

$$S_\chi^m(F, F') := \{x \in A_\chi^*(F \times F')/(p_2)_*(x) \in A_\chi^m(F')\}.$$

As in the previous case, we have a composition

$$\circ : S_\chi^m(F, F') \otimes S_\chi^n(F', F'') \longrightarrow S_\chi^{p+m}(F, F'')$$

given by the formula

$$x \circ y := (p_{13})_*(p_{12}^*(x).p_{23}^*(y)),$$

where the p_{ij} are the natural projections of $F \times F' \times F''$ on two of the three factors.

Definition 3.2 *We define the category of DMC_χ-motives, $\mathcal{M}_\chi^{DM}$ as follows.*

- *Objects of $\mathcal{M}_\chi^{DM}$ are triplets (F, p, m), where F is a DM-stack, $p \in S_\chi^0(F, F)$ an idempotent, and $m \in \mathbb{Z}$.*

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- *The set of morphisms between (F, p, m) and (F', q, n) is defined by*

$$Hom_{\mathcal{M}_\chi^{DM}}((F, p, m), (F', q, n)) := q \circ S_\chi^{n-m}(F, F') \circ p \subset S_\chi^{n-m}(F, F').$$

- *The composition of morphisms in $\mathcal{M}_\chi^{DM}$ is given by composition of χ-correspondences.*

For any morphism $f : F \longrightarrow F'$ between two DM-stacks, we define as usual

$$[f^*] := (\Gamma_f)_*(1) \in S_\chi^0(F', F),$$

as well as its transposed

$$[f_*] := [f^*]^t \in S_\chi^*(F, F').$$

Using this we define a natural functor

$$
\begin{array}{rccc}
h_\chi : & \mathcal{DM}^o & \longrightarrow & \mathcal{M}_\chi^{DM} \\
& F & \mapsto & (F, Id, 0) \\
& f & \mapsto & [f^*]
\end{array}
$$

The same arguments as for motives of varieties show that $\mathcal{M}_\chi^{DM}$ is a $\mathbb{Q}$-tensorial karoubian category. There is alos a tensor product, given as usual by $(F, p, m) \otimes (F', q, n) := (F \times F', p \otimes q, m + n)$.

Note that the compatibility propety of 1.4 implies that there is a natural fully faithfull functor

$$\mathcal{M} \longrightarrow \mathcal{M}_\chi^{DM}.$$

Definition 3.3 *For any DMC_χ-motive M, we define its m-th Chow group by*

$$A_\chi^m(M) := Hom_{\mathcal{M}_\chi^{DM}}(\mathbb{L}^m, M).$$

We will note $A_\chi^*(M) := \bigoplus_m A_\chi^m(M)$.

Remark: Using the Chern character we have

$$Ch^\chi : K_0(F)_{\mathbb{Q}} \simeq A_\chi^*(F).$$

Finally, the indentity principle says that the functor $\mathcal{M}_\chi^{DM} \longrightarrow Hom(\mathcal{DM}^o, Ab)$, which sends M to the functor $F \mapsto A_\chi^*(M \otimes h(F))$ is fully faithfull (it follows immediately from the Yoneda lemma and the fact that every DMC_χ-motive is a direct factor of a $h(F) \otimes \mathbb{L}^m$).

Theorem 3.4 *The natural functor*

$$\mathcal{M} \longrightarrow \mathcal{M}_\chi^{DM}$$

is an equivalence of $\mathbb{Q}$-tensorial categories.

Proof: As for 2.1 it is enough to show that every $h_\chi(F)$ is a direct factor in some $h_\chi(X)$.

For the next lemma recall that for any stack F we can define its inertia stack I_F ([V]), whose objects are pairs (s, h), with s an object of F and h and automorphism of s. It can for example be defined by the formula

$$I_F := F \times_{F \times F} F.$$

Lemma 3.5 *For any Deligne-mumford stack proper over $Spec\, k$ (non necerally smooth), there exist varieties Y_i and finite groups H_i together with a proper representable morphism*

$$F_0 := \coprod_i Y_i \times BH_i \longrightarrow F$$

such that the induced morphism

$$I_{F_0} \longrightarrow I_F$$

is generically finite and surjective.

Proof: By [L-M, 16.6] we can choose a finite and surjective morphism $X \longrightarrow F$, with X a normal scheme. Let $F \longrightarrow M$ be the moduli space of F, and consider F_X, the normalization of the fibre product $F \times_M X$. By definition, the stack F_X is normal and the projection to its moduli space $F_X \longrightarrow X$ possesses a section. It follows from [V, 2.7] that F_X is a neutral gerb. By choosing a finite and etale morphism $Y \longrightarrow X$ and defining $F' := F_X \times_X Y$, we find a trivial gerb $F' \simeq Y \times BH$, together with a morphism $F' \longrightarrow F$. By construction this morphism is generically obtained by a pull back of a etale morphism on M. This implies that there exists a dense open sub-stack U of F, such that $I_U \subset I_F$ is in the image of $I_{F'} \longrightarrow I_F$. Proceding by noetherian induction we find reduced schemes X_i, and finite groups H_i, with a morphism $F' := \coprod_i X_i \times BH_i \longrightarrow F$ such that $I_{F'} \longrightarrow I_F$ is finite and surjective.

We now apply [J] to each X_i and choose generically finite morphism $Y_i \longrightarrow X_i$, with Y_i a variety. Let $F_0 := \coprod_i Y_i \times BH_i$. As $I_{F_0} \simeq I_F \times_F F_0$, the induced morphism

$$I_{F_0} \longrightarrow I_F$$

is still surjective and generically finite. $\square$

Lemma 3.6 *Suppose that k is algebraically closed, and choose an embedding $\mu_\infty(k) \hookrightarrow \mathbb{C}^*$. Let F be a DM-stack, and denote by I_F^t the open and closed sub-stack of I_F whose objects are pairs (s, h), such that the order of h is prime to the characteristic of k. Then there exist an $\mathbb{Q}(\mu_\infty(k))$-algebra isomorphism*

$$A_\chi^*(F) \otimes \mathbb{Q}(\mu_\infty(k)) \simeq A^*(I_F^t) \otimes \mathbb{Q}(\mu_\infty(k)).$$

Furthermore this isomorphism is compatible with inverse and direct images.

Proof: Let $u : I_F^t \longrightarrow C_F^t$ the morphism which sends an object (s, h) to $(s, < h >)$, where $< h >$ is the subgroup generated by h in $Aut(s)$. This is a representable finite et etale morphism. It is easy to see that there is an isomorphism of sheaves of graded $\mathbb{Q}(\mu_\infty(k))$-algebras on $(C_F^t)_{et}$

$$u_*(\mathcal{K}_* \otimes \mathbb{Q}(\mu_\infty(k))) \simeq \mathcal{K}_* \otimes \Lambda_F \otimes \mathbb{Q}(\mu_\infty(k)).$$

This induces the required isomorphism

$$A^*(I_F^t) \otimes \mathbb{Q}(\mu_\infty(k)) \simeq H^*((I_F^t)_{et}, \mathcal{K}_* \otimes \mathbb{Q}(\mu_\infty(k))) \simeq H^*((C_F^t)_{et}, u_*(\mathcal{K}_* \otimes \mathbb{Q}(\mu_\infty(k))))$$

$$\simeq H^*((C_F^t)_{et}, \mathcal{K}_* \otimes \Lambda_F \otimes \mathbb{Q}(\mu_\infty(k))) \simeq A_\chi^*(F) \otimes \mathbb{Q}(\mu_\infty(k)).$$

The compatibility with inverse and direct images is clear by definitions. □

Let $g : F_0 := \coprod_i Y_i \times BH_i \longrightarrow F$ be a morphism as in 3.5.

Lemma 3.7 *The element $\beta := g_*(1) \in A_\chi^*(F)$ is invertible.*

Proof: We first use the projection formula 1.4 to show that for any finite extension k'/k, we have a natural isomorphism of algebras

$$A^*(F)_\chi \simeq A_\chi^*(F \times_{Spec\,k} Spec\,k')^{Gal(k'/k)}.$$

This allows to assume that k is algebraically closed.

Applying the lemma 3.6, it is enough to show that $Ig_*(1) \in A^*(I_F^t)$ is invertible. But, as Ig is generically finite and surjective this is obvious. □

Consider $\Delta_*(\beta) \in A_\chi^*(F \times F) = S_\chi^*(F, F)$, where $\Delta : F \longrightarrow F \times F$ is the diagonal. By the previous lemma β is invertible in the graded ring $S^*(F, F)$. Let $\alpha := [g_*] \circ \beta^{-1} \in S^*(F', F)$. Then we have $[g^*] \circ \alpha = 1$. This shows that the 0-th component of α is a left inverse of $[g^*]$. As the category $\mathcal{M}_\chi^{DM}$ is karoubian, this implies that $[g^*]$ is a direct factor, and so that $h_\chi(F)$ is a direct factor in $h_\chi(F')$.

As $h_\chi(F') \simeq \bigoplus_i h_\chi(X_i) \otimes h_\chi(BH_i)$ it remains to show that for any finite group H, $h_\chi(BH)$ is isomorphic to some power of the trivial motive $h_\chi(Spec\,k)$.

Let $Ch^\chi : K_0(BH) \longrightarrow A_\chi^0(BH)$ the Chern character, $\rho_1, \ldots, \rho_r$ a set of representatives of irreducibles representations of H over k, and $\alpha_i := Ch^\chi(\alpha_i)$. These elements define morphisms of DMC_χ-motives $\alpha_i : h_\chi(Spec\,k) \longrightarrow h_\chi(BH)$ Let us consider the sum

$$\bigoplus_i \alpha_i : h_\chi(Spec\,k)^r \longrightarrow h_\chi(BH),$$

and prove that it is an isomorphism. By the identity principle, we have to show that for every DM-stack F, the induced morphism

$$\bigoplus_i \alpha_i : (A_\chi^*(F))^r \longrightarrow A_\chi^*(F \times BH)$$

is an isomorphism. But as Ch^χ is an isomorphism, the previous morphism is isomorphic to the Kunneth morphism

$$A_\chi^*(F) \otimes A_\chi^*(BH) \longrightarrow A_\chi^*(F \times BH),$$

and so the theorem follows from the following lemma.

Lemma 3.8 *For every DM-stack F and every finite group H, the Kunneth morphism*

$$A_\chi^*(F) \otimes A_\chi^*(BH) \longrightarrow A_\chi^*(F \times BH)$$

is an isomorphism.

Proof: Using galois descent we can suppose that k is algebraically closed. Then, using the lemma 3.6 we reduce the problem to show that the Kunneth morphism

$$A^*(I_F^t) \otimes A^*(I_{BH}^t) \longrightarrow A^*(I_{F \times BH}^t)$$

is an isomorphism.

Let A be a set of representative of conjugacy classes of elements in H with order prime to the characteristic of k. We have

$$I_{BH}^t \simeq \coprod_{h \in A} BZ_h \qquad I_{F \times BH}^t \simeq I_F^t \times I_{BH}^t,$$

where Z_h is the centralisator of h in H. So we only need to prove that the Kunneth morphism

$$A^*(I_F^t) \otimes A^*(BZ_h) \longrightarrow A^*(I_F^t \times BZ_h)$$

is an isomorphism. But this morphism fits into a commutative diagram

$$
\begin{array}{ccc}
A^*(I_F^t) \otimes A^*(BZ_h) & \longrightarrow & A^*(I_F^t \times BZ_h) \\
{\scriptstyle Id \otimes 1} \uparrow & \nearrow {\scriptstyle v^*} & \\
A^*(I_F^t) & &
\end{array}
$$

where $v : I_F^t \times BZ_h \longrightarrow I_F^t$ is the first projection. Now, as $A^*(BZ_h) \simeq \mathbb{Q}$, the vertical morphism is an isomorphism. On the other hand, v has a natural section $u : I_F^t \longrightarrow I_F^t \times BZ_h$ and the projection formula shows that u^* is an isomorphism, which implies that v^* is an isomorphism. $\square$

Inverting the equivalence $\mathcal{M} \longrightarrow \mathcal{M}_\chi^{DM}$ gives a functor

$$h_\chi : \mathcal{DM}^o \longrightarrow \mathcal{M}.$$

As for the case of the first construction this functor has a natural monoidal structure. This implies that for any DM-stack F, the motive $h_\chi(F)$ has a natural structure of a commutative algebra in $\mathcal{M}$. In particular any good cohomology theory for varieties extends trough h_χ to a new theory for stacks.

Proposition 3.9 *The functor h is a direct factor of the functor h_χ.*

Proof: This follows immediately from the natural decomposition $A_\chi^* \simeq A^* \times A_{\chi \neq 1}^*$. $\square$

Remark: For any complex variety V and $\beta \in H_2(V, \mathbb{Z})$, we can define the Gromov-Witten correspondence ([B])

$$I_{g,n}(V, \beta) \in S^*(V^n, \mathcal{M}_{g,n}),$$

which is a morphism of graded DMC-motives ([B-M, 8]). It seems natural to ask if this correspondence extends in a natural way to

$$I_{g,n}^\chi(V, \beta) \in S_\chi^*(V^n, \mathcal{M}_{g,n})$$

(i.e. as a morphism of graded DMC_χ-motives). This question is of course linked to the question of constructing an extended virtual fundamental class $\mathcal{I}_{g,n}^\chi(V, \beta) \in A_*^\chi(\mathcal{M}_{g,n}(V, \beta))$.

4   Examples

We have seen that the two Chow coholomogy theories A^* and A_χ^* give natural functors

$$h, h_\chi : \mathcal{DM}^o \longrightarrow \mathcal{M},$$

such that h is a direct factor of h_χ. In this last chapter we will give some examples of motives associated to certain stacks, and see some expicit relations between h_χ and h.

The proofs of the following three facts are left to the reader (they all follow from the indentity principle and the explicit description of the stacks C_F^t and the sheaves Λ_F). For the sake of simplicity we will suppose that k contains the roots of unity.

If a finite group H acts on a motive M we will denote by M^H the direct factor of M corresponding to the projector $\dfrac{1}{m} . \sum_{h \in H} h$.

1. *Quotients stacks*

    Let H be a finite group acting on a variety X. Let $c(H)$ be a set of representatives of conjugacy classes of cyclic sub-groups of H, whose orders are prime to the characteristic of k. For every $c \in c(H)$ let X^c be the sub-variety of X of fixed points of c, and N_c the normalisator of c in H. For any $c \in c(H)$, let $s(c)$ be the set of injectives characters $c \longrightarrow k^*$.

    Then the group N_c acts on X^c and $s(c)$, and so on the product $X^c \times s(c)$, and there is an isomorphism

    $$h_\chi([X/H]) \simeq \bigoplus_{c \in c(H)} h(X^c \times s(c))^{N_c}.$$

Furthermore, the motive $h([X/H])$ corresponds to the component of the trivial sub-group

$$h([X/H]) \simeq h(X)^H.$$

For example if $X = Spec\, k$ we obtain

$$h_\chi(BH) \simeq h(Spec\, k)^r,$$

where r is the number of irreducible representations of H in k-vetor spaces. But notice that this isomorphism does not preserve the product structures (given on any $h_\chi(F)$ by the diagonal morphism). Indeed, if $\rho_1, \ldots, \rho_r$ are the irreducibles representations of H over k, then we have the mutiplication rules

$$\rho_i \otimes \rho_j \simeq \bigoplus_k \rho_k^{n_k^{i,j}}.$$

Then, the product on $h_\chi(BH)$ corresponds on $h(Spec\, k)^r$ to the morphism

$$h(Spec\, k)^r \otimes h(Spec\, k)^r \simeq h(Spec\, k)^{r^2} \longrightarrow h(Spec\, k)^r$$

given by the r^2 by r matrix $(n_k^{i,j})_{i,j,k}$.

2. *Gerbs*

Let F be a connected DM-stack which is a gerb (i.e. the morphism $C_F^t \longrightarrow F$ is etale), and $F \longrightarrow X$ its projection to its moduli space. Recall that locally for the etale topology of X, F is equivalent to $X \times BH$, for H a finite group. This defines a locally constant sheaf of groups up to inner automorphisms on X_{et}, which is classified by its monodromy

$$\pi_1^{et}(X) \longrightarrow Out(H).$$

Let $cycl(H)$ be the set of cyclic sub-groups of H of order prime to the characteristic of k, and for any $c \in cycl(H)$, $s(c)$ the set the of injectives characters $c \longrightarrow k^*$. The group H acts by conjugaison on $\coprod_{c \in cycl(H)} s(c)$, and let $R(H) := (\coprod_{c \in cycl(H)} s(c))/H$ be the quotient. The group $Aut(H)$ acts naturally on $R(H)$ and any inner automorphisms of H acts trivially, so we deduce a morphism

$$\pi_1^{et}(X) \longrightarrow Aut(R(H)),$$

which it turns corresponds to a finite etale covering $Y \longrightarrow X$.

There is then an isomorphism

$$h_\chi(F) \simeq h(Y).$$

Note that the trivial subgroup with the trivial character induces a section $X \longrightarrow Y$, which gives a decomposition

$$h(Y) \simeq h(X) \oplus h(Y)^{\neq 1}.$$

Furthermore, $h(F)$ corresponds to the factor $h(X)$.

3. 1-*Dimensional complex orbifolds*

Suppose that $k = \mathbb{C}$ is the field of complex number, and that F is a 1-dimensional DM-stack, which is generically a variety (i.e. $C_F^t \longrightarrow F$ is birationnal). Let C be the moduli space of F, which is a smooth projective curve, and note $x_1, \ldots, x_r$ the points of C where F is not a scheme. Locally for the analytic topology around each x_i, F is a quotient stack of a disc by a cyclic group $\mathbb{Z}/n_i$. There is then an isomorphism

$$h_\chi(F) \simeq h(C) \bigoplus_i h(Spec\,\mathbb{C})^{n_i-1},$$

where $h(F)$ corresponds to the factor $h(C)$.

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