Schematic homotopy types and non-abelian Hodge theory

L. Katzarkov[∗]   T. Pantev[†]   B. Toën

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0 Introduction

Modern Hodge theory is concerned with the interaction between the geometry and the topology of complex algebraic varieties. The general idea is to furnish topological invariants of varieties with additional algebraic structures that capture essential information about the algebraic geometry

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of these varieties. The very first example of such a result is the existence of the Hodge decomposition on the Betti cohomology of a complex projective manifold (see e.g. [G-H]), and more generally the existence of Deligne's mixed Hodge structures on the cohomology of arbitrary varieties [D1, D2]. A more sophisticated invariant was developed by P.Deligne, P.Griffiths, J.Morgan and D.Sullivan [DGMS] who combined the Hodge theory of P.Deligne and the rational homotopy theory of D.Quillen and D.Sullivan to construct a Hodge decomposition on the rational homotopy type of a complex projective manifold. Their construction was later extended by J.Morgan who, in the simply connected case, proved the existence of a mixed Hodge structure on the complexified homotopy groups [Mo], and by R.Hain who constructed a mixed Hodge structure on the Mal'cev completion of the fundamental group [Ha1]. More recently C.Simpson constructed an action of the discrete group $\mathbb{C}^{\times \delta}$ on the full pro-algebraic fundamental group of a smooth projective complex variety that can be thought of as a Hodge decomposition [S1].

In [To2] a new homotopy invariant of a space X has been constructed, which is called the *schematization* of X and is denoted by $(X \otimes \mathbb{C})^{\mathrm{sch}}$. The main goal of the present paper is to define a Hodge decomposition on $(X \otimes \mathbb{C})^{\mathrm{sch}}$ when X is a projective manifold, and to show that this structure recovers all of the Hodge structures on cohomology, rational homotopy groups and completions of the fundamental groups mentioned above. For this, the various Hodge decompositions will be viewed as actions of the group $\mathbb{C}^{\times \delta}$. For example, the Hodge decomposition on the cohomology of a projective variety, $H^n(X, \mathbb{C}) \simeq \oplus H^{n-p}(X, \Omega^p_X)$, can be understood via the action of $\mathbb{C}^{\times \delta}$ which has weight p on the direct summand $H^{n-p}(X, \Omega^p_X)$. As a manifestation of this principle we construct an action of the group $\mathbb{C}^{\times \delta}$ on the object $(X \otimes \mathbb{C})^{\mathrm{sch}}$.

In order to state this result, let us recall that for any pointed connected space X, the schematic homotopy type $(X \otimes \mathbb{C})^{\mathrm{sch}}$ is a pointed connected simplicial presheaf on the site of affine complex schemes with the faithfully flat topology. Furthermore, this simplicial presheaf satisfies the following conditions (see §1 for details).

- The sheaf of groups $\pi_1((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ is represented by the pro-algebraic completion of the discrete group $\pi_1(X, x)$.

- There exist functorial isomorphisms $H^n((X \otimes \mathbb{C})^{\mathrm{sch}}, \mathbb{G}_a) \simeq H^n(X, \mathbb{C})$.

- If X is a simply connected finite CW-complex, there exist functorial isomorphisms

$$\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x) \simeq \pi_i(X, x) \otimes \mathbb{G}_a.$$

The object $(X \otimes \mathbb{C})^{\mathrm{sch}}$ is therefore a good candidate for an unificator of the various homotopy invariants previously studied by Hodge theory. Our main result is the following:

Theorem A *For any pointed smooth projective complex variety X, there exist an action of $\mathbb{C}^{\times \delta}$ on $(X \otimes \mathbb{C})^{\mathrm{sch}}$, functorial in X and called the Hodge decomposition. This action recovers the usual Hodge decompositions on cohomology, on completed fundamental groups, and (in the simply connected case) on complexified homotopy groups.*

We will now give an overview of the content of the present work.

Rational homotopy theory and schematic homotopy types

It is well known that the rational homotopy type $X_{\mathbb{Q}}$ of an arbitrary space X is a pro-nilpotent homotopy type. As a consequence, homotopy invariants of the space X which are not of nilpotent nature are not accessible through the space $X_{\mathbb{Q}}$. For example, the fundamental group of $X_{\mathbb{Q}}$ is isomorphic to the Mal'cev completion of $\pi_1(X, x)$, and in particular the full pro-algebraic fundamental group of X is beyond the scope of the techniques of rational homotopy theory.

In order to develop a substitute of rational homotopy theory for non nilpotent spaces, the third author introduced the notion of a *pointed schematic homotopy type* over a field k, as well as a *schematization functor* (see [To2]). When the base field is of characteristic zero, a pointed schematic homotopy type F is essentially a pointed and connected simplicial presheaf on the site of affine k-schemes with the flat topology, whose fundamental group sheaf $\pi_1(F, *)$ is represented by an affine group scheme, and whose homotopy sheaves $\pi_i(F, *)$ are products (possibly infinite) of copies of the additive group $\mathbb{G}_a$. The fundamental group $\pi_1(F, *)$ of a schematic homotopy type can be any affine group scheme, and its action on a higher homotopy group $\pi_i(F, *)$ can be an arbitrary algebraic representation. Furthermore, the homotopy category of augmented and connected commutative differential graded algebras (concentrated in positive degrees) is equivalent to the full sub-category of pointed schematic homotopy types F such that $\pi_1(F, *)$ is a unipotent affine group scheme (see Theorems 1.1.2 and 1.1.5). In view of this, pointed schematic homotopy types are reasonable models for a generalization of rational homotopy theory.

For any pointed connected homotopy type X and any field k, it was proved in [To2, Theorem 3.3.4] that there exist a universal pointed schematic homotopy type $(X \otimes k)^{\mathrm{sch}}$, called the *schematization of X over k*. The schematization is functorial in X. Its universality is explicitly spelled out in following two properties.

- The affine group scheme $\pi_1((X \otimes k)^{\mathrm{sch}}, x)$ is the pro-algebraic completion of the group $\pi_1(X, x)$. In particular, there is a one to one correspondence between finite dimensional linear representations $\pi_1(X, x)$ and finite dimensional linear representations of the affine group scheme $\pi_1((X \otimes k)^{\mathrm{sch}}, x)$.

- For any finite dimensional linear representation V of $\pi_1(X, x)$, also considered as a representation of the group scheme $\pi_1((X \otimes k)^{\mathrm{sch}}, x)$, there is a natural isomorphism on cohomology groups with local coefficients

$$H^{\bullet}(X, V) \simeq H^{\bullet}((X \otimes k)^{\mathrm{sch}}, V).$$

Moreover, for a simply connected finite CW-complex X, one can show [To2, Theorem 2.5.1] that the homotopy sheaves $\pi_i((X \otimes \mathbb{Q})^{\mathrm{sch}}, x)$ are isomorphic to $\pi_i(X, x) \otimes \mathbb{G}_a$, and that the simplicial set of global sections of the simplicial presheaf $(X \otimes \mathbb{Q})^{\mathrm{sch}}$ is a model for the rational homotopy type of X. This fact justifies the use of the schematization functor as a generalization of the rationalization functor to non-nilpotent spaces.

The Hodge decomposition

As shown in [Ka-Pa-To, Section 4.1], when X is the underlying topological space of a compact smooth manifold, its schematization $(X \otimes \mathbb{C})^{\mathrm{sch}}$ can be explicitly described using complexes of differential forms with coefficients in flat connections on X. If X is furthermore a complex projective

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manifold, this description together with Simpson's non-abelian Hodge correspondence gives a model of $(X \otimes \mathbb{C})^{\mathrm{sch}}$ in terms of Dolbeault complexes with coefficients in Higgs bundles. Since the group $\mathbb{C}^{\times \delta}$ acts naturally on the category of Higgs bundles and on their Dolbeault complexes, one gets a natural action of $\mathbb{C}^{\times \delta}$ on $(X \otimes \mathbb{C})^{\mathrm{sch}}$. The main properties of this $\mathbb{C}^{\times \delta}$ action are summarized in the following theorem.

Theorem B *(Theorem 2.4.4) Let X be a pointed smooth projective variety over $\mathbb{C}$, and let*

$$(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$$

denote the schematization of the underlying topological space of X (for the classical topology). Then, there exists an action of $\mathbb{C}^{\times \delta}$ on $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$, called the Hodge decomposition, such that the following conditions are satisfied.

1. *The induced action of $\mathbb{C}^{\times \delta}$ on the cohomology groups $H^n((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, \mathbb{G}_a) \simeq H^n(X^{\mathrm{top}}, \mathbb{C})$ is compatible with the Hodge decomposition in the following sense. For any $\lambda \in \mathbb{C}^{\times \delta}$, and $y \in H^{n-p}(X, \Omega_X^p) \subset H^n(X^{\mathrm{top}}, \mathbb{C})$ one has $\lambda(y) = \lambda^p \cdot y$.*

2. *Let $\pi_1((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)^{\mathrm{red}}$ be the maximal reductive quotient of $\pi_1(X^{\mathrm{top}}, x)^{\mathrm{alg}}$. Then, the induced action of $\mathbb{C}^{\times \delta}$ on $\pi_1((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)^{\mathrm{red}} \simeq \pi_1(X^{\mathrm{top}}, x)^{\mathrm{red}}$ is the one defined in [S1].*

3. *If X^{top} is simply connected, then the induced action of $\mathbb{C}^{\times \delta}$ on*

$$\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)(\mathbb{C}) \simeq \pi_i(X^{\mathrm{top}}, x) \otimes \mathbb{C}$$

   is compatible with the Hodge decomposition defined in [Mo] in the following sense. Let $F^\bullet \pi_i(X^{\mathrm{top}}) \otimes \mathbb{C}$ be the Hodge filtration defined in [DGMS, Mo], then

$$F^p \pi_i(X^{\mathrm{top}}) \otimes \mathbb{C} = \{x \in \pi_i(X^{\mathrm{top}} \otimes \mathbb{C}) | \exists \, q \geq p \text{ so that } \lambda(x) = \lambda^q \cdot x, \, \forall \lambda \in \mathbb{C}^\times\}.$$

4. *Let R_n be the set of isomorphism classes of simple n-dimensional linear representations of $\pi_1((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)$. Then, the induced action of $\mathbb{C}^\times$ on the set*

$$R_n \simeq \mathrm{Hom}(\pi_1(X^{\mathrm{top}}, x), Gl_n(\mathbb{C}))/Gl_n(\mathbb{C})$$

   defines a continuous action of the topological group $\mathbb{C}^\times$ (for the analytic topology).

The weight spectral sequence

For a general space X, the schematization $(X \otimes \mathbb{C})^{\mathrm{sch}}$ turns out to be extremely difficult to compute. For instance, essentially nothing is known about the higher homotopy groups $\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$.

To facilitate computations of this type we propose an approach based on Curtis' spectral sequence in conventional topology [Cu1, Cu2]. For a general pointed schematic homotopy type F, we will construct a *weight tower* $W^{(*)} F^0$, which is an algebraic analog of Curtis' construction from [Cu1, Cu2]. To this tower we associate a spectral sequence, called the *weight spectral sequence*,

going from certain homology invariants of F to its homotopy groups. In general, this spectral sequence will not converge. However, when X is a smooth projective complex manifold, we can use the existence of the Hodge decomposition in order to prove that the weight spectral sequence of $(X^{\text{top}} \otimes \mathbb{C})^{\text{sch}}$ degenerates at E_2 and that its E_∞ term computes the higher homotopy groups of $(X^{\text{top}} \otimes \mathbb{C})^{\text{sch}}$. More precisely, suppose that (X, x) is a pointed smooth and projective complex manifold with schematization $F := (X^{\text{top}} \otimes \mathbb{C}, x)^{\text{sch}}$. Then for any $q \geq 1$ the weight tower of F induces a filtration

$$\cdots \subset F_W^{(p)} \pi_q(F, *) \subset F_W^{(p-1)} \pi_q(F, *) \subset \cdots \subset F_W^{(0)} \pi_q(F, *) = \pi_q(F, *)$$

and a spectral sequence $\{E_r^{p,q+p}(W^{(*)}F^0)\}$, so that

$$E_\infty^{p,p+q} \simeq F_W^{(p-1)} \pi_q(F, *) / F_W^{(p)} \pi_q(F, *),$$

and $E_\bullet^{p,q}$ degenerates at E_2 (see Theorem 3.5.1).

As a corollary we get the following description of the homotopy groups of the schematization. Let G denote the complex pro-reductive completion of $\pi_1(X, x)$ and let $\mathcal{O}(G)$ be the algebra of regular funcions on G. By definition $\pi_1(X, x)$ comes with a Zariski dense representation into G, which can be combined with the left regular representation of G on $\mathcal{O}(G)$ to define a local system of algebras on X which by abuse of notation will be denoted again by $\mathcal{O}(G)$. We will call $\mathcal{O}(G)$ the *universal reductive local system* on X. With this notation we have:

Corollary C *(see Corollary 3.5.3) If $F := (X^{\text{top}} \otimes \mathbb{C}, x)^{\text{sch}}$ is the schematization of a complex projective manifold, then*

1. *$\pi_q(F, *) \simeq \lim \pi_q(F, *) / F_W^{(p)} \pi_q(F, *)$.*

2. *The vector spaces $F_W^{(p-1)} \pi_q(F, *) / F_W^{(p)} \pi_q(F, *)$ only depend on the graded algebra $H^*(X, \mathcal{O}(G))$, where $H^*(X, \mathcal{O}(G))$ is the cohomology algebra of X with coefficients in the universal reductive local system.*

Corollary C provides strong eveidence that the schematization of a smooth and projective complex manifold is much more simple than the schematization of a generic topological manifold. Note that even though the weight spectral sequence $\{E_r^{p,q+p}(W^{(*)}F^0)\}$ is purely topological and exists for any schematic homotopy type F, the proof of the degeneration of this sequence uses some weight properties of the action of $\mathbb{C}^{\times \delta}$ induced by our Hodge decomposition. It is a striking fact that even if this action is *not* an action of the multiplicative group $\mathbb{G}_m$, the notion of weight retains some of its algebraic character and ultimately forces the spectral sequence to degenerate.

Restrictions on homotopy types

For any pointed connected homotopy type X, the schematization $(X \otimes \mathbb{C})^{\text{sch}}$ can be used to define a new homotopy invariant of the space X, which captures information about the action of $\pi_1(X)$ on the higher homotopy groups $\pi_i(X)$. More precisely, we will define $\text{Supp}(\pi_i((X \otimes \mathbb{C})^{\text{sch}}, x))$, the *support* of the sheaf $\pi_i((X \otimes \mathbb{C})^{\text{sch}}, x))$, as the subset of isomorphism classes of all simple representations of $\pi_1(X, x)$ which appear in a finite dimensional sub-quotient of $\pi_i((X \otimes \mathbb{C})^{\text{sch}}, x))$ (see section 4.1.1).

The existence of the Hodge decomposition described in theorem B imposes some restrictions on the homotopy invariants $\mathrm{Supp}(\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x))$. Our first observation is the following corollary.

Corollary D *(Corollary 4.1.3) Let X be a pointed complex smooth projective algebraic variety, and let $R(\pi_1(X,x))$ be the coarse moduli space of simple finite dimensional representations of $\pi_1(X,x)$.*

1. *The subset $\mathrm{Supp}(\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)) \subset R(\pi_1(X,x))$ is stable under the action of $\mathbb{C}^\times$ on $R(\pi_1(X,x))$.*

2. *If $\rho \in \mathrm{Supp}(\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)$ is an isolated point (for the topology induced from the analytic topology of $R(\pi_1(X,x))$), then the local system on X corresponding to ρ underlies a polarizable complex variation of Hodge structures.*

3. *If $\pi_i((X^{\mathrm{top}} \otimes \mathbb{C}), x)$ is an affine group scheme of finite type, then, each simple factor of the semi-simplification of the representation of $\pi_1(X,x)$ on the vector space $\pi_i((X^{\mathrm{top}} \otimes \mathbb{C}), x)$ underlies a polarizable complex variation of Hodge structures on X.*

4. *Suppose that $\pi_1(X,x)$ is abelian. Then, each isolated character $\chi \in \mathrm{Supp}(\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{sch}, x))$ is unitary.*

The previous corollary suggests that one can study the action of the fundamental group of a projective variety on its higher homotopy groups by means of the support invariants. However the reader should keep in mind that the invariant $\mathrm{Supp}(\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x))$ is related to the action of $\pi_1(X,x)$ on $\pi_i(X,x) \otimes \mathbb{C}$ in a highly non-trivial way, which at the moment can be understood only in some very special cases. Nonetheless, one can use Corollary D to produce explicit families of new examples of homotopy types which are not realizable by smooth and projective algebraic varieties (see Theorem 4.1.7).

In the same vein we discuss two other applications. First, in Theorem 4.2.3, we present a *formality result*, asserting that the schematization $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ of a smooth projective complex variety X is completely determined by the pro-reductive fundamental group $\pi_1(X,x)^{\mathrm{red}}$ and the cohomology algebra $H^*(X, \mathcal{O}(\pi_1(X,x)^{\mathrm{red}}))$. This theorem generalizes and extends the formality result of [DGMS]. Finally we provide topological conditions on a smooth projective manifold X under which the image of the Hurewitz morphism $\pi_n(X) \longrightarrow H_n(X)$ is a sub-Hodge structure 4.3.1.

Organization of the paper

The paper is organized in four chapters. In the first one we recall the definitions and main results concerning affine stacks and schematic homotopy types from [To2]. In particular, we recall Theorem 1.4.1, which shows how equivariant co-simplicial algebras are related to the schematization functor. The proofs have not been included, and can be found in [To2, Ka-Pa-To].

In the second chapter we construct the Hodge decomposition on $(X \otimes \mathbb{C})^{\mathrm{sch}}$. For this, we will first review the non-abelian Hodge correspondence between local systems and Higgs bundles of [S1]. We will then explain how to describe the schematization of a space underlying a smooth manifold in terms of differential forms. Finally after introducing the notion of *fixed-point model category* that

will be used to define the Hodge decomposition, we conclude the chapter with a proof of Theorem B.

In the next chapter we define the weight tower of any schematic homotopy type, and define the associated spectral sequence in homotopy. Then, the Hodge decomposition constructed in the previous chapter is used in order to prove the degeneration statement in Corollary C.

In the last chapter we show how the existence of the Hodge decomposition imposes restrictions on homotopy types. In particular, we construct a whole family of examples of homotopy types which are not realizable by smooth projective complex varieties. We also prove the formality statement generalizing the formality theorem of [DGMS] to the schematization, as well as an application of the existence of the Hodge decomposition to the study of the Hurewitz map.

To keep the exposition more focused we concentrate here on the Hodge theoretic aspects of our consturction. This necessitates delegating several technical details (in particular several proofs) concerning background material in schematic homotopy theory to the companion but independent paper [Ka-Pa-To].

Related and future works

The construction used in order to define our Hodge decomposition is similar to the one used by R. Hain in [Ha2]. However, the two approaches differ as the construction of [Ha2] is done relatively to some variations of Hodge structures, whereas we are taking into account all local systems. We do not think that Corollary D can be obtained by the techniques of [Ha2], as it uses in a non trivial way the $\mathbb{C}^{\times}$-equivariant geometry of the whole moduli space of local systems. Note also that the results of [Ha2] do not use at all the non-abelian Hodge correspondence whereas our constructions highly depend on it.

Recently, L.Katzarkov, T.Pantev and C.Simpson, defined a notion of an *abstract non-abelian mixed Hodge structure* [Ka-Pa-S] and discussed the existence of such structures on the non-abelian cohomology of a smooth projective variety. The comparison between the [Ka-Pa-S] approach and the construction of the present work seems very difficult, essentially because both theories still need to be developed before one can even state any conjectures comparing the two points of view. In fact, it seems that a direct comparison of the two approaches is not feasible within our current understanding of the Hodge decomposition on the schematic homotopy type. Indeed, the object $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ is insensitive to the topology on the space of local systems on X, and therefore an important part of the geometry (which is captured in the construction of [Ka-Pa-S]) is lost. This problem is reflected concretely in the fact that the $\mathbb{C}^{\times}$-action on $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ given by our Hodge decomposition is only an action of the discrete group $\mathbb{C}^{\times \delta}$, and therefore does not give rise to a *filtration* in the sense of [Ka-Pa-S]. The fact that the object $(X \otimes \mathbb{C})^{\mathrm{sch}}$ does not vary well in families is another consequence of the same problem. This is similar to the fact that the pro-algebraic fundamental group *does not know that local systems can vary in algebraic famillies* (see [D4, 1.3]). One way to resolve this problem would be to consider the schematization of a space $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ as the $\mathbb{C}$-*points of a schematic shape*, which is a bigger object involving *schematizations* $(X^{\mathrm{top}} \otimes A)^{\mathrm{sch}}$ *over various* $\mathbb{C}$-*algebras* A. We believe that such an object does exist and that it can be endowed with a mixed Hodge structure, extending our construction on $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$.

As explained before, our construction of the Hodge decomposition uses equivariant co-simplicial algebras as algebraic models for schematic homotopy types. These algebraic models are very close to the equivariant differential graded algebras of [Br-Sz, Go-Ha-Ta]. The main difference between the two approaches is that we use co-simplicial algebras equipped with an action of an affine group scheme, whereas in [Br-Sz, Go-Ha-Ta] the authors use algebras equivariant for a discrete group action. In a sense, our approach is an *algebraization* of their approach, adapted for the purpose of Hodge theory. Our formality theorem 4.2.3 can also be considered as a possible answer to §7 Problem 2 of [Go-Ha-Ta].

Originally, a conjectural construction of the Hodge decomposition was proposed in [To3], where the notion of a *simplicial Tannakian category* was used. The reader may notice the Tannakian nature of the construction given in §2.3.

Finally, a crystalline version of our main result has been recently worked out by M. Olsson in [Ol], who has deduced from it some new results on homotopy types of algebraic varieties over fields of positive characteristics (already on the level of the pro-nilpotent fundamental group). M. Olsson is also working out a p-adic version of non-abelian Hodge theory, based in the same way on the theory of schematic homotopy types.

Notations and conventions

Let Δ denote the standard simplicial category. Recall that its objects are the ordered finite sets $[n] := \{0, \ldots, n\}$. For any category C and any functor $F : \Delta \longrightarrow C$ or $F : \Delta^{op} \longrightarrow C$, we write F_n for the object $F([n])$.

We fix an universe $\mathbb{U}$ with $\Delta \in \mathbb{U}$ and let $\mathsf{Aff}/\mathbb{C}$ be the category of those affine schemes over $\mathbb{C}$ which belong to $\mathbb{U}$. In this paper any affine scheme is assumed to be an object in $\mathsf{Aff}/\mathbb{C}$ (i.e. always belongs to $\mathbb{U}$), unless it is explicitly stated otherwise. Throughout the paper we will tacitly identify $\mathsf{Aff}/\mathbb{C}$ with the opposite category of the category of $\mathbb{C}$-algebras belonging to $\mathbb{U}$, and the objects of $\mathsf{Aff}/\mathbb{C}$ will be sometimes considered as algebras via this equivalence.

We will use the Grothendieck site $(\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}}$, whose underlying category is $\mathsf{Aff}/\mathbb{C}$ equipped with the faithfully flat and quasi-compact topology.

We fix a second universe $\mathbb{V}$ such that $\mathbb{U} \in \mathbb{V}$, and let SSet be the category of simplicial sets in $\mathbb{V}$. We denote by $\mathsf{SPr}(\mathbb{C})$ the category of SSet-valued presheaves on the site $(\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}}$. We will always consider the category $\mathsf{SPr}(\mathbb{C})$ together with its local projective model category structure described in [Bl] (see also [To2, Definition 1.1.1]), and the words *equivalence*, *fibration* and *cofibration* will always refer to this model structure. The homotopy category of $\mathsf{SPr}(\mathbb{C})$ will be denoted by $\mathsf{Ho}(\mathsf{SPr}(\mathbb{C}))$, and its objects will be called *stacks*. In the same way *morphism of stacks* will always refer to a

morphism in $\mathsf{Ho}(\mathsf{SPr}(\mathbb{C}))$. When we need to consider objects or morphisms in $\mathsf{SPr}(\mathbb{C})$ we will use the expressions *simplicial presheaves* and *morphism of simplicial presheaves* instead.

Any presheaf of sets on $(\mathsf{Aff}/\mathbb{C})_{\text{ffqc}}$ can be considered as a presheaf of constant simplicial sets, and so as an object in $\mathsf{SPr}(\mathbb{C})$. In particular, the Yoneda embedding gives rise to a functor from $\mathsf{Aff}/\mathbb{C}$ to $\mathsf{SPr}(\mathbb{C})$, which induces a fully faithful functor from $\mathsf{Aff}/\mathbb{C}$ into $\mathsf{Ho}(\mathsf{SPr}(\mathbb{C}))$. Via this embedding, all affine schemes will be considered both as objects in $\mathsf{SPr}(\mathbb{C})$ and as objects in $\mathsf{Ho}(\mathsf{SPr}(\mathbb{C}))$.

We denote by $\mathsf{SPr}_*(\mathbb{C})$ the model category of pointed objects in $\mathsf{SPr}(\mathbb{C})$. For any pointed simplicial presheaf $F \in \mathsf{SPr}_*(\mathbb{C})$ we write $\pi_i(F, *)$ for the homotopy sheaves of F (see for example [To2, 1.1.1]). Objects in $\mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ will be called *pointed stacks*, and those with $\pi_0(F) = *$ *pointed connected stacks*.

Our references for model categories are [Hir, Ho1], and we will always suppose that model categories are $\mathbb{V}$-categories (i.e. the set of morphism between two objects belongs to $\mathbb{V}$).

For any simplicial model category M, $\underline{\mathrm{Hom}}(a, b)$ will be the simplicial set of morphisms between a and b in M, and $\mathbb{R}\underline{\mathrm{Hom}}(a, b)$ will be its derived version. The objects $\mathbb{R}\underline{\mathrm{Hom}}(a, b)$ are well defined and functorial in the homotopy category $\mathsf{Ho}(\mathsf{SSet})$. The simplicial Hom in the model category of pointed objects M_* will be denoted by $\underline{\mathrm{Hom}}_*$, and its derived version by $\mathbb{R}\underline{\mathrm{Hom}}_*$.

All complexes considered in this paper are co-chain complexes (i.e. the differential increases degrees). We will use freely (and often implicitly) the dual Dold-Kan correspondence between positively graded complexes of vector spaces and co-simplicial vector spaces (see [K, 1.4]).

Finally, for a group Γ which belongs to the universe $\mathbb{U}$, we will denote by Γ^{alg} (respectively Γ^{red}) its pro-algebraic completion (respectively its pro-reductive completion). By definition, Γ^{alg} is the universal affine group scheme (respectively affine and reductive group scheme) which admits a morphism (resp. a morphism with Zariski dense image) from the constant sheaf of groups Γ.

1   Review of the schematization functor

In this first chapter, we review the theory of affine stacks and schematic homotopy types introduced in [To2]. The main goal is to recall the theory and fix the notations and the terminology. For further details and proofs the reader may wish to consult [To2, Ka-Pa-To].

1.1   Affine stacks and schematic homotopy types

To begin with, let us recall some basic facts about co-simplicial algebras. For us, an algebra will always mean a commutative unital $\mathbb{C}$-algebra which belongs to $\mathbb{V}$.

We will denote by Alg^Δ the category of co-simplicial algebras belonging to the universe $\mathbb{V}$. By definition Alg^Δ is the category of functors from the standard simplicial category Δ, to the category of $\mathbb{C}$-algebras in $\mathbb{V}$. For any co-simplicial algebra $A \in \mathsf{Alg}^\Delta$, one can consider its underlying co-simplicial $\mathbb{C}$-vector space, and associate to it its normalized co-chain complex $\boldsymbol{N}(A)$ (see [K, 1.4]). Any morphism $f : A \longrightarrow B$ in Alg^Δ, induces a natural morphism of complexes

$$\boldsymbol{N}(f) : \boldsymbol{N}(A) \longrightarrow \boldsymbol{N}(B).$$

We will say that f is an equivalence if $\boldsymbol{N}(f) : \boldsymbol{N}(A) \longrightarrow \boldsymbol{N}(B)$ is a quasi-isomorphism.

The category Alg^Δ is endowed with a simplicial closed model category structure for which the fibrations are epimorphisms, the equivalences are defined above, and the cofibrations are defined

by the usual lifting property with respect to trivial fibrations. This model category is known to be cofibrantly generated, and even finitely generated (see [Ho1, §2.1]).

The category we are really interested in is the full subcategory of Alg^Δ of objects belonging to $\mathbb{U}$. This category is also a finitely generated simplicial closed model category, which is a sub-model category of Alg^Δ in a very strong sense. For example, it satisfies the following stability properties, which will be tacitly used in the rest of this work. Both properties can be deduced easily from the fact that the model categories in question are finitely generated, and the fact that the sets of generating cofibrations and trivial cofibrations (as well as their domains and codomains) of Alg^Δ all belong to $\mathbb{U}$.

1. If A is a co-simplicial algebra which is equivalent to a co-simplicial algebra in $\mathbb{U}$, then there exist a co-simplicial algebra in $\mathbb{U}$, which is cofibrant in Alg^Δ, and which is equivalent to A.

2. Let $\mathrm{Ho}(\mathsf{Alg}^\Delta_\mathbb{U})$ denote the homotopy category of co-simplicial algebras belongings to $\mathbb{U}$. Then the natural functor $\mathrm{Ho}(\mathsf{Alg}^\Delta_\mathbb{U}) \longrightarrow \mathrm{Ho}(\mathsf{Alg}^\Delta)$ is fully faithful. Its essential image consists of co-simplicial algebras which are isomorphic in $\mathrm{Ho}(\mathsf{Alg}^\Delta)$ to co-simplicial algebras in $\mathbb{U}$.

Let CDGA denote the model category of commutative differential (positively) graded $\mathbb{C}$-algebras belonging to $\mathbb{V}$ (see [Bo-Gu]). Let

$$Th : \mathrm{Ho}(\mathsf{Alg}^\Delta) \longrightarrow \mathrm{Ho}(\mathsf{CDGA}),$$

denote the functor of Thom-Sullivan cochains introduced in [Hi-Sc, Theorem 4.1]. In modern language, the functor Th is equivalent to the functor of homotopy limits along the category Δ in the model category CDGA (see [Hir, §20]). The functor Th has an inverse

$$D : \mathrm{Ho}(\mathsf{CDGA}) \longrightarrow \mathrm{Ho}(\mathsf{Alg}^\Delta)$$

which is defined as follows. Let $\boldsymbol{D} : C^+ \longrightarrow Vect^\Delta$ denote the *denormalization functor* (described e.g. in [K, 1.4]) from the category C^+ of co-chain complexes of $\mathbb{C}$-vector spaces concentrated in positive degrees to the category $Vect^\Delta$ of co-simplicial vector spaces. Recall from [K, 1.4] that $\boldsymbol{D}$ admits functorial morphisms (the *shuffle product*)

$$\boldsymbol{D}(E) \otimes \boldsymbol{D}(F) \longrightarrow \boldsymbol{D}(E \otimes F)$$

which are unital associative and commutative. In particular, if $A \in \mathsf{CDGA}$ is a commutative differential graded algebra, one can use the shuffle product to define a natural commutative algebra structure on $\boldsymbol{D}(A)$ by the composition $\boldsymbol{D}(A) \otimes \boldsymbol{D}(A) \longrightarrow \boldsymbol{D}(A \otimes A) \longrightarrow \boldsymbol{D}(A)$. This defines a functor

$$\boldsymbol{D} : \mathsf{CDGA} \longrightarrow \mathsf{Alg}^\Delta,$$

which preserves equivalences, and thus induces a functor on the level of homotopy categories

$$D : \mathrm{Ho}(\mathsf{CDGA}) \longrightarrow \mathrm{Ho}(\mathsf{Alg}^\Delta).$$

Now Theorem [Hi-Sc, 4.1] implies that Th and D are inverse equivalences.

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Since the functors D and Th induce equivalences of homotopy categories, readers who are not comfortable with co-simplicial objects can replace co-simplicial algebras by commutative differential graded algebras in the discussion below. However, in order to emphasize the homotopy-theoretic context of our construction we choose to work systematically with co-simplicial algebras rather than commutative differential graded algebras.

Next we define the geometric *spectrum* of a co-simplicial algebra. More precisely we define a functor

$$\mathrm{Spec} : (\mathsf{Alg}^\Delta)^{op} \longrightarrow \mathsf{SPr}(\mathbb{C}),$$

by the following formula

$$\mathrm{Spec}\, A \; : \quad (\mathsf{Aff}/\mathbb{C})^{op} \longrightarrow \mathsf{SSet}$$
$$\mathrm{Spec}\, B \longmapsto \underline{\mathrm{Hom}}(A, B),$$

where as usual $\underline{\mathrm{Hom}}(A, B)$ denotes the simplicial set of morphisms from the co-simplicial algebra A to the co-simplicial algebra B. In other words, if A is given by a co-simplicial object $[n] \mapsto A_n$, then the presheaf of n-simplices of $\mathrm{Spec}\, A$ is given by $(\mathrm{Spec}\, B) \mapsto \mathrm{Hom}(A_n, B)$.

The functor Spec is a right Quillen functor. Its left adjoint functor $\mathcal{O}$ associates to each simplicial presheaf F the co-simplicial algebra of *functions* on F. Explicitly

$$\mathcal{O}(F)_n := \mathrm{Hom}(F, \mathbb{G}_a),$$

where $\mathbb{G}_a := \mathrm{Spec}\,\mathbb{C}[T] \in \mathsf{Aff}/\mathbb{C}$ is the additive group scheme.

The right derived functor of Spec induces a functor on the level of homotopy categorie

$$\mathbb{R}\,\mathrm{Spec} : \mathrm{Ho}(\mathsf{Alg}^\Delta)^{op} \longrightarrow \mathrm{Ho}(\mathsf{SPr}(\mathbb{C})),$$

whose restriction to the full sub-category of $\mathrm{Ho}(\mathsf{Alg}^\Delta)$ consisting of objects isomorphic to a co-simplicial algebra in $\mathbb{U}$ is fully faithful. Furthermore, for any object $A \in \mathrm{Ho}(\mathsf{Alg}^\Delta)$, isomorphic to some co-simplicial algebra belonging of $\mathbb{U}$, the adjunction morphism

$$A \longrightarrow \mathbb{L}\mathcal{O}(\mathbb{R}\,\mathrm{Spec}\, A),$$

in $\mathrm{Ho}(\mathsf{Alg}^\Delta)$, is an isomorphism. We are now ready to define affine stacks:

Definition 1.1.1 *([To2, Definition 2.2.4]) An* affine stack *is a stack $F \in \mathrm{Ho}(\mathsf{SPr}(\mathbb{C}))$ isomorphic to $\mathbb{R}\,\mathrm{Spec}\, A$, for some co-simplicial algebra A belonging to $\mathbb{U}$.*

The following important result characterizes pointed connected affine stacks, and relates it to homotopy theory over the complex numbers.

Theorem 1.1.2 *([To2, Theorem 2.4.1, 2.4.5]) Let $F \in \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ be a pointed stack. The following three conditions are equivalent.*

1. *The pointed stack F is affine and connected.*

2. *The pointed stack F is connected and for all $i > 0$ the sheaf $\pi_i(F, *)$ is represented by an affine unipotent group scheme (see [De-Ga, IV §2]).*

3. *There exist a cohomologically connected co-simplicial algebra A (i.e. $H^0(A) \simeq \mathbb{C}$), which belongs to $\mathbb{U}$, and such that $F \simeq \mathbb{R}\operatorname{Spec} A$.*

Recall that a commutative and unipotent affine group scheme over $\mathbb{C}$ is the same thing as a linearly compact vector space (see [De-Ga, IV, §2 Proposition 4.2 (b)] and [Sa, II.1.4]). This implies that for F a pointed connected affine stack, and $i > 1$, the sheaves $\pi_i(F, *)$ are associated to well defined $\mathbb{C}$-vector spaces $\pi^i(F, *) \in \mathbb{U}$, by the formula

$$\pi_i(F, *) \ : \quad (\mathsf{Aff}/\mathbb{C})_{\mathsf{ffqc}} \xrightarrow{\hspace{3cm}} \mathsf{Ab}$$
$$\operatorname{Spec} B \longmapsto \operatorname{Hom}_{\mathbb{C}-\mathsf{Vect}}(\pi^i(F, *), B).$$

If $F \simeq \mathbb{R}\operatorname{Spec} A$, the vector spaces $\pi^i(F, *)$ are isomorphic to the co-homotopy groups $\pi^i(Th(A))$ of the corresponding commutative differential graded algebra as defined in [Bo-Gu, 6.12].

Let us recall now that for a pointed simplicial presheaf F, one can define its simplicial presheaf of loops $\Omega_* F$. The functor $\Omega_* : \mathsf{SPr}_*(\mathbb{C}) \longrightarrow \mathsf{SPr}(\mathbb{C})$ is right Quillen, and can be derived to a functor defined on the level of homotopy categories

$$\mathbb{R}\Omega_* F : \operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C})) \longrightarrow \operatorname{Ho}(\mathsf{SPr}(\mathbb{C})).$$

Definition 1.1.3 *A pointed and connected stack $F \in \operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ is called a* pointed affine ∞-gerbe *if the loop stack $\mathbb{R}\Omega_* F \in \operatorname{Ho}(\mathsf{SPr}(\mathbb{C}))$ is affine.*

By definition, a pointed affine ∞-gerbe is a pointed stack, and will always be considered in the category of pointed stacks $\operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$. In the same way, a morphism between pointed affine ∞-gerbes will always mean a morphism in $\operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$.

A pointed schematic homotopy type will be a pointed affine ∞-gerbe which in addition satisfies a cohomological condition. Before we state this condition, let us recall that for any algebraic group G, for any finite dimensional linear representation V of G and any integer $n > 1$ one can define a stack $K(G, V, n)$ (see [To2, §1.2]), which is the unique pointed and connected stack having

$$\pi_1(K(G, V, n), *) \simeq G \qquad \pi_n(K(G, V, n), *) \simeq V \qquad \pi_i(K(G, V, n), *) \simeq 0 \ for \ i \neq 1, n,$$

together with the given action of G on V, and with trivial associated Postnikov invariant in $H^{n+1}(G, V)$. With this notation we have:

Definition 1.1.4 • *A morphism of pointed stacks $f : F \longrightarrow F'$ is a **P**-equivalence if for any algebraic group G, any linear representation of finite dimension V and any integer $n > 1$, the induced morphism*

$$f^* : \mathbb{R}\underline{\operatorname{Hom}}_*(F', K(G, V, n)) \longrightarrow \mathbb{R}\underline{\operatorname{Hom}}_*(F, K(G, V, n))$$

is an isomorphism.

• *A pointed stack H is **P**-local, if for any **P**-equivalence $f : F \longrightarrow F'$ the induced morphism*

$$f^* : \mathbb{R}\underline{\operatorname{Hom}}_*(F', H) \longrightarrow \mathbb{R}\underline{\operatorname{Hom}}_*(F, H)$$

is an isomorphism.

- *A* pointed schematic homotopy type *is a pointed affine ∞-gerbe which is $\boldsymbol{P}$-local.*

By definition, a pointed schematic homotopy type is a pointed stack, and will always be considered in the category of pointed stacks $\mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C}))$. In the same way, a morphism between pointed schematic homotopy types will always mean a morphism in $\mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C}))$.

The following theorem is a partial analogue for pointed schematic homotopy types of Theorem 1.1.2.

Theorem 1.1.5 *([To2, Theorem 3.2.4], [Ka-Pa-To, Cor. 3.6]) Let F be a pointed and connected stack. Then, F is a pointed schematic homotopy type if and only if it satisfies the following two conditions.*

1. *The sheaf $\pi_1(F, *)$ is represented by an affine group scheme.*

2. *For any $i > 1$, the sheaf $\pi_i(F, *)$ is represented by a unipotent group scheme. In other words, the sheaf $\pi_i(F, *)$ is represented by a linearly compact vector space in $\mathbb{U}$ (see [Sa, II.1.4]).*

We finish this section by recalling the main existence theorem of schematic homotopy theory. The category SSet can be embedded into the category $\mathsf{SPr}(\mathbb{C})$ by viewing a simplicial set X as a constant simplicial presheaf on $(\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}}$. With this convention we have the following important definition:

Definition 1.1.6 *([To2, Definition 3.3.1]) Let X be a pointed and connected simplicial set in $\mathbb{U}$. A* schematization *of X over $\mathbb{C}$ is a pointed schematic homotopy type $(X \otimes \mathbb{C})^{\mathrm{sch}}$, together with a morphism in $\mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C}))$*

$$u : X \longrightarrow (X \otimes \mathbb{C})^{\mathrm{sch}}$$

which is a universal for morphisms from X to pointed schematic homotopy types.

We have stated Definition 1.1.6 only for simplicial sets in order the simplify the exposition. However, by using the *singular functor Sing* (see for example [Ho1]), from the category of topological spaces to the category of simplicial sets, one can define the schematization of a pointed connected topological space. In what follows we will always assume implicitly that the functor *Sing* has been applied when necessary and we will generally not distinguish between topological spaces and simplicial sets when considering the schematization functor.

Theorem 1.1.7 *([To2, Theorem 3.3.4]) Any pointed and connected simplicial set (X, x) in $\mathbb{U}$ possesses a schematization over $\mathbb{C}$.*

As already mentioned, specifying a commutative and unipotent affine group scheme over $\mathbb{C}$ is equivalent to specifying a linearly compact vector space. This implies that the sheaves

$\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ are associated to well defined $\mathbb{C}$-vector space $\pi^i((X \otimes \mathbb{C})^{\mathrm{sch}}, x) \in \mathbb{U}$, by the formula

$$\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x) : \quad (\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}} \xrightarrow{\hspace{3cm}} Ab$$
$$\mathrm{Spec}\, B \longmapsto \mathrm{Hom}_{\mathbb{C}-\mathsf{Vect}}(\pi^i((X \otimes \mathbb{C})^{sch}, x), B).$$

Furthermore, the natural action of the affine group scheme $\pi_1((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ on $\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ is continuous in the sense that it is induced by an algebraic action on the vector space $\pi^i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$.

In general the homotopy sheaves $\pi_i((X \otimes \mathbb{C})^{sch}, x)$ are relatively big (they are not of finite type over $\mathbb{C}$, even when X is a finite homotopy type) and are hard to compute. The only two general cases where one knows something are the following.

Proposition 1.1.8 *([To2, Corollaries 3.3.7, 3.3.8, 3.3.9]) Let X be a pointed connected simplicial set in $\mathbb{U}$, and let $(X \otimes \mathbb{C})^{\mathrm{sch}}$ be its schematization.*

1. *The affine group scheme $\pi_1((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ is naturally isomorphic to the pro-algebraic completion of the discrete group $\pi_1(X, x)$ over $\mathbb{C}$.*

2. *There is a natural isomorphism*

$$H^\bullet(X, \mathbb{C}) \simeq H^\bullet((X \otimes \mathbb{C})^{\mathrm{sch}}, \mathbb{G}_a).$$

3. *If X is simply connected and of finite type (i.e. the homotopy type of a simply connected and finite CW complex), then for any $i > 1$, the group scheme $\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$ is naturally isomorphic to the pro-unipotent completion of the discrete groups $\pi_i(X, x)$. In other words, for any $i > 1$*

$$\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x) \simeq \pi_i(X, x) \otimes_{\mathbb{Z}} \mathbb{G}_a.$$

1.2 Equivariant stacks

For the duration of this section we fix a presheaf of groups G on $(\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}}$, which will be considered as a group object in $\mathsf{SPr}(\mathbb{C})$. We will make the assumption that G is cofibrant as an object of $\mathsf{SPr}(\mathbb{C})$. For example, G could be representable (i.e. an affine group scheme), or a constant presheaf associated to a group in $\mathbb{U}$.

Let $G\text{-}\mathsf{SPr}(\mathbb{C})$ be the category of simplicial presheaves equipped with a left action of G, which is again a closed model category (see [S-S]). Recall that the fibrations (respectively equivalences) in $G\text{-}\mathsf{SPr}(\mathbb{C})$ are defined to be the morphisms inducing fibrations (respectively equivalences) between the underlying simplicial presheaves. The model category $G\text{-}\mathsf{SPr}(\mathbb{C})$ will be called the *model category of G-equivariant simplicial presheaves*, and the objects in $\mathrm{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C}))$ will be called *G-equivariant stacks*. For any G-equivariant stacks F and F' we will denote by $\underline{\mathrm{Hom}}_G(F, F')$ the simplicial set of morphisms in $G\text{-}\mathsf{SPr}(\mathbb{C})$, and by $\mathbb{R}\underline{\mathrm{Hom}}_G(F, F')$ its derived version.

Next recall that to any group G one can associate its classifying simplicial presheaf $BG \in \mathsf{SPr}_*(\mathbb{C})$ (see [To2, §1.3]). The object $BG \in \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ is well defined up to a unique isomorphism by the following properties

$$\pi_0(BG) \simeq * \qquad \pi_1(BG, *) \simeq G \qquad \pi_i(BG, *) = 0 \ for \ i > 1.$$

Consider the coma category $\mathsf{SPr}(\mathbb{C})/BG$, of objects over the classifying simplicial presheaf BG, endowed with its natural simplicial closed model structure (see [Ho1]). Recall that fibrations, equivalences and cofibrations in $\mathsf{SPr}(\mathbb{C})/BG$ are defined on the underlying objects in $\mathsf{SPr}(\mathbb{C})$. We set $BG := EG/G$, where EG is a cofibrant model (fixed once and for all) of $*$ in $G\text{-}\mathsf{SPr}(\mathbb{C})$.

Next we define a pair of adjoint functors

$$G\text{-}\mathsf{SPr}(\mathbb{C}) \xrightarrow[\underset{Mo}{\longleftarrow}]{De} \mathsf{SPr}(\mathbb{C})/BG,$$

where De stands for *descent* and Mo for *monodromy*. If F is a G-equivariant simplicial presheaf, then $De(F)$ is defined to be $(EG \times F)/G$, where G acts diagonally on $EG \times F$. Note that there is a natural projection $De(F) \longrightarrow EG/G = BG$, and so $De(F)$ is naturally an object in $\mathsf{SPr}(\mathbb{C})/BG$.

The functor Mo which is right adjoint to De can be defined in the following way. For an object $F \longrightarrow BG$ in $\mathsf{SPr}(\mathbb{C})/BG$, the simplicial presheaf underlying $Mo(F)$ is defined by

$$
\begin{aligned}
Mo(F) \; : \quad & (\mathsf{Aff}/\mathbb{C})^{op} \xrightarrow{\hspace{2cm}} \mathsf{SSet} \\
& Y \longmapsto \underline{\mathrm{Hom}}_{BG}(EG \times Y, F),
\end{aligned}
$$

where $\underline{\mathrm{Hom}}_{BG}$ denotes the simplicial set of morphisms in the coma category $\mathsf{SPr}(\mathbb{C})/BG$, and EG is considered as an object in $\mathsf{SPr}(\mathbb{C})/BG$ by the natural projection $EG \longrightarrow EG/G = BG$. The action of G on $Mo(F)$ is then defined by making G act on EG. We have the following useful

Lemma 1.2.1 *([Ka-Pa-To, Lemma 3.10]) The Quillen adjunction (De, Mo) is a Quillen equivalence.*

The previous lemma implies that the derived Quillen adjunction induces an equivalence of categories

$$\mathrm{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C})) \simeq \mathrm{Ho}(\mathsf{SPr}(\mathbb{C})/BG).$$

Definition 1.2.2 *For any G-equivariant stack $F \in \mathrm{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C}))$ define the* quotient stack $[F/G]$ *of F by G as the object $\mathbb{L}\,De(F) \in \mathrm{Ho}(\mathsf{SPr}(\mathbb{C})/BG)$ corresponding to $F \in \mathrm{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C}))$.*

The construction also implies that the homotopy fiber of the natural projection

$$p : [F/G] \longrightarrow BG$$

is naturally isomorphic to the underlying stack of the G-equivariant stack F.

An important example of a quotient stack to keep in mind is the following. Suppose that G acts on a sheaf of groups V. Then, G acts also naturally on the simplicial presheaf $K(V, n)$. The quotient $[K(V, n)/G]$ of $K(V, n)$ by G is naturally isomorphic to $K(G, V, n)$ described in the previous section.

1.3 Equivariant co-simplicial algebras and equivariant affine stacks

Suppose that G is an affine group scheme and consider the category of linear representations of G. By definition this is the category of quasi-coherent sheaves of $\mathcal{O}$-modules in $\mathbb{V}$, on the big site $(\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}}$, which are equipped with a linear action of the presheaf of groups G. Equivalently, it is

the category of co-modules in $\mathbb{V}$ over the co-algebra $\mathcal{O}(G)$ of regular functions on G. This category will be denoted by $\mathsf{Rep}(G)$. Note that it is an abelian $\mathbb{C}$-linear tensor category, which admits all $\mathbb{V}$-limits and $\mathbb{V}$-colimits. The category of co-simplicial G-modules is defined to be the category $\mathsf{Rep}(G)^\Delta$, of co-simplicial objects in $\mathsf{Rep}(G)$.

Recall from [Ka-Pa-To, §3.2] that there exists a simplicial finitely generated closed model structure on the category $\mathsf{Rep}(G)^\Delta$, such that the following properties are satisfied

- A morphism $f : E \longrightarrow E'$ is an equivalence if and only if, for any i, the induced morphism $H^i(f) : H^i(E) \longrightarrow H^i(E')$ is an isomorphism.

- A morphism $f : E \longrightarrow E'$ is a cofibration if and only if, for any $n > 0$, the induced morphism $f_n : E_n \longrightarrow E'_n$ is a monomorphism.

- A morphism $f : E \longrightarrow E'$ is a fibration if and only if it is an epimorphism whose kernel K is such that for any $n \geq 0$, K_n is an injective object in $\mathsf{Rep}(G)$.

The category $\mathsf{Rep}(G)$ is endowed with a symmetric monoidal structure, given by the tensor product of co-simplicial G-modules (defined levelwise). In particular we can consider the category $G\text{-}\mathsf{Alg}^\Delta$ of commutative unital monoids in $\mathsf{Rep}(G)^\Delta$. It is reasonable to view the objects in $G\text{-}\mathsf{Alg}^\Delta$ as co-simplicial algebras equipped with an action of the group scheme G. Motivated by this remark we will refer to the category $G\text{-}\mathsf{Alg}^\Delta$ as the *category of G-equivariant co-simplicial algebras*. From another point of view, the category $G\text{-}\mathsf{Alg}^\Delta$ is also the category of simplicial affine schemes of $\mathbb{V}$ equipped with an action of G.

Every G-equivariant co-simplicial algebra A has an underlying co-simplicial G-module again denoted by $A \in \mathsf{Rep}(G)^\Delta$. This defines a forgetful functor

$$G\text{-}\mathsf{Alg}^\Delta \longrightarrow \mathsf{Rep}(G)^\Delta$$

which has a left adjoint L, given by the free commutative monoid construction.

As proved in [Ka-Pa-To, §3.2], there exists a simplicial cofibrantly generated closed model structure on the category $G\text{-}\mathsf{Alg}^\Delta$, such that the following two properties are satisfied

- A morphism $f : A \longrightarrow A'$ is an equivalence if and only if the induced morphism in $\mathsf{Rep}(G)^\Delta$ is an equivalence.

- A morphism $f : A \longrightarrow A'$ is a fibration if and only if the induced morphism in $\mathsf{Rep}(G)^\Delta$ is a fibration.

Similarly to the non-equivariant case, we could have defined a model structure of G-equivariant commutative differential graded algebras. This is the category $G\text{-}\mathsf{CDGA}$, of commutative monoids in $C^+(\mathsf{Rep}(G))$ - the symmetric monoidal model category of positively graded co-chain complexes in $\mathsf{Rep}(G)$.

There exists a cofibrantly generated closed model structure on the category $G\text{-}\mathsf{CDGA}$, such that the following two properties are satisfied

- A morphism $f : A \longrightarrow A'$ in G-CDGA is an equivalence if and only if the induced morphism in $C^+(\mathsf{Rep}(G))$ is a quasi-isomorphism.

- A morphism $f : A \longrightarrow A'$ in G-CDGA is a fibration if and only if the induced morphism in $C^+(\mathsf{Rep}(G))$ is a fibration (i.e. $A_n \longrightarrow A'_n$ is surjective for any $n \geq 0$).

As in the non-equivariant case, there exist a denormalization functor

$$(1.3.1) \qquad\qquad D : \mathrm{Ho}(G\text{-}\mathsf{CDGA}) \longrightarrow \mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta).$$

Indeed, the denormalization functor and the shuffle products exist over any $\mathbb{C}$-linear tensor abelian base category, and in particular over the tensor category $\mathsf{Rep}(G)$. Therefore one can repeat the definition of the functor Th in [Hi-Sc, 4.1] and produce a functor of G-equivariant Thom-Sullivan co-chains:

$$(1.3.2) \qquad\qquad Th : \mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta) \longrightarrow \mathrm{Ho}(G\text{-}\mathsf{CDGA})$$

which is an inverse of D. This allows us to view any G-equivariant commutative differential graded algebra as a well defined object in $\mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta)$ and vise-versa. We will make a frequent use of this point of view in what follows.

For any G-equivariant co-simplicial algebra A, one can define its (geometric) spectrum $\mathrm{Spec}_G\, A \in G\text{-}\mathsf{SPr}(\mathbb{C})$, by taking the usual spectrum of its underlying co-simplicial algebra and keeping track of the G-action. Explicitly, if A is given by a morphism of co-simplicial algebras $A \longrightarrow A \otimes \mathcal{O}(G)$, one finds a morphism of simplicial schemes

$$G \times \mathrm{Spec}\, A \simeq \mathrm{Spec}\,(A \otimes \mathcal{O}(G)) \longrightarrow \mathrm{Spec}\, A,$$

which induces a well defined G-action on the simplicial scheme $\mathrm{Spec}\, A$. Hence, by passing to the simplicial presheaves represented by G and $\mathrm{Spec}\, A$, one gets the G-equivariant simplicial presheaf $\mathrm{Spec}_G(A)$.

This procedure defines a functor

$$\mathrm{Spec}_G : (G\text{-}\mathsf{Alg}^\Delta)^{op} \longrightarrow G\text{-}\mathsf{SPr}(\mathbb{C}),$$

which by [Ka-Pa-To, §3.2] is a right Quillen functor. The left adjoint of Spec_G will be denoted by $\mathcal{O}_G : G\text{-}\mathsf{SPr}(\mathbb{C}) \longrightarrow G\text{-}\mathsf{Alg}^\Delta$.

One can form the right derived functor of Spec_G:

$$\mathbb{R}\,\mathrm{Spec}_G : \mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta)^{op} \longrightarrow \mathrm{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C})),$$

which possesses a left adjoint $\mathbb{L}\mathcal{O}_G$. One can then compose this functor with the quotient stack functor $[-/G]$, and obtain a functor

$$[\mathbb{R}\,\mathrm{Spec}_G(-)/G] : \mathrm{Ho}(G\text{-}\mathsf{Alg}^\Delta)^{op} \longrightarrow \mathrm{Ho}(\mathsf{SPr}(\mathbb{C})/\,BG),$$

which still possesses a left adjoint due to the fact that $[-/G]$ is an equivalence of categories. We will denote this left adjoint again by

$$\mathbb{L}\mathcal{O}_G : \text{Ho}(\mathsf{SPr}(\mathbb{C})/BG) \longrightarrow \text{Ho}(G\text{-}\mathsf{Alg}^\Delta)^{op}.$$

As proved in [Ka-Pa-To, Proposition 3.14], If $A \in \text{Ho}(G\text{-}\mathsf{Alg}^\Delta)^{op}$ is isomorphic to some G-equivariant co-simplicial algebra in $\mathbb{U}$, then the adjunction morphism

$$A \longrightarrow \mathbb{L}\mathcal{O}_G(\mathbb{R}\,\text{Spec}_G\; A)$$

is an isomorphism. In particular, the functors $\mathbb{R}\,\text{Spec}_G$ and $[\mathbb{R}\,\text{Spec}_G(-)/G]$ become fully faithful when restricted to the full sub-category of $\text{Ho}(G\text{-}\mathsf{Alg}^\Delta)$ consisting of G-equivariant co-simplicial algebras isomorphic to some object in $\mathbb{U}$. This justifies the following definition.

Definition 1.3.1 *[Ka-Pa-To, §3.2] An equivariant stack $F \in \text{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C}))$ is a G-equivariant affine stack if it is isomorphic to some $\mathbb{R}\,\text{Spec}_G(A)$, with A being a G-equivariant co-simplicial algebra in $\mathbb{U}$.*

We conclude this section with a proposition showing that stacks of the form $[\mathbb{R}\,\text{Spec}_G(A)/G]$ are often pointed schematic homotopy types. Therefore the theory of equivariant differential graded algebras gives a way to produce schematic homotopy types.

Proposition 1.3.2 *[Ka-Pa-To, §3.2] Let $A \in G\text{-}\mathsf{Alg}^\Delta$ be a G-equivariant co-simpicial algebra in $\mathbb{U}$, such that the underlying algebra of A has an augmentation $x : A \longrightarrow \mathbb{C}$ and is connected (i.e. $H^0(A) \simeq \mathbb{C}$). Then, the quotient stack $[\mathbb{R}\,\text{Spec}_G(A)/G]$ is a pointed schematic homotopy type. Furthermore, one has*

$$\pi_i([\mathbb{R}\,\text{Spec}_G(A)/G], x) \simeq \pi_i(\mathbb{R}\,\text{Spec}\; A, x) \text{ for } i > 1,$$

and the fundamental group $\pi_1([\mathbb{R}\,\text{Spec}_G(A)/G], x)$ is an extension of G by the pro-unipotent group $\pi_1(\mathbb{R}Spec\, A, x)$.

1.4 An explicit model for $(X \otimes \mathbb{C})^{\text{sch}}$

Let X be a pointed and connected simplicial set in $\mathbb{U}$. In this section we describr an explicit model for $(X \otimes \mathbb{C})^{\text{sch}}$ which is based on the notion of equivariant affine stacks.

Let G be the complex pro-reductive completion of the discrete group $\pi_1(X, x)$. By definition G comes with a universal homomorphism $\pi_1(X, x) \longrightarrow G$ with a Zariski dense image. The universal homomorphism induces a morphism $X \longrightarrow B(G(\mathbb{C}))$ be the of simplicial sets. This latter morphism is well defined up to homotopy, and we choose a representative once and for all. Let $p : P \longrightarrow X$ be the corresponding G-torsor in $\mathsf{SPr}(\mathbb{C})$. More precisely, P is the simplicial presheaf sending an affine scheme $\text{Spec}\, A \in \mathsf{Aff}/\mathbb{C}$ to the simplicial set $P(A) := (EG(A) \times_{BG(A)} X)$. The morphism $p : P \longrightarrow X$ is then a well defined morphism in $\text{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C}))$, the group G acting on $P = (EG \times_{BG} X)$ by its action on EG, and trivially on X. Alternatively we can describe P by the formula

$$P \simeq (\widetilde{X} \times G)/\pi_1(X, x),$$

where $\widetilde{X}$ is the universal covering of X, and $\pi_1(X, x)$ acts on $\widetilde{X} \times G$ by the diagonal action (our convention here is that $\pi_1(X, x)$ acts on G by left translation). We assume a this point that $\widetilde{X}$ is chosen to be cofibrant in the model category of $\pi_1(X, x)$-equivariant simplicial sets. For example, we may assume that $\widetilde{X}$ is a $\pi_1(X, x)$-equivariant cell complex.

We consider now the G-equivariant affine stack $\mathbb{R}\operatorname{Spec}_G \mathcal{O}_G(P) \in \mathrm{Ho}(G\text{-}\mathsf{SPr}(\mathbb{C}))$, which comes naturally equipped with its adjunction morphism $P \longrightarrow \mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)$. This induces a well defined morphism in $\mathrm{Ho}(\mathsf{SPr}(\mathbb{C}))$:

$$X \simeq [P/G] \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G].$$

Furthermore, as X is pointed, this morphism induces a natural morphism in $\mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C}))$

$$u : X \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G].$$

With this notation we now have the following important

Theorem 1.4.1 *([Ka-Pa-To, Theorem 3.20]) The natural morphism*

$$u : X \longrightarrow [\mathbb{R}\operatorname{Spec}_G \mathbb{L}\mathcal{O}_G(P)/G]$$

is a model for the schematization of X.

Let (X, x) be a pointed connected simplicial set in $\mathbb{U}$, let G be the pro-reductive completion of the group $\pi_1(X, x)$, and let $\widetilde{X}$ be the universal covering of X. Again, we assume that $\widetilde{X}$ is chosen to be cofibrant as a $\pi_1(X, x)$-simplicial set. Consider $\mathcal{O}(G)$ as a locally constant sheaf of algebras on X via the natural action of $\pi_1(X, x)$. Let

$$C^\bullet(X, \mathcal{O}(G)) := (\mathcal{O}(G)^{\widetilde{X}})^{\pi_1(X,x)}$$

be the co-simplicial algebra of co-chains on X with coefficients in $\mathcal{O}(G)$ (see [Ka-Pa-To, Section 3.2] for details). This co-simplicial algebra is equipped with a natural G-action, induced by the regular representation of G. One can thus consider $C^\bullet(X, \mathcal{O}(G))$ as an object in $G\text{-}\mathsf{Alg}^\Delta$. We have the following immediate

Corollary 1.4.2 *([Ka-Pa-To, Corollary 3.21]) With the previous notations, one has*

$$(X \otimes \mathbb{C})^{\mathrm{sch}} \simeq [\mathbb{R}\operatorname{Spec}_G C^\bullet(X, \mathcal{O}(G))/G].$$

Remarks: The previous corollary shows that the schematic homotopy type $(X \otimes \mathbb{C})^{\mathrm{sch}}$ can be explicitly described by the G-equivariant co-simplicial algebra of co-chains on X with coefficients in $\mathcal{O}(G)$. For example, it is possible to describe the $\mathbb{C}$-vector spaces $\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)(\mathbb{C})$ for $i > 1$ by using the minimal model for the corresponding commutative differential graded algebra $Th(C^\bullet(X, \mathcal{O}(G)))$. This was the original description given by P. Deligne in his letter [D5].

The previous corollary can be restated in terms of the pro-algebraic completion of $\pi_1(X, x)$ rather than the pro-reductive one. In fact if in the statement in Corollary 1.4.2 we take G to be $\pi_1(X, x)^{\mathrm{alg}}$, the statement remains valid and the proof is exactly the same. In summary: if G is the pro-algebraic completion of $\pi_1(X, x)$, and $C^\bullet(X, \mathcal{O}(G))$ is the G-equivariant co-simplicial algebra of co-chains on X with coefficients in G, one again has $(X \otimes \mathbb{C})^{\mathrm{sch}} \simeq [\mathbb{R}\operatorname{Spec}_G C^\bullet(X, \mathcal{O}(G))/G]$.

2 The Hodge decomposition

In this section, we will construct the Hodge decomposition on $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$, when X^{top} is the underlying homotopy type of a smooth projective complex algebraic variety. This Hodge decomposition is a higher analogue of the *Hodge filtration* on the pro-algebraic fundamental group defined by C.Simpson in [S1]. More precisely, it is an action of the discrete group $\mathbb{C}^{\times \delta}$ on the pointed stack $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$, such that the induced actions on its fundamental group, cohomology and homotopy groups coincide with the various previously defined Hodge filtrations of [Mo, Ha1, S1].

The construction we propose here is based on two results. The first is the explicit description (reviewed in section 1.4) of $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ in terms of the equivariant co-simplicial algebra of co-chains on X^{top} with coefficients in the universal reductive local system. The second is the non-abelian Hodge theorem of [S1] establishing a correspondence between local systems and Higgs bundles as well as their cohomology.

Here is a short outline of the construction. Let G be the pro-reductive completion of $\pi_1(X^{\mathrm{top}}, x)$, and let $C^{\bullet}(X^{\mathrm{top}}, \mathcal{O}(G))$ be the G-equivariant co-simplicial algebra of co-chains on X^{top} with coefficients in the local system of algebras $\mathcal{O}(G)$ (see section 1.4). First we use the non-abelian Hodge correspondence to show that this G-equivariant co-simplicial algebra can be constructed from the Dolbeault cohomology complexes of certain Higgs bundles. Since the category of Higgs bundles is equipped with a natural $\mathbb{C}^{\times}$-action, this will induce a well defined action of $\mathbb{C}^{\times \delta}$ on the pointed stack $[\mathbb{R} \operatorname{Spec}_G C^{\bullet}(X^{\mathrm{top}}, \mathcal{O}(G))/G]$ which, as we have seen, is isomorphic to $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$.

In order to implement this program we first recall the non-abelian Hodge correspondence between local systems and Higgs bundles in the form presented by C.Simpson in [S1]. In particular we explain briefly how this correspondence extends to the relevant categories of *Ind*-objects. Next we make precise the relation between the explicit model for the schematization presented in Corollary 1.4.2 and certain algebras of differential forms. We also define the notion of a fixed-point model category, which generalizes the notion of a model category of objects together with a group action. Finally we use these results to endow $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ with an action of the group $\mathbb{C}^{\times \delta}$. This action is what we call the *Hodge decomposition of* $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$.

2.1 Review of the non-abelian Hodge correspondence

In this section we recall some results concerning the non-abelian Hodge correspondence of [S1], that will be needed for the proof of the formality theorem, and for the construction of the Hodge decomposition.

Let X be a smooth projective algebraic variety over $\mathbb{C}$, and let X^{top} be the underlying topological space (in the complex topology). As we explained in the first chapter the functor *Sing* allows us to also view X^{top} as a simplicial set. Fix a base point $x \in X^{\mathrm{top}}$. Let $L_B(X)$ be the category of semi-simple local systems of finite dimensional $\mathbb{C}$-vector spaces on X^{top}. It is a rigid $\mathbb{C}$-linear tensor category which is naturally equivalent to the category of finite dimensional semi-simple representations of the fundamental group $\pi_1(X^{\mathrm{top}}, x)$.

The category of semi-simple C^{∞}-bundles with flat connections on X will be denoted by $L_{DR}(X)$. The objects in $L_{DR}(X)$ are pairs (V, ∇), where V is a C^{∞}-bundle, $\nabla : V \longrightarrow V \otimes A^1$ is an integrable connection, and A^1 is the sheaf of C^{∞}-differential forms on X. $L_{DR}(X)$ is also a rigid $\mathbb{C}$-linear tensor category with monoidal structure given by

$$(V, \nabla_V) \otimes (W, \nabla_W) := (V \otimes W, \nabla_V \otimes \operatorname{Id}_W + \operatorname{Id}_V \otimes \nabla_W).$$

The functor which maps a flat bundle to its monodromy representations at x, induces an equivalence (the Riemann-Hilbert correspodence) of tensor categories $L_B(X) \simeq L_{DR}(X)$.

Recall next that a Higgs bundle on X is a C^∞-vector bundle V, together with an operator $D'' : V \longrightarrow V \otimes A^1$, satisfying $(D'')^2 = 0$, and the Leibniz's rule $D''(a \cdot s) = \overline{\partial}(a) \cdot s + a \cdot D''(s)$, for any section s of V and any function a. We will denote by $L_{Dol}(X)$ the category of poly-stable Higgs bundles (V, D'') with vanishing rational Chern classes (see [S1, §1]). Again, $L_{Dol}(X)$ is a rigid $\mathbb{C}$-linear tensor category, with monoidal structure given by

$$(V, D''_V) \otimes (W, D''_W) := (V \otimes W, D''_V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes D''_W).$$

A harmonic bundle on X, is a triple (V, ∇, D''), where V is a C^∞-bundle such that (V, ∇) is a flat bundle, (V, D'') is a Higgs bundle, and the operators ∇ and D'' are related by a harmonic metric (see [S1, §1]). A morphism of harmonic bundles is a morphism of C^∞-bundles which preserves ∇ and D'' (actually preserving ∇ or D'' is enough, see [S1, Lemma 1.2]). The category of such harmonic bundles will be denoted by $L_{D'}(X)$, and is again a $\mathbb{C}$-linear tensor category. The existence of the harmonic metric implies that (V, ∇) is semi-simple, and that (V, D'') is poly-stable with vanishing chern classes. Therefore, there exist natural projections

$$
\begin{array}{ccc}
 & L_{D'}(X) & \\
 \swarrow & & \searrow \\
L_{DR}(X) & & L_{Dol}(X).
\end{array}
$$

The essence of the non-abelian Hodge correspondence is captured in the following theorem.

Theorem 2.1.1 ([S1, Theorem 1]) *The natural projections*

$$
\begin{array}{ccc}
 & L_{D'}(X) & \\
 \swarrow & & \searrow \\
L_{DR}(X) & & L_{Dol}(X).
\end{array}
$$

are equivalence of $\mathbb{C}$-linear tensor categories.

Remark: The two projection functors of the previous theorem are functorial in X. As a consequence of the functoriality of the equivalence $L_{DR}(X) \simeq L_{D'}(X) \simeq L_{Dol}(X)$, one obtains that it is compatible with the fiber functors at $x \in X$:

$$L_{DR}(X) \longrightarrow L_{DR}(\{x\}) \simeq \mathsf{Vect},$$
$$L_{D'}(X) \longrightarrow L_{D'}(\{x\}) \simeq \mathsf{Vect},$$
$$L_{Dol}(X) \longrightarrow L_{Dol}(\{x\}) \simeq \mathsf{Vect}.$$

For an object $(V, \nabla) \in L_{DR}(X)$, one can form its de Rham complex of C^∞-differential forms

$$(A^\bullet_{DR}(V), \nabla) := A^0(V) \xrightarrow{\nabla} A^1(V) \xrightarrow{\nabla} \dots \xrightarrow{\nabla} A^n(V) \xrightarrow{\nabla} \dots,$$

where $A^n(V)$ is the space of global sections of the C^∞-bundle $V \otimes A^n$, and A^n is the sheaf of (complex valued) smooth differential forms of degree n on X. These complexes are functorial

in (V, ∇), and compatible with the tensor products, in the sense that there exists morphims of complexes

$$(A^\bullet_{DR}(V), \nabla_V) \otimes (A^\bullet_{DR}(W), \nabla_W) \longrightarrow (A^\bullet_{DR}(V \otimes W), \nabla_V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \nabla_W),$$

which are functorial, associative, commutative and unital in the arguments (V, ∇_V) and (W, ∇_W). In other words, the functor $(V, \nabla) \mapsto (A^\bullet_{DR}(V), \nabla)$, from $L_{DR}(X)$ to the category of complexes, is a symmetric pseudo-monoidal functor. We denote by

$$H^\bullet_{DR}(V) := H^\bullet(A^\bullet_{DR}(V), \nabla),$$

the cohomology of the de Rham complex of (V, ∇).

For an object (V, D'') of $L_{Dol}(X)$, one can define its Dolbeault complex of C^∞-differential forms as

$$(A^\bullet_{Dol}(V), D'') := A^0(V) \xrightarrow{D''} A^1(V) \xrightarrow{D''} \ldots \xrightarrow{D''} A^n(V) \xrightarrow{D''} \ldots,$$

Again, these complexes are functorial and compatible with the monoidal structure, in the sense explained above. We denote by

$$H^\bullet_{Dol}(V) := H^\bullet(A^\bullet_{Dol}(V), D''),$$

the cohomology of the Dolbeault complex of (V, D'').

Finally, for a harmonic bundle $(V, \nabla, D'') \in L_{D'}(X)$, one can consider the operator $D' := \nabla - D''$ on V, and the sub-complex $(\mathrm{Ker}(D'), D'')$ of the Dolbeault complex $(A^\bullet_{Dol}(V), D'')$ consisting of differential forms α with $D'(\alpha) = 0$. Note that $(\mathrm{Ker}(D'), D'') = (\mathrm{Ker}(D'), \nabla)$ is also the sub-complex of the de Rham complex $(A^\bullet_{DR}(V), \nabla)$ of forms α with $D'(\alpha) = 0$. This complex will be denoted by

$$(A^\bullet_{D'}(V), \nabla) = (A^\bullet_{D'}(V), D'') := (\mathrm{Ker}(D'), D'') = (\mathrm{Ker}(D'), \nabla).$$

Again, the functor $(V, \nabla, D'') \mapsto (A_{D'}(V), \nabla)$ is compatible with the tensor product in the sense explained above. In this way we obtain a natural diagram of complexes

$$(A^\bullet_{D'}(V), \nabla) = (A^\bullet_{D'}(V), D'')$$

$$(A^\bullet_{DR}(V), \nabla) \qquad\qquad\qquad (A^\bullet_{Dol}(V), D'').$$

An important property of the correspondence 2.1.1 is the following compatibility and formality results with respect to de Rham and Dolbeault complexes. In the following theorem, $(H^\bullet_{DR}(V), 0)$ (respectively $(H^\bullet_{Dol}(V), 0)$) denotes the complex which underlying graded vector space is $H^\bullet_{DR}(V)$ (respectively $H^\bullet_{Dol}(V)$) and with zero differential.

Theorem 2.1.2 ([S1, Lemma 2.2]) *Let $(V, \nabla) \in L_{DR}(X)$, and $(V, D'') \in L_{Dol}(X)$ be the corresponding Higgs bundle via Theorem 2.1.1. Then, there exists a functorial diagram of quasi-isomorphisms of complexes*

$$(\textbf{formality}) \quad (A^\bullet_{DR}(V), \nabla) \longleftarrow (A^\bullet_{D'}, D) = (A^\bullet_{D'}(V), D'') \longrightarrow (A^\bullet_{Dol}(V), D'')$$

$$(H^\bullet_{Dol}(V), 0) \qquad\qquad\qquad\qquad\qquad (H^\bullet_{DR}(V), 0)$$

*The diagram (**formality**) is moreover functorial for pull-backs along morphisms of pointed smooth and projective manifolds $Y \to X$.*

In the next section we will use theorem 2.1.2 extended to the case of *Ind*-objects. The reader may find definitions and general properties of categories of *Ind*-objects in [Sa, SGA4-I]. The categories of *Ind*-objects (belonging to the universe $\mathbb{U}$) in $L_B(X)$, $L_{DR}(X)$, $L_{Dol}(X)$ and $L_{D'}(X)$ will be denoted respectively by $T_B(X)$, $T_{DR}(X)$, $T_{Dol}(X)$, and $T_{D'}(X)$. The equivalences of theorem 2.1.1 extend verbatim to $\mathbb{C}$-linear tensor equivalences

$$T_B(X) \simeq T_{DR} \simeq T_{D'}(X) \simeq T_{Dol}(X).$$

Similarly the pseudo-monoidal functors

$$L_{DR}(X) \longrightarrow \mathsf{C}^+$$
$$(V, \nabla) \longmapsto (A^\bullet_{DR}(V), \nabla),$$

$$L_{Dol}(X) \longrightarrow \mathsf{C}^+$$
$$(V, D'') \longmapsto (A^\bullet_{Dol}(V), D''),$$

$$L_{D'}(X) \longrightarrow \mathsf{C}^+$$
$$(V, \nabla, D') \longmapsto (A^\bullet_{D'}(V), D''),$$

to the category C^+ of complexes of $\mathbb{C}$-vector spaces concentrated in non-negative degrees, extend naturally to pseudo-monoidal functors from the categories of *Ind*-objects

$$T_{DR}(X) \longrightarrow \mathsf{C}^+$$
$$\{(V_i, \nabla_i)\}_{i \in I} \longmapsto \mathrm{colim}_{i \in I}(A^\bullet_{DR}(V_i), \nabla_i)$$

$$T_{Dol}(X) \longrightarrow \mathsf{C}^+$$
$$\{(V_i, D''_i)\}_{i \in I} \longmapsto \mathrm{colim}_{i \in I}(A^\bullet_{Dol}(V_i), D''_i),$$

$$T_{D'}(X) \longrightarrow \mathsf{C}^+$$
$$\{(V_i, \nabla_i, D''_i)\}_{i \in I} \longmapsto \mathrm{colim}_{i \in I}(A^\bullet_{D'}(V_i), D''_i).$$

By convention, objects in T_{DR} (respectively T_{Dol}, respectively $T_{D'}$) will again be denoted by (V, ∇) (respectively (V, D''), respectively (V, ∇, D'')). If (V, ∇) (respectively (V, D''), respectively (V, ∇, D'')) is the *Ind*-object $\{(V_i, \nabla_i)\}_{i \in I} \in T_{DR}(X)$ (respectively $\{(V_i, D''_i)\}_{i \in I} \in T_{Dol}(X)$, respectively $\{(V_i, \nabla_i, D''_i)\}_{i \in I} \in T_{D'}(X)$), we will also put

$$(A^\bullet_{DR}(V), \nabla) := \mathrm{colim}_{i \in I}(A^\bullet_{DR}(V_i), \nabla_i)$$
$$(A^\bullet_{Dol}(V), D'') := \mathrm{colim}_{i \in I}(A^\bullet_{Dol}(V_i), D''_i)$$
$$(A^\bullet_{D'}(V), D'') := \mathrm{colim}_{i \in I}(A^\bullet_{D'}(V_i), D''_i).$$

Theorem 2.1.2 then extends to the following formality result for *Ind*-objects.

Corollary 2.1.3 *Let $(V, \nabla) \in T_{DR}(X)$, and $(V, D'') \in T_{Dol}(X)$ be the corresponding Higgs bundle via Theorem 2.1.1 for Ind-objects. Then, there exists a functorial diagram of quasi-isomorphisms of complexes*

$$(\textit{Ind-}\textbf{formality}) \quad (A^\bullet_{DR}(V), \nabla) \longleftarrow (A^\bullet_{D'}, D) = (A^\bullet_{D'}(V), D'') \longrightarrow (A^\bullet_{Dol}(V), D'')$$

$$(H^\bullet_{Dol}(V), 0) \qquad\qquad\qquad\qquad\qquad (H^\bullet_{DR}(V), 0)$$

The diagram ($\textit{Ind-}\textbf{formality}$) is moreover functorial for pullbacks along morphisms of pointed smooth projective manifolds $Y \to X$.

The categories $T_B(X)$, $T_{DR}(X)$, $T_{Dol}(X)$ and $T_{D'}(X)$ are naturally $\mathbb{C}$-linear tensor abelian categories. Furthermore, they possess all $\mathbb{U}$-limits and $\mathbb{U}$-colimits. This last property allows one to define an action of the monoidal category Vect, of vector spaces in $\mathbb{U}$, on the categories $T_B(X)$, $T_{DR}(X)$, $T_{Dol}(X)$ and $T_{D'}(X)$, making them into closed module categories over Vect (see [Ho1, §4.2]). Precisely, this means that there exists bilinear functors (external products)

$$\otimes : \mathsf{Vect} \times T_?(X) \longrightarrow T_?(X),$$

where $T_?(X)$ is one of the categories $T_B(X)$, $T_{DR}(X)$, $T_{Dol}(X)$ and $T_{D'}(X)$. Moreover these functors satisfy the usual adjunction property

$$\mathrm{Hom}(A \otimes V, W) \simeq \underline{\mathrm{Hom}}_{\mathsf{Vect}}(A, \underline{\mathrm{Hom}}(V, W)),$$

where $\underline{\mathrm{Hom}}(V, W) \in \mathsf{Vect}$ is the vector space of morphisms coming from the linear structure on $T_?(X)$.

Using these external products, one can define the notion of an action of an affine group scheme G on an object $V \in T_?(X)$. Indeed, let $\mathcal{O}(G)$ be the Hopf algebra of functions on G. Then, a G-action on $V \in T_?(X)$ is a morphism

$$V \longrightarrow \mathcal{O}(G) \otimes V,$$

which turns V into a co-module over the co-algebra $\mathcal{O}(G)$. Dually an action of G on V is the data of a factorization of the functor

$$h_V : T_?(X) \longrightarrow \mathsf{Vect}$$
$$V' \longmapsto \underline{\mathrm{Hom}}(V', V)$$

through the category of linear representations of G. These definitions will be used in the next section, and we will talk freely about action of an affine group scheme G on an object of $T_B(X)$, $T_{DR}(X)$, $T_{Dol}(X)$ and $T_{D'}(X)$.

Remark 2.1.4 Let $\mathcal{C}$ be a Tannakian category with $\mathcal{C} = \mathsf{Rep}(G)$ for some pro-algebraic group G. There are two natural actions (by algebra automorphisms) of the group G on the algebra of functions $\mathcal{O}(G)$. These actions are induced respectively by the rigtht and left translation action of G on itself. If we now view $\mathcal{O}(G)$ as an algebra object in $\mathcal{C}$ via say the left action, then the right action of G turns $\mathcal{O}(G)$ into a G-equivariant algebra object in $\mathcal{C}$.

2.2 Schematization and differential forms

In this section (X, x) will denote a pointed connected compact smooth manifold and X^{top} will denote the underlying topological space of X. Let $L_B(X)$ be the category of semi-simple local systems of finite dimensional $\mathbb{C}$-vector spaces on X^{top}. It is a rigid $\mathbb{C}$-linear tensor category which is naturally equivalent to the category of finite dimensional semi-simple representations of the fundamental group $\pi_1(X^{\text{top}}, x)$.

The category of semi-simple C^∞ complex vector bundles with flat connections on X will be denoted as before by $L_{DR}(X)$. Recall that the category $L_{DR}(X)$ is a rigid $\mathbb{C}$-linear tensor category, and the functor which maps a flat bundle to its monodromy representations at x, induces an equivalence of tensor categories $L_B(X) \simeq L_{DR}(X)$ (this again is the Riemann-Hilbert correspondence).

Let $G_X := \pi_1(X^{\text{top}}, x)^{\text{red}}$ be the pro-reductive completion of the group $\pi_1(X^{\text{top}}, x)$. Note that it is the Tannaka dual of the category $L_B(X)$. The algebra $\mathcal{O}(G_X)$ of regular functions on G_X can be viewed as the left regular representation of G_X. Through the universal morphism $\pi_1(X^{\text{top}}, x) \longrightarrow G_X$, we can also consider $\mathcal{O}(G_X)$ as a linear representation of $\pi_1(X^{\text{top}}, x)$. This linear representation is not finite dimensional, but it is admissible in the sense that it equals the union of its finite dimensional sub-representations. Therefore, the algebra $\mathcal{O}(G_X)$ corresponds to an object in the $\mathbb{C}$-linear tensor category $T_B(X)$, of Ind-local systems on X^{top}. By convention all of our Ind-objects are labeled by $\mathbb{U}$-small index categories.

Furthermore, the algebra structure on $\mathcal{O}(G_X)$, gives rise to a map

$$\mu : \mathcal{O}(G_X) \otimes \mathcal{O}(G_X) \longrightarrow \mathcal{O}(G_X),$$

which is easily checked to be a morphism in $T_B(X)$. This means that if we write $\mathcal{O}(G_X)$ as the colimit of finite dimensional local systems $\{V_i\}_{i \in I}$, then the product μ will be given by a compatible system of morphisms in $L_B(X)$

$$\mu_{i,k} : V_i \otimes V_i \longrightarrow V_k,$$

for some index $k \in I$ with $i \leq k \in I$. The morphism μ (or equivalently the collection of morphisms $\mu_{i,k}$), endows the object $\mathcal{O}(G_X) \in T_B(X)$ with a structure of a commutative unital monoid. Through the Riemann-Hilbert correspondence $T_B(X) \simeq T_{DR}(X)$, the algebra $\mathcal{O}(G_X)$ can also be considered as a commutative monoid in the tensor category $T_{DR}(X)$ of Ind-objects in $L_{DR}(X)$.

Let $\{(V_i, \nabla_i)\}_{i \in I} \in T_{DR}(X)$ be the object corresponding to $\mathcal{O}(G_X)$. For any $i \in I$, one can form the de Rham complex of C^∞-differential forms

$$(A^\bullet_{DR}(V_i), \nabla_i) := A^0(V_i) \xrightarrow{\nabla_i} A^1(V_i) \xrightarrow{\nabla_i} \ldots \xrightarrow{\nabla_i} A^n(V_i) \xrightarrow{D_i} \ldots.$$

In this way we obtain an inductive system of complexes $\{(A^\bullet_{DR}(V_i), D_i)\}_{i \in I}$ whose colimit complex was defined to be the de Rham complex of $\mathcal{O}(G_X)$

$$(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) := \text{colim}_{i \in I}(A^\bullet_{DR}(V_i), \nabla_i).$$

The complex $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$ has a natural structure of a commutative differential graded algebra, coming from the commutative monoid structure on $\{(V_i, \nabla_i)\}_{i \in I} \in T_{DR}(X)$. Using wedge products of differential forms, these morphisms induce in the usual fashion morphisms of complexes

$$(A^\bullet_{DR}(V_i), \nabla_i) \otimes (A^\bullet_{DR}(V_j), \nabla_j) \longrightarrow (A^\bullet_{DR}(V_k), \nabla_k)$$

which, after passing to the colimit along I, turn $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$ into a commutative differential graded algebra.

The affine group scheme G_X acts via the right regular representation on the *Ind*-local system $\mathcal{O}(G_X)$. As explained in Remark 2.1.4 this action is compatible with the algebra structure. By functoriality, we get an action of G_X on the corresponding object in $T_{DR}(X)$. Furthermore, if G_X acts on an inductive system of flat bundles (V_i, ∇_i), then it acts naturally on its de Rham complex $\mathrm{colim}_{i \in I}(A^\bullet_{DR}(V_i), \nabla_i)$, by acting on the spaces of differential forms with coefficients in the various V_i. Indeed, if the action of G_X is given by a co-module structure

$$\{(V_i, \nabla_i)\}_{i \in I} \longrightarrow \{\mathcal{O}(G_X) \otimes (V_i, \nabla_i)\}_{i \in I},$$

then one obtains a morphism of *Ind-C^∞*-bundles by tensoring with the sheaf A^n of differential forms on X

$$\{V_i \otimes A^n\}_{i \in I} \longrightarrow \{\mathcal{O}(G_X) \otimes (V_i \otimes A^n)\}_{i \in I}.$$

Taking global sections on X, one has a morphism

$$\mathrm{colim}_{i \in I} A^n(V_i) \longrightarrow \mathrm{colim}_{i \in I} A^n(V_i) \otimes \mathcal{O}(G_X),$$

which defines an action of G_X on the space of differential forms with values in the *Ind-C^∞*-bundle $\{V_i\}_{i \in I}$. Since this action is compatible with the differentials ∇_i, one obtains an action of G_X on the de Rham complex $\mathrm{colim}_{i \in I}(A^\bullet_{DR}(V_i), \nabla_i)$. Furthermore since the action is compatible with the algebra structure on $\mathcal{O}(G_X)$ it follows that G_X acts on $\mathrm{colim}_{i \in I}(A^\bullet_{DR}(V_i), \nabla_i)$ by algebra automorphisms. Thus, the group scheme G_X acts in a natural way on the complex $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$, turning it into a well defined G_X-equivariant commutative differential graded algebra.

Using the denormalization functor (see (1.3.1)) $D : \mathrm{Ho}(G_X\text{-}\mathsf{CDGA}) \longrightarrow \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta)$, we obtain a well defined G_X-equivariant co-simplicial algebra denoted by

$$C^\bullet_{DR}(X, \mathcal{O}(G_X)) := D(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta).$$

To summarize:

Definition 2.2.1 *Let (X, x) be a pointed connected smooth manifold, and let $G_X := \pi_1(X, x)^{\mathrm{red}}$ be the pro-reductive completion of its fundamental group. The G_X-equivariant commutative differential graded algebra of de Rham cochains of X with coefficients in $\mathcal{O}(G_X)$ will be denoted by*

$$(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\mathsf{CDGA}).$$

Its denormalization will be denoted by

$$C^\bullet_{DR}(X, \mathcal{O}(G_X)) := D(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) \in \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta).$$

Any smooth map $f : (Y, y) \longrightarrow (X, x)$ of pointed connected smooth manifolds induces a morphism $G_Y := \pi_1(Y, y)^{\mathrm{red}} \longrightarrow G_X := \pi_1(X, x)^{\mathrm{red}}$, and therefore a well defined functor

$$f^* : \mathrm{Ho}(G_X\text{-}\mathsf{Alg}^\Delta) \longrightarrow \mathrm{Ho}(G_Y\text{-}\mathsf{Alg}^\Delta).$$

Clearly the pull-back of differential forms via f induces a well defined morphism in $\mathsf{Ho}(G_Y\text{-}\mathsf{Alg}^\Delta)$

$$f^* : f^* C^\bullet_{DR}(X, \mathcal{O}(G_X)) \longrightarrow C^\bullet_{DR}(Y, \mathcal{O}(G_Y)).$$

Since this morphism depends functorially (in an obvious fashion) on the morphism f, we get a well defined functor

$$(X, x) \mapsto [\mathbb{R}\operatorname{Spec}_{G_X} C^\bullet_{DR}(X, \mathcal{O}(G_X))/G_X],$$

from the category of pointed connected smooth manifolds to the category of pointed schematic homotopy types. To simplify notation, we will denote this functor by $(X, x) \mapsto (X \otimes \mathbb{C})^{\mathrm{diff}}$.

Proposition 2.2.2 *([Ka-Pa-To, Proposition 4.8]) For a pointed connected smooth and compact manifold (X, x), there exists a functorial isomorphism in $\mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C}))$*

$$(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}} \simeq (X \otimes \mathbb{C})^{\mathrm{diff}}.$$

2.3 Fixed-point model categories

Before we define the Hodge decomposition on the schematization $((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)$, we will need some preliminary results about fixed-point model categories.

Let M be a cofibrantly generated model category with $\mathbb{U}$-limits and colimits, and let Γ be a group in $\mathbb{U}$. Suppose that the group Γ acts on M by auto-equivalences. This means that the action is given by a monoidal functor $\Gamma \longrightarrow \underline{\mathrm{End}}(M)$ (see [S1, §6] for a precise definition). For $\gamma \in \Gamma$ and $x \in \mathrm{ob}(M)$, let $\gamma \cdot x$ denote the image of x under the auto-equivalence γ. Similarly for a morphism f in M we will use the notation $\gamma \cdot f$ for the image of f by the auto-equivalence γ.

Define a new category M^Γ of Γ-fixed points in M as follows.

- An object in M^Γ is the data consisting of an object $x \in M$, together with isomorphisms $u_\gamma : \gamma \cdot x \longrightarrow x$ specified for each $\gamma \in \Gamma$, satisfying the relation $u_{\gamma_2} \circ (\gamma_2 \cdot u_{\gamma_1}) = u_{\gamma_2 \cdot \gamma_1}$. Such an object will be denoted by (x, u).

- A morphism $(x, u) \longrightarrow (y, v)$, between two objects in M^Γ, is a morphism $f : x \longrightarrow y$ in M, such that for any $\gamma \in \Gamma$ one has $(\gamma \cdot f) \circ u_\gamma = v_\gamma \circ f$.

The following proposition is a particular case of the existence of a model structure on *the category of sections of a left Quillen presheaf* introduced in [H-S, Theorem 17.1]. The proof is a straitgforward generalization of [Hir, Theorem 13.8.1], and is left to the reader.

Proposition 2.3.1 *There exist a unique structure of a cofibrantly generated model category on M^Γ such that a morphism $f : (x, u) \longrightarrow (y, v)$ is an equivalence (respectively a fibration) if and only if the underlying morphism in M, $f : x \longrightarrow y$ is an equivalence (respectively a fibration).*

From the definition of a fixed-point model category it follows immediately that any Γ-equivariant left Quillen functor $F : M \longrightarrow N$ induces a left Quillen functor F^Γ on the model category of fixed points $F^\Gamma : M^\Gamma \longrightarrow N^\Gamma$. Furthermore, if G^Γ is the right adjoint to F^Γ, then the functors

$$\mathbb{L}F^\Gamma : \mathsf{Ho}(M^\Gamma) \longrightarrow \mathsf{Ho}(N^\Gamma), \qquad \mathbb{R}G^\Gamma : \mathsf{Ho}(N^\Gamma) \longrightarrow \mathsf{Ho}(M^\Gamma)$$

commute with the forgetful functors

$$\mathrm{Ho}(M^\Gamma) \longrightarrow \mathrm{Ho}(M), \qquad \mathrm{Ho}(N^\Gamma) \longrightarrow \mathrm{Ho}(N).$$

Our main example of a fixed-point model category will be the following. Fix an affine group scheme G, and let Γ be a group acting on G, by group automorphisms. We will suppose that Γ and G both belong to $\mathbb{U}$. The group Γ acts naturally (by auto-equivalences) on the category $G\text{-}\mathsf{Alg}^\Delta$, of G-equivariant co-simplicial algebras. To be more precise, the group Γ acts on the Hopf algebra B corresponding to G. Then, for any co-module $u : M \to M \otimes B$ over B, and $\gamma \in \Gamma$, one defines $\gamma \cdot M$ to be the B-co-module with co-action

$$M \xrightarrow{\;u\;} M \otimes B \xrightarrow{\mathrm{Id} \otimes \gamma} M \otimes B.$$

This defines an action of the group Γ on the category $\mathsf{Rep}(G)$ of linear representations of G. By functoriality, this action extends to an action on the category of co-simplicial objects, $\mathsf{Rep}(G)^\Delta$. Finally, as this action is by monoidal auto-equivalences, it induces a natural action on the category of G-equivariant co-simplicial algebras, $G\text{-}\mathsf{Alg}^\Delta$.

In the same way, the group Γ acts on the category $G\text{-}\mathsf{SPr}(\mathbb{C})$, of G-equivariant simplicial presheaves. For any $F \in G\text{-}\mathsf{SPr}(\mathbb{C})$ and $\gamma \in \Gamma$, we write $\gamma \cdot F$ for the G-equivariant simplicial presheaf whose underlying simplicial presheaf is the same as the one for F and whose G-action is defined by composing the G-action on F with the automorphism $\gamma : G \longrightarrow G$.

The right Quillen functor $\mathrm{Spec}_G : (G\text{-}\mathsf{Alg}^\Delta)^{op} \longrightarrow G\text{-}\mathsf{SPr}(\mathbb{C})$ commutes with the action of Γ, and therefore induces a right Quillen functor on the model categories of fixed points:

$$\mathrm{Spec}_G^\Gamma : ((G\text{-}\mathsf{Alg}^\Delta)^\Gamma)^{op} \longrightarrow (G\text{-}\mathsf{SPr}(\mathbb{C}))^\Gamma.$$

The group Γ also acts on the comma category $\mathsf{SPr}(\mathbb{C})/BG$. For $\gamma \in \Gamma$, and $u : F \to BG$ an object in $\mathsf{SPr}(\mathbb{C})/BG$, $\gamma \cdot u : F \to BG$ is defined by the composition

$$F \xrightarrow{\;u\;} BG \xrightarrow{\;\gamma\;} BG\,.$$

One can then consider the left Quillen functor $De : G\text{-}\mathsf{SPr}(\mathbb{C}) \longrightarrow \mathsf{SPr}(\mathbb{C})/BG$ defined in section 1.2. Since this functor is Γ-equivariant, it induces a functor of the categories of fixed points

$$De^\Gamma : (G\text{-}\mathsf{SPr}(\mathbb{C}))^\Gamma \longrightarrow (\mathsf{SPr}(\mathbb{C})/BG)^\Gamma,$$

which is a Quillen equivalence (see Lemma 1.2.1). For $F \in \mathrm{Ho}((G\text{-}\mathsf{SPr}(\mathbb{C}))^\Gamma)$, the corresponding object in $\mathrm{Ho}(\mathsf{SPr}(\mathbb{C})/BG)^\Gamma)$ will be denoted by $[F/G]^\Gamma$. By composing with the right derived functor of Spec_G^Γ one obtains a functor

$$[\mathbb{R}\,\mathrm{Spec}_G^\Gamma(-)/G]^\Gamma : \mathrm{Ho}((G\text{-}\mathsf{Alg}^\Delta)^\Gamma)^{op}) \longrightarrow \mathrm{Ho}((\mathsf{SPr}(\mathbb{C})/BG)^\Gamma).$$

Finally, the forgetful functor $\mathsf{SPr}(\mathbb{C})/BG \longrightarrow \mathsf{SPr}(\mathbb{C})$ becomes Γ-equivariant if we endow $\mathsf{SPr}(\mathbb{C})$ with the trivial Γ-action. The forgetful functor induces a well defined functor

$$\mathrm{Ho}((\mathsf{SPr}(\mathbb{C})/BG)^\Gamma) \longrightarrow \mathrm{Ho}(\mathsf{SPr}(\mathbb{C})^\Gamma).$$

By composing with $[\mathbb{R}\operatorname{Spec}_G^\Gamma(-)/G]^\Gamma$ one gets a new functor, which will is the one we are ultimately interested in

$$[\mathbb{R}\operatorname{Spec}_G^\Gamma(-)/G] : \operatorname{Ho}((G\text{-}\mathsf{Alg}^\Delta)^\Gamma)^{op} \longrightarrow \operatorname{Ho}(\mathsf{SPr}(\mathbb{C})^\Gamma).$$

This last functor sends a G-equivariant co-simplicial algebra fixed under the Γ action to a Γ-equivariant stack.

2.4 Construction of the Hodge decomposition

First recall the notion of an action of a group Γ on a pointed stack $F \in \operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$.

Definition 2.4.1 *An action of a group Γ on a pointed stack $F \in \operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$, is a pair (F_0, u), consisting of a Γ-equivariant pointed stack $F_0 \in \operatorname{Ho}(\Gamma\text{-}\mathsf{SPr}_*(\mathbb{C}))$, and an isomorphism u of pointed stacks (i.e. an isomorphism in $\operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$) between F and the pointed stack underlying F_0.*

A morphism $(F, F_0, u) \longrightarrow (F', F_0', u')$ between two stacks F and F' equipped with Γ-actions, is the data of two morphisms $a : F \to F'$ and $b : F_0 \to F_0'$, in $\operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ and $\operatorname{Ho}(\Gamma\text{-}\mathsf{SPr}_*(\mathbb{C}))$, such that the following diagram

$$\begin{array}{ccc} F & \xrightarrow{\ a\ } & F' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ F_0 & \xrightarrow{\ b\ } & F_0' \end{array}$$

commutes in $\operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$.

With these definitions, the pointed stacks equipped with a Γ action form a category. By definition, this category is also the 2-fiber product of the categories $\operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$ and $\operatorname{Ho}(\Gamma\text{-}\mathsf{SPr}_*(\mathbb{C}))$ over $\operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C}))$, and is therefore equivalent to $\operatorname{Ho}(\Gamma\text{-}\mathsf{SPr}_*(\mathbb{C}))$ via the functor $(F, F_0, u) \mapsto F_0$. Using this equivalence, we will view the pointed stacks equipped with a Γ action as objects in $\operatorname{Ho}(\Gamma\text{-}\mathsf{SPr}_*(\mathbb{C}))$.

Now, fix a complex smooth projective algebraic variety X, a point $x \in X$ and let X^{top} be the underlying topological space for the classical topology. Consider the fiber-at-x functor

$$\omega_x : L_{Dol}(X) \longrightarrow \mathsf{Vect}.$$

With this choice $(L_{Dol}(X), \otimes, \omega_x)$ becomes a neutral Tannakian category, whose affine group scheme of tensor automorphisms will be denoted by G_X. In [S1] C.Simpson introduced and studied and action of $\mathbb{C}^{\times \delta}$ on G_X. The discrete group $\mathbb{C}^{\times \delta}$ acts on the category $L_{Dol}(X)$ as follows. For $\lambda \in \mathbb{C}^{\times \delta}$, and (V, D''), write $D'' = \overline{\partial} + \theta$, where θ is the Higgs field of (V, D''). Then, we define $\lambda \cdot (V, D'')$ to be $(V, \overline{\partial} + \lambda \cdot \theta)$. This defines an action of $\mathbb{C}^{\times \delta}$ by tensor auto-equivalences of $L_{Dol}(X)$. Since ω_x is invariant under the action of the group $\mathbb{C}^{\times \delta}$ on $L_{Dol}(X)$ it follows that group scheme G_X is endowed with a natural $\mathbb{C}^{\times \delta}$-action.

Let T_{Dol} be the category of those *Ind*-objects in $L_{Dol}(X)$, which belong to $\mathbb{U}$. Recall that T_{Dol} has a natural structure of a $\mathbb{C}$-linear tensor category. Furthermore, the action of $\mathbb{C}^\times$ extends to an action by tensor auto-equivalences of T_{Dol}. The resulting fiber functor $\omega_x : T_{Dol} \longrightarrow \mathsf{Vect}$ has a right adjoint $p : \mathsf{Vect} \longrightarrow T_{Dol}$, which is still $\mathbb{C}^\times$-invariant. The image of the trivial algebra **1**

under p is therefore a monoid in T_{Dol} (commutative and unital, as usual). Moreover, as p is fixed under the action of $\mathbb{C}^\times$, $p(\mathbf{1})$ is naturally fixed under $\mathbb{C}^\times$ as a commutative monoid in T_{Dol}. In other words

$$p(\mathbf{1}) \in (\mathsf{Comm}(T_{Dol}))^{\mathbb{C}^\times},$$

where $\mathsf{Comm}(T_{Dol})$ denotes the category of commutative monoids in T_{Dol}. Through the equivalence of T_{Dol} with the category of linear representations of G_X, the commutative monoid $p(\mathbf{1})$ corresponds to the left regular representation of G_X on the algebra of functions $\mathcal{O}(G_X)$.

Consider now the adjunction morphism

$$c : p(\mathbf{1}) \longrightarrow p(\omega_x(p(\mathbf{1}))) \simeq p(\mathcal{O}(G)) \simeq \mathcal{O}(G_X) \otimes p(\mathbf{1}),$$

where the tensor product on the right is the external tensor product of $p(\mathbf{1})$ with the vector space $\mathcal{O}(G_X)$. The morphism c defines a structure of $\mathcal{O}(G_X)$-co-module on the commutative monoid $p(\mathbf{1})$. The fact that p is fixed under $\mathbb{C}^{\times\delta}$ implies that $p(\mathbf{1})$ is naturally a fixed point of the category of commutative monoids in T_{Dol}, which are equipped with an action of the affine group scheme G_X. Indeed, this is equivalent to the fact that the multiplication morphism $G_X \times G_X \longrightarrow G_X$ is $\mathbb{C}^{\times\delta}$-equivariant.

The object $p(\mathbf{1}) \in T_{Dol}$ is an inductive system of objects in $L_{Dol}(X)$, i.e. an inductive system of Higgs bundles $\{(V_i, D_i'')\}_{i \in I}$. One can consider for any $i \in I$, the Dolbeault complex $(A_{Dol}^\bullet(V_i), D_i'')$ (see § 2.1), and thus define an inductive system of complexes $(A_{Dol}^\bullet(V_i), D_i'')_{i \in I}$. The colimit along I of this inductive system was denoted by $(A_{Dol}^\bullet(\mathbf{1}), D'')$. The commutative monoid structure on $p(\mathbf{1})$ induces a well defined commutative differential graded algebra structure on $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$.

Now, as $p(\mathbf{1})$ is a commutative monoid fixed under the $\mathbb{C}^{\times\delta}$-action, $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$ possesses a natural action of $\mathbb{C}^{\times\delta}$ defined in the following way. For $\lambda \in \mathbb{C}^{\times\delta}$, let $\lambda \cdot p(\mathbf{1}) \in T_{Dol}(X)$ be the image of $p(\mathbf{1})$ under the action of $\mathbb{C}^{\times\delta}$ on $T_{Dol}(X)$, and

$$u_\lambda : \lambda \cdot p(\mathbf{1}) \simeq p(\mathbf{1})$$

be the isomorphism coming from the structure of being a fixed object. We define an isomorphism ϕ_λ of $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$ by the following commutative diagram

$$
\begin{array}{ccc}
(A_{Dol}^\bullet(p(\mathbf{1})), D'') & \xrightarrow{u_\lambda} & (A_{Dol}^\bullet(\lambda \cdot p(\mathbf{1})), D'') \\
\phi_\lambda \downarrow & & \downarrow [\lambda] \\
(A_{Dol}^\bullet(p(\mathbf{1})), D'') & \xrightarrow[u_\lambda]{} & (A_{Dol}^\bullet(\lambda \cdot p(\mathbf{1})), D''),
\end{array}
$$

where $[\lambda]$ is the automorphism of $A_{Dol}^\bullet(\lambda \cdot p(\mathbf{1})), D''$ which is multiplication by λ^p on the differential forms of type (p, q). The assignment $\lambda \mapsto \phi_\lambda$ defines an action of $\mathbb{C}^{\times\delta}$ on the complex $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$. Moreover, this action is compatible with the wegde product of differential forms, and is therefore an action of $\mathbb{C}^{\times\delta}$ on the commutative differential graded algebra $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$.

Finally, the action of G_X on $p(\mathbf{1})$ induces a well defined action of G_X on the commutative differential graded algebra $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$. Here the argument is the same as in the definition of the G_X-equivariant commutative differential graded algebra $(A_{DR}^\bullet(\mathcal{O}(G_X)), D)$ given in § 2.2. The group scheme G_X acts on the underlying Ind-C^∞-bundle of $p((1))$, and therefore on the spaces of differential forms with coefficients in this bundle. This action preserves the operator D'' coming

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from the structure of Higgs bundle, and therefore induces an action of G_X on the Dolbeault complex $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$.

This action is compatible with the action of $\mathbb{C}^{\times\delta}$ on $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$ and on G_X, and gives rise to a fixed point

$$(A_{Dol}^\bullet(p(\mathbf{1})), D'') \in (G_X\text{-}CDGA)^{\mathbb{C}^\times}.$$

In summary we have defined the following object. Let $\mathsf{CRep}(G_X)$ be the category of (positively graded) complexes of the linear representations of G_X which belong to $\mathbb{U}$. The group $\mathbb{C}^{\times\delta}$ acts on G_X, and therefore on the category $G_X\text{-}CDGA$ of commutative and unital algebras in $CRep(G_X)$. The algebra $(A_{Dol}^\bullet(\mathbf{1}), D'')$ is then an object in the fixed category $(G_X\text{-}CDGA)^{\mathbb{C}^\times}$. Using the denormalization functor which sends a differential graded algebra to a co-simplicial algebra, one can then consider

$$C_{Dol}^\bullet(X, \mathcal{O}(G_X)) := D(A_{Dol}^\bullet(p(\mathbf{1})), D'') \in \mathrm{Ho}((G_X\text{-}\mathsf{Alg}^\Delta)^{\mathbb{C}^{\times\delta}}).$$

This G_X-equivariant co-simplicial algebra is the Dolbeault counter-part of $C_{DR}^\bullet(X, \mathcal{O}(G_X))$ appearing in definition 2.2.1.

Proposition 2.4.2 Let $A := C_{Dol}^\bullet(X, \mathcal{O}(G_X)) \in \mathrm{Ho}((G_X\text{-}\mathsf{Alg}^\Delta)^{\mathbb{C}^{\times\delta}}).$

(a) After forgetting the action of G_X, there exist a natural $\mathbb{C}^{\times\delta}$-equivariant augmentation morphism in $\mathrm{Ho}((\mathsf{Alg}^\Delta)^{\mathbb{C}^{\times\delta}})$

$$e : A \longrightarrow \mathbb{C}.$$

(b) The underlying pointed stack of the $\mathbb{C}^{\times\delta}$-equivariant pointed stack

$$\mathrm{Spec}\,\mathbb{C} \xrightarrow{e^*} \mathbb{R}\,\mathrm{Spec}^{\mathbb{C}^{\times\delta}}(A) \longrightarrow [\mathbb{R}\,\mathrm{Spec}_{G_X}^{\mathbb{C}^{\times\delta}}(A)/G_X]$$

is functorially isomorphic to $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ in $\mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C}))$.

Proof: For the proof of (a) note that there is a natural morphism of $\mathbb{C}^{\times\delta}$-fixed commutative differential graded algebras $A \longrightarrow A^0$, obtained by projection on the 0-th terms of the co-simplicial object A. By definition, A^0 is the space of C^∞-sections of the Ind-object $p(\mathbf{1}) \in T_{Dol}$. In particular we can can compose the above morphism with the evaluation at $x \in X$, followed by the evaluation at $e \in G_X$ to get a $\mathbb{C}^\times$-equivariant morphism

$$A \longrightarrow A_0 \longrightarrow p(\mathbf{1})_x \simeq \mathcal{O}(G_X) \longrightarrow \mathbb{C}.$$

This proves part (a) of the proposition.

For the proof of (b) recall first that the G_X-equivariant commutative differential graded algebra $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$ admits an interpretation as the colimit of the inductive system $\{(A_{Dol}^\bullet(V_i), D_i'')\}_{i \in I}$ of complexes with G_X-action. Theorem 2.1.1 and Theorem 2.1.2 (or Corollary 2.1.3) imply that $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$ is naturally isomorphic in $\mathrm{Ho}(G_X\text{-}CDGA)$ to the colimit of the corresponding inductive system of connections $\{(A_{DR}^\bullet(V_i), \nabla_i)\}_{i \in I}$. On the other hand this inductive system of connections corresponds in turn to the Ind-local system $\mathcal{O}(\pi_1(X, x)^{\mathrm{red}}) \simeq \mathcal{O}(\pi_1(X, x)^{\mathrm{red}})$. Therefore, $(A_{Dol}^\bullet(p(\mathbf{1})), D'')$ is naturally isomorphic in $\mathrm{Ho}(G_X\text{-}\mathsf{CDGA})$ to $(A_{DR}^\bullet(\mathcal{O}(G_X), \nabla)$ defined in the last paragraph. Proposition 2.2.2 implies the existence of a functorial isomorphism in $\mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C}))$

$$(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}} \simeq [\mathbb{R}\,\mathrm{Spec}_{G_X}\,A/G_X].$$

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In conclusion, we have constructed a $\mathbb{C}^{\times \delta}$-equivariant stack $[\mathbb{R}\operatorname{Spec}_{G_X}^{\mathbb{C}^{\times \delta}} A/G_X] \in \operatorname{Ho}(\mathbb{C}^{\times \delta}\text{-}\mathsf{SPr}_*(\mathbb{C}))$, and a functorial isomorphism between its underlying pointed stack and $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$. By defintion, this is the data of a functorial $\mathbb{C}^{\times \times}$-action on the pointed stack $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$. $\qquad \square$

We are now ready to define the Hodge decomposition:

Definition 2.4.3 *The* Hodge decomposition *on the schematic homotopy type* $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ *is the* $\mathbb{C}^{\times \delta}$*-action defined by Proposition 2.4.2. By abuse of notation we will write*

$$(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}} \in \operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{C}^{\times}})$$

for the schematic homotopy type of X^{top} *together with its Hodge decomposition.*

The functoriality statement in Theorem 2.1.1 and 2.1.2 immediately implies that the Hodge decomposition is functorial. Note however that the construction depends on the choice of the point $x \in X^{\mathrm{top}}$ and so the functoriality is well defined only for pointed morphisms of algebraic varieties. Finally we have the following natural compatibility result:

Theorem 2.4.4 *Let X be a pointed smooth and projective algebraic variety over $\mathbb{C}$, and let*

$$(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}} \in \operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{C}^{\times \delta}})$$

be the schematization of X^{top} together with its Hodge decomposition.

(i) *The induced action of $\mathbb{C}^{\times \delta}$ on the cohomology groups $H^n((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, \mathbb{G}_a) \simeq H^n(X^{\mathrm{top}}, \mathbb{C})$ is compatible with the Hodge decomposition in the following sense. For any $\lambda \in \mathbb{C}^{\times}$, and $y \in H^{n-p}(X, \Omega_X^p) \subset H^n(X^{\mathrm{top}}, \mathbb{C})$ one has $\lambda(y) = \lambda^p \cdot y$.*

(ii) *Let $\pi_1((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)^{\mathrm{red}}$ be the maximal reductive quotient of $\pi_1((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)$. The induced action of $\mathbb{C}^{\times \delta}$ on $\pi_1((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)^{\mathrm{red}} \simeq \pi_1(X^{\mathrm{top}}, x)^{\mathrm{red}}$ coincides with the one defined in [S1].*

(iii) *If X^{top} is simply connected, then the induced action of $\mathbb{C}^{\times}$ on*

$$\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x) \simeq \pi_i(X^{\mathrm{top}}, x) \otimes \mathbb{C}$$

is compatible with the Hodge decomposition defined in [Mo]. More precisely, suppose that $F^\bullet \pi_i(X^{\mathrm{top}}) \otimes \mathbb{C}$ is the Hodge filtration defined in [Mo], then

$$F^p \pi_i(X^{\mathrm{top}}) \otimes \mathbb{C} = \left\{ x \in \pi_i(X^{\mathrm{top}} \otimes \mathbb{C}) \, \middle| \, \exists \, q \geq p, \text{ such that } \lambda(x) = \lambda^q \cdot x, \ \forall \lambda \in \mathbb{C}^{\times} \right\}.$$

(iv) *Let R_n be the set of isomorphism classes of n-dimensional simple linear representations of $\pi_1((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)$ Then, the induced action of $\mathbb{C}^{\times}$ on the set*

$$R_n \simeq \operatorname{Hom}(\pi_1(X^{\mathrm{top}}, x), Gl_n(\mathbb{C}))/Gl_n(\mathbb{C})$$

defines a continuous action of the topological group $\mathbb{C}^{\times}$ (for the analytic topology).

Proof: (i) Since the group scheme $G_X := \pi_1(X^{\mathrm{top}}, x)^{\mathrm{red}}$ is reductive, its Hochschild cohomology with coefficients in finite dimensional local systems vanishes. Therefore, one has

$$H^n((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, \mathbb{G}_a) \simeq H^n(\mathbb{R}\operatorname{Spec}_{G_X} A, \mathbb{G}_a)^{G_X} \simeq H^n(B)^{G_X} \simeq H^n(B^{G_X}),$$

where $B = (A^\bullet_{Dol}(p(\mathbf{1})), D'')$ is the G_X-equivariant co-simplicial algebra defined during the construction of the Hodge decomposition. Clearly B^{G_X} is isomorphic to the commutative differential graded algebra of Dolbeault cochains $A^\bullet_{Dol}(\mathbf{1}, \overline{\partial})$, of the trivial Higgs bundle $(\mathbf{1}, \overline{\partial})$. Therefore,

$$B^{G_X} \simeq \bigoplus_p (A^{p,\bullet}(X), \overline{\partial})[-p].$$

Moreover, the action of $\lambda \in \mathbb{C}^\times$ on $x \in A^{p,q}(X)$ is given by $\lambda(x) = \lambda^p \cdot x$. This implies that the induced action of $\mathbb{C}^\times$ on $H^n((X \otimes \mathbb{C})^{\mathrm{sch}}, \mathbb{G}_a)$ is the one required.

(ii) This is clear by construction.

(iii) If X^{top} is simply connected, then $\pi_1(X^{\mathrm{top}}, x)^{\mathrm{red}} = *$, and therefore one has that $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ is naturally isomorphic to

$$\mathbb{R}\operatorname{Spec}(C^\bullet_{Dol}(X, \mathbf{1})),$$

where $C^\bullet_{Dol}(X, \mathbf{1})$ is the dernormalization of $(A^\bullet_{Dol}(\mathbf{1}), \overline{\partial}))$. The action of $\mathbb{C}^\times$ has weight p on $A^{p,q}(X)$. This action also correspond to a decreasing filtration $F^\bullet$ of $(A^\bullet_{Dol}(\mathbf{1}), \overline{\partial})$ by sub-CDGA, where F^p consists of differential forms of type (p', q) with $p' \geq p$.

The vector space $\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)(\mathbb{C}) \simeq \pi_i(X, x) \otimes \mathbb{C}$ is the i-th homotopy group of the mapping space $\mathbb{R}\underline{\operatorname{Hom}}_{cdga}((A^\bullet_{Dol}(\mathbf{1}), \overline{\partial}), \mathbb{C})$, which is in turn naturally isomorphic to the i-th homotopy group of the commuttative dga $(A^\bullet_{Dol}(\mathbf{1}), \overline{\partial})$, as defined in [Bo-Gu]. In other words, it is the dual of the space of indecomposable elements of degree i in the minimal model of $(A^\bullet_{Dol}(\mathbf{1}), \overline{\partial})$. As the action of $\mathbb{C}^\times$ on $(A^\bullet_{Dol}(\mathbf{1}), \overline{\partial})$ is compatible with the Hodge filtration $F^\bullet$, the induced action of $\mathbb{C}^\times$ on $\pi_i(X, x) \otimes \mathbb{C}$ is compatible with the filtration induced by $F^\bullet$ on the space of indecomposables. This filtration being the Hodge filtration defined in [Mo], this proves (iii).

(iv) This follows from (iii) and [S1, Proposition 1.5]. $\qquad\square$

3 Weights

As we have already mentioned, the Hodge decomposition on $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ constructed in the previous section is only a part of a richer structure, which includes a *weight filtration*. In the simply connected case, the weight filtration is constructed in [Mo], and its existence reflects the fact that the Hodge structures defined on the complexified homotopy groups are only mixed Hodge structures. The existence of the weight filtration on the rational homotopy type of a projective X is related to the fact that any simply connected homotopy type (or more generally any nilpotent homotopy type) is a *successive extensions of abelian homotopy types*.

3.1 Linear stacks and homology theory

In this section we present a linear version of the schematic homotopy types, and use it to define homology of schematic homotopy types.

Let $\mathcal{O}$ be the sheaf on $(\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}}$ represented by the affine line $\mathbb{A}^1$. By definition $\mathcal{O}$ is a sheaf of $\mathbb{C}$-algebras and can be viewed as an object in $\mathsf{SPr}(\mathbb{C})$. We consider the category $\mathsf{SMod}(\mathcal{O})$, of objects in $\mathsf{SPr}(\mathbb{C})$ equipped with a structure of $\mathcal{O}$-modules (and morphisms preserving this structure). In other words, $\mathsf{SMod}(\mathcal{O})$ is the category of simplicial objects in the category $\mathsf{Mod}(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules on $\mathsf{Aff}/\mathbb{C}$ (note that we do not impose any sheaf conditions and consider all presheaves of $\mathcal{O}$-modules).

Forgetting the $\mathcal{O}$-module structure defines a forgetful functor

$$\mathsf{SMod}(\mathcal{O}) \longrightarrow \mathsf{SPr}(\mathbb{C}).$$

This functor has a left adjoint

$$- \otimes \mathcal{O} : \mathsf{SPr}(\mathbb{C}) \longrightarrow \mathsf{SMod}(\mathcal{O}).$$

The category $\mathsf{SMod}(\mathcal{O})$ has a natural model structure for which the fibrations and equivalences are defined in $\mathsf{SPr}(\mathbb{C})$ through the forgetful functor. We will call the model category $\mathsf{SMod}(\mathcal{O})$ the category of *linear stacks*. It is Quillen equivalent (via the Dold-Kan correspondence) to the model category of negatively graded co-chain complexes of $\mathcal{O}$-modules. In particular its homotopy category $\mathrm{Ho}(\mathsf{SMod}(\mathcal{O})$ is equivalent to $D^-((\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}}, \mathcal{O})$, the bounded above derived category of $\mathcal{O}$-modules on the ringed site $((\mathsf{Aff}/\mathbb{C})_{\mathrm{ffqc}}, \mathcal{O})$.

Let E be a co-simplicial $\mathbb{C}$-vector space. We define a linear stack $\mathrm{Spel}\,E$ (*Spel* stands for **s**pectre **lin**éaire) by the formula

$$\mathrm{Spel}\,E : \quad \mathsf{Aff}/\mathbb{C} \longrightarrow \mathsf{SSet}$$
$$A \longmapsto \underline{\mathrm{Hom}}(E, A),$$

where $\underline{\mathrm{Hom}}(E, A)$ is the simplicial set having $\mathrm{Hom}(E_n, A)$ as sets of n-simplicies. The $\mathcal{O}$-module structure on $\mathrm{Spel}\,E$ is given by the natural A-module structure on each simplicial set $\underline{\mathrm{Hom}}(E, A)$. Note that the simplicial presheaf underlying $\mathrm{Spel}\,E$ is isomorphic to $\mathrm{Spec}\,(\mathrm{Sym}^\bullet(E))$, where $\mathrm{Sym}^\bullet(E)$ is the free commutative co-simplicial $\mathbb{C}$-algebra generated by the co-simplicial vector space E.

The functor Spel is a right Quillen functor

$$\mathrm{Spel} : \mathsf{Vect}^\Delta \longrightarrow \mathsf{SMod}(\mathcal{O}),$$

from the model category of co-simplicial vector spaces (with the usual model structure for which equivalences and fibrations correspond to quasi-isomorphisms and epimorphisms through the Dold-Kan correspondence) to the model category of linear stacks. Therefore the functor Spel can be derived into a functor

$$\mathbb{R}\,\mathrm{Spel} : \mathrm{Ho}(\mathsf{Vect}^\Delta) \longrightarrow \mathsf{SMod}(\mathcal{O}).$$

(actually Spel preserves equivalences, so $\mathrm{Spel} \simeq \mathbb{R}\,\mathrm{Spel}$ Its is easy to check that this functor is fully faithful when restricted to the full sub-category of $\mathbb{U}$-small co-simplicial vector spaces.

Definition 3.1.1 *The essential image of the functor* $\mathbb{R}\,\mathrm{Spel}$*, restricted to* $\mathbb{U}$*-small co-simplicial vector spaces, is called the category of* linear affine stacks.

Note that as $\mathbb{R}\operatorname{Spel} E \simeq \mathbb{R}\operatorname{Spec}(\operatorname{Sym}^\bullet(E))$, the forgetful functor

$$\operatorname{Ho}(\mathsf{SMod}(\mathcal{O})) \longrightarrow \operatorname{Ho}(\mathsf{SPr}(\mathbb{C})),$$

sends linear affine stacks to affine stacks.

Let F be an affine stack, and let $A = \mathbb{L}\mathcal{O}(F)$ be the co-simplicial algebra of cohomology on F, which can be chosen to belong to $\mathbb{U}$. Viewing A as a co-simplicial vector space by forgetting the multiplicative structure one gets a linear affine stack

$$\mathcal{H}(F, \mathcal{O}) := \operatorname{Spel} A,$$

which we will call the *homology type of F*. The functor $F \mapsto \mathcal{H}(F, \mathcal{O})$ is left adjoint to the forgetful functor from linear affine stacks to affine stacks. In particular, one has an adjunction morphism of affine stacks

$$F \longrightarrow \mathcal{H}(F, \mathcal{O}),$$

and it is reasonable to interpret this morphism as the *abelianization* of the schematic homotopy type F.

By construction, the i-th homotopy sheaf of $\mathcal{H}(F, \mathcal{O})$ is the linearly compact vector space dual to $H^i(F, \mathcal{O})$ (a linearly compact vector space is an object in the category of pro-vector spaces of finite dimension). It is denoted by $H_i(F, \mathcal{O})$ and is called the *i-th homology sheaf of F*. The adjunction morphism above induces Hurewitz maps

$$\pi_i(F) \longrightarrow H_i(F, \mathcal{O})$$

which are well defined for all i.

3.2   The weight tower

First we introduce a general notion of a weight filtration on an affine stack.

The weight tower of an affine stack

Let $\mathbb{T}$ be the category corresponding to the poset of natural numbers $\mathbb{N}$. The category $\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}}$, of functors from $\mathbb{T}$ to $\mathsf{SPr}_*(\mathbb{C})$ is a model category for which equivalences and fibrations are defined levelwise (see e.g. [Hir]). We will call the model category $\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}}$ the *model category of towers in* $\mathsf{SPr}_*(\mathbb{C})$. We will call the objects in the homotopy category $\operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}})$ *towers of pointed stacks*.

We will define a functor

$$W^* : \operatorname{Ho}(\mathsf{AS}^c_*) \longrightarrow \operatorname{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}}),$$

from the homotopy category of pointed and connected affine stacks to the homotopy category of towers of pointed stacks. This construction is an algebraic counter-part of a well known topological construction (see [Cu1, Cu2]), and will be achieved using the homotopy theory of simplicial affine group schemes discussed in detail in [Ka-Pa-To].

As explained in [Ka-Pa-To], there exists a model category of ($\mathbb{U}$-small) simplicial affine group schemes sGAff, whose equivalences are defined as the morphisms inducing quasi-isomorphisms on

the corresponding co-simplicial Hopf algebras. Recall also, that the natural 'classifying stack' functor

$$\mathsf{sGAff} \longrightarrow \mathsf{SPr}_*(\mathbb{C})$$
$$G_* \longmapsto BG_*,$$

can be derived on the right to a functor

$$\mathrm{Ho}(\mathsf{sGAff}) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C}))$$
$$G_* \longmapsto BR(G_*)$$

(here, R is a fibrant replacement functor in the model category sGAff). The functor BR is fully faithful and its essential image consists precisely of all pointed schematic homotopy types (see [Ka-Pa-To, Theorem 3.5]).

The functor of complex points $H_* \mapsto H_*(\mathbb{C})$ is an exact conservative (see [De-Ga, III §3 Corollary 7.6]) functor from the category sGAff to the category SSet of simplicial sets in $\mathbb{U}$. Furthermore, this functor possesses a left adjoint, sending a simplicial set to the free simplicial affine group scheme it generates[1]. Using this adjunction, one can construct (as explained e.g. in [Il]) a standard free resolution $L_*G_* \longrightarrow G_*$. The object L_*G_* is a bi-simplicial affine group scheme, such that each L_*G_n is the standard free resolution of G_n. We denote by LG_* the diagonal of L_*G_*, and consider the induced morphism of simplicial affine group schemes $LG_* \longrightarrow G_*$. The morphism $LG_*(\mathbb{C}) \longrightarrow G_*(\mathbb{C})$ is a homotopy equivalence (see [Il]), which implies that the morphism of pointed simplicial presheaves

$$BLG_* \longrightarrow BG_*$$

induces an isomorphisms on all homotopy presheaves π_i^{pr}. In particular it induces an isomorphism of pointed stacks. This construction gives a functorial morphism of simplicial affine group schemes $LG_* \longrightarrow G_*$ such that the induced morphism

$$BLG_* \longrightarrow BG_*$$

is an equivalence of pointed stacks, and furthermore each affine group scheme LG_n is free.

To any $G_* \in \mathsf{sGAff}$, we can now associate a new simplicial affine group scheme $H \in \mathsf{sGAff}$ which is defined to be the maximal unipotent quotient $H := LG_*^{\mathrm{uni}}$ of LG_*. The assignment $G_* \mapsto H_*$ can be thought of as the *left derived functor of the pro-unipotent completion functor*.

We now consider the lower central series

$$\cdots \subset H_*^{(i)} \subset H_*^{(i-1)} \subset \cdots \subset H_*^{(1)} \subset H_*^{(0)} = H_*,$$

given for any $i \geq 1$ by

$$H_*^{(i)} := [H_*^{(i-1)}, H_*] \subset H_*.$$

The filtration $H_*^{(i)}$ on H_* gives rise to a tower of morphisms of simplicial affine group schemes

$$H_* \longrightarrow \cdots \longrightarrow H_*/H_*^{(i)} \longrightarrow H_*/H_*^{(i-1)} \longrightarrow \cdots \longrightarrow H_*/H_*^{(1)}.$$

Passing to classifying stacks gives a tower of morphisms of pointed simplicial presheaves

$$BH_* \longrightarrow \ldots \longrightarrow B\left(H_*/H_*^{(i)}\right) \longrightarrow B\left(H_*/H_*^{(i-1)}\right) \longrightarrow \ldots \longrightarrow B\left(H_*/H_*^{(1)}\right).$$

---

[1]The free affine group scheme generated by a set I is the pro-algebraic completion of the free group over I

Combining all these we get a functor

$$(3.2.1) \qquad\qquad \mathsf{sGAff} \longrightarrow \mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}}$$
$$G_* \longmapsto \left\{ B\left(H_*/H_*^{(i)} \right) \right\}_i .$$

Lemma 3.2.1 *The functor* (3.2.1) *preserves equivalences and induces a well defined functor*

$$\mathsf{Ho}(\mathsf{sGAff}) \longrightarrow \mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}}).$$

Proof: First of all, by construction each H_n is a unipotent group scheme. Since sheaves represented by affine and unipotent group schemes are stable under passage to finite limits and finite colimits, we conclude that the homotopy sheaves $\pi_j(BH_*, *)$ are unipotent. Therefore, Theorem 1.1.2 implies that the pointed stacks BH_* and BH'_* are affine stacks.

Now suppose $G_* \longrightarrow G'_*$ is an equivalence in the model category sGAff. First we will check that the induced morphism $BH_* \longrightarrow BH'_*$ is an isomorphism of pointed stacks. Indeed, since these stacks are affine stacks, it is enough to show that the morphism $BH_* \longrightarrow BH'_*$ induces an isomorphism on the cohomology with coefficient in $\mathbb{G}_a$ (see [To2, Theorem 2.2.9]). As $G_* \longrightarrow G'_*$ is an equivalence, it induces an isomorphism on the Hochschild cohomologies, and in particular it induces a quasi-isomorphism $\mathcal{O}(BG_*) \longrightarrow \mathcal{O}(BG'_*)$. Similarly the natural morphism

$$\mathcal{O}(BLG_*) \longrightarrow \mathcal{O}(BLG'_*)$$

is a quasi-isomorphism. Due to the fact that

$$BLG_* \simeq \operatorname{hocolim} BLG_n,$$

the problem reduces to showing that for each integer n the induced morphism on Hochschild cohomology

$$(3.2.2) \qquad\qquad H^*(LG_n, \mathcal{O}) \longrightarrow H^*(H_n, \mathcal{O})$$

is an isomorphism. But LG_n is a free affine group scheme, and H_n are free unipotent affine group schemes, and therefore

$$H^i(LG_n, \mathcal{O}) = H^i(H_n, \mathcal{O}) = 0 \quad \forall\, i > 1.$$

Furthermore $H^1(K, \mathcal{O}) \simeq H^1(K^{\mathrm{uni}}, \mathcal{O})$ for any affine group scheme K and hence (3.2.2) is an isomorphism.

This implies that $BH_* \longrightarrow BH'_*$ and so, Lemma 3.2.1 reduces to the following result:

Lemma 3.2.2 *Let H_* and K_* be objects in sGAff, such that each H_n and K_n is a free unipotent affine group scheme. Let $f : H_* \longrightarrow K_*$ be an equivalence in sGAff. Then, for all $i > 0$, the induced morphism*

$$BH_*/H_*^{(i)} \longrightarrow BK_*/K_*^{(i)}$$

is an equivalence of pointed stacks.

Proof: By induction it is enough to check that each morphism

$$BH_*^{(i)}/H_*^{(i+1)} \longrightarrow BK_*^{(i)}/K_*^{(i+1)}$$

is an equivalence. To see this, we consider the induced morphism

$$\prod_i H_*^{(i)}/H_*^{(i+1)} \longrightarrow \prod_i K_*^{(i)}/K_*^{(i+1)}.$$

This morphism can be seen as a morphism of simplicial filtered linearly compact Lie algebras, where the filtration is given by the product decomposition,

$$\cdots \subset \prod_{i>i_0+1} \subset \prod_{i>i_0} \subset \cdots \prod_i,$$

and the Lie algebra structure is given by the commutator bracket. Set $H_*^{\mathrm{ab}} := H_*/H_*^{(1)}$ and $K_*^{\mathrm{ab}} := K_*/K_*^{(1)}$. One has a commutative diagram of simplicial linearly compact vector spaces

$$
\begin{array}{ccc}
\prod_i H_*^{(i)}/H_*^{(i+1)} & \longrightarrow & \prod_i K_*^{(i)}/K_*^{(i+1)} \\
\uparrow & & \uparrow \\
H_*^{\mathrm{ab}} & \longrightarrow & K_*^{\mathrm{ab}}.
\end{array}
$$

Furthermore, as each H_n and K_n is free, the vertical morphisms induce isomorphisms

$$L(H_*^{\mathrm{ab}}) \simeq \prod_i H_*^{(i)}/H_*^{(i+1)} \qquad L(K_*^{\mathrm{ab}}) \simeq \prod_i K_*^{(i)}/K_*^{(i+1)},$$

where $L(V)$ denotes the free simplicial linearly compact Lie algebra generated by a simplicial linearly compact vector space V. Moreover, these isomorphisms are compatible with the natural filtrations on $L(H_*^{\mathrm{ab}})$ and $L(K_*^{\mathrm{ab}})$ defined by the iterated brackets. Hence, it suffices to prove that the induced morphism of linear affine stacks $BH_*^{\mathrm{ab}} \longrightarrow BK_*^{\mathrm{ab}}$ is an isomorphism. But, this follows immediately by observing that BH_*^{ab} is the homology type of BH_*, and BK_*^{ab} is the homology type of BK_*. Thus

$$BH_*^{\mathrm{ab}} \simeq \mathcal{H}(BH_*, \mathcal{O}) \simeq \mathcal{H}(BK_*, \mathcal{O}) \simeq BK_*^{\mathrm{ab}}$$

and so the lemma is proven.   □

This completes the proof of Lemma 3.2.1   □

Lemma 3.2.1, combined with the equivalence between the homotopy category $\mathsf{Ho}(\mathsf{SHT}_*)$ of pointed schematic homotopy types and $\mathsf{Ho}(\mathsf{sGAff})$, yield a functor

$$W^{(*)} : \mathsf{Ho}(\mathsf{SHT}_*) \longrightarrow \mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}}).$$

By restricting this functor to the full sub-category of pointed and connected affine stacks one gets the *weight tower functor*

$$W^{(*)} : \mathsf{Ho}(\mathsf{AS}_*^c) \longrightarrow \mathsf{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}}),$$

from the homotopy category of pointed and connected affine stacks to the homotopy category of towers of pointed stacks.

Let now F be a pointed schematic homotopy type. Consider the natural projection

$$F \longrightarrow K(\pi_1(F, *)^{\mathrm{red}}, 1),$$

where $\pi_1(F)^{\mathrm{red}}$ is the maximal reductive quotient of $\pi_1(F, *)$. Let F^0 denote the homotopy fiber of this morphism. By theorems 1.1.2 and 1.1.5 the stack F^0 is a pointed and connected affine stack. We can therefore consider its weight tower $W^{(*)}F^0$:

Definition 3.2.3 *For a schematic homotopy type F, the weight tower of F is the object $W^{(*)}F^0 \in \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}})$ defined above. For any $i \geq 0$, the i-th graded piece of the weight tower of F, denoted by $\mathrm{Gr}_W^{(i)} F^0$, is defined to be the homotopy fiber of the morphism $W^{(i+1)}F^0 \longrightarrow W^{(i)}F^0$.*

Caution: The construction $F \mapsto W^{(*)}F^0$ is not fully functorial in F, simply because the assignment $F \mapsto \pi_1(F, *)^{\mathrm{red}}$ is not functorial in F. However, $F \mapsto \pi_1(F, *)^{\mathrm{red}}$ and hence $F \mapsto W^{(*)}F^0$ are functorial with respect to *reductive morphisms*.

Definition 3.2.4 *A morphism $f : F \longrightarrow G$ of pointed schematic homotopy types is* reductive *if the image of the induced morphism $\pi_1(F, *) \longrightarrow \pi_1(G, *)^{\mathrm{red}}$ is a reductive affine group scheme.*

Equivalently, the morphism f is reductive if and only if the induced functor from the category of linear representations of $\pi_1(G, *)$ to the category of linear representations of $\pi_1(F, *)$ preserves semi-simple objects. In particular, any isomorphisms of pointed schematic homotopy types is a reductive morphism.

Clearly, the construction $F \mapsto W^{(*)}F^0$ induces a functor

$$W^{(*)} : \mathrm{Ho}(\mathsf{SHT}_*)^{\mathrm{red}} \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}}),$$

from the category $\mathrm{Ho}(\mathsf{SHT}_*)^{\mathrm{red}}$ of pointed schematic homotopy types and reductive morphisms to the homotopy category of towers of pointed stacks.

Remark 3.2.5 If $f : (X, x) \longrightarrow (Y, y)$ is a morphism of pointed smooth projective complex manifolds, then the induced morphism $(X^{\mathrm{top}} \otimes \mathbb{C}, x)^{\mathrm{sch}} \longrightarrow (Y^{\mathrm{top}} \otimes \mathbb{C}, y)^{\mathrm{sch}}$ is a reductive morphism. Indeed, we can factor every such f as the composition $X \to \mathbb{P}^N \times Y \to Y$ of a closed immersion and a projection. The projection induces an isomorphism on fundamental groups and so is obviously a reductive morphism. For closed immersions the reductivity follows from the pluriharmonicity of the equivariant harmonic map associated to a reductive local system, see [S3, Proposition 2.2].

By construction, there exists a natural morphism of towers of pointed stacks

$$F^0 \longrightarrow W^{(*)}F^0,$$

corresponding to the projections

$$LG_* \longrightarrow H_* \longrightarrow H_*/H_*^{(i)}$$

(here $F^0 = BG_* \simeq BLG_*$ is considered as the constant tower). By adjunction, this morphism correspond to a well defined morphism of pointed stacks

$$F^0 \longrightarrow W^{(\infty)} := \operatorname*{holim}_{i \in \mathbb{T}} W^{(i)} F^0.$$

The essential properties of this morphism are summarized in the following propodition.

Proposition 3.2.6 *Let F be a pointed schematic homotopy type.*

(1) *For any i, the pointed stack $W^{(i)} F^0$ is a pointed and connected affine stack.*

(2) *The natural morphism*

$$F^0 \longrightarrow W^{(\infty)} := \operatorname*{holim}_{i \in \mathbb{T}} W^{(i)} F^0$$

is an isomorphism of pointed stacks.

(3) *For any $i \geq 0$, $\operatorname{Gr}_W^{(i)} F^0$ is the underlying stack of a linear schematic homotopy type.*

(4) *There is an isomorphism of affine stacks*

$$\operatorname{Gr}_W^{(0)} F^0 \simeq \mathcal{H}(F^0, \mathcal{O}).$$

(5) *Let L be the (linearly compact) free Lie algebra generated by the (linearly compact) vector space $H_{\bullet>0}(F^0, \mathcal{O})$. For any $p \geq 1$ and $q \geq 2$, let $L_{p,q}$ be the (closed) sub-vector space of L consisting of elements generated by all brackets $[x_1, [x_2, \ldots, [x_q]\ldots], with $x_j \in H_{d_j}(F^0, \mathcal{O})$ and $\sum d_j = p$.*

Then, for any $i \geq 1$ and any $p \geq 1$ there is an isomorphism

$$\pi_p(\operatorname{Gr}_W^{(i)} F^0, *) \simeq L_{p+i, i+1}.$$

Proof: We already proved items (1), (3), (4) and (5) in the process of proving lemmas 3.2.1 and 3.2.2. For the proof of (2) we will keep the same notation as for the construction of $W^{(*)} F^0$ (i.e. $F^0 = BG_*$ and $H_* = LG_*^{\mathrm{uni}}$). First of all, as we saw in the proof of lemma 3.2.1 the natural morphism

$$F^0 = BG_* \longrightarrow BH_*$$

induces an isomorphism on cohomology with coefficients in $\mathbb{G}_a$. Since F^0 and BH_* are affine stacks this implies that $F^0 \simeq BH_*$.

Furtermore, each $W^{(i)} F^0$ is an affine stack and in particular a pointed schematic homotopy type. Therefore, in order to prove that

$$F^0 \simeq BH_* \longrightarrow \operatorname*{holim}_{i \in \mathbb{T}} W^{(i)} F^0$$

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is an isomorphism, it is enough to prove that the natural morphism of co-simplicial Hopf algebras

$$(3.2.3) \qquad\qquad \operatorname*{colim}_{i\in\mathbb{T}} \mathcal{O}(H_*/H_*^{(i)}) \longrightarrow \mathcal{O}(H_*)$$

is a quasi-isomorphism. However H_* is unipotent, and so $\cap H_*^{(i)} = \{*\}$. Thus (3.2.3) is even an isomorphism. The proposition is proven. $\qquad\square$

<center>Equivariant weight filtration</center>

All of the constructions presented above have enough built-in functoriality to be compatible with the extra structure of a discrete group action. Indeed, suppose $F \in \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C})^\Gamma)$ is a Γ-equivariant schematic homotopy type, where Γ is a discrete group. The action of Γ can be understood as morphism of simplicial monoids

$$\Gamma \longrightarrow \underline{\mathrm{End}}(F).$$

Without a loss of generality we can assume that F is fibrant as a pointed simplicial presheaf, and therefore the last morphism can be written as morphism of simplicial monoids

$$\Gamma \longrightarrow \mathbb{R}\underline{\mathrm{End}}(F),$$

which is well defined in the homotopy category of simplicial monoids.

Let G_* be a fibrant object in sGAff such that $F^0 \simeq BG_*$. According to [Ka-Pa-To, Theorem 3.5], we have a natural isomorphism of simplicial monoids

$$\Gamma \longrightarrow \mathbb{R}\underline{\mathrm{End}}_{\mathsf{sGAff}}(G_*).$$

Let $\widetilde{\Gamma}$ be a cofibrant replacement of Γ in the model category of simplicial monoids. The above morphism can then be represented as an actual morphism of simplicial monoids

$$\widetilde{\Gamma} \longrightarrow \underline{\mathrm{End}}_{\mathsf{sGAff}}(G_*),$$

giving rise to a $\widetilde{\Gamma}$-action on G_*. Applying the constructions $G_* \mapsto LG_* \mapsto LG_*^{\mathrm{uni}} \mapsto W^{(*)}F$ preserves the $\widetilde{\Gamma}$-action, and gives rise to a $\widetilde{\Gamma}$-object in the model category $\mathsf{SPr}_*(\mathbb{C})^\mathbb{T}$. This yields a natural functor

$$W^{(*)} : \mathrm{Ho}(\mathsf{SHT}_*^\Gamma)^{\mathrm{red}} \longrightarrow \mathrm{Ho}((\mathsf{SPr}_*(\mathbb{C})^\mathbb{T})^{\widetilde{\Gamma}}),$$

from the homotopy category of Γ-equivariant pointed schematic homotopy types and reductive morphisms to the homotopy category of Γ-equivariant towers. This functor is a Γ-equivariant refinement of the functor $W^{(*)}$ constructed before. Finally, it is well known that the natural functor

$$\mathrm{Ho}((\mathsf{SPr}_*(\mathbb{C})^\mathbb{T})^\Gamma) \longrightarrow \mathrm{Ho}((\mathsf{SPr}_*(\mathbb{C})^\mathbb{T})^{\widetilde{\Gamma}})$$

is an equivalence of categories (see [D-K]). Therefore, we get a functor

$$W^{(*)} : \mathrm{Ho}(\mathsf{SHT}_*^\Gamma)^{\mathrm{red}} \longrightarrow \mathrm{Ho}((\mathsf{SPr}_*(\mathbb{C})^\mathbb{T})^\Gamma),$$

which is a Γ-equivariant version of our previous construction.

Remark 3.2.7 In the construction of the weight tower of a pointed schematic homotopy type F, we ignored the natural action of the group $\pi_1(F, *)^{\mathrm{red}}$ on F^0. Yet another refinement of the construction should take this action into account, and would produce a tower of $\pi_1(F, *)^{\mathrm{red}}$-equivariant pointed stacks. We did not investigate this issue carefully but strongly believe that such a refinement should exists. In particular, the schematic homotopy type F should be reconstructed from its $\pi_1(F, *)^{\mathrm{red}}$-equivariant tower by the formula

$$F \simeq \left[\underset{i \in \mathbb{T}}{\mathrm{holim}}\, W^{(i)} F^0 \middle/ \pi_1(F, *)^{\mathrm{red}} \right].$$

3.3 Real structures

For a field extension L/K, one has a base change functor

$$- \otimes_K L : \mathrm{Ho}(\mathsf{SPr}_*(K)) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(L)),$$

which is right adjoint to the forgetful functor

$$\mathrm{Ho}(\mathsf{SPr}_*(L)) \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(K)).$$

The functor $- \otimes_K L$ preserves pointed schematic homotopy types, and so the universal property of the schematization implies that for any pointed and connected simplicial set (X, x) there exists a natural morphism

$$(X \otimes L, x)^{\mathrm{sch}} \longrightarrow (X \otimes K, x)^{\mathrm{sch}} \otimes_K L.$$

This morphism is in general not an isomorphism, but it is known to be an isomorphism when L/K is a finite extension.

Proposition 3.3.1 *Let L/K be a finite extension of fields and let X be a pointed and connected simplicial set. The natural morphism*

$$(X \otimes L, x)^{\mathrm{sch}} \longrightarrow (X \otimes K, x)^{\mathrm{sch}} \otimes_K L$$

is an isomorphism of pointed stacks.

Proof: For any discrete group Γ, the natural morphism $\Gamma^{\mathrm{alg}, L} \longrightarrow \Gamma^{\mathrm{alg}, K} \otimes_K L$ is an isomorphism of affine group schemes (here $\Gamma^{\mathrm{alg}, k}$ denotes the pro-algebraic completion of Γ over the field k). As the schematization of X over a field k can be written (see [Ka-Pa-To]) as $BG_*^{\mathrm{alg}, k}$ for a simplicial group G_* this implies the proposition. $\square$

We will use roposition 3.3.1 to endow the stack $(X \otimes \mathbb{C}, x)^{\mathrm{sch}}$ with a natural real structure

$$(X \otimes \mathbb{C}, x)^{\mathrm{sch}} \longrightarrow (X \otimes \mathbb{R}, x)^{\mathrm{sch}} \otimes_\mathbb{R} \mathbb{C}.$$

In the case where X is a smooth compact manifold, this real structure can be directly seen at the level of the equivariant algebra of differential forms $(A_{DR}^\bullet(\mathcal{O}(G_X)), \nabla)$. Indeed, the *Ind*-flat bundle $\mathcal{O}(G_X)$ has a natural real form given as the Tannakian dual of the category of real flat bundles over X. This induces a natural real structure on the de Rham complex $(A_{DR}^\bullet(\mathcal{O}(G_X)), \nabla)$ which is the real structure of $(X \otimes \mathbb{C}, x)^{\mathrm{sch}}$ discussed above.

Of course, for a pointed schematic homotopy type F over $\mathbb{R}$, one can construct a real weight tower $W^{(*)}F^0 \in \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{R})^{\mathbb{T}})$, compatible with the base change from $\mathbb{R}$ to $\mathbb{C}$. This can be seen as follows.

Let F be a pointed schematic homotopy type over $\mathbb{R}$, and let F^0 be the homotopy fiber of the projection $F \longrightarrow K(\pi_1(F, *)^{\mathrm{red}}, 1)$, where $\pi_1(F, *)^{\mathrm{red}}$ is the real maximal reductive quotient of $\pi_1(F, *)$. Clearly, one has $\pi_1(F, *)^{\mathrm{red}} \otimes_{\mathbb{R}} \mathbb{C} \simeq (\pi_1(F, *) \otimes_{\mathbb{R}} \mathbb{C})^{\mathrm{red}}$, and therefore

$$F^0 \otimes_{\mathbb{R}} \mathbb{C} \simeq (F \otimes_{\mathbb{R}} \mathbb{C})^0.$$

Hence $F^0 \simeq BG_*$, for a simplicial real affine group scheme G_* (see [Ka-Pa-To, Theorem 3.5]). Consider now the functor $H_* \mapsto H_*(\mathbb{C})$, from the category of simplicial real affine group schemes to the category $\mathbb{Z}/2 - \mathsf{SSet}$ of simplicial sets with an action of $\mathbb{Z}/2$. This functor is exact, conservative and has a left adjoint. It can therefore be used to construct a resolution $LG_* \longrightarrow G_*$, whichin turns gives rise to the standard free resolution of $G_* \otimes_{\mathbb{R}} \mathbb{C}$ used in the definition of the weight tower of $F \otimes_{\mathbb{R}} \otimes \mathbb{C}$. Finally, as the lower central serie and the construction $H \mapsto H^{\mathrm{uni}}$ are compatible with base change from $\mathbb{R}$ to $\mathbb{C}$, this shows that the weight tower $W^{(*)}(F \otimes_{\mathbb{R}} \mathbb{C}^0)$ has a natural real structure $W^{(*)}(F^0) \in \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{R})^{\mathbb{T}})$ given by

$$W^{(i)}(F^0) := B(LH_*/LH_*^{(i)}) \qquad H_* = LG_*^{\mathrm{uni}}.$$

This construction gives a functor

$$W^{(*)} : \mathrm{Ho}(\mathsf{SHT}_*(\mathbb{R}))^{\mathrm{red}} \longrightarrow \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{R})^{\mathbb{T}})$$

from the homotopy category of pointed schematic homotopy types over $\mathbb{R}$ and reductive morphisms, to the homotopy category of towers of real pointed stacks. The compatibility with the base change from $\mathbb{R}$ to $\mathbb{C}$ is an isomorphism of towers of pointed stacks (functorial with respect to reductive morphisms)

$$(W^{(*)}F^0) \otimes_{\mathbb{R}} \mathbb{C} \simeq W^{(*)}((F \otimes_{\mathbb{R}} \mathbb{C})^0).$$

3.4 The spectral sequence of a tower

Let $\{F_i\} \in \mathrm{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}})$ be a tower of pointed stacks:

$$\cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 = \bullet,$$

and let W_i denote the homotopy fiber of the morphism $F_{i+1} \longrightarrow F_i$. The associated long exact sequences of homotopy groups give rise to an exact couple of sheaves

and hence defines a spectra sequence $\{E_r^{p,q}(F_*)\}$ of sheaves of groups (abelian when $q - p > 1$). To explicate, consider first the groups

$$Z_r^{p,q} = \mathrm{Ker}\left(\pi_{q-p}(W_{p-1}) \longrightarrow \frac{\pi_{q-p}(F_p)}{\mathrm{Im}\,(\pi_{q-p}(F_{p+r-1}) \to \pi_{q-p}(F_p))}\right)$$

$$B_r^{p,q} = \mathrm{Ker}\,(\pi_{q-p+1}(F_{p-1}) \longrightarrow \pi_{q-p+1}(F_{p-r})).$$

The boundary operator $\pi_{q-p+1}(F_{p-1}) \longrightarrow \pi_{q-p}(W_{p-1})$ induces a morphism

$$\partial : B_r^{p,q} \longrightarrow Z_r^{p,q},$$

and we set

$$E_r^{p,q} = \frac{Z_r^{p,q}}{\partial B_r^{p,q}}.$$

The differential

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+r-1}$$

is given by the composition

$$E_r^{p,q} \longrightarrow \text{Im}\left(\pi_{q-p}(F_{p+r-1}) \to \pi_{q-p}(F_p)\right) \longrightarrow E_r^{p+r,q+r-1}.$$

We refer to [G-J, VI §2] for more details on this spectral sequence.

The convergence of this spectral sequence can be quite subtle in general. However it is known that under certain conditions it converges and its limit computes the groups $\pi_*(\text{holim } F_i)$. For our purposes the following simple case of the *complete convergence lemma* of Bousfield and Kan will suffice.

Lemma 3.4.1 *Let $\{F_i\} \in \text{Ho}(\mathsf{SPr}_*(\mathbb{C})^{\mathbb{T}})$ be a tower of pointed and connected affine stacks, and let $F := \text{holim } F_i$. Let $\{E_r^{p,q}(F_*)\}$ be the associated spectral sequence in homotopy, and assume that there is an integer N such that $d_r = 0$ for any $r \geq N$. Then the following two conditions are satisfied.*

1. *The limiting tableu of $\{E_r^{p,q}(F_*)\}$ is given by*

$$E_\infty^{p,q} \simeq \frac{\text{Ker}\left(\pi_{q-p}(F) \to \pi_{q-p}(F_{p-1})\right)}{\text{Ker}\left(\pi_{q-p}(F) \to \pi_{q-p}(F_p)\right)}$$

2. *The natural morphism*

$$\pi_*(F) \longrightarrow \lim \pi_*(F_i)$$

   is an isomorphism.

Proof: Consider the global section functor $\mathbb{R}\Gamma : \text{Ho}(\mathsf{SPr}_*(\mathbb{C})) \longrightarrow \text{Ho}(\mathsf{SSet}_*)$. The fact that $H^i(\text{Spec } A, H) = 0$ for any $i > 0$ and any affine unipotent group scheme H implies that $\mathbb{R}\Gamma(K(H,n)) \simeq K(H(\mathbb{C}),n)$, for any unipotent affine group scheme H. Postnikov induction combined with [To2, Proposition 1.2.2] and Theorem 1.1.2 implies that for any pointed and connected affine stack F the natural morphism

$$\pi_i(\mathbb{R}\Gamma(F), *) \longrightarrow \pi_i(F, *)(\mathbb{C})$$

is always an isomorphism. In particular, $\mathbb{R}\Gamma$ is a conservative functor which commutes with taking homotopy groups. Recall also that the functor $H \mapsto H(\mathbb{C})$ is an exact and conservative functor from the category of sheaves of groups represented by affine group schemes to the category of groups.

Now, since each stack F_i is a pointed and connected affine stack, the spectral sequence $\{E_r^{p,q}(F_*)\}$ consists of affine unipotent group schemes. Therefore, the spectral sequence $\{E_r^{p,q}(F_*)\}(\mathbb{C})$ is the

spectral sequence of the tower of pointed simplicial sets $\{F_i(\mathbb{C})\}$ described in [G-J, VI §2]. The proposition now follows from [G-J, VI Corollary 2.22]. □

By construction the spectral sequence $\{E_r^{p,q}(F_*)\}$ is functorial in the tower $\{F_i\}$. Therefore, if a discrete group acts on the tower of pointed stacks $\{F_i\}$, then it also acts on the spectral sequence $\{E_r^{p,q}(F_*)\}$. Similarly, if the tower $\{F_i\}$ is defined over $\mathbb{R}$, then so is the spectral sequence $\{E_r^{p,q}(F_*)\}$.

3.5 Purity and degeneration of the weight spectral sequence

Let F be a pointed schematic homotopy type. The weight tower $W^{(*)}F^0$ gives rise to the *weight spectral sequence* $\{E_r^{p,q}(W^{(*)}F^0)\}$ of F. By construction this spectral sequence is functorial in F with respect to reductive morphisms.

Theorem 3.5.1 *Let (X, x) be a pointed complex projective manifold and $F = (X^{\mathrm{top}} \otimes \mathbb{C}, x)^{\mathrm{sch}}$ be its schematization. Then the weight spectral sequence $\{E_r^{p,q}(W^{(*)}F^0)\}$ of F degenerates at E_2 (i.e. $d_r = 0$ for all $r \geq 2$).*

Proof: The proof is based on the standard purity argument.

Consider a category $\mathcal{C}$ defined as follows. The objects of $\mathcal{C}$ are linearly compact $\mathbb{R}$-vector spaces V (i.e. abelian unipotent affine group schemes over $\mathrm{Spec}\ \mathbb{R}$) together with an action of the discrete group $\mathbb{C}^{\times\delta}$ on the linearly compact $\mathbb{C}$-vector space $V \otimes_{\mathbb{R}} \mathbb{C}$. The morphisms in $\mathcal{C}$ are morphisms of linearly compact $\mathbb{R}$-vector spaces whose base extension to $\mathbb{C}$ commutes with the action of $\mathbb{C}^{\times}$. The category $\mathbb{C}$ is clearly an abelian category. For an object V in $\mathcal{C}$, the action of an element $\lambda \in \mathbb{C}^{\times}$ on $v \in V \otimes_{\mathbb{R}} \mathbb{C}$ will be denoted by $\lambda(v)$, while $\lambda \cdot v$ will denote the scaling action of $\mathbb{C}^{\times}$ coming from the $\mathbb{C}$-vector space structure.

Definition 3.5.2 *An object $V \in \mathcal{C}$ is pure of weight $n \in \mathbb{Z}$ if for any $\lambda \in U(1) \subset \mathbb{C}^{\times}$ and any $v \in V \otimes_{\mathbb{R}} \mathcal{C}$ one has*

$$\lambda^n \cdot \overline{\lambda(v)} = \lambda(\overline{v}),$$

where $x \mapsto \overline{x}$ denotes complex conjugation.

Clearly, the full subcategory of objects in $\mathcal{C}$ which are pure of weight n is stable under passing to extensions, sub-objects and cokernels. Furthermore, it is clear that there are no non-zero morphisms between two objects in $\mathcal{C}$ which are pure of different weights. Finally, the (completed) tensor products of linearly compact vector spaces, turns $\mathcal{C}$ into a symmetric monoidal category, and the tensor product of two pure objects of weights p and q gives a pure object of weight $p + q$.

Going back to the proof of theorem 3.5.1, note that the Hodge decomposition on $(X^{\mathrm{top}} \otimes \mathbb{C}, x)^{\mathrm{sch}}$, induces an action of $\mathbb{C}^{\times\delta}$ on the tower $W^{(*)}F^0$ and hence on the weight spectral sequence $\{E_r^{p,q}(W^{(*)}F^0)\}$. Furthermore, the natural real structure $(X^{\mathrm{top}} \otimes \mathbb{R}, x)^{\mathrm{sch}}$ on the schematic homotopy type $(X^{\mathrm{top}} \otimes \mathbb{C}, x)^{\mathrm{sch}}$ gives rise to a natural real structure on the weight spectral sequence $\{E_r^{p,q}(W^{(*)}F^0)\}$ of F. This shows that the spectral sequence $\{E_r^{p,q}(W^{(*)}F^0)\}$ can be viewed as a spectral sequence in the abelian category $\mathcal{C}$ described above.

Now, in order to check that $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+r-1}$ vanish for all $r \geq 2$, it suffices to check that $E_r^{p,q}$, as an object of $\mathcal{C}$, is pure of weight $-q$. This, in turn, will follow if we can show that the object $E_1^{p,q} \in \mathcal{C}$ is pure of weight $-q$. By Proposition 3.2.6(5) it is enough to check that $H_q(F^0, \mathcal{O})$ is pure

of weight $-q$ (recall that $L_{p,q}$ can be identified with a sub-space of $H_{d_1}(F^0, \mathcal{O}) \otimes \cdots \otimes H_{d_q}(F^0, \mathcal{O})$ with $\sum d_j = p$).

Finally, $H_q(F^0, \mathcal{O})$ is the dual of the vector space $H^q(X, \mathcal{O}(G_X))$, where G_X is the pro-reductive completion of the group $\pi_1(X, x)$. As an *Ind*-local system on X, $\mathcal{O}(G_X)$ is isomorphic to $\oplus_L L^{\dim L}$, where the sum is over the set of isomorphism classes of simple $\mathbb{C}$-linear representations of $\pi_1(X, x)$. Using the non-abelian Hodge correspondence this *Ind*-local system correspond to the *Ind*-Higgs bundle $\oplus_L V^{\dim V}$ where V runs trough the set of isomorphism classes of stable Higgs bundles of degree 0. The space $H^q(X, \mathcal{O}(G_X))$ can therefore be described as

$$H^q(X, \mathcal{O}(G_X)) \simeq \bigoplus_{(V, \theta)} H^q(A^\bullet_{Dol}(V^{dim V}, D'')),$$

where the right hand side is the Dolbeault cohomology as recalled in Section 2.1. Now, if $(V, D'') = (V, \overline{\partial} + \theta)$ is a stable Higgs bundle of degree 0 corresponding to a local system L on X, the Higgs bundle $(\overline{V}, \overline{\partial} - \theta)$ corresponds to the local system $\overline{L}$ complex conjugate of L (see [S1]). Therefore, the complex conjugation acts on $H^q(X, \mathcal{O}(G_X))$ by sending a differential form $v \in A^{i,j}_{Dol}(V, \overline{\partial} + \theta)$ to the form $\overline{v} \in A^{j,i}_{Dol}(\overline{V}, \overline{\partial} - \theta)$. On the other hand, $\mathbb{C}^\times$ acts on $H^q(X, \mathcal{O}(G_X))$ by sending a differential form $v \in A^{i,j}_{Dol}(V, \overline{\partial} + \theta)$ to the form $\lambda^j \cdot v \in A^{i,j}_{Dol}(V, \overline{\partial} + \lambda \cdot \theta)$. Therefore, for $\lambda \in \mathbb{C}^\times$, one has

$$\lambda^q \cdot \overline{\lambda(v)} = \lambda(\overline{v}) \in A^{j,i}_{Dol}(\overline{V}, \overline{\partial} - \lambda \cdot \theta).$$

This shows that for any $v \in H^q(X, \mathcal{O}(G_X))$ and any $\lambda \in \mathbb{C}^\times$, one has $\lambda^q \cdot \overline{\lambda(v)} = \lambda(\overline{v})$. Dualizing, one gets that $H_q(F^0, \mathcal{O}) \simeq H^q(X, \mathcal{O}(G_X))^\vee$ is pure of weight $-q$. The theorem is proven. $\qquad\square$

Theorem 3.5.1 and lemma 3.4.1 have the following important corollary.

Corollary 3.5.3 *Let (X, x) be a pointed complex projective manifold. Let $F = (X^{\text{top}} \otimes \mathbb{C}, x)^{\text{sch}}$ be its schematization, $W^{(*)} F^0$ be the corresponding weight tower and set*

$$F^{(p)}_W \pi_q(F^0, *) := \text{Ker}\left(\pi_q(F^0, *) \longrightarrow \pi_q(W^{(p)} F^0, *)\right).$$

Then, one has

(1) $\pi_1(F^0, *) = \text{Ker}\left(\pi_1(F, *) \longrightarrow \pi_1(F, *)^{\text{red}}\right).$

(2) $\pi_q(F^0, *) \simeq \pi_q(F, *), \qquad \forall\, q > 1.$

(3) $\pi_q(F^0, *) \simeq \lim \pi_q(F^0, *)/F^{(p)}_W \pi_q(F^0, *).$

(4) $F^{(p-1)}_W \pi_q(F^0, *)/F^{(p)}_W \pi_q(F^0, *) \simeq E^{p,q+p}_\infty \simeq E^{p,q+p}_2(W^{(*)} F^0).$

Corollary 3.5.3 gives a concrete way of computing the homotopy groups of the schematization $(X^{\text{top}} \otimes \mathbb{C}, x)^{\text{sch}}$ of a projective manifold, while very little is known of these groups for a general

topological space X. Indeed, by Proposition 3.2.6 (5) the complex $(E_1^{p,q}, d_1)$ can be described explictly. For instance, the groups $E_1^{p,q}$ are given by

$$E_1^{p,q} \simeq L_{q-1,p} \subset \bigoplus_{d_1+\cdots+d_p=q-1} H_{d_1}(F^0, \mathcal{O}) \widehat{\otimes} \ldots \widehat{\otimes} H_{d_p}(F^0, \mathcal{O}).$$

The differential d_1 is given by the (co-)products listed in Proposition 3.2.6 (5), whereas the vanishing of d_2 can be interpreted as the vanishing of Massey (co-)products (see [Ar]).

4   Restrictions on homotopy types

In this section we will give some applications of the existence of a mixed Hodge structure on the schematization of a projective manifold.

4.1   An example

In this section we will use the existence of the Hodge decomposition on $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ in order to give examples of homotopy types which are not realizable as homotopy types of smooth projective varieties. These examples are obtained after defining new homotopy invariants of a space X in terms of the stack $(X \otimes \mathbb{C})^{\mathrm{sch}}$ and the action of $\pi_1(X, x)$ on the spaces $\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x)$. The existence of the Hodge decomposition implies strong restrictions on these invariants, and it is relatively easy to find explicit examples of homotopy types violating these restrictions.

   We should note also that our invariants are trivial as soon as one restricts to the case when the action of $\pi_1(X, x)$ on $\pi_i((X \otimes \mathbb{C})^{sch}, x)$ is nilpotent. Therefore, it appears that our examples can not be ruled out by using Hodge theory on rational homotopy types in the way it is done for example in [DGMS, Mo].

   Let us start with a pointed schematic homotopy type F such that $\pi_1(F, *)$ is an affine group scheme. According to [To2, Proposition 3.2.9] the group scheme $\pi_i(F, *)$ is an abelian unipotent affine group scheme for $i > 1$. Hence $\pi_i(F, *)$ is isomorphic to a (possibly infinite) product of $\mathbb{G}_a$'s.

   Consider now the maximal reductive quotient of the affine group scheme $\pi_1(F, *)$

$$\pi_1(F, *) \longrightarrow \pi_1(F, *)^{\mathrm{red}}.$$

Using the Levy decomposition, let us choose a section of this morphism $s : \pi_1(F, *)^{\mathrm{red}} \longrightarrow \pi_1(F, *)$. The morphism s allows us to consider the induced action of $\pi_1(F, *)^{\mathrm{red}}$ on $\pi_i(F, *)$. But, since $\pi_i(F, *)$ is a linearly compact vector space, and $\pi_1(F, *)$ is a reductive affine group scheme acting on it, there exists a decomposition of $\pi_i(F, *)$ as a (possibly infinite) product

$$\pi_i(F, *) \simeq \prod_{\rho \in R(\pi_1(F, *))} \pi_i(F, *)^{\rho},$$

where $R(\pi_1(F, *))$ is the set of isomorphism classes of finite dimensional simple linear representations of $\pi_1(F, *)$, and $\pi_i(F, *)^{\rho}$ is a product (possibly infinite) of representations in the class ρ. Using the fact that the Levy decomposition is unique up to an inner automorphism one can check that the set $\{\rho \in R(\pi_1(F, *)) \mid \pi_i(F, *)^{\rho} \neq 0\}$ is independent of the choice of the section s. With this notation we have the following:

Definition 4.1.1 *Let F be a pointed schematic homotopy type. The subset*

$$\operatorname{Supp}(\pi_i(F, *)) = \{\rho \in R(\pi_1(F, *)) \mid \pi_i(F, *)^\rho \neq 0\} \subset R(\pi_1(F, *))$$

is called the support of $\pi_i(F, *)$ *for every $i > 1$.*

Note that for a pointed and connected simplicial set in $\mathbb{U}$ the supports $\operatorname{Supp}(\pi_i((X \otimes \mathbb{C})^{\mathrm{sch}}, x))$ of X are homotopy invariants of X.

Suppose now that X^{top} is the underlying pointed space of a smooth projective algebraic variety over $\mathbb{C}$. Then, the stack $(X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}$ comes equipped with the Hodge decomposition defined in the previous section. The naturality of the construction of the Hodge decomposition implies the following:

Lemma 4.1.2 *For any $i > 1$, the subset $\operatorname{Supp}(\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x))$ is invariant under the $\mathbb{C}^\times$-action on $R(\pi_1((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)) \simeq R(\pi_1(X^{\mathrm{top}}, x))$.*

As a consequence of this lemma we get.

Corollary 4.1.3 *Let X be a pointed smooth projective complex algebraic variety.*

1. *If $\rho \in \operatorname{Supp}(\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)$ is an isolated point (for the induced topology of $R(\pi_1(X, x))$), then its corresponding local system on X underlies a polarizable complex variation of Hodge structure.*

2. *If $\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)$ is an affine group scheme of finite type, then each simple factor of the semi-simplification of the representation of $\pi_1(X, x)$ to the vector space $\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x)$ underlies a polarizable complex variation of Hodge structure on X.*

3. *Suppose that $\pi_1(X, x)$ is abelian. Then each isolated character $\chi \in \operatorname{Supp}(\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x))$ is unitary.*

Proof: Clearly Parts (2) and (3) of the corollary are direct consequences of (1). For the proof of part (1) note that the fact that the action of $\mathbb{C}^\times$ on the space $R(\pi_1(X, x))$ is continuous yields that $\rho \in \operatorname{Supp}(\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x))$ is fixed by the action of $\mathbb{C}^\times$. Part (1) of the corollary now follows from [S1, Corollary 4.2]. $\qquad\square$

The previous corollary shows that the existence of the Hodge decomposition imposes severe restrictions on the supports $\operatorname{Supp}(\pi_i((X^{\mathrm{top}} \otimes \mathbb{C})^{\mathrm{sch}}, x))$, and therefore on the homotopy type of X^{top}. The next theorem will provide a family of examples of homotopy types which are not realizable as homotopy typer of smooth projective varieties.

Before stating the theorem let us recall that the notion of a good group (see [To2, §3.4]). Recall that for a group Γ we denote by Γ^{alg} its pro-algebraic completion.

Definition 4.1.4 *A group Γ (in $\mathbb{U}$) is called algebraically good (relative to $\mathbb{C}$) if the natural morphism of pointed stacks*

$$(K(\Gamma, 1) \otimes \mathbb{C})^{\mathrm{sch}} \longrightarrow K(\Gamma^{\mathrm{alg}}, 1)$$

is an isomorphism.

Remark: An immediate but important remark is that a group Γ is algebraically good if and only if for any linear representation of finite dimension V of Γ^{alg} the natural morphism $\Gamma \longrightarrow \Gamma^{\mathrm{alg}}$ induces an isomorphism

$$H_H^\bullet(\Gamma^{\mathrm{alg}}, V) \simeq H^\bullet(\Gamma, V),$$

where $H_H^\bullet$ denotes Hochschild cohomology of the affine group scheme Γ^{alg}.

Besides finite groups, there are three known classes of examples of algebraically good groups (see [Ka-Pa-To, §4.5] for proofs and details):

- Any finitely generated abelian group is an algebraically good group.

- Any fundamental group of a compact Riemann surface is an algebraically good group.

- A successive extension of a finitely presented free groups is an algebraically good group.

In addition to the goodness property we will often need the following finiteness condition:

Definition 4.1.5 *Let Γ be an abstract group. We will say that the group Γ is of type ($\boldsymbol{F}$) (over $\mathbb{C}$) if:*

(a) For every n and every finite dimensional complex representation V of Γ the group $H^n(\Gamma, V)$ is finite dimensional;

(b) $H^\bullet(\Gamma, -)$ commutes with inductive limits of finite dimensional complex representations of Γ.

Remark 4.1.6 (i) The condition ($\boldsymbol{F}$) is a slight relaxation of a standard finiteness condition in combinatorial group theory. Recall [Br, Chapter VIII] that a group Γ is defined to be of type $\boldsymbol{FP}_\infty$ if the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a resolution by free $\mathbb{Z}\Gamma$-modules of finite type. It is well known, [Br, Section VIII.4] that if Γ is of type $\boldsymbol{FP}_\infty$, then all cohomologies of Γ with finite type coefficients are also of finite type and that the cohomology of Γ commutes with direct limits. In particular such a Γ will be of type ($\boldsymbol{F}$).

(ii) As usual, it is convenient to try and study the cohomology of a group Γ trough their topological incarnation as the cohomology of the classifying space of Γ. In particular, if $K(\Gamma, 1)$ admits a realization as a CW complex having only finitely many cells in each dimension, it is clear that both $\boldsymbol{FP}_\infty$ and ($\boldsymbol{F}$) hold for Γ

(iii) If Γ is an algebraically good group of type ($\boldsymbol{F}$), then for any linear representation V of Γ^{alg}, maybe of infinite dimension, one has

$$H^*(\Gamma, V) \simeq H^*(\Gamma^{\mathrm{alg}}, V).$$

Indeed, every linear representation of Γ^{alg} is the inductive limit of its finite dimensional sub-representations, and the Hochschild cohomology of Γ^{alg} always commutes with inductive limits.

In the next theorem we use the notion of a group of Hodge type, which can be found in e.g. [S1, §4].

Theorem 4.1.7 *Let $n > 1$ be an integer. Let Y be a pointed and connected simplicial set in $\mathbb{U}$ such that: $\pi_1(Y, y) = \Gamma$ is an algebraically good group of type $(\boldsymbol{F})$; $\pi_i(Y, y)$ is of finite type for any $1 < i < n$, and $\pi_i(Y, y) = 0$ for $i \geq n$.*

Let $\rho : \Gamma \longrightarrow Gl_m(\mathbb{Z})$ be an integral representation and let $\rho_{\mathbb{C}} : \Gamma \longrightarrow Gl_m(\mathbb{C})$ be the induced complex linear representation . Denote by $\rho_1, \ldots, \rho_r$ the simple factors of the semi-simplification of $\rho_{\mathbb{C}}$.

Let X be the homotopy type defined by the following homotopy cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ K(\Gamma, \mathbb{Z}^m, n) & \longrightarrow & K(\Gamma, 1). \end{array}$$

Suppose that there exists a smooth and projective complex algebraic variety X, such that the n-truncated homotopy types $\tau_{\leq n} X^{\text{top}}$ and $\tau_{\leq n} Z$ are equivalent, then the real Zariski closure of the image of each ρ_j is a group of Hodge type.

Proof: The theorem is based of the following lemma, describing the homotopy groups of $(Z \otimes \mathbb{C})^{\text{sch}}$.

Lemma 4.1.8 *([Ka-Pa-To, Prop. 4.14]) For any $i > 1$ there are isomorphisms of affine group schemes*

$$\pi_i((Z \otimes \mathbb{C})^{\text{sch}}, x) \simeq \pi_i(Z, x) \otimes \mathbb{G}_a.$$

Theorem 4.1.7 now follows from the previous lemma, Corollary 4.1.3 and [S1, Lemma 4.4]. □

As an immediate consequence we get:

Corollary 4.1.9 *If in theorem 4.1.7, the real Zariski closure of the image of one of the representations ρ_j is not a group of Hodge type, then Z is not the n-truncation of the homotopy type of a smooth and projective algebraic variety defined over $\mathbb{C}$.*

A list of examples of representations ρ satisfying the hypothesis of the previous corollary can be obtained using the list of [S1, §4]. Here are two explicit examples:

A. Let $\Gamma = \mathbb{Z}^{2g}$, and ρ be any reductive integral representation such that its complexification $\rho_{\mathbb{C}}$ is non-unitary. Then, one of the characters ρ_j is not unitary, which implies by [S1, 4.4.3] that the real Zariski closure of its image is not of Hodge type.

B. Let Γ be the fundamental group of compact Riemann surface of genus $g > 2$, and let $m > 2$. Consider any surjective morphism $\rho : \Gamma \longrightarrow Sl_m(\mathbb{Z}) \subset Gl_m(\mathbb{Z})$. Then the real Zariski closure of ρ is $Sl_n(\mathbb{R})$ which is not of Hodge type (see [S1, §4]).

4.2 The formality theorem

In this paragraph we will show that for a pointed smooth and projective manifold X the pointed schematic homotopy type $(X^{\text{top}} \otimes \mathbb{C})^{\text{sch}}$ is formal. This result is a generalization of the formality result of [DGMS], and could possibly be used to obtain other restrictions on homotopy types of projective manifolds. We leave this question for the future, and restrict ourselves to present a proof of the formality theorem.

To state the main definition of this paragraph, let us recall that for any affine group scheme G, and any G-equivariant commutative differential graded algebra A, or any G-equivariant co-simplicial algebra A, one has the cohomology algebra $H^{\bullet}(A)$. This algebra is in a natural way a graded algebra with an action of G, and therefore can be considered as a G-equivariant commutative differential graded algebra with trivial differential.

Definition 4.2.1 *Let G be an affine group scheme.*

- *A G-equivariant commutative differential graded algebra A is G-formal if it is isomorphic to $H^{\bullet}(A)$ in $\text{Ho}(G\text{-}\mathsf{CDGA})$.*

- *A G-equivariant co-simplicial algebra A is G-formal if it is isomorphic in $\text{Ho}(G\text{-}\mathsf{Alg}^{\Delta})$ to the denormalization $D(H^{\bullet}(A))$.*

- *A pointed schematic homotopy type F is formal, if it is isomorphic in $\text{Ho}(\mathsf{SPr}(\mathbb{C}))$ to an object of the form $[\mathbb{R}\operatorname{Spec}_G A/G]$, whith A a G-formal G-equivariant co-simplicial algebra, and G an affine reductive group scheme.*

Let X be a pointed and connected simplicial set in $\mathbb{U}$, and $G_X := \pi_1(X,x)^{\text{red}}$ be the pro-reductive completion of its fundamental group. One can consider the G_X-equivariant commutative differential graded algebra $H^{\bullet}(C^{\bullet}(X, \mathcal{O}(G_X)))$, of cohomology of X with coefficients in the local system of algebras $\mathcal{O}(G_X)$. As this cohomology algebra is such that $H^0(C^{\bullet}(X, \mathcal{O}(G_X))) \simeq \mathbb{C}$, Proposition 1.3.2 implies that $[\mathbb{R}\operatorname{Spec}_{G_X} H^{\bullet}(X, \mathcal{O}(G_X))/G_X]$ is a pointed schematic homotopy type.

Definition 4.2.2 *The formal schematization of the pointed and connected simplicial set X is the pointed schematic homotopy type*

$$(X \otimes \mathbb{C})^{\text{for}} := [\mathbb{R}\operatorname{Spec}_{G_X} H^{\bullet}(X, \mathcal{O}(G_X))/G_X] \in \text{Ho}(\mathsf{SPr}_*(\mathbb{C})).$$

The main theorem of this section is the following formality statement, which in particular answers Problem 2 in [Go-Ha-Ta, §7].

Theorem 4.2.3 *Let X be a pointed smooth and projective complex manifold and X^{top} its underlying topological space. Then, there exist an isomorphism in $\text{Ho}(\mathsf{SPr}_*(\mathbb{C}))$, functorial in X*

$$(X^{\text{top}} \otimes \mathbb{C})^{\text{sch}} \simeq (X^{\text{top}} \otimes \mathbb{C})^{\text{for}}.$$

Proof: By using Proposition 2.2.2, it is enough to produce a functorial isomorphism in $\text{Ho}(G_X\text{-}\mathsf{CDGA})$

$$(A^{\bullet}_{DR}(\mathcal{O}(G_X), \nabla) \simeq H^{\bullet}_{DR}(X, \mathcal{O}(G_X)),$$

where $H^\bullet_{DR}(X, \mathcal{O}(G_X)) \in \mathrm{Ho}(G_X\text{-}\mathsf{CDGA})$ is the cohomology algebra of $(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla)$.

But, writing $\mathcal{O}(G_X)$ as an Ind-object $(V, \nabla) \in T_{DR}(X)$, corresponding to $(V, \nabla, D'') \in T_{D'}(X)$, and applying Corollary 2.1.3 one obtains a diagram of quasi-isomorphisms

$$(A^\bullet_{DR}(\mathcal{O}(G_X)), D) \longleftarrow (A^\bullet_{D'}(V), D'') \longrightarrow (H^\bullet_{DR}(X, \mathcal{O}(G_X)), 0)).$$

By the compatibility of the functor $(V, \nabla) \mapsto (A^\bullet_{D'}(V), D'')$ with the tensor products, this diagram is actually a diagram of G_X-equivariant commutative differential graded algebras. Passing to the homotopy category, one obtains well defined and functorial isomorphisms in $\mathrm{Ho}(G_X\text{-}\mathsf{CDGA})$

$$(A^\bullet_{DR}(\mathcal{O}(G_X)), \nabla) \simeq H^\bullet_{DR}(X, \mathcal{O}(G_X)).$$

The theorem is proven. $\qquad \square$

Remark 4.2.4 The above theorem implies in particular that for a smooth projective variety X the algebraic-geometric invariants we have considered in this paper are captured not only by the cochain dg algebra by also by the corresponding graded algebra of cohomology. More precisely, if G denotes the pro-reductive completion of $\pi_1(X)$, and $\mathcal{O}(G)$ denotes the algebra of functions on G viewed as a local system on X, then the schematization $(X \otimes \mathbb{C})^{\mathrm{sch}}$ and in particular the support invariants of X are all determined by the G-equivariant graded algebra of cohomology $H^\bullet(X, \mathcal{O}(G))$. We do not know if, as for the rational homotopy type, the $\mathbb{C}^\times$ action on $(X \otimes \mathbb{C})^{\mathrm{sch}}$ is also determined by the its action on the group G and the G-equivariant graded algebra $H^\bullet(X, \mathcal{O}(G))$.

The functoriality assumption in theorem 4.2.3 has the following striking consequence.

Corollary 4.2.5 *Let $f, g : (X, x) \longrightarrow (Y, y)$ two morphisms between pointed smooth and projective complex manifolds. Suppose that*

1. *The induced morphisms $f, g : \pi_1(X, x)^{\mathrm{red}} \longrightarrow \pi_1(Y, y)^{\mathrm{red}}$ are equal.*

2. *For any simple local system L on Y the induces morphisms*

$$f, g : H^*(Y, L) \longrightarrow H^*(X, f^*(L) = g^*(L))$$

   are equal.

Then the two morphisms of pointed stacks $f, g : (X^{\mathrm{top}} \otimes \mathbb{C}, x)^{\mathrm{sch}} \longrightarrow (Y^{\mathrm{top}} \otimes \mathbb{C}, y)^{\mathrm{sch}}$ are equal as morphisms in $\mathrm{Ho}(\mathsf{Spr}_*(\mathbb{C}))$.

4.3 Hurewitz maps

In this section we use the Hodge decomposition on the schematic homotopy type of a smooth projective X in order to show that under certain conditions the image of the Hurewitz map

$$\pi_n(X) \to H_n(X, \mathbb{Z})$$

is a sub Hodge structure. For $n = 2$ this answers a question of P.Eyssidieux.

Theorem 4.3.1 *Suppose X is smooth and projective over $\mathbb{C}$ satisfying*

  (a) $\pi_1(X)$ is good,

  (b) $\pi_i(X)$ is finitely generated for $1 < i < n$.

Then $\mathrm{Im}[\pi_n(X) \to H_n(X, \mathbb{Z})]$ is a Hodge substructure.

Proof. The natural map $\mathsf{s} : X \to (X \otimes \mathbb{C})^{\mathrm{sch}}$ induces a commutative diagram

$$(4.3.1)$$

$$
\begin{array}{ccc}
\pi_n(X) \otimes \mathbb{C} & \xrightarrow{\ \mathsf{s}\ } & \pi_n((X \otimes \mathbb{C})^{\mathrm{sch}}) \\
{\scriptstyle \mathrm{Hu}}\Big\downarrow & \searrow{\scriptstyle \psi} & \Big\downarrow{\scriptstyle \mathrm{Hu}^{sch}} \\
H_n(X, \mathbb{C}) & \xrightarrow[\ \cong\ ]{} & H_n((X \otimes \mathbb{C})^{\mathrm{sch}}, \mathcal{O})
\end{array}
$$

Since the schematic Hurewitz map is functorial for morphisms in the homotopy category of simplicial presheaves and since $(X \otimes \mathbb{C})^{\mathrm{sch}}$ can be considered as $\mathbb{C}^\times$-equivariant object in this category, it follows that the image of $\mathrm{Hu}^{\mathrm{sch}}$ is preserved under the $\mathbb{C}^\times$ action on $H_n(X, \mathbb{C})$. Therefore it suffices to show that the maps $\mathrm{Hu}^{\mathrm{sch}}$ and ψ have the same image.

    By the commutativity of (4.3.1) we know that $\mathrm{im}(\psi) \subset \mathrm{im}(\mathrm{Hu}^{\mathrm{sch}})$ and so we will be done if we can show the opposite inclusion $\mathrm{im}(\psi) \supset \mathrm{im}(\mathrm{Hu}^{\mathrm{sch}})$. For this we will need the following

Lemma 4.3.2 *Let X be a smooth complex projective variety satisfying conditions (a) and (b). For every complex finite dimensional representation V of $\pi_1(X)$, the morphism s induces a bijection*

$$\mathsf{s}^* : \mathrm{Hom}^{\mathrm{cont}}_{\pi_1(X)}(\pi_n((X \otimes \mathbb{C})^{\mathrm{sch}}), V) \widetilde{\to} \mathrm{Hom}_{\pi_1(X)}(\pi_n(X), V).$$

where $\mathrm{Hom}_{\pi_1(X)}(\pi_n(X), V)$ denotes the space of $\pi_1(X)$-equivariant morphisms between the abelian groups $\pi_n(X)$ and V, and $\mathrm{Hom}^{\mathrm{cont}}_{\pi_1(X)}(\pi_n((X \otimes \mathbb{C})^{\mathrm{sch}}), V)$ denotes the space of $\pi_1(X)$-equivariant cotinuous morphisms between the linearly compact vector spaces $\pi_n((X \otimes \mathbb{C})^{\mathrm{sch}})$ and V.

Proof. Let $\tau_{\leq n-1} : X \to X_{\leq n-1}$ be the Postnikov truncation. Consider the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \tau_{\leq n-1}\ } & X_{\leq n-1} \\
{\scriptstyle \mathsf{s}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathsf{s}} \\
(X \otimes \mathbb{C})^{\mathrm{sch}} & \xrightarrow{\ \tau_{\leq n-1}\ } & ((X \otimes \mathbb{C})^{\mathrm{sch}})_{\leq n-1}
\end{array}
$$

The vertical maps in this diagram induce a morphism between the five term long exact sequences coming from to the Leray spectral sequence of the truncation maps:

$$
\begin{array}{ccccc}
H^n(((X \otimes \mathbb{C})^{\mathrm{sch}})_{\leq n-1}, V) & \longrightarrow & H^n((X \otimes \mathbb{C})^{\mathrm{sch}}, V) & \longrightarrow & \mathrm{Hom}^{\mathrm{cont}}_{\pi_1(X)}(\pi_n((X \otimes \mathbb{C})^{\mathrm{sch}}), V) \to \\
\Big\downarrow{\scriptstyle (1)} & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \mathsf{s}^*} \\
H^n(X_{\leq n-1}, V) & \longrightarrow & H^n(X, V) & \longrightarrow & \mathrm{Hom}_{\pi_1(X)}(\pi_n(X, V) \longrightarrow
\end{array}
$$

$$
\begin{array}{ccccc}
\longrightarrow & H^{n+1}((X \otimes \mathbb{C})^{\mathrm{sch}})_{\leq n-1}, V) & \longrightarrow & H^{n+1}((X \otimes \mathbb{C})^{\mathrm{sch}}, V) \\
& \Big\downarrow{\scriptstyle (2)} & & \Big\downarrow{\scriptstyle \cong} \\
\longrightarrow & H^{n+1}(X_{\leq n-1}, V) & \longrightarrow & H^{n+1}(X, V)
\end{array}
$$

To show that s^* is an isomorphism, it suffices to show that (1) and (2) are isomorphisms. This will follow if we know that the natural map $X_{\leq n-1} \to ((X \otimes \mathbb{C})^{\mathrm{sch}})_{\leq n-1}$ induces an isomorphism between the schematic homotopy types $(X_{\leq n-1} \otimes \mathbb{C})^{\mathrm{sch}}$ and $((X \otimes \mathbb{C})^{\mathrm{sch}})_{\leq n-1}$. This is equivalent to showing that the stack $(X_{\leq n-1} \otimes \mathbb{C})^{\mathrm{sch}}$ is $(n-1)$-truncated, which in turn follows from the hypothesis (a) and (b) and [Ka-Pa-To, Proposition 4.21] applied to the moprphism $X_{\leq n-1} \to X_{\leq 1}$. The lemma is proven. $\square$

To finish the proof of the theorem take $V = H_n(X, \mathbb{C})/\operatorname{im}(\psi)$. Now the composition

$$\pi_n(X) \otimes \mathbb{C} \xrightarrow{\mathsf{s}} \pi_n((X \otimes \mathbb{C})^{\mathrm{sch}}) \xrightarrow{\mathrm{Hu}^{\mathrm{sch}}} H_n(X, \mathbb{C}) \longrightarrow V$$

is zero by construction, the lemma implies that the composition

$$\pi_n((X \otimes \mathbb{C})^{\mathrm{sch}}) \xrightarrow{\mathrm{Hu}^{\mathrm{sch}}} H_n(X, \mathbb{C}) \longrightarrow V$$

is zero as well. In particular $\operatorname{im}(\mathrm{Hu}^{\mathrm{sch}}) \subset \operatorname{im}(\psi)$ which completes the proof of the theorem. $\square$

Remark 4.3.3 It is not hard to construct interesting examples of varieties X satisfying the hypothesis of Theorem 4.3.1. For instance we can start with a variety Z which is a $K(\pi, 1)$ with good fundamental group, e.g. Z can be a product of curves or and abelian variety, and take X to be a smooth hyperplane section of dimension n.

Remark 4.3.4 Note that the lemma implies that $\pi_n((X \otimes \mathbb{C})^{\mathrm{sch}})$ is the $\pi_1(X)$-equivariant unipotent completion of $\pi_n(X)$. In particular, the image of $\pi_n(X)$ in $\pi_n((X \otimes \mathbb{C})^{\mathrm{sch}})$ is Zariski dense.

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L. Katzarkov: UCI, Irvine, CA 92612, e-mail: lkatzark@math.uci.edu

T. Pantev: University of Pennsylvania, Philadelphia, PA 19104, e-mail: tpantev@math.upenn.edu

B. Toen: Laboratoire Emile Picard, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France, e-mail: toen@picard.ups-tlse.fr