Trace formula for dg-categories and Bloch's conductor conjecture I

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Abstract

We present an ℓ-adic trace formula for smooth and proper admissible dg-categories over a base monoidal dg-category. As an application, we prove (a version of) Bloch's conductor conjecture ([Bl, Conjecture, p. 423]) under the additional hypothesis that the inertia group acts with unipotent monodromy.

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1 Introduction

In his seminal paper [Bl] Bloch introduced the so-called Bloch's intersection number $[\Delta_X, \Delta_X]_S$ for a proper regular scheme $X \longrightarrow S$ over a henselian trait S. This number can be seen as the degree of the localized top Chern class of the coherent sheaf $\Omega^1_{X/S}$ and measures the relative singularities of X over S. In the same paper Bloch introduced the famous conductor formula, which can be seen as a computation of the Bloch's intersection number in termes of the arithmetic geometry of X/S. It reads as follows.

Conjecture 1.0.1 *We have*

$$[\Delta_X, \Delta_X]_S = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell) - Sw(X_{\bar{K}}),$$

where $X_{\bar{k}}$ and $X_{\bar{K}}$ denotes the special and generic geometric fibers of X over S, $\chi(-, \ell)$ denotes ℓ-adic Euler characteristics and $Sw(-)$ is the Swan conductor.

In [Bl] the above formula is proven in relative dimension 1. Further results implying special cases of the above has been obtained since then (see section §5 for a more precise discussion about the known cases), the most recent one being a proof in the geometric case (see [Sai]) . In the mixed characteristic case, the conjecture is open in general oustide the cases covered by [Ka-Sa]. In particular, for isolated singularities the above conjecture already appeared in Deligne's exposé [SGA7-I, Exp. XVI], and remains open.

The purpose of the present paper is to make a first step towards a new comprehension of the Bloch's conductor formula but bringing ideas from non-commutative and derived algebraic geometry. Before explaining briefly these ideas let us mention the main result of the present work. We start by definition another Bloch's intersection number, called the *categorical Bloch's intersection number* (see definition 5.0.2) and denoted by $[\Delta_X, \Delta_X]^{cat}_S$. It is defined as an intersection number in the setting of non-commutative algebraic geometry, or more precisely as the Euler characteristic of the Hochschild complex of the (dg-)category of singularities of the special fiber X_k. The precise comparison with the original Bloch's number is not covered in this work and will appear in a forthcoming paper. Having introduced this number we prove the following theorem.

Theorem 1.0.2 *Assume that the monodromy action on $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$ is unipotent, then we have*

$$[\Delta_X, \Delta_X]^{cat}_S = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell).$$

The above theorem covers new cases which are not covered by [Ka-Sa, Sai], as neither we assume that S is of equicharacteristic nor that X is semi-stable over S. The unipotency condition is of course restrictive, but we also present in our last sections ideas of how to deal with the general case.

Before finishing this introduction we would like to explain the main ideas leading to theorem 1.0.2 as we feel that they are new to the subject and can be useful for other questions of geometrico-algebraic nature. The key idea behind theorem 1.0.2 is that it is a direct consequence of a trace formula for dg-categories

(a.k.a. non-commutative schemes). The general philosophy is explained in the authors survey [To-Ve]: briefly, a scheme X over S provides a function π on X, simply by pulling-back a uniformizer on S. This function can in turn be used to produce the non-commutative scheme $MF(X/S)$ of matrix factorizations on X. According to [BRTV] any non-commutative scheme has an ℓ-cohomology theory, and the ℓ-cohomology of $MF(X/S)$ is the inertia invariant part of vanishing cohomology. The theorem 1.0.2 is then a direct consequence of the trace formula for non-commutative schemes announced in [To-Ve] and proved in this work.

We want to add here that the trace formula for dg-categories (or non-commutative schemes) is a rather formal statement which essentially consists of a formal use of symmetric monoidal ∞-categories. However, the fact that the dg-category $MF(X/S)$ is nice enough for the trace formula to make sense consists of several non-trivial statements. The first result is that $MF(X/S)$ is not only a dg-category but comes equiped with an action of some monoidal category $\mathcal{B}$ (see section §4.1). This monoidal category is obtained as a convolution dg-category of a derived groupoid and is not an object that can be described by classical algebraic geometry. The second statement is that, when considered over $\mathcal{B}$, $MF(X/S)$ is a smooth and proper dg-category. This is, in our opinion, a deep and surprising result, which in characteristic zero is a well known fact (see for instance [Pr]). The smooth and proper character of $MF(X/S)$ insures for instance that our categorical Bloch's intersection number $[\Delta_X, \Delta_X]_S^{cat}$ is well defined (see definition 5.0.2). Finally, a third result which is of independant interset, is a Kunneth formula for inertia invariant vanishing cohomology (see Prop. 3.4.2). This formula is close to the Thom-Sebastanni theorem of [Il2], without being equivalent. It sounds new to us particularly in the mixed characteristic situation. It has many important consequences for us, one of them being that $MF(X/S)$ is *admissible* to our trace formula when the monodromy is unipotent (admissiblity is a technical condition that must be checked case by case). Another consequence is that it paves the way to the general form of the Bloch's conductor formula, as this explained in our section §6 and will be investigated in a future work.

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Notations. [

- Throughout the text, A will denote a (discrete) commutative noetherian ring. When needed, A will be required to satisfy further properties (such as being excellent, local and henselian) that will be made precise in due course.
- $L(A)$ will denote the A-linear dg-category of (fibrant and) cofibrant complexes localized with respect to quasi-isomorphisms.
- $\mathbf{dgCat}_A$ will denote the Morita ∞-category of dg-categories over A (see §2.1).
- **Top** denotes the ∞-category of spaces (obtained, e.g. as the coherent nerve of the Dwyer-Kan localization of the category of simplicial sets along weak homotopy equivalences). **Sp** denotes the ∞-category of spectra.

2 A non-commutative trace formalism

2.1 ∞-Categories of dg-categories

We denote by A a commutative ring. We remind here some basic facts about the ∞-category of dg-categories, its monoidal structure and its theory of monoids and modules.

We consider the category $dgCat_A$ of small A-linear dg-categories and A-linear dg-functors. We remind that an A-linear dg-functor $T \longrightarrow T'$ is a Morita equivalence if the induced functor of the corresponding derived categories of dg-modules $f^* : D(T') \longrightarrow D(T)$ is an equivalence of categories (see [To1] for details). The ∞-category of dg-categories over S is defined to be the localisation of $dgCat_A$ along these Morita equivalences, and will be denoted by $\mathbf{dgCat}_S$ or $\mathbf{dgCat}_A$. Being the ∞-category associated to a model category, $\mathbf{dgCat}_A$ is a presentable ∞-category. As in [To1, § 4], the tensor product of A-linear dg-categories can be derived to a symmetric monoidal structure on the ∞-category $\mathbf{dgCat}_A$. This symmetric monoidal structure moreover distributes over colimits making $\mathbf{dgCat}_A$ into a presentable symmetric monoidal ∞-category. We have a notion of rigid, or dualizable, objects in $\mathbf{dgCat}_A$. It is a well known fact that dualizable objects in $\mathbf{dgCat}_S$ are precisely smooth and proper dg-categories over A (see [To2, Prop. 2.5]).

The compact objects in $\mathbf{dgCat}_S$ are the dg-categories of finite type over A in the sense of [To-Va]. We denote their full sub-∞-category by $\mathbf{dgCat}_S^{ft} \subset \mathbf{dgCat}_S$. The full sub-category $\mathbf{dgCat}_S^{ft}$ is preserved by the monoidal structure, and moreover any dg-category is a filtered colimit of dg-categories of finite type. We thus have a natural equivalence of symmetric monoidal ∞-categories

$$\mathbf{dgCat}_S \simeq \mathbf{Ind}(\mathbf{dgCat}_S^{ft}).$$

We will from time to time have to work in a bigger ∞-category, denoted by $\widetilde{\mathbf{dgCat}}_A$, which contains $\mathbf{dgCat}_A$ as a non-full sub-∞-category. By [To2] we have a symmetric monoidal ∞-category $\mathbf{dgCat}_A^{lp}$ of presentable dg-categories over A. We define $\widetilde{\mathbf{dgCat}}_A$ has being the full sub-∞-category consisting of all compactly generated dg-categories. The ∞-category $\mathbf{dgCat}_A$ can be identified with the non-full sub-∞-category of $\widetilde{\mathbf{dgCat}}_A$ which consists of compact objects preserving dg-functors. This provides a faithful embedding of symmetric monoidal ∞-categories

$$\mathbf{dgCat}_A \hookrightarrow \widetilde{\mathbf{dgCat}}_A.$$

On the level of objects this embedding sends a small dg-category T to the compactly generated dg-category $\hat{T}$ of dg-modules over T^o. An equivalent description of $\widetilde{\mathbf{dgCat}}_A$ is as the ∞-category of small dg-categories together with the mapping spaces given by the classifying space of all bi-dgmodules between small dg-categories.

Definition 2.1.1 *A* monoidal A-dg-category *is a unital and associative monoid in the symmetric monoidal ∞-category* $\mathbf{dgCat}_A$*. A* module over a monoidal A-dg-category B *will, by definition, mean a left B-module in* $\mathbf{dgCat}_A$ *in the sense of [Lu-HA].*

The ∞-category of left B-modules will be denoted by $\mathbf{dgCat}_B$. For such a B-module T, we have a morphism $\mu : B \otimes_A T \to T$ in $\mathbf{dgCat}_A$, that will be simply denoted by $(b, x) \mapsto r \otimes x$. For a weak-monoidal

A-dg-category B, we will denote by $B^{\otimes\text{-op}}$ the monoidal A-dg-category where the monoid structure is the opposite to the one of B, i.e. $b \otimes^{op} b' := b' \otimes b$. Note that $B^{\otimes\text{-op}}$ should not be confused with B^o (which is still a monoidal A-dg-category), where the "arrows" and not the monoid structure have been reversed, i.e. $B^o(b, b') := B(b', b)$. By definition, a *right* B-module is a (left) $B^{\otimes\text{-op}}$-module. The ∞-category of right B-module is simply denoted by $\mathbf{dgCat}_{B^{\otimes\text{-op}}}$ or $\mathbf{dgCat}^B$. If B is a monoidal A-dg-category, then $B^{\otimes\text{-op}} \otimes_A B$ is again a monoidal A-dg-category, and B can be considered either as left $B^{\otimes\text{-op}} \otimes_A B$ (denoted by B^L) or as a right $B^{\otimes\text{-op}} \otimes_A B$-module (denoted by B^R). For T a B-module, and T' a right B-module, then $T' \otimes_A T$ is naturally a right $B^{\otimes\text{-op}} \otimes_A B$-module, and we define

$$T' \otimes_B T := (T' \otimes_A T) \otimes_{B^{\otimes\text{-op}} \otimes_A B} B^L$$

which is an object in $\mathbf{dgCat}_A$.

Let B be a monoidal A-dg-category. We can consider the symmetric monoidal embedding $\mathbf{dgCat}_A \hookrightarrow \widetilde{\mathbf{dgCat}}_A$ and thus $\widehat{B}$ as a monoid in $\widetilde{\mathbf{dgCat}}_A$. The ∞-category of $\widehat{B}$-modules in $\widetilde{\mathbf{dgCat}}_A$ is denoted by $\widetilde{\mathbf{dgCat}}_B$, and its objects are called *big B-modules*. The natural ∞-functor $\mathbf{dgCat}_B \longrightarrow \widetilde{\mathbf{dgCat}}_B$ is faithful and its image consists of all big B-modules $\widehat{T}$ such that the morphism $\widehat{B} \hat{\otimes}_A \widehat{T} \longrightarrow \widehat{T}$ is a small morphism.

It is known that the symmetric monoidal ∞-category $\widetilde{\mathbf{dgCat}}_A$ is rigid (see [To2]), and that for any $\widehat{T}$ its dual is given by $\widehat{T^o}$, and the evaluation and coevaluation morphisms are defined by T considered as $T^o \otimes_a T$-module. This formally implies that if $\widehat{T}$ is a big B-module, then its dual $\widehat{T^o}$ is naturally a *right* big B-module. We thus have two big morphisms

$$\mu : \widehat{B} \hat{\otimes}_A \widehat{T} \longrightarrow \widehat{T} \quad \mu^o : \widehat{T^o} \hat{\otimes}_A \widehat{B} \longrightarrow \widehat{T^o}.$$

By duality these morphisms also provides a third big morphism

$$h : \widehat{T^o} \widehat{\otimes}_A \widehat{T} \longrightarrow \widehat{B}.$$

This last big morphism h is obtained by duality from

$$\mu^* : \widehat{T} \longrightarrow \widehat{B} \hat{\otimes}_A \widehat{T}$$

the right adjoint to μ. We now make the following definitions.

Definition 2.1.2 *Let B be a monoidal dg-category and T a B-module. We say that*

1. *T is* cotensored *(over B) if the big morphism μ^o defined above is a small morphism*
2. *T is* proper *(or* enriched*) (over B) if the big morphism h defined above is a small morphism.*

We can make the above definition more explicit in the following manner. Let B and T be as above. For two objects $b \in B$ and $x \in T$, we can define consider the dg-functor

$$x^b : T^o \longrightarrow L(A)$$

sending $y \in T$ to $T(\mu(b, y), x)$, where $\mu : B \otimes_A T \longrightarrow T$ is the B-module structure on T. Then, T is cotensored over B if and only if for all b and x the above dg-module $T^o \longrightarrow L(A)$ is compact in the derived

category $D(T^o)$ of all T^o-dg-modules. When T is assumed triangulated then this is equivalent to ask for the dg-module to be representable by an object $x^b \in T$. In a similar manner, we can phrase the enriched condition by requiring the for any $x \in T$ and $y \in T$, the dg-module

$$B^o \longrightarrow L(A)$$

sending b to $T(\mu(b,x),y)$ to be compact (or representable if B is already assumed to be triangulated).

An important comment: by definition, when T is cotensored, the big right B-module $\widehat{T^o}$ is in fact a small right B-module. To put things differently, when T is cotensored then T^o is naturally a right B-module in $\mathbf{dgCat}_A$. This right module structure is the morphism in $\mathbf{dgCat}_A$

$$\mu^o : T^o \otimes_A B \longrightarrow T^o$$

which sends $(x,b) \in T^o \otimes_A B$ to the cotensor $x^b \in T^o$.

We finish this part with some facts about existence of tensor products of modules over monoidal dg-categories. As a general fact, $\mathbf{dgCat}_A$ is a presentable symmetric monoidal ∞-category, and as such for any monoidal dg-category B there exists a tensor product ∞-functor

$$\otimes_B : \mathbf{dgCat}_B \times \mathbf{dgCat}^B \longrightarrow \mathbf{dgCat}_A,$$

sending a left B-module T and a right B-module T' to $T \otimes_B T'$ (see [Lu-HA]). Assume furthermore that T is a B-module which is also cotensored in the sense of definition 2.1.2, we thus have that T^o is a right B-module. In this case we can form

$$T^o \otimes_B T \in \mathbf{dgCat}_A.$$

When T is not cotensored, the object $T \otimes_B T^o$ does not exist anymore. However, we can always consider the presentable dg-categories $\widehat{T}$ and $\widehat{T^o}$ as left and righht modules over $\widehat{B}$. Their tensor product $\widehat{T^o}\widehat{\otimes}_{\widehat{B}}\widehat{T}$ now only make sense as a presentable dg-category which has no reason to be compactly generated. Of course, when T is cotensored, this presentable dg-category is compactly generated and we have

$$\widehat{T^o \otimes_B T} \simeq \widehat{T^o}\widehat{\otimes}_{\widehat{B}}\widehat{T}.$$

To finish, we pass the following easy but useful observation. Let B be a monoidal dg-category, and assume that B is generated, as triangulated dg-category, but its unit object $1 \in B$. Then, any big B-module is small, and also cotensored.

2.2 The ℓ-adic realization of dg-categories

We denote by $\mathcal{SH}_S$ the stable $\mathbb{A}^1$-homotopy ∞-category of schemes over S (see [Vo, Def. 5.7] and [Ro, § 2]). It is a presentable symmetric monoidal ∞-category whose monoidal structure will be denoted by $\wedge_S$. Homotopy invariant algebraic K-theory defines an $\mathbb{E}_\infty$-ring object in $\mathcal{SH}_S$ that we denote by $\mathbf{B}\mathbb{U}_S$ (a more standard notation is KGL). We denote by $\mathbf{B}\mathbb{U}_S - \mathbf{Mod}$ the ∞-category of $\mathbf{B}\mathbb{U}_S$-modules in $\mathcal{SH}_S$. It is a presentable symmetric monoidal ∞-category whose monoidal structure will be denoted by $\wedge_{\mathbf{B}\mathbb{U}_S}$.

As proved in [BRTV], there exists a lax symmetric monoidal ∞-functor

$$\mathsf{M}^- : \mathbf{dgCat}_S \longrightarrow \mathbf{B}\mathbb{U}_S - \mathbf{Mod},$$

which is denoted by $T \mapsto \mathsf{M}^T$. The precise construction of the ∞-functor M^- is rather involved and uses in an essential manner the theory of non-commutative motives of [Ro] as well as the comparison with the stable homotopy theory of schemes. Intuitively, the ∞-functor M^- sends a dg-category T to the homotopy invariant K-theory functor $S' \mapsto \mathsf{HK}(S' \otimes_S T)$. To be more precise, there is an obvious forgetful ∞-functor

$$\mathsf{U} : \mathbf{B}\mathbb{U}_S - \mathbf{Mod} \longrightarrow \mathbf{Fun}^{\infty}(Sm_S^{op}, \mathbf{Sp}),$$

to the ∞-category of presheaves of spectra on the category Sm_S of smooth S-schemes. For a given dg-category T over S, the presheaf $\mathsf{U}(\mathsf{M}^T)$ is defined by sending a smooth S-scheme $S' = \mathbf{Spec}\, A' \to \mathbf{Spec}\, A = S$ to $\mathsf{HK}(A' \otimes_A T)$, the homotopy invariant non-connective K-theory spectrum of $A' \otimes_A T$ (see [Ro, 4.2.3]).

The ∞-functor M^- satisfies some basic properties which we recall here.

1. The ∞-functor M^- is a localizing invariant, i.e. for any short exact sequence $T_0 \hookrightarrow T \longrightarrow T/T_0$ of dg-categories over A, the induced sequence

$$\mathsf{M}^{T_0} \longrightarrow \mathsf{M}^T \longrightarrow \mathsf{M}^{T/T_0}$$

exhibits M^{T_0} has the fiber of the morphism $\mathsf{M}^T \to \mathsf{M}^{T/T_0}$ in $\mathbf{B}\mathbb{U}_S - \mathbf{Mod}$.

2. The natural morphism $\mathbf{B}\mathbb{U}_S \longrightarrow \mathsf{M}^A$, induced by the lax monoidal structure of M^-, is an equivalence of $\mathbf{B}\mathbb{U}_S$-modules.

3. The ∞-functor $T \mapsto \mathsf{M}^T$ commutes with filtered colimits.

4. For any quasi-compact and quasi-separated scheme X, and any morphism $p : X \longrightarrow S$, we have a natural equivalence of $\mathbf{B}\mathbb{U}_S$-modules

$$\mathsf{M}^{\mathsf{L}_{\mathrm{Perf}}(X)} \simeq p_*(\mathbf{B}\mathbb{U}_X),$$

where $p_* : \mathbf{B}\mathbb{U}_X - \mathbf{Mod} \longrightarrow \mathbf{B}\mathbb{U}_S - \mathbf{Mod}$ is the direct image of $\mathbf{B}\mathbb{U}$-modules, and $\mathsf{L}_{\mathrm{Perf}}(X)$ is the dg-category of perfect complexes on X.

We now let ℓ be a prime number invertible in A. We denote by $\mathsf{L}_{\mathsf{ct}}(S_{ét}, \ell)$ the ∞-category of constructible $\mathbb{Q}_\ell$-complexes on the étale site $S_{ét}$ of S. It is a symmetric monoidal ∞-category, and we denote by

$$\mathsf{L}(S_{ét}, \ell) := \mathbf{Ind}(\mathsf{L}_{\mathsf{ct}}(S_{ét}, \ell))$$

its completion under filtered colimits (see [Ga-Lu, Def. 4.3.26]). According to [Ro, Cor. 2.3.9], there exists an ℓ-adic realization ∞-functor $r_\ell : \mathcal{SH}_S \longrightarrow \mathsf{L}(S_{ét}, \ell)$. By construction, r_ℓ is a symmetric monoidal ∞-functor sending a smooth scheme $p : X \longrightarrow S$ to $p_! p^!(\mathbb{Q}_\ell)$, or, in other words, to the relative ℓ-adic homology of X over S.

We let $\mathbb{T} := \mathbb{Q}_\ell[2](1)$, and we consider the $\mathbb{E}_\infty$-ring object in $\mathsf{L}(S_{ét}, \ell)$

$$\mathbb{Q}_\ell(\beta) := \oplus_{n \in \mathbb{Z}} \mathbb{T}^{\otimes n}.$$

In this notation, β stands for $\mathbb{T}$, and $\mathbb{Q}_\ell(\beta)$ for the algebra of Laurent polynomials in β, so we could also have written

$$\mathbb{Q}_\ell(\beta) = \mathbb{Q}_\ell[\beta, \beta^{-1}].$$

As shown in [BRTV], there exists a canonical equivalence $r_\ell(\mathbf{B}\mathbb{U}_S) \simeq \mathbb{Q}_\ell(\beta)$ of $\mathbb{E}_\infty$-ring objects in $\mathsf{L}(S_{ét}, \ell)$, that is induced by the Chern character from algebraic K-theory to étale cohomology. We thus obtain a well-defined symmetric monoidal ∞-functor

$$r_\ell : \mathbf{B}\mathbb{U}_S - \mathbf{Mod} \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod},$$

from $\mathbf{B}\mathbb{U}_S$-modules in $\mathcal{SH}_S$ to $\mathbb{Q}_\ell(\beta)$-modules in $\mathsf{L}(S_{ét}, \ell)$. By pre-composing with the functor $T \mapsto \mathsf{M}^T$, we obtain a lax monoidal ∞-functor

$$r_\ell : \mathbf{dgCat}_S \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod}.$$

Definition 2.2.1 *The ∞-functor defined above*

$$r_\ell : \mathbf{dgCat}_S \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod}$$

is called the ℓ-adic realization functor for dg-categories over S.

From the standard properties of the functor $T \mapsto \mathsf{M}^T$, recalled above, we obtain the following properties for the ℓ-adic realization functor $T \mapsto r_\ell(T)$.

1. The ∞-functor r_ℓ is a localizing invariant, i.e. for any short exact sequence $T_0 \hookrightarrow T \longrightarrow T/T_0$ of dg-categories over A, the induced sequence

$$r_\ell(T_0) \longrightarrow r_\ell(T) \longrightarrow r_\ell(T/T_0)$$

   is a fibration sequence in $\mathbb{Q}_\ell(\beta) - \mathbf{Mod}$.

2. The natural morphism

$$\mathbb{Q}_\ell(\beta) \longrightarrow r_\ell(A),$$

   induced by the lax monoidal structure, is an equivalence in $\mathbb{Q}_\ell(\beta) - \mathbf{Mod}$.

3. The ∞-functor r_ℓ commutes with filtered colimits.

4. For any separated morphism of finite type $p : X \longrightarrow S$, we have a natural morphism of $\mathbb{Q}_\ell(\beta)$-modules

$$r_\ell(\mathsf{L}_{\mathrm{Perf}}(X)) \longrightarrow p_*(\mathbb{Q}_\ell(\beta)),$$

   where $p_* : \mathbb{Q}_\ell(\beta) - \mathbf{Mod} \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod}$ is induced by the direct image $L_{\mathsf{ct}}(X_{ét}, \ell) \longrightarrow L_{\mathsf{ct}}(S_{ét}, \ell)$ of constructible $\mathbb{Q}_\ell$-complexes. If p is proper, or if A is a field, this morphism is an equivalence.

2.3 Chern character

As explained in [BRTV], there is a symmetric monoidal ∞-category $\mathcal{SH}^{nc}_B$ of non-commutative motives over B. As an ∞-category it is the full sub-∞-category of ∞-functors of (co)presheaves of spectra

$$\mathbf{dgCat}^{ft}_A \longrightarrow \mathbf{Sp},$$

satisfying Nisnevich descent and $\mathbb{A}^1$-homotopy invariance.

We consider $\Gamma : \mathsf{L}(S_{ét}, \ell) \longrightarrow \mathbf{dg}_{\mathbb{Q}_\ell}$, the global section ∞-functor, taking an ℓ-adic complex on $S_{ét}$ to its hyper-cohomology. Composing this with the Dold-Kan construction $\mathbb{R}\mathbf{Map}_{\mathbf{dg}_{\mathbb{Q}_\ell}}(\mathbb{Q}_\ell, -) : \mathbf{dg}_{\mathbb{Q}_\ell} \longrightarrow \mathbf{Sp}$, we obtain an ∞-functor

$$| - | : \mathsf{L}(S_{ét}, \ell) \longrightarrow \mathbf{Sp},$$

which morally computes hyper-cohomology of $S_{ét}$ with ℓ-adic coefficients, i.e. for any $E \in \mathsf{L}(S_{ét}, \ell)$, we have natural isomorphisms

$$H^i(S_{ét}, E) \simeq \pi_{-i}(|E|)\,, i \in \mathbb{Z}.$$

By what we have seen in our last paragraph, the composite functor $T \mapsto |r_\ell(T)|$ provides a (co)presheaves of spectra

$$\mathbf{dgCat}_A^{ft} \longrightarrow \mathbf{Sp},$$

satisfying Nisnevich descent and $\mathbb{A}^1$-homotopy invariance. It thus defines an object $|r_\ell| \in \mathcal{SH}_A^{nc}$. The fact that r_ℓ is lax symmetric monoidal implies moreover that $|r_\ell|$ is endowed with a natural structure of a $\mathbb{E}_\infty$-ring object in $\mathcal{SH}_B^{nc}$.

Each $T \in \mathbf{dgCat}_A^{ft}$ defines a corepresentable object $h^T \in \mathcal{SH}_A^{nc}$, characterized by the ($\infty$-)functorial equivalences

$$\mathbb{R}\mathbf{Map}_{\mathcal{SH}_B^{nc}}(h^T, F) \simeq F(T),$$

for any $F \in \mathcal{SH}_A^{nc}$. The existence of h^T is a formal statement, however the main theorem of [Ro] implies that we have natural equivalences of spectra

$$\mathbb{R}\mathbf{Map}_{\mathcal{SH}_A^{nc}}(h^T, h^A) \simeq \mathsf{HK}(T),$$

where $\mathsf{HK}(T)$ stands for non-connective homotopy invariant algebraic K-theory of the dg-category T. In other words, $T \mapsto \mathsf{HK}(T)$ defines an object in $\mathcal{SH}_A^{nc}$ which is isomorphic to h^B. By Yoneda lemma, we thus obtain an equivalence of spaces

$$\mathbb{R}\mathbf{Map}^{lax-\otimes}(\mathsf{HK}, |r_\ell|) \simeq \mathbb{R}\mathbf{Map}_{\mathbb{E}_\infty-\mathbf{Sp}}(\mathbb{S}, |r_\ell(A)|) \simeq *.$$

In other words, there exists a unique (up to a contractible space of choices) lax symmetric monoidal natural transformation

$$\mathsf{HK} \longrightarrow |r_\ell|,$$

between lax monoidal ∞-functors from $\mathbf{dgCat}_A^{ft}$ to $\mathbf{Sp}$. We extend this to all dg-categories over A as usual by passing to Ind-completion $\mathbf{dgCat}_A \simeq \mathbf{Ind}(\mathbf{dgCat}_A^{ft})$.

Definition 2.3.1 *The natural transformation defined above is called the* ℓ-adic Chern character*. It is denoted by*

$$Ch_\ell : \mathsf{HK}(-) \longrightarrow |r_\ell(-)|.$$

Definition 2.3.1 contains a formal Grothendieck-Riemann-Roch formula. Indeed, for any B-linear dg-functor $f : T \longrightarrow T'$, the square of spectra

$$\begin{array}{ccc} \mathsf{HK}(T) & \xrightarrow{f_!} & \mathsf{HK}(T') \\ {\scriptstyle Ch_{\ell,T}}\downarrow & & \downarrow{\scriptstyle Ch_{\ell,T'}} \\ |r_\ell(T)| & \xrightarrow[f_!]{} & |r_\ell(T')| \end{array}$$

commutes up to a natural equivalence.

2.4 Trace formula for dg-categories

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category ([To-Ve], [Lu-HA, Definition 2.0.0.7]).

Hypothesis 2.4.1 *The underlying ∞-category $\mathcal{C}$ has small sifted colimits, and the tensor product preserves small colimits in each variable.*

Definition 2.4.2 *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category satisfying hypo thesis 2.4.1. We denote by $\underline{\mathbf{Alg}}(\mathcal{C})$ the $(\infty, 2)$-category of algebras in $\mathcal{C}^{\otimes}$ denoted by $\mathbf{Alg}_{(1)}(\mathcal{C}^{\otimes})$ in* [Lu-COB, Definition 4.1.11].

Informally, one can describe $\underline{\mathbf{Alg}}(\mathcal{C})$ as the $(\infty, 2)$-category with:

- objects: associative unital monoids (=: E_1-algebras) in $\mathcal{C}$.
- $Map_{\underline{\mathbf{Alg}}(\mathcal{C})}(B, B') := \mathbf{Bimod}_{B',B}(\mathcal{C}^{\otimes})$, the ∞-category of (B', B)-bimodules.
- The composition of 1-morphisms (i.e. of bimodules) is given by tensor product.
- The composition of 2-morphisms (i.e. of mrophisms between bimodules) is the usual composition.

Definition 2.4.3 *Let B be an algebra in $\mathcal{C}$ and X a left B-module. Identify X with a 1- morphism $\underline{X} : 1_{\mathcal{C}} \to B$ in $\underline{\mathbf{Alg}}(\mathcal{C})$. A right B-dual of X is a right adjoint $\underline{Y} : B \to 1_{\mathcal{C}}$ to $\underline{X}$.*

Unraveling the definition, we get that a right dual of X is a left $B^{\otimes\text{-op}}$-module Y, the unit of adjunction (or *coevaluation*) is a map $coev : 1_{\mathcal{C}} \to Y \otimes_B X$ in $\mathcal{C}$, the counit of adjunction (or *evaluation*) is a map $ev : X \otimes Y \to 1_{\mathcal{C}}$ of (B, B)-bimodules; u and v satisfy usual compatibilities.
Note that, if a right B-dual of X exists then it is "unique" (i.e. unique up to a contractible space of choices).

If the right B-module Y is the right B-dual of the left B-module X, then we can define the trace of any map $f : X \to X$ of left B-modules, as follows.
Recall that we have a coevaluation map in $\mathcal{C}$ $coev : 1_{\mathcal{C}} \to Y \otimes_B X$ in $\mathcal{C}$ and an evaluation map of (B, B)-bimodules $ev : X \otimes Y \to 1_{\mathcal{C}}$.
Consider the graph Γ_f defines as the composite

$$1_{\mathcal{C}} \xrightarrow{coev} Y \otimes_B X \xrightarrow{id \otimes f} Y \otimes_B X.$$

We now elaborate on the evaluation map. Observe that

- $B \in \mathcal{C}$ has a left $B^{\otimes\text{-op}} \otimes B$-module structure that we will denote by B^L.
- $B \in \mathcal{C}$ has a right $B^{\otimes\text{-op}} \otimes B$-module structure (i.e. a left $B \otimes B^{\otimes\text{-op}}$-module), that we will denote by B^R.
- $ev : X \otimes Y \to B^L$ is a map of left $B \otimes B^{\otimes\text{-op}}$- modules.
- the composite $ev' : Y \otimes X \xrightarrow{\sigma} X \otimes Y \xrightarrow{ev} B^R$ is a map of left $(B^{\otimes\text{-op}} \otimes B)$-modules

Apply $(-)\otimes_{B^{\otimes\text{-op}}\otimes B} B^l$ to the composite

$$ev' : Y\otimes X \xrightarrow{\sigma} X\otimes Y \xrightarrow{ev} B^R$$

to get

$$ev_{HH} :\ (Y\otimes X)\otimes_{B^{\otimes\text{-op}}\otimes B} B^L \longrightarrow B^R\otimes_{B^{\otimes\text{-op}}\otimes B} B^l =: \mathsf{HH}_{\mathcal{C}}(B)\ .$$

Now observe that $(Y\otimes X)\otimes_{B^{\otimes\text{-op}}\otimes B} B^L \simeq Y\otimes_B X$ in $\mathcal{C}$.

Definition 2.4.4 *The* non-commutative trace of $f : X\to X$ over B *is defined as the composite*

$$Tr_B(f) :\ 1_{\mathcal{C}} \xrightarrow{\Gamma_f} Y\otimes_B X \simeq (Y\otimes X)\otimes_{B^{\otimes\text{-op}}\otimes B} B^L \xrightarrow{ev_{HH}} B^R\otimes_{B^{\otimes\text{-op}}\otimes B} B^L =: \mathsf{HH}_{\mathcal{C}}(B)\ .$$

$Tr_B(f)$ is a morphism in $\mathcal{C}$.

Remark 2.4.5 Let $B\in \mathbf{CAlg}(\mathcal{C}^{\otimes})$, and let us still denote by B its image via the canonical map $\mathbf{CAlg}(\mathcal{C}^{\otimes})\to \mathbf{Alg}_{E_1}(\mathcal{C}^{\otimes})$. In this case, $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$ is a symmetric monoidal ∞-category, and if $X\in \mathbf{Mod}_B(\mathcal{C}^{\otimes})$ is a dualizable object (in the usual sense), then its (left and right) dual in $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$ is also a right-dual of X according to Definition 2.4.3. Thus, in his case, any $f : X\to X$ in $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$, has *two possible traces*, a non-commutative one (as in Definition 2.4.4

$$Tr_B(f) : 1_{\mathcal{C}} \longrightarrow \mathsf{HH}_{\mathcal{C}}(B)$$

which is a morphism in $\mathcal{C}$, and a more standard, commutative one

$$Tr^{\mathrm{c}}_B(f) : B \longrightarrow B$$

which is a morphism on $\mathbf{Mod}_B(\mathcal{C}^{\otimes})$. The two traces are related by the following commutative diagram

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "u_B"] \arrow[d, "Tr_B(f)"'] & B \arrow[d, "Tr^{\mathrm{c}}_B(f)"] \\ \mathsf{HH}_{\mathcal{C}}(B) \arrow[r, "a"'] & B \end{tikzcd}$$

where $a : \mathsf{HH}_{\mathcal{C}}(B)\to B$ is the canonical augmentation (which exists since B is commutative), and $u_B : 1_{\mathcal{C}}\to B$ is the unit map of the algebra B in $\mathcal{C}$.

The case of dg-categories. Let us specialize the previous discussion to dg-categories.

Let $\mathcal{B}$ be a monoidal dg-category, i.e. an associative and unital algebra in the symmetric monoidal ∞-category $\mathcal{C} = \mathbf{dgCat}_A$, and we can apply the previous theory here.

Proposition 2.4.6 *For any B-module T which is cotensored in the sense of definition, the big B-module $\widehat{T}$ has a right dual in the symmetric monoidal ∞-category $\widetilde{\mathbf{dgCat}}_A$. The underlying big dg-category of the right dual is $\widehat{T^o}$.*

Proof. This is very similar to the argument used in [To2, Prop. 2.5 (1)]. We consider $\widehat{T^o}$ and we define evaluation and coevaluation maps as follows.

We consider the big morphism h introduced before definition 2.1.2

$$h : \widehat{T^o}\widehat{\otimes}_A\widehat{T} \longrightarrow \widehat{B}.$$

The domain of this morphism is naturally a $\widehat{B}$-bimodule, and the morphism h has a canonical promotion to a morphism of bimodules. This morphism h is then chosen to be our evalution morphism.

The coevaluation is obtained by duality. We start by the diagonal bimodule

$$T : (T^o \otimes_A T)^o \longrightarrow L(A) = \widehat{A}.$$

sending (x, y) to $T(y, x)$. This morphism naturally descends to $(T^o \otimes_B T)^o$ and provides a dg-functor

$$(T^o \otimes_B T)^o \longrightarrow \widehat{A}.$$

Note that T^o is naturally a right B-module because T is assumed to be cotensored (unless $T^o \otimes_B T$ would not make sense). This dg-functor is an object in $\widehat{T^o \otimes_B T} \simeq \widehat{T^o}\widehat{\otimes}_A\widehat{T}$, and thus defines a coevaluation morphism

$$\widehat{A} \longrightarrow \widehat{T^o}\widehat{\otimes}_A\widehat{T}.$$

These two evaluation and coevaluation morphisms satisfy the required triangular identities and make $\widehat{T^o}$ a right dual to $\widehat{T}$. □

According to the previous proposition, and cotensored B-module T has a big right dual, so comes equiped with big evaluation and coevaluation maps.

Definition 2.4.7 *For a monoidal dg-category B and a B-module $T \in \mathbf{dgCat}_B$ is saturated over $\mathcal{B}$ if*

1. *T is cotensored.*
2. *the evaluation and coevaluation maps are small.*

In particular, if T saturated over $\mathcal{B}$, and $f : T \to T$ is a morphism in $\mathbf{dgCat}_{\mathcal{B}}$ (!), then the trace

$$Tr_{\mathcal{B}}(f : T \to T) : A \to \mathsf{HH}(\mathcal{B}/A) = \mathcal{B}^R \otimes_{\mathcal{B}^{\otimes\text{-op}}\otimes_A\mathcal{B}} \mathcal{B}^L$$

is also small, i.e. it is a morphism *inside* $\mathbf{dgCat}_A$.

Definition 2.4.8 *A saturated $T \in \mathbf{dgCat}_{\mathcal{B}}$ is $\ell^{\otimes}$-admissible if the canonical map*

$$r_\ell(T^{op}) \otimes_{r_\ell(\mathcal{B})} r_\ell(T) \to r_\ell(T^{op} \otimes_{\mathcal{B}} T)$$

is an equivalence in $Sh_{\mathbb{Q}_\ell}(S)$.

The trace $Tr_{\mathcal{B}}(f)$ is a map $A \to \mathsf{HH}(\mathcal{B}/A)$, hence it induces a map in $\mathbf{Sp}$

$$K(Tr_{\mathcal{B}}(f)) : K(A) \to K(\mathsf{HH}(\mathcal{B}/A))$$

which is actually a map of $K(A)$-modules (in spectra), since K is lax-monoidal. Hence it corresponds to an element denoted as

$$[\mathsf{HH}(T/\mathcal{B}, f)] \equiv tr_{\mathcal{B}}(f) \in K_0(\mathsf{HH}(\mathcal{B}/A)).$$

Therefore, its image by the ℓ-adic Chern character $Ch_{\ell,0} := \pi_0(Ch_\ell)$

$$Ch_{\ell,0} : K_0(\mathsf{HH}(\mathcal{B}/A)) \to Hom_{D(r_\ell(A))}(r_\ell(A), r_\ell(\mathsf{HH}(\mathcal{B}/A))) \simeq H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A))),$$

is an element $Ch_{\ell,0}([\mathsf{HH}(T/\mathcal{B}; f)]) \in H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A)))$.

On the other hand, the trace of $r_\ell(f)$ is, by definition, a morphism

$$r_\ell(A) \simeq \mathbb{Q}_\ell(\beta) \to \mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A))$$

in $\mathbf{Mod}_{r_\ell(A)}(Sh_{\mathbb{Q}_\ell}(S))$.
We may further compose this with the canonical map

$$\mathsf{HH}(r_\ell(\mathcal{B})/r_\ell(A)) \to r_\ell(\mathsf{HH}(\mathcal{B}/A))$$

(given by lax-monoidality of $r_\ell(-)$), to get a map

$$r_\ell(A) \to r_\ell(\mathsf{HH}(\mathcal{B}/A))$$

in $\mathbf{Mod}_{r_\ell(A)}(Sh_{\mathbb{Q}_\ell}(S))$. This is the same thing as an element denoted as

$$tr_{r_\ell(\mathcal{B})}(r_\ell(f)) \in \pi_0(|r_\ell(\mathsf{HH}(\mathcal{B}/A))|) \simeq H^0(S_{\text{ét}}, r_\ell(\mathsf{HH}(\mathcal{B}/A))).$$

Theorem 2.4.9 *Let $\mathcal{B}$ a monoidal dg-category over A, $T \in \mathbf{dgCat}_{\mathcal{B}}$ a saturated and $\ell^{\otimes}$-admissible B-module, and $f : T \to T$ map in $\mathbf{dgCat}_{\mathcal{B}}$. Then, we have*

$$Ch_{\ell,0}([\mathsf{HH}(T/\mathcal{B}, f)]) = tr_{r_\ell(\mathcal{B})}(r_\ell(f))$$

in $H^0(S_{\acute{e}t}, r_\ell(\mathsf{HH}(\mathcal{B}/A), r_\ell))$.

Proof. This is statement is a formal statement using uniqueness of right duals and its consequence: traces are preserved by symmetric monoidal ∞-functors or lax symmetric monoidal ∞-functors satisfying our admissibility condition. The key statement is the following lemma, left as an exercice to the reader, and applied to our ℓ-adic realization functor.

Lemma 2.4.10 *Let F be a lax symmetric monoidal ∞-functors between presentable symmetric monoidal ∞-categories*

$$F : \mathcal{C} \longrightarrow \mathcal{D}.$$

Let B be a monoid in $\mathcal{C}$, M a left B-module and $f : M \longrightarrow M$ an endomorphism of B-modules. We assume that M has a right dual M^o and that the natural morphism

$$F(M^o) \otimes_{F(B)} F(M) \longrightarrow F(M^o \otimes_B M)$$

is an equivalence. Then, $F(M)$ has a right dual, and we have

$$F(Tr(f)) = i(Tr(F(f)))$$

as elements in $\pi_0(Hom_{\mathcal{D}}(\mathbf{1}, F(HH(B)))$, and i is induced by the natural morphism $HH(F(B)) \longrightarrow F(HH(B))$.

□

3 Invariant vanishing cycles

This section gathers general results about invariant vanishing cycles (*I-vanishing cycles*, for short), their relations with dg-categories of singularities, and their behaviour under products. These results are partially taken from [BRTV], and the only original result is Proposition 3.4.2 that can be seen as a form of Thom-Sebastiani formula in the mixed-characteristic setting.

All along this section, A will be a strictly henselian excellent dvr with fraction field $K = \mathsf{Frac}(A)$, and perfect (hence algebraically closed) residue field k. We let $S = Spec\, A$. All schemes over S are assumed to be separated and of finite type over S. We denoted by $i : s := Spec\, k \longrightarrow S$ the closed point of S, and $j : \eta := Spec\, K \longrightarrow S$ its generic point. For an S-scheme X, we denote by $X_s := X \times_S s$ its special fiber, and $X_\eta = X \times_S \eta$ its generic fiber. Accordingly, we write $X_{\bar{s}} := X \times_S Spec\, \bar{k}$ and $X_{\bar{\eta}} := X \times_S Spec\, K^{sp}$ for the geometric special and geometric generic fiber, respectively.

3.1 Trivializing the Tate twist

We let ℓ be a prime invertible in k, and we denote by p the characteristic exponent of k. As k is algebraically closed, we may, and will, choose once for all a group isomorphism

$$\mu_\infty(k) \simeq \mu_\infty(K) \simeq (\mathbb{Q}/\mathbb{Z})[p^{-1}]$$

between the group of roots of unity in k and the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$. Equivalently, we have chosen a given group isomorphism

$$\lim_{(n,p)=1} \mu_n(k) \simeq \hat{\mathbb{Z}}',$$

where $\hat{\mathbb{Z}}' := \lim_{(n,p)=1} \mathbb{Z}/n$. In particular, we have selected a topological generator of $\lim_{(n,p)=1} \mu_n(k)$, corresponding to the image of $1 \in \mathbb{Z}$ inside $\hat{\mathbb{Z}}'$. The choice of the isomorphism above also provides a chosen isomorphism $\mathbb{Q}_\ell(1) \simeq \mathbb{Q}_\ell$ of $\mathbb{Q}_\ell$-sheaves on S, where (1) denotes, as usual, the Tate twist. By tensoring this isomorphism, we get various chosen isomorphisms $\mathbb{Q}_\ell(i) \simeq \mathbb{Q}_\ell$ for all $i \in \mathbb{Z}$.

We remind that the absolute Galois group I of K (which coincides with the *inertia group* in our case) sits in an extension of pro-finite groups[1]

$$1 \longrightarrow P \longrightarrow I \longrightarrow I_t \simeq \hat{\mathbb{Z}}' \longrightarrow 1\,,$$

where P is a pro-p-group (the *wild inertia* subgroup). For any continuous finite dimensional $\mathbb{Q}_\ell$-representation V of I, the group P acts by a finite quotient G_V on V. Moreover, the Galois cohomology of V can then be explicitly identified with the two-terms complex

$$V^G \xrightarrow{1-T} V^G$$

where T is the action of the chosen topological generator of I_t. This easily implies that for any $\mathbb{Q}_\ell$-representation V of I, the natural pairing on Galois cohomology

$$H^i(I,V) \otimes H^{1-i}(I,V^\vee) \longrightarrow H^1(I,\mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$$

is non-degenerate. In other words, if we denote by V^I the complex of cohomology of I with coefficients in V, we have a natural quasi-isomorphism $(V^I)^\vee \simeq (V^\vee)^I[1]$.

3.2 Reminders on actions of the inertia group

Let $X \longrightarrow S$ be an S-scheme (separated and of finite type, according to our conventions). We recall from [SGA7-II, Exp. XIII, 1.2] that we can associate to X a *vanishing topos* $(X/S)^\nu_{et}$ which is defined as (a 2-)fiber product of toposes

$$(X/S)^\nu_{et} := (X_s)^\sim_{et} \times_{s^\sim_{et}} \eta^\sim_{et}.$$

Since S is strictly henselian, $s^\sim_{et}$ is in fact the punctual topos, and the fiber product above is in fact a product of topos. The topos $\eta^\sim_{et}$ is equivalent to the topos of sets with continuous action of $I = \mathrm{Gal}(K^{sp}/k)$, where K^{sp} denotes a seprable closure of K. Morally, $(X/S)^\nu_{et}$ is the topos of étale sheaves on X_s, endowed with a continuous action of I (see [SGA7-II, Exp. XIII, 1.2.4]).

As explained in [BRTV] we have an ℓ-adic ∞-category $\mathcal{D}((X/S)^\nu_{et},\mathbb{Z}_\ell)$. Morally speaking, objects of this ∞-category consist of the data of an object $E \in \mathcal{D}(\bar{X}_s,\mathbb{Z}_\ell)$ together with a continuous action of I. We say that such an object is *constructible* if E is a constructible object in $\mathcal{D}(\bar{X}_s,\mathbb{Z}_\ell)$, and we denote by $\mathcal{D}_c((X/S)^\nu_{et},\mathbb{Z}_\ell)$ the full sub-∞-category of constructible objects.

Definition 3.2.1 *The* ∞-category of ind-constructible I-equivariant ℓ-adic complexes on X_s *is defined by*

$$\mathcal{D}^I_{ic}(X_s,\mathbb{Q}_\ell) := \mathsf{Ind}(\mathcal{D}_c((X/S)^\nu_{et},\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell).$$

The full sub-∞-category of constructible objects is $\mathcal{D}^I_c(X_s,\mathbb{Q}_\ell) := \mathcal{D}_c((X/S)^\nu_{et},\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Note that since we have chosen trivialisations of the Tate twists, $\mathbb{Q}_\ell(\beta)$ is identified with $\mathbb{Q}_\ell[\beta,\beta^{-1}]$ where β is a free variable in degree 2. This is a graded algebra object in $\mathcal{D}^I_{ic}(X_s,\mathbb{Q}_\ell)$, and we define the ∞-category $\mathcal{D}^I_{ic}(X_s,\mathbb{Q}_\ell(\beta))$ as the ∞-category of $\mathbb{Q}_\ell(\beta)$-modules in $\mathcal{D}^I_{ic}(X_s,\mathbb{Q}_\ell)$, or equivalently, the 2-periodic ∞-category of ind-constructible $\mathbb{Q}_\ell$-adic complexes on $(X/S)^\nu_{et}$.

[1] Note that the tame inertia quotient I_t is canonically isomorphic to $\hat{\mathbb{Z}}'(1)$, and it becomes isomorphic to $\hat{\mathbb{Z}}'$ through our choice.

Definition 3.2.2 *An object $E \in \mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell(\beta))$ is* constructible *if it belongs to the thick triangulated sub-∞-category generated by objects of the form $E_0(\beta) = E_0 \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(\beta)$ for E_0 a constructible object in $D^I_{ic}(X_s, \mathbb{Q}_\ell)$.*

The full sub-∞-category of $\mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell(\beta))$ consisting of constructible objects is denoted $\mathcal{D}^I_c(X_s, \mathbb{Q}_\ell(\beta))$. Similarly, for any S-scheme X, we define $\mathcal{D}_c(X, \mathbb{Q}_\ell(\beta))$ as the full sub-∞-category of objects in $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell(\beta))$ generated by $E_0(\beta)$ for E_0 constructible.

Note that strictly speaking an object of $\mathcal{D}^I_c(X_s, \mathbb{Q}_\ell(\beta))$ is not constructible in the usual sense, as its underlying object in $D_{ic}(X_s, \mathbb{Q}_\ell)$ is 2-periodic.

The topos $(X/S)^\nu_{et}$ comes with a natural projection $(X/S)^\nu_{et} \longrightarrow (X_s)^\sim_{et}$ whose direct image is an ∞-functor denoted by

$$(-)^I : \mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$$

called the *I-invariants* functor. This ∞-functor preserves constructibility. The ∞-categories $\mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell)$ and $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$ carries natural symmetric monoidal structures and the ∞-functor $(-)^I$ comes equipped with a natural lax symmetric monoidal structure (being induced by the direct image of a morphism of toposes). Moreover, $(-)^I$ is the right adjoint of the symmetric monoidal ∞-functor $U : \mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell)$ endowing objects in $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$ with the trivial action of I. This gives $\mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell)$ the structure of a $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$-module via $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell) \times \mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell) \to \mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell) : (E, F) \mapsto U(E) \otimes F$. This bi-functor distributes over colimits, thus by the adjoint theorem, we get an enrichment of $\mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell)$ over $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$. Note that $\mathcal{D}^I_{ic}(X_s, \mathbb{Q}_\ell)$ is also enriched over itself.

It is important to notice that the I-invariants functor commutes with base change in the following sense. Let $f : \operatorname{Spec} k \longrightarrow X_s$ be a geometric point. The morphism f defines a geometric point of $(X_s)^\sim_{et}$ and thus induces a geometric morphism of toposes

$$\eta^\sim_{et} \longrightarrow (X/S)^\nu_{et}.$$

We thus have an inverse image functor

$$f^* : \mathcal{D}^I_c(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c(\eta, \mathbb{Q}_\ell) = D_c(I, \mathbb{Q}_\ell).$$

where $\mathcal{D}_c(I, \mathbb{Q}_\ell)$ is the ∞-category of finite dimensional complexes of ℓ-adic representations of I. As usual, the square of ∞-functors

$$\begin{tikzcd}
\mathcal{D}^I_c(X_s, \mathbb{Q}_\ell) \arrow[r, "(-)^I"] \arrow[d, "f^*"'] & \mathcal{D}_c(X_s, \mathbb{Q}_\ell) \arrow[d, "f^*"] \\
\mathcal{D}_c(I, \mathbb{Q}_\ell) \arrow[r, "(-)^I"'] & \mathcal{D}_c(\mathbb{Q}_\ell)
\end{tikzcd}$$

comes equipped with a natural transformation

$$f^*((-)^I) \Rightarrow (f^*(-))^I.$$

It can be checked that this natural transformation is always an equivalence. In particular, for any geometric point x in X_s, we have a natural equivalence of ℓ-adic complexes $(E)^I_x \simeq (E_x)^I$, for any $E \in \mathcal{D}^I_c(X_s, \mathbb{Q}_\ell)$.

The dualizing complex ω of the scheme X_s can be used in order to obtain a dualizing object in $D_c^I(X_s, \mathbb{Q}_\ell)$ as follows. We consider ω as an object in $D_c^I(X_s, \mathbb{Q}_\ell)$ endowed with the trivial I-action. We then have an equivalence of ∞-categories

$$\mathbb{D}_I : \mathcal{D}_c^I(X_s, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c^I(X_s, \mathbb{Q}_\ell)^{op}$$

sending E to $\mathbb{R}\underline{Hom}(E, \omega)$, where $\mathbb{R}\underline{Hom}$ denotes the natural enrichment of $\mathcal{D}_c^I(X_s, \mathbb{Q}_\ell)$ over itself. We obviously have a canonical biduality equivalence $\mathbb{D}_I^2 \simeq id$. The duality functor $\mathbb{D}_I$ is compatible with the usual Grothendieck duality functor $\mathbb{D}$ for the scheme X_s up to a shift, as explained by the following lemma.

Lemma 3.2.3 *For any object $E \in \mathcal{D}_c^I(X_s, \mathbb{Q}_\ell)$, there is a functorial equivalence in $\mathcal{D}_c(X_s, \mathbb{Q}_\ell)$*

$$d : \mathbb{D}(E^I)[-1] \simeq (\mathbb{D}_I(E))^I.$$

Proof. Taking I-invariants is a lax monoidal ∞-functor, so we have a natural map $E^I \otimes \mathbb{D}_I(E)^I \longrightarrow (E \otimes \mathbb{D}_I(E))^I$, that can be composed with the evaluation morphism $E \otimes \mathbb{D}_I(E) \longrightarrow \omega$ to obtain $E^I \otimes \mathbb{D}_I(E)^I \longrightarrow \omega^I$. As the action of I on ω is trivial, we have a canonical equivalence $\omega^I \simeq \omega \otimes \mathbb{Q}_\ell^I \simeq \omega \oplus \omega[-1]$. By projection on the second factor we get a pairing $E^I \otimes \mathbb{D}_I(E)^I \longrightarrow \omega[-1]$, and thus a map

$$\mathbb{D}_I(E)^I \longrightarrow \mathbb{D}(E^I)[-1].$$

We claim that the above morphism is an equivalence in $\mathcal{D}_c(X_s, \mathbb{Q}_\ell)$. For this it is enough to check that the above morphism is a stalkwise equivalence. Now, the stalk of the above morphism at a geometric point x of X_s can be written as

$$(E(x)^\vee)^I \longrightarrow (E(x)^I)^\vee[-1]$$

where $E(x) := H_x^*(X, E) \in D_c(\mathbb{Q}_\ell)$ is the local cohomology of E at x, and $(-)^\vee$ is now the standard linear duality over $\mathbb{Q}_\ell$. The result now follows from the following well-known duality for $\mathbb{Q}_\ell$-representations of I: for any finite dimensional Ql-representation V of I, the fundamental class in $H^1(I, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$, induces a canonical isomorphism of Galois cohomologies

$$H^*(I, V^\vee) \simeq H^{1-*}(I, V)^\vee.$$

□

3.3 Invariant vanishing cycles and dg-categories

From [SGA7-II, Exp. XIII] and [BRTV, 4.1], the vanishing cycles construction provides an ∞-functor

$$\phi : \mathcal{D}_c(X, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c^I(X_s, \mathbb{Q}_\ell).$$

Applied to the constant sheaf $\mathbb{Q}_\ell$, we get this way an object denoted by $\nu_{X/S}$ (or simply ν_X if S is clear) in $\mathcal{D}_c^I(X_s, \mathbb{Q}_\ell)$.

Definition 3.3.1 *The I-*invariant vanishing cycles *of X relative to S (or I-*vanishing cycles*, for short) is the object*

$$\nu_X^I := (\nu_X)^I \in \mathcal{D}_c(X_s, \mathbb{Q}_\ell).$$

There are several possible descriptions of invariant vanishing cycles. First of all, by its very definition, ν_X^I is related to the I-invariant nearby cycles $\psi_X^I := (\psi_X)^I$ by means of an exact triangle in $D_c(X_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}_\ell^I \longrightarrow \psi_X^I \longrightarrow \nu_X^I. \tag{1}$$

Another description, in terms of *local cohomology*, is the following. We let $U = X_K$ be the open complement of X_s inside X, and $j_X : U \hookrightarrow X$ and $i_X : X_s \hookrightarrow X$ the corresponding immersions. Then, the I-vanishing cycles enters in an exact triangle in $D_c(X_s, \mathbb{Q}_\ell)$

$$\nu_X^I \longrightarrow \mathbb{Q}_\ell \longrightarrow i_X^!(\mathbb{Q}_\ell)[2]. \tag{2}$$

Triangle (2) follows from the octahedral axiom applied to the triangles (1) and

$$\mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell^I \simeq \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[-1] \longrightarrow \mathbb{Q}_\ell[-1],$$

taking also into account the triangle

$$i_X^! \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow i_X^*(j_X)_* j_X^* \mathbb{Q}_\ell \simeq \psi_X^I.$$

We get one more description of ν_X^I (or rather, of $\nu_X^I(\beta) := \nu_X^I \otimes \mathbb{Q}_\ell(\beta)$) using the *$\ell$-adic realization of the dg-category of singularities* studied in [BRTV], at least when X is a regular scheme with smooth generic fiber. Let $\mathsf{Sing}(X_s) = \mathsf{Coh}^b(X_s)/\mathsf{Perf}(X_s)$ be the dg-category of singularities of the scheme X_s. This dg-category is naturally linear over the dg-category $\mathsf{Perf}(X_s)$, and thus we can take its ℓ-adic realization $r_\ell(\mathsf{Sing}(X_s))$ (see [BRTV]) which is a $\mathbb{Q}_\ell(\beta)$-module in $\mathcal{D}_{ic}(X_s, \mathbb{Q}_\ell)$. When X is a regular scheme and X_K is smooth over K, we have from [BRTV] a canonical equivalence in $\mathcal{D}_c(X_s, \mathbb{Q}_\ell(\beta))$

$$\nu_X^I(\beta)[1] \simeq r_\ell(\mathsf{Sing}(X_s)) \tag{3}$$

where $\nu_X^I(\beta)$ stands for $\nu_X^I \otimes \mathbb{Q}_\ell(\beta)$.

We conclude this section with another description of $\nu_X^I(\beta)$, see equivalence (5), this time in terms of *sheaves of singularities*. We need a preliminary result.

Lemma 3.3.2 *We assume that S is excellent. Let $p : X \to S$ be a separated and finite type morphism. Then, we have*

1. *$r_\ell(\mathsf{Perf}(X)) \simeq \mathbb{Q}_{\ell,X}(\beta)$ in $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell)$.*
2. *$r_\ell(\mathsf{Coh}^b(X)) \simeq \omega_X(\beta)$ in $\mathcal{D}_{ic}(X, \mathbb{Q}_\ell)$, where $\omega_X \simeq p^!(\mathbb{Q}_\ell)$ is the ℓ-adic dualizing complex of X*[2]*.*
3. *There exists a canonical map $\eta_X : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_X(\beta)$ in $\mathcal{D}_c(X, \mathbb{Q}_\ell(\beta))$, called the* 2-periodic ℓ-adic fundamental class *of X.*

[2]Note that since S is excellent, X is excellent so that ω_X exists by a theorem of Gabber ([ILO, Exp. XVII, Th. 0.2]).

Proof: First of all, finite type and separated morphisms of neotherian schemes are compactifiable (Nagata theorem) and we can thus assume that X is proper over S.

(1) follows immediately from [BRTV, Prop. 3.9 and formula (3.7.13))]. In order to prove (2) we first produce a map $\alpha : r_\ell(\mathsf{Coh}^b(X)) \to \omega_X(\beta)$. In the notations of [BRTV, §3], we first construct a map $\alpha^{\mathrm{mot}} : \mathcal{M}_X^\vee(\mathsf{Coh}^b(X)) \to p^!(\mathbf{B}\mathbb{U}_S) =: \omega_X^{\mathrm{mot}}$ in $\mathrm{SH}(X)$, whose étale ℓ-adic realization will be α. Since p is proper, α^{mot} is the same thing, by adjunction, as a map $p_*(\mathcal{M}_X^\vee(\mathsf{Coh}^b(X))) \to \mathbf{B}\mathbb{U}_S$ in $\mathrm{SH}(S)$. Now, $p_*(\mathcal{M}_X^\vee(\mathsf{Coh}^b(X)))$ is just $\mathcal{M}_S^\vee(\mathsf{Coh}^b(X)))$, where $\mathsf{Coh}^b(X)$ is viewed as a dg-category over S, via p. If Y is smooth over S, we have by [Pr, Prop. B.4.1], an equivalence of S-dg-categories

$$\mathsf{Coh}^b(X) \otimes_S \mathsf{Coh}^b(Y) \simeq \mathsf{Coh}^b(X \times_S Y) \tag{4}$$

Through this identification, $\mathcal{M}_S^\vee(\mathsf{Coh}^b(X)) \in \mathrm{SH}(S)$ is the ∞-functor $Y \mapsto \mathrm{KH}(\mathsf{Coh}^b(X \times_S Y))$, and $\mathrm{KH}(\mathsf{Coh}^b(X \times_S Y))$ is equivalent to the G-theory spectrum $\mathrm{G}(X \times_S Y)$ of $X \times_S Y$, by $\mathbb{A}^1$-invariance of G-theory. Since S is regular, $\mathbf{B}\mathbb{U}_S \simeq \mathrm{G}_S := \mathrm{G}(-/S)$ canonically in $\mathrm{SH}(S)$, and we can take the map $\mathcal{M}_S^\vee(\mathsf{Coh}^b(X))) \to \mathbf{B}\mathbb{U}_S \simeq \mathrm{G}_S$ to be the push forward p_* on G-theories $\mathrm{G}(X \times_S -) \to G(-/S)$. This gives us a map $\alpha^{\mathrm{mot}} : \mathcal{M}_X^\vee(\mathsf{Coh}^b(X)) \to p^!(\mathbf{B}\mathbb{U}_S) =: \omega_X^{\mathrm{mot}}$. Now observe that by [BRTV, formula (3.7.13)], the étale ℓ-adic realization of $p^!(\mathbf{B}\mathbb{U}_S)$ is canonically equivalent to $p^!(\mathbb{Q}_\ell(\beta)) \simeq \omega_X(\beta)$ (since étale ℓ-adic realization commutes with six operations, [BRTV, Rmk. 3.23]). Therefore, we get our map $\alpha : r_\ell(\mathsf{Coh}^b(X)) \to \omega_X(\beta)$. Checking that α is an equivalence is a local statement, i.e. it is enough to show that if $j : V = \mathrm{Spec}, A \hookrightarrow X$ is an open affine subscheme, then $j^*(\alpha)$ is an equivalence. Now, $j^* r_\ell(\mathsf{Coh}^b(X)) \simeq r_\ell(j^*\mathsf{Coh}^b(X)) \simeq r_\ell(\mathsf{Coh}^b(V))$, and $j^*\omega_X \simeq j^!\omega_X \simeq \omega_V$, so $j^*\alpha$ identifies with a map $r_\ell(\mathsf{Coh}^b(V)) \to \omega_V(\beta)$. Since V is affine and of finite type over S, we can choose a closed immersion $i : V \hookrightarrow V'$, with V' affine and smooth (hence regular) over S. Let $h : V' \setminus V \hookrightarrow V'$ be the complementary open immersion. Since V' and $V' \setminus V$ are regular, by Quillen localization and the properties of the nc realization functor $\mathcal{M}^\vee$ (see [BRTV]), we get a cofiber sequence

$$\mathcal{M}_{V'}^\vee(\mathsf{Coh}^b(V)/V') \to \mathbf{B}\mathbb{U}_{X'} \to h_*\mathbf{B}\mathbb{U}_{V'\setminus V},$$

where the notation $\mathsf{Coh}^b(V)/V'$ means that $\mathsf{Coh}^b(V)$ is viewed as a dg-catgeory over V', via i. In other words, $\mathcal{M}_{V'}^\vee(\mathsf{Coh}^b(V)/V') \simeq i_*\mathcal{M}_V^\vee(\mathsf{Coh}^b(V))$. If we apply i^* to this cofiber sequence, and compare what we obtain to the application of i^* to the standard localization sequence

$$i_* i^! \mathbf{B}\mathbb{U}_{V'} \to \mathbf{B}\mathbb{U}_{V'} \to h_* h^* \mathbf{B}\mathbb{U}_{V'} = h_*\mathbf{B}\mathbb{U}_{V'\setminus V},$$

we finally get, after étale ℓ-adic realization, that $\omega_V(\beta) \simeq i^!\mathbb{Q}_\ell(\beta) \simeq r_\ell(\mathsf{Coh}^b(V))$. This implies that $j^*\alpha$ is also an equivalence.

By (1) and (2), the map in (3) is finally obtained by applying r_ℓ to the inclusion $\mathsf{Perf}(X) \to \mathsf{Coh}^b(X)$.

□

Remark 3.3.3 Note that the proof of Lemma 3.3.2 also shows that a 2-periodic ℓ-adic fundamental class map $\eta_U : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_U(\beta)$ is also defined for any open subscheme $U \hookrightarrow X$ over S, for X proper over S.

Definition 3.3.4 *Let X/S be a proper S-scheme, as above. The* sheaf of singularities of X *is defined to be the cofiber of the 2-periodic ℓ-adic fundamental class morphism η_X (Lemma 3.3.2 (3))*

$$\omega_X^o := \mathit{Cofib}(\eta_X : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_X(\beta)).$$

By construction of η_X in Lemma 3.3.2, ω^o_X is then the ℓ-adic realization $r_\ell(\mathsf{Sing}(X))$ of the dg-category $\mathsf{Sing}(X)$, considered as a dg-category over X.

Remark 3.3.5 If $p : \mathcal{X} \to S$ is a proper lci map from a *derived* scheme $\mathcal{X}$, we can still define a 2-periodic ℓ-adic fundamental class map $\eta_{\mathcal{X}}$, as in Lemma 3.3.2. This can be done by observing that the pushforward on G-theories along the inclusion of the truncation $\mathrm{t}_0\mathcal{X} \to \mathcal{X}$ is an equivalence, and that p being lci we have a natural inclusion $\mathsf{Perf}(\mathcal{X}) \to \mathsf{Coh}^b(\mathcal{X})$. We further observe that in this case, while the ∞-category $\mathcal{D}_c(\mathcal{X}, \mathbb{Q}_\ell(\beta))$ only depends on the reduced subscheme $(\mathrm{t}_0\mathcal{X})_{red}$, and the same is true for the objects $\mathbb{Q}_{\ell,\mathcal{X}}(\beta)$ and $\omega_{\mathcal{X}}(\beta)$, in contrast, the morphism $\eta_{\mathcal{X}}$ *does* depend on the derived structure on $\mathcal{X}$, and thus it is not a purely topological invariant.

Let us come back to X a regular scheme, proper over S. We have a canonical equivalence in $\mathcal{D}_c(X_s, \mathbb{Q}_\ell(\beta))$

$$\nu^I_X(\beta)[1] \simeq \omega^o_{X_s}. \tag{5}$$

This is a reformulation of the equivalence (3), in view of Lemma 3.3.2.

Remark 3.3.6 When X is not regular anymore, but still proper and lci[3] over S, there is nonetheless a natural morphism $\nu^I_X(\beta)[1] \longrightarrow \omega^o_{X_s}$, constructed as follows. Consider again the triangle (2)

$$\nu^I_X \longrightarrow \mathbb{Q}_\ell \longrightarrow i^!_X(\mathbb{Q}_\ell)[2].$$

On X, we do have the 2-periodic ℓ-adic fundamental class $\eta_X : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_X(\beta)$, and by taking its !-pullback by $i^!_X$, we get a morphism $i^!_X(\mathbb{Q}_\ell)(\beta) \longrightarrow \omega_{X_s}(\beta)$. This produces a sequence of morphisms

$$\nu^I_X(\beta) \longrightarrow \mathbb{Q}_\ell(\beta) \longrightarrow i^!_X(\mathbb{Q}_\ell)[2](\beta) = i^!_X(\mathbb{Q}_\ell)(\beta) \longrightarrow \omega_{X_s}(\beta).$$

The resulting composite morphism $\mathbb{Q}_\ell(\beta) \to \omega_{X_s}(\beta)$ is the 2-periodic ℓ-adic fundamental class of X_s. Moreover, by construction, the composition $\nu^I_X(\beta) \longrightarrow \mathbb{Q}_\ell(\beta) \longrightarrow \omega_{X_s}(\beta)$ is canonically the zero map, and this induces the natural morphism

$$\alpha_X : \nu^I_X(\beta)[1] \longrightarrow \omega^o_{X_s}$$

we were looking for. Summing up, the morphism α_X always exists for any proper, lci scheme X over S, and is an equivalence whenever X is regular.

3.4 A Künneth theorem for invariant vanishing cycles

In this section, we consider two S-schemes X and Y (separated of finite type), such that that X_K and Y_K are smooth over K, and both X and Y are regular and connected. For simplicity[4] we also assume that X and Y are flat over S. We set $Z := X \times_S Y$, and consider this as a scheme over S. We have $Z_s \simeq X_s \times_s Y_s$, and the ∞-category $D^I_c(Z_s, \mathbb{Q}_\ell)$ comes equipped with pull-back functors

$$p^* : D^I_c(X_s, \mathbb{Q}_\ell) \longrightarrow D^I_c(Z_s, \mathbb{Q}_\ell) \longleftarrow D^I_c(Y_s, \mathbb{Q}_\ell) : q^*.$$

[3]Note that a morphism of finite type between regular schemes is lci, since we can check that its relative cotangent complex has perfect amplitude in $[-1, 0]$.

[4]In the non-flat case, the fiber product of X and Y over S, to be considered below, should be replaced by the derived fiber product.

By taking their tensor product, we get an external product functor

$$\boxtimes := p^*(-) \otimes q^*(-) : D_c^I(X_s, \mathbb{Q}_\ell) \times D_c(Y_s, \mathbb{Q}_\ell) \longrightarrow D_c^I(Z_s, \mathbb{Q}_\ell).$$

For two objects $E \in D_c^I(X_s, \mathbb{Q}_\ell)$ and $F \in D_c^I(Y_s, \mathbb{Q}_\ell)$, we can define the Künneth morphism in $D_c(Z_s, \mathbb{Q}_\ell)$

$$\mathsf{k} : (E \boxtimes F)^I[-1] \longrightarrow E^I \boxtimes F^I$$

as follows. Since Grothendieck duality on Z_s is compatible with external products, in order to define k it is enough to define its dual

$$\mathbb{D}(E^I) \boxtimes \mathbb{D}(F^I) \longrightarrow \mathbb{D}((E \boxtimes F)^I)[1].$$

By Lemma 3.2.3, the datum of such a morphism is equivalent to that of a morphism

$$\mathbb{D}_I(E)^I \boxtimes \mathbb{D}_I(F)^I \longrightarrow \mathbb{D}_I(E \boxtimes F)^I.$$

We now define k as the map induced by the composite

$$\mathbb{D}_I(E)^I \boxtimes \mathbb{D}_I(F)^I \xrightarrow{\mu_{(-)^I}} (\mathbb{D}_I(E) \boxtimes \mathbb{D}_I(F))^I \xrightarrow{(\mu_{D_I})^I} \mathbb{D}_I(E \boxtimes F)^I$$

where $\mu_{(-)^I}$ is the lax monoidal structure on $(-)^I$, and μ_{D_I} the one on D_I[5].

Definition 3.4.1 *With the above notations, the I-*invariant convolution *of the two objects $E \in D_c^I(X_s, \mathbb{Q}_\ell)$ and $F \in D_c^I(Y_s, \mathbb{Q}_\ell)$ is defined to be the cone of the Künneth morphism, and denoted by $(E \circledast F)^I$. By definition it sits in a triangle*

$$(E \boxtimes F)^I[-1] \xrightarrow{\mathsf{k}} E^I \boxtimes F^I \longrightarrow (E \circledast F)^I.$$

The main result of this section is the following proposition, relating the I-invariant convolution of vanishing cycles on X and Y to the dualizing complex of Z. It can be also considered as a computation of the ℓ-adic realization of the ∞-category $\mathsf{Sing}(Z)$ of singularities of Z.
Note that, as X and Y are generically smooth over S, so is Z, and thus the 2-periodic ℓ-adic fundamental class map $\eta_Z : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_Z(\beta)$ of Lemma 3.3.2 is an equivalence over the generic fiber. Therefore, ω_Z^o is supported on Z_s, and can be considered canonically as an object in $\mathcal{D}_c(Z_z, \mathbb{Q}_\ell)$.

Proposition 3.4.2 *With the above notations and assumptions, there is a canonical equivalence*

$$\omega_Z^o \simeq (\nu_X \circledast \nu_Y)^I(\beta)$$

in $D_c(Z_s, \mathbb{Q}_\ell(\beta))$.

Proof. The proof of this proposition will combine various exact triangles together with an application of Gabber's Künneth formula for nearby cycles.

To start with, the vanishing cycles ν_Z of Z sits in an exact triangle in $\mathcal{D}_c^I(Z_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}_\ell \longrightarrow \psi_Z \longrightarrow \nu_Z \ ,$$

[5]Note that μ_{D_I} is in fact an equivalence.

where $\psi_Z = \psi_Z(\mathbb{Q}_\ell)$ is the complex of nearby cycles of Z over S. According to [Be-Be, Lemma 5.1.1] or [Il1, 4.7]), we have a natural equivalence in $\mathcal{D}_c^I(Z_s, \mathbb{Q}_\ell)$, induced by external product

$$\psi_Z \simeq \psi_X \boxtimes \psi_Y.$$

The object ν_Z then becomes the cone of the tensor product of the two morphisms in $\mathcal{D}_c^I(Z_s, \mathbb{Q}_\ell)$

$$\mathbb{Q}_\ell \longrightarrow p^*(\psi_X) \qquad \mathbb{Q}_\ell \longrightarrow q^*(\psi_Y)$$

where p and q are the two projections from Z down to X and Y, respectively. Now, cones of tensor products are computed via the following well known lemma (see B.0.1 for a proof).

Lemma 3.4.3 *Let $\mathcal{C}$ be a stable symmetric monoidal ∞-category, and*

$$u : x \to y \qquad v : x' \to y'$$

two morphisms. Let $C(u)$ be the cone of u, $C(v)$ be the cone of v, and $C(u \otimes v)$ the cone of the tensor product $u \otimes v : x \otimes x' \to y \otimes y'$. Then, there exists a natural exact triangle

$$C(u) \otimes x' \bigoplus x \otimes C(v) \longrightarrow C(u \otimes v) \longrightarrow C(u) \otimes C(v).$$

The above lemma implies the existence of a natural exact triangle in $\mathcal{D}_c^I(Z_s, \mathbb{Q}_\ell)$

$$\nu_X \boxplus \nu_Y \longrightarrow \nu_Z \longrightarrow \nu_X \boxtimes \nu_Y,$$

which, by taking I-invariants, yields an exact triangle in $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell)$

$$(\mathbf{T1}) \qquad \nu_X^I \boxplus \nu_Y^I \longrightarrow \nu_Z^I \longrightarrow (\nu_X \boxtimes \nu_Y)^I.$$

The dualizing complex $\omega_{Z_s} = (Z_s \to s)^! \mathbb{Q}_\ell$ of $Z_s \simeq X_s \times_s Y_s$ is canonically equivalent to $\omega_{X_s} \boxtimes \omega_{Y_s}$ (see Lemma B.0.2). Moreover, the virtual fundamental class of Z_s

$$\eta_{Z_s} : \mathbb{Q}_\ell(\beta) \longrightarrow \omega_{Z_s}(\beta) \simeq \omega_{X_s} \boxtimes \omega_{Y_s}(\beta)$$

simply is the external tensor product of the virtual fundamental classes of X_s and Y_s. By Lemma 3.4.3 we thus get a second exact triangle in $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell(\beta))$

$$(\mathbf{T2}) \qquad \omega_{X_s}^o \boxplus \omega_{Y_s}^o \longrightarrow \omega_{Z_s}^o \longrightarrow \omega_{X_s}^o \boxtimes \omega_{Y_s}^o.$$

There is a morphism from the triangle (T1) to the triangle (T2) which is defined using the natural morphism

$$\alpha_Z : \nu_Z^I(\beta)[1] \longrightarrow \omega_{Z_s}^o$$

introduced in Remark 3.3.6. In fact, Z is proper and lci over S (since X/S is flat and lci[6], and being lci is stable under flat base change and composition), and the map α_Z is defined for any proper, lci scheme Z

[6] Again, note that a morphism of finite type between regular schemes is lci, since we can check that its relative cotangent complex has perfect amplitude in $[-1, 0]$.

over S, being an equivalence when Z is regular with smooth generic fiber. Using the compatible maps α_X, α_Y and α_Z we get a commutative square

$$\begin{tikzcd} \nu_X^I(\beta)[1] \boxplus \nu_Y^I(\beta)[1] \arrow[r] \arrow[d, "\alpha_Z \oplus \alpha_Y"'] & \nu_Z^I(\beta)[1] \arrow[d, "\alpha_Z"] \\ \omega_{X_s}^o \boxplus \omega_{Y_s}^o \arrow[r] & \omega_{Z_s}^o. \end{tikzcd}$$

This produces a morphism from triangle $(T1)$ (tensored by $\mathbb{Q}_\ell(\beta)[1]$) to triangle $(T2)$

$$\begin{tikzcd} \nu_X^I(\beta)[1] \boxplus \nu_Y^I(\beta)[1] \arrow[r] \arrow[d] & \nu_Z^I(\beta)[1] \arrow[r] \arrow[d, "\alpha_Z"] & (\nu_X \boxtimes \nu_Y)^I(\beta)[1] \arrow[d] \\ \omega_{X_s}^o \boxplus \omega_{Y_s}^o \arrow[r] & \omega_{Z_s}^o \arrow[r] & \omega_{X_s}^o \boxtimes \omega_{Y_s}^o. \end{tikzcd} \tag{6}$$

Since X and Y are regular with smooth generic fibers, the maps α_X and α_Y are equivalences, therefore the leftmost vertical morphism is also an equivalence. Thus the right hand square is a cartesian square.

Now, the rightmost vertical morphism can be written, again using the equivalences α_X and α_Y, as

$$(\nu_X \boxtimes \nu_Y)^I(\beta)[1] \longrightarrow (\nu_X^I(\beta)[1]) \boxtimes_{\mathbb{Q}_\ell(\beta)} (\nu_Y^I(\beta)[1]) \simeq (\nu_X^I \boxtimes \nu_Y^I)[2](\beta)$$

This morphism is the Künneth map k of Definition 3.4.1 tensored by $\mathbb{Q}_\ell[2](\beta) \simeq \mathbb{Q}_\ell(\beta)$, and thus its cone is $(\nu_X \circledast \nu_Y)^I(\beta)$. In order to finish the proof of the proposition it then remains to show that the cone of the middle vertical morphism in (6)

$$\alpha_Z : \nu_Z^I(\beta)[1] \longrightarrow \omega_{Z_s}^o$$

can be canonically identified with ω_Z^o.

For this, we remind the exact triangle (2) tensored by $\mathbb{Q}_\ell(\beta)$

$$\mathbb{Q}_\ell(\beta) \longrightarrow i_Z^!(\mathbb{Q}_\ell(\beta)) \longrightarrow \nu_Z^I(\beta)[1].$$

Using the fundamental class $\eta_Z : \mathbb{Q}_\ell(\beta) \to \omega_Z(\beta)$, we get a morphism of triangles

$$\begin{tikzcd} \mathbb{Q}_\ell(\beta) \arrow[r] \arrow[d, "id"'] & i_Z^!(\mathbb{Q}_\ell(\beta)) \arrow[r] \arrow[d, "i_Z^!(\eta_Z)"] & \nu_Z^I(\beta)[1] \arrow[d] \\ \mathbb{Q}_\ell(\beta) \arrow[r] & i_Z^!(\omega_Z(\beta)) \arrow[r] & \omega_{Z_s}^o. \end{tikzcd}$$

The right hand square is thus cartesian, so that the cone of the vertical morphism on the right is canonically identified with the cone of the vertical morphism in the middle. By definition, this cone is $i_Z^!(\omega_Z^o)$. Since Z_K is smooth over K, the ℓ-complex ω_Z^o is supported on Z_s, and thus $i_Z^!(\omega_Z^o)$ is canonically equivalent to ω_Z^o, and we conclude. □

Corollary 3.4.4 *We keep the same notations and assumptions as in Proposition 3.4.2, and we further assume one of the following conditions:*

1. *the I-action on ν_X and on ν_Y is tame, or*

2. *the reduced scheme $(X_s)_{red}$ is smooth over k.*

Then, there is a canonical equivalence

$$\omega_Z^o \simeq (\nu_X \boxtimes \nu_Y)^I(\beta)$$

in $\mathcal{D}_c(Z_s, \mathbb{Q}_\ell(\beta))$.

Proof. It is enough to prove that under any one of the two assumptions, $(\nu_X \circledast \nu_Y)^I$ is canonically equivalent to $(\nu_X \boxtimes \nu_Y)^I$. If the scheme $(X_s)_{red}$ is smooth over k, then we have $\nu_X^I(\beta) = 0$. Indeed, triangle (2) can be then re-written

$$\nu_X^I \longrightarrow \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell[2n+2]$$

where n is the dimension of X_s. By tensoring by $\mathbb{Q}_\ell(\beta)$, we get a triangle

$$\nu_X^I(\beta) \longrightarrow \mathbb{Q}_\ell(\beta) \xrightarrow{b} \mathbb{Q}_\ell[2n+2](\beta) \simeq \mathbb{Q}_\ell(\beta)$$

where b is an equivalence. Therefore, $\nu_X^I(\beta) = 0$, and, by definition of I-invariant convolution, this implies that $(\nu_X \circledast \nu_Y)^I \simeq (\nu_X \boxtimes \nu_Y)^I$.

Assume now that the action of I on ν_X and ν_Y is tame. This means that the action of I factors through the natural quotient $I \longrightarrow I_t$, where I_t is the tame inertia group, which is canonically isomorphic to $\hat{\mathbb{Z}}'$, the prime-to-p part of the profinite completion of $\mathbb{Z}$. As we have chosen a topological generator T of I_t (see Section 3.1), the actions of I are then completely caracterized by the automorphisms T on ν_X and ν_Y. Moreover, ν_X^I is then naturally equivalent to the homotopy fiber of $(1-T) : \nu_X \to \nu_X$, and similarly for ν_Y^I. From this it is easy to see that the Künneth map

$$(\nu_X \boxtimes \nu_Y)^I[-1] \longrightarrow \nu_X^I \boxtimes \nu_Y^I$$

fits in an exact triangle

$$(\nu_X \boxtimes \nu_Y)^I[-1] \longrightarrow \nu_X^I \boxtimes \nu_Y^I \longrightarrow (\nu_X \boxtimes \nu_Y)^I$$

where the second morphism is induced by the lax monoidal structure on $(-)^I$. We conclude that there is a natural equivalence $(\nu_X \circledast \nu_Y)^I \simeq (\nu_X \boxtimes \nu_Y)^I$. □

4 Dg-categories of singularities

4.1 The monoidal dg-category $\mathcal{B}$ and its action

We keep our standing assumptions: A is an excellent strictly henselian dvr with perfect residue field k and fraction field K. We let $S = \operatorname{Spec}, A$ and $s = \operatorname{Spec} k$, as usual, and choose an uniformizer π of A.

We let $G := s \times_S^h s$ (derived fiber product), considered as a derived scheme over S. The derived scheme G has a canonical structure of groupoid in derived schemes acting on s. The composition in the groupoid G induces a convolution monoidal structure on the dg-category of coherent complexes on G

$$\odot : \mathsf{Coh}^b(G) \otimes_A \mathsf{Coh}^b(G) \longrightarrow \mathsf{Coh}^b(G).$$

More explicitly, we have a map of derived schemes

$$G \times_s G \xrightarrow{q} G,$$

defined as the projection on the first and third components $s \times_S s \times_S s \to s \times_S s$. We then define $\odot$ by the formula

$$E \odot F := q_*(E \boxtimes_s F)$$

for two coherent complexes E and F on G. More generally, if $X \longrightarrow S$ is any scheme, with special fiber X_s (possibly a derived scheme, by taking the derived fiber at s), the groupoid G acts naturally on X_s via the natural projection

$$q_X : G \times_s X_s \simeq (s \times_S s) \times_s (s \times_S X) \simeq s \times_S X_s \longrightarrow X_s.$$

This defines an external action

$$\odot : \mathsf{Coh}^b(G) \otimes_A \mathsf{Coh}^b(X_s) \longrightarrow \mathsf{Coh}^b(X_s)$$

by $E \odot M := (q_X)_*(E \boxtimes_s M)$.

The homotopy coherences issues for the above $\odot$-structures can be handled using the fact that the construction $Y \mapsto \mathsf{Coh}^b(Y)$ is in fact a symmetric lax monoidal ∞-functor from a certain ∞-category of correspondences between derived schemes to the ∞-category of dg-categories over A. As a result, $\mathsf{Coh}^b(G)$ is endowed with a natural structure of a monoid in the symmetric monoidal ∞-category $\mathbf{dgCat}_A$, and that, for any X/S, $\mathsf{Coh}^b(X_s)$ is naturally a module over $\mathsf{Coh}^b(G)$ in $\mathbf{dgCat}_A$. However, for our purposes it will be easier and more efficient to provide explicit models for both $\mathsf{Coh}^b(G)$ and its action on $\mathsf{Coh}^b(X_s)$. This will be done locally in the Zariski topology in a similar spirit to [BRTV, Section 2]; the global construction will then be obtained by a rather straightforward gluing procedure.

4.1.1 The weak-monoidal dg-categories $\mathcal{B}^+$ and $\mathcal{B}$

Let K_A be the Koszul commutative A-dg-algebra of A with respect to π

$$K_A : \ A \xrightarrow{\pi} A$$

sitting in degrees $[-1, 0]$. The canonical generator of K_A in degree -1 will be denoted by h. In the same way, we define the commutative A-dg-algebra

$$K_A^2 := K_A \otimes_A K_A$$

which is the Koszul dg-algebra of A with respect to the sequence (π, π). As a commutative graded A-algebra, K_A^2 is $Sym_A(A^2[1])$, and it is endowed with the unique multiplicative differential sending the two generators h and h' in degree -1 to π (and hh' to $\pi \cdot h' - \pi \cdot h$).

Moreover, K_A^2 has a canonical structure of *Hopf algebroid* over K_A, in which the source and target map are the two natural inclusions $K_A \longrightarrow K_A^2$, whereas the unit is given by the multiplication $K_A^2 \to K_A$. The composition (or coproduct) in this Hopf algebroid structure is given by

$$\Delta := \mathrm{id} \otimes 1 \otimes \mathrm{id} : K_A \otimes_A K_A = K_A^2 \longrightarrow K_A^2 \otimes_{K_A} K_A^2 = K_A \otimes_A K_A \otimes_A K_A.$$

Finally, the antipode is the automorphism of K_A^2 exchanging the two factors K_A. This structure of Hopf algebroid endows the dg-category $\mathsf{Mod}(K_A^2)$, of $K^2(A)$-dg-modules, with a unital and associative monoidal structure $\odot$. It is explicitely given for two object E and F, by the formula

$$E \odot F := E \otimes_{K_A} F$$

where the $K_A \otimes_A K_A \otimes_A K_A$-module on the rhs is considered as a K_A^2 module via the map Δ. The unit of this monoidal structure is the object K_A, viewed as a K_A^2-module by the multiplication $K_A^2 \to K_A$. It is not hard to see that $\odot$ preserves cofibrant K_A^2-dg-modules; more generally it makes $\mathsf{Mod}(K_A^2)$ into a monoidal model category in the sense of [Ho, Ch. 4]. Note however that the unit K_A is *not* cofibrant in this model structure.

Definition 4.1.1 *The* monoidal dg-category $\mathcal{B}^+_{str}$ *is defined to be* $\mathsf{Mod}^c(K_A^2)$*, the dg-category of all cofibrant dg-modules over* K_A^2 *which are perfect over* A*, together with the unit object* K_A*. It is endowed with the monoidal structure* $\odot$ *described above.*

By Appendix B, the localization of the dg-category $\mathcal{B}^+_{str}$ along all quasi-isomorphisms, defines a weak-monoidal dg-category (in the sense of Appendix A).

Definition 4.1.2 *The* weak-monoidal dg-category $\mathcal{B}^+$ *is defined to be the localization*

$$W_{eq}^{-1}(\mathcal{B}^+_{str}),$$

where W_{eq} *is the set of quasi-isomorphisms. It is naturally a unital and associative monoid in the symmetric monoidal* ∞*-category* $\mathbf{dgCat}_A$*.*

Remark 4.1.3 Note that $\mathcal{B}^+$ defined above is a model for $\mathsf{Coh}^b(G)$, for our derived groupoid $G = s \times_S s$ above. Indeed, the commutative dga K_A^2 is quasi-isomorphic to the normalization of the simplicial algebra $k \otimes_A^{\mathbb{L}} k$. Now, since $G \to S$ is a closed immersion, a quasi-coherent complex E on G is coherent iff its direct image on S is coherent, hence perfect, S being regular. In particular, we have that $\mathsf{Coh}^b(G)$ is equivalent to the dg-category of all cofibrant K_A^2-dg-modules which are perfect over A The latter dg-category is also naturally equivalent to the localization of $\mathcal{B}^+_{str}$ along quasi-isomorphisms[7]

We now introduce the weak monoidal dg-category $\mathcal{B}$, defined as a further localization of $\mathcal{B}^+$. This will be our main "base monoid" for the module dg-categories we will be interested in.

Definition 4.1.4 *The* weak monoidal dg-category $\mathcal{B}$ *is defined to be the localization*

$$\mathcal{B} := L_W(\mathcal{B}^+),$$

where W *is the set of morphisms in* $\mathcal{B}^+$ *whose cone is perfect as a* K_A^2*-dg-module.*

As $\mathcal{B}^+$ it itself defined as a localization of $\mathcal{B}^+_{str}$, $\mathcal{B}$ can also be realized as localization of $\mathcal{B}^+_{str}$ directly. Inside $\mathcal{B}^+_{str}$ we have the quasi-isomorphisms W_{eq} of dg-modules, and also the morphisms W_{pe} between dg-modules whose cones are perfect over K_A^2. Then, by definition, we have a natural equivalence

$$\mathcal{B} \simeq L_{W_{\mathrm{eq}} \cup W_{\mathrm{pe}}} \mathcal{B}^+_{str}.$$

[7] We leave to the reader to check that adding the unit objet K_A does not change the localization.

Since the monoidal structure $\odot$ is compatible with both $W_{\rm eq}$ and $W_{\rm pe}$, this presentation implies that $\mathcal{B}$ comes equipped with a natural structure of an associative and unital monoid in $\mathbf{dgCat}_A$. Note, moreover, that $\mathcal{B}$ comes equipped with a natural morphism of monoids

$$\mathcal{B}^+ \longrightarrow \mathcal{B}$$

given by the localization map.

4.1.2 The local actions

Let now $X = \operatorname{Spec} R$ be a regular and flat scheme over A. As done above for the monoid structure on $\mathcal{B}^+$, we will define a *strict model* for $\mathsf{Coh}^b(X_s)$, together with a *strict model* for the $\mathsf{Coh}^b(G)$-action on $\mathsf{Coh}^b(X_s)$. In order to do this, let K_R be the Koszul dg-algebra of R with respect to π, which comes equipped with a natural map $K_A \longrightarrow K_R$ of cdga's over A. We consider $\mathsf{Mod}^c(K_R)$, the dg-category of all cofibrant K_R-dg-modules which are perfect as R-modules (note that R is regular, and see Remark 4.1.3). The same argument as in Remark 4.1.3 then shows that this dg-category is naturally equivalent to $\mathsf{Coh}^b(X_s)$. Moreover, $\mathsf{Mod}^c(K_R)$ has a structure of a $\mathcal{B}^+_{str}$-module dg-category defined as follows. For $E \in \mathcal{B}^+_{str}$, and $M \in \mathsf{Mod}^c(K_R)$, we can define

$$E \odot M := E \otimes_{K_A} M,$$

where, in the rhs, we have used the "right" K_A-dg-module structure on E, i.e. the one induced by the composition

$$K_A \xrightarrow{\sim} A \otimes_A K_A \xrightarrow{\mathrm{id}\otimes u} K_A \otimes_A K_A \ ,$$

$u : A \to K_A$ being the canonical map. As E is either the unit or it is cofibrant over K_A^2 (and thus cofibrant over K_A), $E \otimes_{K_A} M$ is again a cofibrant K_R-module, and again perfect over R, i.e. $E \odot M \in \mathsf{Mod}^c(K_R)$. By localization along quasi-isomorphisms, (see Proposition 4.1.5 for details) we obtain that $\mathsf{Coh}^b(X_s)$ carries a natural $\mathcal{B}^+$-module structure as an object in the symmetric monoidal ∞-category $\mathbf{dgCat}_A$.

We now apply a similar argument in order to define a $\mathcal{B}$-*action on* $\mathsf{Sing}(X_s)$. Let again $X = Spec\, R$ be a regular scheme over A, and consider $\mathsf{Mod}^c(K_R)$ as a $\mathcal{B}^+_{str}$-module dg-category as above. Let $W_{R,\rm pe}$ be the set of morphisms in $\mathsf{Mod}^c(K_R)$ whose cones are perfect dg-modules over K_R. By localization we then get a $\mathcal{B}$-module structure on $L_{W_{R,\rm pe}}\mathsf{Mod}^c(K_R)$. Note that the localization $L_{W_{R,\rm pe}}\mathsf{Mod}^c(K_R)$ is a model for the dg-category $\mathsf{Sing}(X_s)$, which therefore comes equipped with the structure of a $\mathcal{B}$-module in $\mathbf{dgCat}_A$.

We gather the details of above constructions in the following

Proposition 4.1.5 *Let $X = \operatorname{Spec} R$ be a regular scheme, flat over $S = \operatorname{Spec} A$, and X_s its special fiber. Then there is a canonical $\mathcal{B}^+$-module structure (resp., $\mathcal{B}$-module structure) on $\mathsf{Coh}^b(X_s)$ (resp., on $\mathsf{Sing}(X_s)$), inside $\mathbf{dgCat}_A$.*

Proof. This is an easy application of the localization results presented in Appendix B.
We first treat the case of $\mathcal{B}^+$ and $T := \mathsf{Coh}^b(X_s)$. If $T^{\rm str} := \mathsf{Mod}^c(X_r)$ and $W_{T,\rm eq}$ denotes the quasi-isomorphisms in $T^{\rm str}$, we have $W_{T,\rm eq}^{-1}T^{\rm str} \simeq T$ in $\mathbf{dgCat}_A$. Analogously, $W_{\rm eq}^{-1}\mathcal{B}^+_{\rm str} \simeq \mathcal{B}^+$ in $\mathbf{dgCat}_A$. In order to apply the localization result of Appendix B, we need to prove that the tensor product $\odot : \mathcal{B}^+_{\rm str} \otimes_A T^{\rm str} \to$

T^{str} (defined in 4.1.2) sends $W_{\mathrm{eq}} \otimes id \cup id \otimes W_{T,\mathrm{eq}}$ to $W_{T,\mathrm{eq}}$. If $L, L' \in \mathcal{B}^+_{\mathrm{str}}$, $E, E' \in T^{\mathrm{str}}$, and $w' : L \to L'$, $w : E \to E'$ are quasi-isomorphisms, then $w' \odot id_E$ is again a quasi-isomorphism (because L, and L' are cofibrant over K_A^2, hence over K_A, and thus w' is in fact a homotopy equivalence), and the same is true for $id_L \odot w$ (since L is cofibrant over K_A). Therefore, $\odot$ does send $W_{\mathrm{eq}} \otimes id \cup id \otimes W_{T,\mathrm{eq}}$ to $W_{T,\mathrm{eq}}$, and there is an induced canonical map $(W_{\mathrm{eq}} \otimes id \cup id \otimes W_{T,\mathrm{eq}})^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to W_{T,\mathrm{eq}}^{-1} T^{\mathrm{str}} \simeq \mathsf{Coh}^b(X_s)$. By composing this with the natural equivalence $(W_{\mathrm{eq}} \otimes id \cup id \otimes W_{T,\mathrm{eq}})^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to W_{\mathrm{eq}}^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A W_{T,\mathrm{eq}}^{-1} T^{\mathrm{str}}$ (Appendix B), we finally get our $\mathcal{B}^+$-module structure on $\mathsf{Coh}^b(X_s)$ inside $\mathbf{dgCat}_A$.
We now treat the case of $\mathcal{B}$ and $T = \mathsf{Sing}(X_s)$. Here, we consider the pairs $(T^{\mathrm{str}} = \mathsf{Mod}^c(X_r), W_T)$ where W_T are the maps in T^{str} whose cones are perfect over K_R, and $(\mathcal{B}^*_{\mathrm{str}}, W)$, where W are the maps in $\mathcal{B}^+_{\mathrm{str}}$ whose cones are perfect over K_A^2. We have $W^{-1}\mathcal{B}^*_{\mathrm{str}} \simeq \mathcal{B}$, and $W_T^{-1}T^{\mathrm{str}} \simeq \mathsf{Sing}(X_s)$, and we need to prove that both $W \odot id$ and $id \odot W_T$ are contained in W_T. Let $u : L \to L' \in W$ and $v : E \to E' \in W_T$, and $C(-)$ denote the cone construction. We have $C(id_L \odot v) \simeq L \otimes_{K_A} C(v)$ and $C(u \odot id_E) \simeq C(u) \otimes_{K_A} E$. By hypothesis, L is perfect over A hence over K_A (since K_A is perfect over A), and $C(v)$ is perfect over K_A, since $X \to S$ is lci (as a map of finite type between regular schemes), and thus $K_A \to K_R$ is derived lci (recall that X/S is flat so that X_s is also the derived fiber), and pushforward along a lci map preserves perfect complexes. So, $C(id_L \odot v) \in W_T$. On the other hand, the "right-hand" map $K_A \to K_A^2$ (with respect to which L and L' are viewed as K_A-dg-modules in the definition of $\odot$) is derived lci, hence $C(u)$ is perfect over K_A, being perfect over K_A^2 by hypothesis. Moreover, since $s \to S$ is a closed immersion, E is perfect over K_A iff $(X \to S)_* E$ is perfect (= coherent, S being regular) over S; but $(X_s \to X)_* E$ is perfect by hypothesis, and pushforward along $X \to S$ preserves perfect complexes, since X/S is lci. Therefore $C(u \odot id_E) \in W_T$, and we deduce that $\odot$ does send $W \otimes id \cup id \otimes W_T$ to W_T. This gives us an induced canonical map $(W \otimes id \cup id \otimes W_T)^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to W_T^{-1}T^{\mathrm{str}} \simeq \mathsf{Sing}(X_s)$. By composing this with the natural equivalence $(W \otimes id \cup id \otimes W_T)^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A T^{\mathrm{str}} \to W^{-1}\mathcal{B}^+_{\mathrm{str}} \otimes_A W_T^{-1}T^{\mathrm{str}}$ (Appendix B), we finally get our $\mathcal{B}$-module structure on $\mathsf{Sing}(X_s)$ inside $\mathbf{dgCat}_A$.

□

Lemma 4.1.6 *The natural morphism*

$$\mathsf{Coh}^b(X_s) \otimes_{\mathcal{B}^+} \mathcal{B} \longrightarrow \mathsf{Sing}(X_s)$$

is an equivalence of $\mathcal{B}$-modules.

Proof. This is a reformulation of [BRTV, Proposition 2.31]. □

4.1.3 The global actions

We now let X be a regular scheme, not necessarily affine anymore. We have by Zariski descent

$$\mathsf{Coh}^b(X_s) \simeq \lim_{Spec\, R \subset X} \mathsf{Mod}^c(K_R),$$

where the limit is taken over all affine opens $Spec\, R$ of X. The right hand side of the above equivalence is a limit of dg-categories underlying $\mathcal{B}^+$-module structures (Proposition 4.1.5). As the forgetful functor from $\mathcal{B}^+$-modules to dg-categories reflects limits, this endows $\mathsf{Coh}^b(X_s)$ with a unique structure of $\mathcal{B}^+$-module.

In the same way, we have Zariski descent for $\mathsf{Sing}(X_s)$ in the sense that

$$\mathsf{Sing}(X_s) \simeq \lim_{Spec\, R \subset X} L_{W_{R,\mathrm{pe}}}(\mathsf{Mod}^c(K_R)).$$

The right hand side of the above equivalence is a limit of dg-categories underlying $\mathcal{B}$-module structures (Proposition 4.1.5). As the forgetful functor from $\mathcal{B}$-modules to dg-categories reflects limits, this makes the dg-category $\mathsf{Sing}(X_s)$ into a $\mathcal{B}$-module in a natural way.

An important property of these $\mathcal{B}^+$ and $\mathcal{B}$-module structures is given in the following proposition.

Proposition 4.1.7 *Let X be a flat regular scheme over S.*

1. *The $\mathcal{B}^+$-module structure on $\mathsf{Coh}^b(X_s)$ is cotensored.*
2. *The $\mathcal{B}$-module structure on $\mathsf{Sing}(X_s)$ is cotensored.*

Proof. This follow formally from the fact that the monoidal dg-categories $\mathcal{B}^+$ and $\mathcal{B}$ are generated by their unit objects. □

4.2 Künneth formula for dg-categories of singularities

In the previous section we have seen that, for any regular scheme X over S, the dg-category $\mathsf{Sing}(X_s)$ are equipped with a natural $\mathcal{B}$-module structure. In this section we compute tensor products of dg-categories of singularities over $\mathcal{B}$.

From a general point of view, let $T \in \mathbf{dgCat}_A$ be a $\mathcal{B}$-module, and assume that T is also *co-tensored* over $\mathcal{B}$ (Def. 4.1.7). Then T^o has a natural structure of a $\mathcal{B}^{\otimes -op}$-module given by co-tensorisation (see Appendix A). By Proposition 4.1.7, we may take $T = \mathsf{Sing}(X_s)$, so that T^o is a $\mathcal{B}^{\otimes -op}$-module. In particular, if we have another regular scheme Y, we are entitled to take the tensor product

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s),$$

which is a well defined object in $\mathbf{dgCat}_A$. We further assume, for simplicity, that X and Y are also flat over S. The main result of this section is the following proposition, which is a categorical version of our Künneth formula for vanishing cycles (Proposition 3.4.2).

Proposition 4.2.1 *Let X and Y be two regular schemes, flat over S. There is a natural equivalence in* $\mathbf{dgCat}_A$

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s) \simeq \mathsf{Sing}(X \times_S Y).$$

Proof. First of all, we claim that the result is local on $X \times_S Y$. Indeed, we have two prestacks of dg-categories on the small Zariski site Z_{zar}:

$$U \times_S V \mapsto \mathsf{Sing}(U_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(V_s) \qquad U \times_S V \mapsto \mathsf{Sing}(U \times_S V)$$

for any two affine opens $U \subset X$ and $V \subset Y$. These two prestacks are stacks of dg-categories, and thus we have equivalences in $\mathbf{dgCat}_A$

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s) \simeq \lim_{U \subset X, V \subset Y} \mathsf{Sing}(U_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(V_s) \tag{7}$$

$$\mathsf{Sing}(X \times_S Y) \simeq \lim_{U \subset X, V \subset Y} \mathsf{Sing}(U \times_S V). \tag{8}$$

The stack property (8) is proved in [BRTV, 2.3] (where it is moreover shown that this is a stack for the h-topology). The stack property (7) is indeed a consequence of the same descent argument for dg-categories of singularities. Indeed, we have the following lemma.

Lemma 4.2.2 *Let Z be an S-scheme, and F be a stack of $\mathcal{O}_Z$-linear dg-categories. Assume that F is a $\mathcal{B}^{\otimes -op}$-module stack*[8]*, and let T_0 be a $\mathcal{B}$-module dg-category. Then, the prestack $F \otimes_{\mathcal{B}} T_0$ of dg-categories of Z_{Zar}, sending $W \subset Z$ to $F(W) \otimes_{\mathcal{B}} T_0$ is a stack.*

Proof of Lemma 4.2.2. It is an application of the main result of [To2]. We denote, as usual, by $\widehat{T} := \mathbb{R}\underline{\mathcal{H}om}(T, \widehat{A})$ the (non-small) dg-category of all T^o-dg-modules. By the main result of [To2], the dg-category

$$\lim_{W \subset Z} (\widehat{F(W) \otimes_{\mathcal{B}} T_0})$$

is compactly generated, and its dg-category of compact objects is equivalent in $\mathbf{dgCat}_A$ to $\lim_{W \subset Z} F(W) \otimes_{\mathcal{B}} T_0$. Moreover, we have

$$(\widehat{F(W) \otimes_{\mathcal{B}} T_0}) \simeq \widehat{F(W)} \widehat{\otimes}_{\widehat{\mathcal{B}}} \widehat{T_0},$$

where $\widehat{\otimes}$ is the symmetric monoidal structure on presentable dg-categories. As $\widehat{\otimes}$ is rigid when restricted to compactly generated dg-categories, we have that $\widehat{\otimes}_{\widehat{\mathcal{B}}}$ distributes over limits on both factors. We thus have

$$(\widehat{F(Z) \otimes_{\mathcal{B}} T_0}) \simeq \lim_{W \subset Z} \widehat{F(W)} \widehat{\otimes}_{\widehat{\mathcal{B}}} \widehat{T_0}.$$

Passing to the sub-dg-categories of compact objects, we find that

$$F(Z) \otimes_{\mathcal{B}} T_0 \simeq \lim_{W \subset Z} F(W) \otimes_{\mathcal{B}} T_0$$

which is the statement of the lemma. ◇

The previous lemma immediately implies the stack property (7).
We are thus reduced to the case where X and Y are both affine, and we have to produce an equivalence

$$\mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(Y_s) \simeq \mathsf{Sing}(X \times_S Y)$$

that is compatible with Zariski localization on X and Y.

In this case we start by the following.

[8] I.e., F is a stack on Z_{Zar} with values in the ∞-category of $\mathcal{B}^{\otimes -op}$-modules in $\mathbf{dgCat}_A$.

Lemma 4.2.3 *There is a natural equivalence of dg-categories over A*

$$L_{coh}(X_s)^o \otimes_{\mathcal{B}^+} L_{coh}(Y_s) \simeq L_{coh}^{Z_s}(Z),$$

where $Z := X \times_S Y$, and the right hand side is the dg-category of coherent complexes on Z with cohomology supported on the special fiber Z_s. T his equivalence is furthermore functorial in X and Y.

Proof of lemma 4.2.3. We use the strict models introduced in our last section. Let $X := Spec\, B$ and $Y := Spec\, C$, and K_B and K_C the Koszul dg-algebras of B and C with respect to the element π. As in our last section we have the Hopf dg-algebroid K_A^2 and its monoidal dg-category of modules $K_A^2 - Mod^c$. It acts on both, $K_B - Mod^c$ and $K_C - Mod^c$, the dg-categories of cofibrant K_B (resp. K_C) dg-modules which are perfect over B (resp. over C).

We define a dg-functor

$$\phi : (K_B - Mod^c)^o \otimes_{K_A^2 - Mod^c} K_C - Mod^c \longrightarrow B \otimes_A C - Mod^c,$$

where $B \otimes_A C - Mod^c$ is the category of $B \otimes_A C$-dg-modules which are also perfect over $B \otimes_A C$. This dg-functor sends a pair of objects (E, F) to the object $\mathbb{D}(E) \otimes_{K_A} F$, where $\mathbb{D}(E) = \underline{Hom}_{K_B}(E, K_B)$ is the K_B-linear dual of E over K_B. After localization with respect to quasi-isomorphism we get a well defined morphism in $\mathbf{dgCat}_A$

$$\phi : L_{coh}(X_s)^o \otimes_{\mathcal{B}^+} L_{coh}(Y_s) \longrightarrow L_{coh}(Z).$$

In order to finish the proof, we have to check two conditions.

1. The image of ϕ generates (by shifts, sums, cones and retracts) the full sub-dg-category $L_{coh}^{Z_s}(Z)$.
2. The dg-functor ϕ above is fully faithful.

Now, on the level of objects the dg-functor ϕ sends a pair (E, F), of coherent sheaves on X_s and Y_s, to the coherent sheaf on Z

$$j_*(E \boxtimes_k F),$$

where $j : Z_s \hookrightarrow Z$ is the closed embedding and $E \boxtimes_k F$ is the external product of E by F on $Z_s = X_s \times_s Y_s$. It is known that coherent complexes of the form $E \boxtimes_k F$ generate $L_{coh}(Z_s)$. As the coherent sheaves of the form $j_*(G)$, for $G \in L_{coh}(Z_s)$, clearly generate the dg-category $L_{coh}^{Z_s}(Z)$, we see that condition (1) above is satisfied.

It now remains to show that ϕ is fully faithful. Given two pairs of objects $(E, F), (E', F') \in L_{coh}(X_s) \times L_{coh}(Y_s)$, we have the induced morphism by ϕ

$$\mathbb{R}\underline{Hom}(E', E) \otimes_{\mathcal{B}^+(1)} \mathbb{R}\underline{Hom}(F, F') \longrightarrow \mathbb{R}\underline{Hom}(j_*(\mathbb{D}(E) \boxtimes F), j_*(\mathbb{D}(E') \boxtimes F')),$$

where here $\mathcal{B}^+(1)$ denotes the algebra of endomorphism of the unit in $\mathcal{B}^+$. As we have already seen $\mathcal{B}^+ \simeq k[u]$ as an $\mathbb{E}_1$-algebra. As X_s and Y_s are Gorenstein scheme, the structure sheaf $\mathcal{O}$ is dualizing complex, and the above morphism can also be written as

$$\mathbb{R}\underline{Hom}(E, E') \otimes_{k[u]} \mathbb{R}\underline{Hom}(F, F') \longrightarrow \mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')).$$

Both sides of the above morphisms enter in a distinguished triangle. On the left hand side, for any two $k[u]$-dg-modules M and N, we have a triangle of A-dg-modules

$$M \otimes_k N \longrightarrow M \otimes_k N \longrightarrow M \otimes_k [u]N,$$

where the morphism on the right is the natural projection. This exact triangle comes from the usual exact triangle of $k[u]$ bi-dgmodules

$$k[u] \otimes_k k[u] \longrightarrow k[u] \otimes k[u] \longrightarrow B.$$

On the right hand side, we have by adjunction

$$\mathbb{R}\underline{Hom}(j_*(\mathbb{D}(E) \boxtimes F), j_*(\mathbb{D}(E') \boxtimes F')) \simeq \mathbb{R}\underline{Hom}(j^* j_*(\mathbb{D}(E) \boxtimes F), \mathbb{D}(E') \boxtimes F')).$$

The adjunction map $j^* j_* \to id$, provides an exact triangle of coherent sheaves on Z_s

$$E \boxtimes_k F[1] \longrightarrow j^* j_*(E \boxtimes_k F) \longrightarrow E \boxtimes_k F.$$

The coboundary of this triangle is a map

$$E \boxtimes_k F \longrightarrow E \boxtimes_k F[2]$$

is precisely given by the action of $k[u]$. We thus have another exact triangle

$$\mathbb{R}\underline{Hom}(E, F) \otimes_k \mathbb{R}\underline{Hom}(E', F') \longrightarrow \mathbb{R}\underline{Hom}(E, F) \otimes_k \mathbb{R}\underline{Hom}(E', F') \longrightarrow$$

$$\longrightarrow \mathbb{R}\underline{Hom}(j_*(\mathbb{D}(E) \boxtimes F), j_*(\mathbb{D}(E') \boxtimes F')).$$

By inspection, the morphism ϕ is compatible with these two triangles and provides an equivalence as wanted

$$\mathbb{R}\underline{Hom}(E, E') \otimes_{k[u]} \mathbb{R}\underline{Hom}(F, F') \longrightarrow \mathbb{R}\underline{Hom}(j_*(E \boxtimes F), j_*(E' \boxtimes F')).$$

□

4.3 Saturatedness

As a consequence of the Kunneth formula for dg-categories of singularities we prove the following result.

Proposition 4.3.1 *Let X be a regular and flat S-scheme.*

1. *If X is proper over S, then the dg-category $\mathsf{Coh}^b(X_s)$ is proper over $\mathcal{B}^+$.*
2. *The dg-category $\mathsf{Sing}(X_s)$ is always smooth over $\mathcal{B}$.*

Proof: (1) We have to show that the big morphism

$$h : \mathsf{Coh}^b(X_s)^o \widehat{\otimes_A} \mathsf{Coh}^b(X_s) \longrightarrow \widehat{\mathcal{B}^+} \simeq \widehat{Coh(G)}.$$

is small. Here $G = s \times_S s$ and the morphism above is obtained as follows. The derived scheme G is a derived groupoid which acts on X_s by means of the natural projection on the last two factors

$$\mu : G \times_s X_s = s \times_S s \times_S X \longrightarrow X_s.$$

The projection on the first and third factors provides another morphism

$$p : G \times_s X_s \longrightarrow X_s.$$

The morphism p and μ put together define a morphism of derived schemes

$$q : G \times_s X_s \longrightarrow X_s \times_S X_s.$$

Finally, we have the projection $r : G \times_s X_s \longrightarrow G$. The dg-functor

$$h : \mathsf{Coh}^b(X_s)^o \widehat{\otimes_A} \mathsf{Coh}^b(X_s) \longrightarrow \widehat{\mathsf{Coh}^b(G)}$$

is then obtained as follows. For two coherent complexes E and F on X_s, we form the external Hom $\mathcal{H}om_A(E, F)$ which is a coherent complex on $X_s \times_S X_s$. We have

$$h(E, F) \simeq r_*(q^* \mathcal{H}om_A(E, F)).$$

This is a quasi-coherent complex on G. It turns out that q and r are local complete intersection morphisms of derived schemes and moreover r is proper. This implies that q^* and r_* preserve coherent complexes, and thus that $h(E, F)$ is coherent on G.

(2) We have $\mathsf{Sing}(X_s) \simeq \mathsf{Coh}^b(X_s) \otimes_{\mathcal{B}^+} \mathcal{B}$, thus (1) implies that $Sing(X_s)$ is proper over $\mathcal{B}$. To prove it is smooth we need to prove that the coevaluation big morphism $A \longrightarrow \mathsf{Sing}(X_s)^o \otimes_{\mathcal{B}} \mathsf{Sing}(X_s)$ is a small morphism. Using our Kunneth for dg-category of singularities 4.2.1 this morphism corresponds to the data of an ind-object in $\mathsf{Sing}(X \times_S X)$. This object is the structure sheaf of the diagonal Δ_X inside $X \times_S X$ which is an object in $\mathsf{Sing}(X \times_S X)$. This shows that the coevaluation morphism is a small morphism and thus that $\mathsf{Sing}(X_s)$ is smooth over $\mathcal{B}$. □

5 Bloch's conductor formula with unipotent monodromy

Our base scheme is a discrete valuation ring $S = \mathbf{Spec}\, A$, with perfect residue field k, and fraction field K. Let $p : X \longrightarrow S$ be proper and flat morphism of finite type, and of relative dimension n. We assume that the generic fiber X_K is smooth over K, and that X is a regular scheme. We write $\bar{K}$ for the separable closure of K (inside a fixed algebraic closure).

In his 1985 paper [Bl], Bloch formulated the following conductor formula conjecture which is a kind of vast arithmetic generalization of Gauss-Bonnet formula, where an intersection theoretic (coherent) term, the *Bloch's number*, is conjectured to be equal to an arithmetic (étale) term, the *Artin conductor*. We address the reader to [Bl] and [Ka-Sa] for more detailed definitions of the various objects involved in the statement.

Conjecture 5.0.1 [Bloch's conductor Conjecture] *Under the above hypotheses on $p : X \to S$, we have an equality*

$$[\Delta_X, \Delta_X]_S = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell) - \mathsf{Sw}(X_{\bar{K}}),$$

where $\chi(Y,\ell)$ denotes the $\mathbb{Q}_\ell$-adic Euler characteristic *of a variety Y, for ℓ prime to the characteristic of k,* $\mathsf{Sw}(X_{\bar{K}})$ *is the* Swan conductor *of the* $\mathsf{Gal}(\bar{K}/K)$*-representation $H^*(X_{\bar{K}},\mathbb{Q}_\ell)$, and $[\Delta_X,\Delta_X]_S$ is* Bloch's number *of X/S, i.e. the degree in $\mathsf{CH}_0(k)\simeq\mathbb{Z}$ of Bloch's localised self-intersection $(\Delta_X,\Delta_X)_S\in\mathsf{CH}_0(X_k)$ of the diagonal in X. The (negative of the) rhs is called the* Artin conductor *of X/S, and denoted by* $\mathsf{Art}(X/S)$.

It is easy to see that the conjecture above can be reduced to the strictly henselian case (k algebraically closed). We will thus assume from now that k is algebraically closed.
The conjecture 5.0.1 is known in several special cases that we remind below.

1. When k is of characteristic zero, the formula follows from the work of [Kap]. When furthermore X_s has only isolated singularities the formula was known as the Milnor formula stating that the dimension of the space of vanishing cycles equals the dimension of the Jacobian ring.

2. When S is of equicharacteristic the formula has been proved recently in [Sai], based on Beilinson's theory of singular support of ℓ-adic sheaves. The case of isolated singularities already appeared in [SGA7-I, Exp. XVI].

3. When X is semi-stable over S, that is the reduced divisor $(X_s)_{red}\subset X$ is simple normal crossing, the formula has been proved in [Ka-Sa].

In view of the previous known case, one of the major open case is for isolated singularities in mixed characteristic, which is the conjecture appearing in Deligne's exposé [SGA7-I, Exp. XVI].

We now introduce a new definition for the Bloch number $[\Delta_X.\Delta_X]_S$ in terms of dg-categories of singularities. We will denote this new number differently, as we do not make a precise comparison in the present work.

We start by considering $\mathsf{Sing}(X_s)$, the dg-category of singularities ox the special fiber. It comes equiped with its canonical $\mathcal{B}$-module structure described in proposition 4.1.7. As X is proper over S we know by proposition 4.3.1 that $\mathsf{Sing}(X_s)$ is saturated over $\mathcal{B}$. We can thus form the trace of the identity of $\mathsf{Sing}(X_s)$, which is a morphism in $\mathbf{dgCat}_A$

$$A\longrightarrow HH(\mathcal{B}).$$

This morphism is by definition determined by a perfect $HH(\mathcal{B})^o$-dg-module, and thus provides a class in K-theory

$$[HH(\mathsf{Sing}(X_s))]\in K_0(HH(\mathcal{B})).$$

The Chern character of this element belongs to $H^0(S,r_\ell(HH(\mathcal{B})))$ which by lax monoidality has a natural morphism

$$H^0(S,r_\ell(HH(\mathcal{B})))\longrightarrow H^0(S,HH(r_\ell(\mathcal{B}))).$$

We have already seen that the realization $r_\ell(\mathcal{B})$ is naturally equialent to $i_*(\mathbb{Q}_\ell(\beta)\oplus\mathbb{Q}_\ell(\beta)[1])$ which is a commutative monoid in $\mathcal{SH}_S$. As such it has a canonical projection

$$HH(r_\ell(\mathcal{B}))\longrightarrow i_*(\mathbb{Q}_\ell(\beta)).$$

All together, the composition of the Chern map and this projection induces a well defined morphism

$$K_0(HH(\mathcal{B}))\longrightarrow H^0(S,i_*(\mathbb{Q}_\ell(\beta)))\simeq\mathbb{Q}_\ell.$$

The above morphism is by definition the *rank function* and will simpy de denoted by χ.

Definition 5.0.2 *With the notations as above, the* categorical Bloch number *of X/S is defined by*

$$[\Delta_X, \Delta_X]_S^{cat} := \chi([HH(\mathsf{Sing}(X_s)]) \in \mathbb{Q}_\ell.$$

The number defined above is a priori only an ℓ-number. We will not investigate this in the present paper but it can be shown that it is in fact an integer and that it always coincides with the original definition of the Bloch's intersection number appearing in the conjecture 5.0.1.

We now arrive at our main theorem.

Theorem 5.0.3 *With X/S as above, and assume that the inertia subgroup* $\mathrm{I} := \mathsf{Gal}(\bar{K}/K^{unr}) \subseteq \mathsf{Gal}(\bar{K}/K)$ *acts unipotently on $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$. Then we have*

$$[\Delta_X, \Delta_X]_S^{cat} = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell).$$

Proof: We apply our trace formula 2.4.9 for $T = \mathsf{Sing}(X_s)$ and $f = id$. For this we need to check that the conditions of theorem 2.4.9 are satisfied.

By Prop. 4.3.1, we know that T is saturated over $\mathcal{B}$. Moreover, because the action of I is unipotent it is also tame, and thus by Cor. 3.4.4 we have the Kunneth formula

$$r_\ell(\mathsf{Sing}(X \times_S X)) \simeq (\mathbb{H}(X_s, \nu_X) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X)^I(\beta),$$

where as usual ν_X denotes vanishing cycles on X_s. Moreover, by Kunneth for dg-categories of singularities we have the canonical equivalence of Prop. 4.2.1

$$T^o \otimes_{\mathcal{B}} T \simeq \mathsf{Sing}(X \times_S X).$$

Putting these together, and using the main theorem of [BRTV] we see that the morphism

$$r_\ell(T^o) \otimes_{r_\ell(\mathcal{B})} r_\ell(T) \longrightarrow r_\ell(T^o \otimes_{\mathcal{B}} T)$$

is equivalent to the Kunneth map (tensored with $\mathbb{Q}_\ell(beta)$)

$$\mathbb{H}(X_s, \nu_X[-1])^I \otimes_{\mathbb{Q}_\ell^I} \mathbb{H}(X_s, \nu_X[-1])^I \longrightarrow (\mathbb{H}(X_s, \nu_X) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X))^I[-2].$$

The fact that this morphism is an equivalence is now a consequence of the following lemma.

Lemma 5.0.4 *Let $\mathcal{D}_{uni}(I, \mathbb{Q}_\ell)$ be the full sub-∞-category of $\mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell)$ consisting of all object E for which the action of I on $H^i(E)$ is unipotent. Then the invariant ∞-functor induces an equivalence of symmetric monoidal ∞-categories*

$$(-)^I : \mathcal{D}_{uni}(I, \mathbb{Q}_\ell) \simeq \mathcal{D}_{pe}(\mathbb{Q}_\ell[\epsilon_1]),$$

where $\mathbb{Q}_\ell[\epsilon_1]$ is the free commutative dg-algebra generated by ϵ_1 is degree 1, and $\mathcal{D}_{pe}(\mathbb{Q}_\ell[\epsilon_1])$ is its ∞-category of perfect dg-modules.

Proof: This is a well known fact. The ∞-functor

$$(-)^I : \mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}_c(\mathbb{Q}_\ell)$$

is lax symmetric monoidal so induces a lax monoidal ∞-functor

$$(-)^I : \mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}(\mathbb{Q}_\ell^I).$$

It is easy to see that $\mathbb{Q}_\ell^I$ is canonically equivalent to $\mathbb{Q}_\ell[\epsilon_1]$, and the choice a such an equivalence only depends on the choice of a generator of $H^1(I, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$. We thus have an induced lax symmetric monoidal ∞-functor

$$(-)^I : \mathcal{D}_c(Spec\, K, \mathbb{Q}_\ell) \longrightarrow \mathcal{D}(\mathbb{Q}_\ell[\epsilon_1]).$$

The above ∞-functor is in fact symmetric monoidal, as it preserves unit objects and $D_c(Spec\, K, \mathbb{Q}_\ell)$ is generated by the unit object $\mathbb{Q}_\ell$. Finally, $(-)^I$ is also an equivalence for the same reason. $\square$

The above lemma implies that the Kunneth map

$$\mathbb{H}(X_s, \nu_X[-1])^I \otimes_{\mathbb{Q}_\ell^I} \mathbb{H}(X_s, \nu_X[-1])^I \longrightarrow (\mathbb{H}(X_s, \nu_X[-1]) \otimes_{\mathbb{Q}_\ell} \mathbb{H}(X_s, \nu_X[-1]))^I$$

is an equivalence and thus that T is admissible over $\mathcal{B}$ as wanted.

We have checked the conditions for the trace formula 2.4.9 so we have

$$Ch(HH(T/B)) = Tr(id : r_\ell(T))$$

in $H^0(S, r_\ell(HH(\mathcal{B})))$. We use our morphism χ in order to get an equality in $\mathbb{Q}_\ell$

$$\chi(Ch(HH(T/B)) = \chi(Tr(id : r_\ell(T))).$$

We now have to identify the two sides of the above formula. On the left hand side, we have by definition our Bloch number $[\Delta_X, \Delta_X]_S^{cat}$. Unfolding the definition, and using lemma 5.0.4 above the right hand side can be described as follows. The dg-algebra $\mathbb{Q}_\ell^I$ is such that $K_0(\mathbb{Q}_\ell^I) \simeq \mathbb{Z}$. Viewing $\mathbb{Z}$ inside $\mathbb{Q}_\ell$, this isomorphism is induced by sending the class of dg-module E to the trace of the identity inside $HH_0(\mathbb{Q}_\ell^I) \simeq \mathbb{Q}_\ell$. Using the functoriality of the trace for the morphism of commutative dg-algebras $\mathbb{Q}_\ell^I \longrightarrow \mathbb{Q}_\ell^I(beta)$, we see that the right hand side of our formula simply is the trace of the identity on $\mathbb{H}(X_s, \nu_X[-1])$ as an object inside $\mathcal{D}_{uni}(I, \mathbb{Q}_\ell)$. This trace is easy to compute, as it equals 1 on the unit object $\mathbb{Q}_\ell$. As the unit object generates $\mathcal{D}_{uni}(I, \mathbb{Q}_\ell)$, we have that the trace of the identity on any object E equals the Eule characteristic of the underlying complex of $\mathbb{Q}_\ell$-spaces.

We thus have shown that

$$[\Delta_X, \Delta_X]_S^{cat} = \sum_i (-1)^i dim_{\mathbb{Q}_\ell} H^{i-1}(X_s, \nu_X).$$

However, by proper base change the complex $\mathbb{H}(X_s, \nu_X)$ appears in an exact triangle

$$H(X_{\bar{k}}, \mathbb{Q}_\ell) \longrightarrow H(X_{\bar{K}}, \mathbb{Q}_\ell) \longrightarrow \mathbb{H}(X_s, \nu_X).$$

We therefore have shown the equality

$$[\Delta_X, \Delta_X]_S^{cat} = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell)$$

as required. $\square$

6 Further comments

We end this paper by few comments about our theorem 5.0.3 and the general strategy. A first comment is of course that unipotent monodromy is a rather restrictive condition as it implies in particular tameness and thus our theorem does not see any interesting arithmetic aspects such as the Swan conductor. However, we would like to convince the reader here that this is only the beginning of the story and that the our Kunneth formula for invariant vanishing cycles Prop. 3.4.2 will most probably allow to also treat non-tame situation.

As a first comment, we note that the dg-category $T := \mathsf{Sing}(X_s)$ together with its $\mathcal{B}$-module recovers the entire Galois action on vanishing cohomology as follows. For each totally ramified finite cover $S' \longrightarrow S$ of group G, we consider $\mathcal{D}_{S'} := \mathsf{Sing}(S'_s)$, the dg-category of singularities of the special fiber of S' over S. By Prop. 3.4.2, we have

$$r_\ell(T \otimes_{\mathcal{B}} \mathcal{D}^o_{S'}) \simeq (\mathbb{H}(X_s, \nu_X) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(G))^I(\beta),$$

where $\mathbb{Q}_\ell(G) = r_\ell(\mathcal{D}_{S'})$ is the reduced group algebra of G. If we denote by $I' \subset I$ the kernel of the quotient map $I \to G$ we thus obtained

$$r_\ell(T \otimes_{\mathcal{B}} \mathcal{D}^o_{S'}) \simeq cofib(\mathbb{H}(X_s, \nu_X)^I \to \mathbb{H}(X_s, \nu_X)^{I'})(\beta).$$

By passing to the limit this produces a manner to recover the cofiber (tensor $\mathbb{Q}_\ell(\beta)$ as usual) of the morphism

$$\mathbb{H}(X_s, \nu_X)^I \to \mathbb{H}(X_s, \nu_X).$$

Moreover, as each G acts on $\mathcal{D}_{S'}$ we can also reconstruct the action of I on this cofiber.

This comment makes very plausible the fact that theorem 5.0.3 can be pursued beyond the unipotent case, by making use of the tensor product $T \otimes_{\mathcal{B}} \mathcal{D}^o_{S'}$ (for a cover $S' \to S$ making the monodromy unipotent) and then applying our trace formula for the G action on it. This idea is currently under investigation.

A Localizations of monoidal dg-categories

In this appendix we remind some basic facts about localizations of dg-categories introduced in [To1]. The purpose of the section is to explain the multiplicative properties of the localization construction. In particular, we explain how localization of strict monoidal dg-categories gives rise to monoids in $\mathbf{dgCat}_A$, and thus to monoidal dg-categories in the sense of our definition 2.1.1.

Let T be a dg-category over A, together with W a set of morphisms in $Z^0(T)$, the underlying category of T (this is the category of 0-cycles in T, i.e. $Z^0(T)(x,y) := Z^0(T(x,y))$). For the sake of brevity, we will just say that W *is set of maps in* T. In other words, we allow W not to be strictly speaking a *subset* of the set of morphisms in T, but just a set together with a map $W \to Mor(T)$ from W to the set of morphisms in T. Recall that a localization of T with respect to W, is a dg-category $L_W T$ together with a morphism in $\mathbf{dgCat}_A$

$$l : T \longrightarrow L_W T$$

such that, for any $U \in \mathbf{dgCat}_A$, map induced by l on mapping spaces

$$Map(L_W T, U) \longrightarrow Map(T, U)$$

is fully faithful and its image consists of all $T \to U$ sending W to equivalences in U (i.e. the induced functor $[T] \to [U]$ sends elements of W to isomorphisms in $[U]$).

As explained in [To1], localization always exists, and are unique up to a contractible space of choices (because they represents an obvious ∞-functor). We will describe here a model for $(T, W) \mapsto L_W T$ which will have nice properties with respect to tensor products of dg-categories. For this, let $dgcat_A^{W,c}$ be the category of pairs (T, W), where T is a dg-category with cofibrant hom's over A, and W a set of maps in T. Morphisms $(T, W) \longrightarrow (T', W')$ in $dgcat_A^{W,c}$ are dg-functors $T \longrightarrow T'$ sending W to W'.

We fix once for all a factorization

$$\Delta^1_A \xrightarrow{j} \tilde{I} \xrightarrow{p} \overline{\Delta}^1_A,$$

with j a cofibration and p a trivial fibration. Here Δ^1_A is the A-linearisation of the category Δ^1 that classifies morphisms, and $\overline{\Delta}^1_A$ is the linearisation of the category that classifies isomorphisms. For an object $(T, W) \in dgcat_A^{W,c}$ we define $W^{-1}T$ by the following cocartesian diagram in dg-categories

$$\begin{array}{ccc} \coprod_W \Delta^1_A & \longrightarrow & T \\ \downarrow & & \downarrow \\ \coprod_W \tilde{I} & \longrightarrow & W^{-1}T, \end{array}$$

where $\coprod_W \Delta^1_A \longrightarrow T$ is the canonical dg-functor corresponding to the set W of morphisms in T.

Lemma A.0.1 *The canonical morphism $l : T \longrightarrow W^{-1}T$ defined above is a localization of T along W.*

Proof. According to [To1], the localization of T can be construced as the homotopy push-out of dg-categories

$$\begin{array}{ccc} \coprod_W \Delta^1_A & \longrightarrow & T \\ \downarrow & & \downarrow \\ \coprod_W A & \longrightarrow & L_W T. \end{array}$$

The lemma then follows from the observation that when T has cofibrant hom's, then the push-out diagram defining $W^{-1}T$ is in fact a homotopy push-out diagram. $\square$

The construction $(T, W) \longrightarrow W^{-1}T$ clearly defines a functor

$$dgcat_A^{W,c} \longrightarrow dgcat_A^c$$

from $dgcat_A^{W,c}$ to $dgcat_A^c$, the category of dg-categories with cofibrant hom's. Moreover, this functor comes equipped with a natural symmetric colax monoidal structure. Indeed, $dgcat_A^{W,c}$ is a symmetric monoial category, where the tensor product is given by

$$(T, W) \otimes (T', W') := (T \otimes_A T', W \otimes id \cup id \otimes W').$$

We have a natural map

$$T \otimes_A T' \longrightarrow (W^{-1}T) \otimes_A ((W')^{-1}T'),$$

which by construction has a canonical extension

$$(W \otimes id \cup id \otimes W')^{-1}(T \otimes_A T') \longrightarrow (W^{-1}T) \otimes_A ((W')^{-1}T').$$

The unit in $dgcat_A^{W,c}$ is $(A, \emptyset)$, which provides an canonical isomorphism $(\emptyset)^{-1}A \simeq A$. These data endow the functor $(T, W) \mapsto W^{-1}T$ with a symmetric colax monoidal structure. By composing with the canonical symmetric monoidal ∞-functor $dgcat_A^c \longrightarrow \mathbf{dgCat}_A$, we get a symmetric colax monoidal ∞-functor

$$dgcat_A^{W,c} \longrightarrow \mathbf{dgCat}_A,$$

which sends (T, W) to $W^{-1}T$. By [To3, Ex. 4.3.3], this colax symmetric monoidal ∞-functor is in fact monoidal. We thus have a symmetric monoidal localization ∞-functor

$$W^{-1}(-) : dgcat_A^{W,c} \longrightarrow \mathbf{dgCat}_A.$$

As a result, if T is a (strict) monoid in $dgcat_A^{W,c}$, then $W^{-1}T$ carries a canonical structure of a monoid in $\mathbf{dgCat}_A$. This applies particularly to stict monoidal dg-categories endowed with a compatible notion of equivalences. By MacLane coherence theorem any such a structure can be turned into a strict monoid in $dgcat_A^{W,c}$, and by localization into a monoid in $\mathbf{dgCat}_A$. In other words, the localization of a monoidal dg-categorie along a set of maps W that is compatible with the monoidal structure, is a monoid in $\mathbf{dgCat}_A$. The same is true for dg-categories which are modules over a given monoidal dg-category.

B Auxiliary results

In this Appendix, for the readers' convenience, we simply collect the proofs of some technical results that are used in the main text. Most of them are easy and/or probably well-known, and we claim no originality, but we were not able not locate them in the literature.

Lemma B.0.1 *Let $\mathcal{C}$ be a stable symmetric monoidal ∞-category, and*

$$u : x \to y \qquad v : x' \to y'$$

two morphisms. Let $C(u)$ be the cone of u, $C(v)$ be the cone of v, and $C(u \otimes v)$ the cone of the tensor product $u \otimes v : x \otimes x' \to y \otimes y'$. Then, there exists a natural exact triangle

$$C(u) \otimes x' \bigoplus x \otimes C(v) \longrightarrow C(u \otimes v) \longrightarrow C(u) \otimes C(v).$$

Proof. Factor $u \otimes v$ as $x \otimes x' \xrightarrow{u \otimes id} y \otimes x' \xrightarrow{id \otimes v} y \otimes y'$, and apply the octahedral axiom to the triangles

$$x \otimes x' \xrightarrow{u \otimes id} y \otimes x' \xrightarrow{f} C(u) \otimes x'$$

$$y \otimes x' \xrightarrow{id \otimes v} y \otimes y' \longrightarrow y \otimes C(v) \xrightarrow[{[1]}]{d'} y \otimes x'[1] \ ,$$

to get a triangle

$$C(u)\otimes x' \longrightarrow C(u\otimes v) \longrightarrow y\otimes C(v) \xrightarrow[{[1]}]{\theta} C(u)\otimes x'[1]$$

together with the compatibility $\theta = f[1]\circ d'$. Now observe that $\theta\circ(u\otimes \mathrm{id}_{C(v)}) = 0$, and apply the octahedral axiom to the triangles

$$x\otimes C(v) \xrightarrow{u\otimes id} y\otimes C(v) \longrightarrow C(u)\otimes C(v)\ ,$$

$$y\otimes C(v) \xrightarrow{\theta} C(u)\otimes x'[1] \xrightarrow{f} C(u\otimes v)[1]$$

to conclude. □

Lemma B.0.2 *Let k be an algebraically closed field, $p_X : X \to s := \operatorname{Spec} k$, $p_Y : Y \to s$ be proper morphisms of schemes, and $p_1 : Z := X \times_s Y \to X$, $p_2 : Z := X \times_s Y \to Y$ the natural projections. If ω_Z, ω_X, ω_Y denote the $\mathbb{Q}_\ell$-adic dualizing complexes of Z, X, and Y, respectively, there is a canonical equivalence*

$$a : p_1^*\omega_X \otimes p_2^*\omega_Y \longrightarrow \omega_Z.$$

Proof. We first exhibit the map a. We denote simply by $\underline{\mathrm{Hom}}_T(-,-)$ the derived internal hom in $\mathcal{D}_c(T,\mathbb{Q}_\ell)$ (so that $D := \underline{\mathrm{Hom}}_Z(-,\omega_Z)$ is the $\mathbb{Q}_\ell$-adic duality on Z). By adjunction, giving a map a is the same thing as giving a map $\omega_X \to (p_1)_*\underline{\mathrm{Hom}}_Z(p_2^*\omega_Y, p_2^!\omega_Y)$. Since p_X is proper, by [SGA4-III, Exp XVIII, 3.1.12], we have a canonical equivalence

$$(p_1)_*\underline{\mathrm{Hom}}_Z(p_2^*\omega_Y, p_2^!\omega_Y) \longrightarrow p_X^!(p_Y)_*\underline{\mathrm{Hom}}_Y(\omega_Y,\omega_Y).$$

Therefore, we are left to define a map

$$\omega_X \simeq p_X^!\mathbb{Q}_\ell \to p_X^!(p_Y)_*\underline{\mathrm{Hom}}_Y(\omega_Y,\omega_Y),$$

and we take $p_X^!(\alpha)$ for this map, where α is the adjoint to the canonical map $\mathbb{Q}_\ell \simeq p_Y^*\mathbb{Q}_\ell \to \underline{\mathrm{Hom}}_Y(\omega_Y,\omega_Y)$. One can then prove that a is an equivalence, by checking it stalkwise. □

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