Derived Algebraic Geometry and Deformation Quantization

Bertrand Toën (CNRS, Université de Montpellier 2)

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Aim : Study the moduli spaces of sheaves on higher dimensional compact oriented manifolds, from the point of views of homotopy theory and deformation quantization.

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Purpose of the present talk : Explain and comment this result.

Plan of the lecture

1. Deformation quantization

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2. Low dimensions

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3. Derived algebraic stacks

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4. Existence of quantizations

Deformation quantization

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- "less commutative" = "less symmetric" (e.g. braided, associative ...).

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- The non-commutative deformation $A_\hbar$ of A is (a part of) the 0-quantization of our theorem.

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- $Rep_\hbar(G)$ is the 2-quantization of $Bun_G(X) = BG$ of our theorem.

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  which is not quite an algebraic variety : it is a derived algebraic stack (explanations later on).
- $Bun_G(X)$ is very singular, but is locally the critical locus of a polynomial function

$$f : \mathbb{C}^n \longrightarrow \mathbb{C} \qquad Bun_G(X) \simeq Crit(f) = \{x \in \mathbb{C}^n / f'(x) = 0\}.$$

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  $\chi(Bun_G(X), \mathcal{E})$ gives the *Donaldson-Thomas invariants of* X.
- The sheaf $\mathcal{E}$ is very closely related to the (-1)-quantization of our theorem (explanations later on).

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Important fact : X behaves as a formal thickening of $H^0(X)$. Very similar to $Y_{red} \subset Y$.

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Derived algebraic stacks can have easy algebraic description (G-equivariant dg-algebras) but live in a very complicated (∞-)category.

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- Derived stacks of stable maps, derived (higher) stacks of objects in nice derived categories.

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- Cellular decomposition of X allows to describe $Bun_G(X)$.

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Important for us : nicer infinitesimal behaviour.

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Warning : not true for underived mapping stack.

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- If $n < 0$, deformation of E_n-monoidal structures makes sense as E_{-n}-monoidal structure with formal parameter $\hbar_n$ of degre $-2n$.

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Theorem (T.)

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- for $n = 1$ and Z a smooth variety it was a conjecture of Kapustin (ICM 2010). For $n > 0$ conjectured by Kontsevich-Voronov.
- Requires understanding the deformations of E_n-$\otimes$-∞-categories (already $n = 0$ was open, see Lurie's ICM 2010).

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Formality $\Rightarrow$ Poisson structures of degree n on $Bun_G(X)$ provides an n-quantization of $Bun_G(X)$.

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on $Crit(f)$ $\mathcal{M} = MF(f)$, the category of matrix factorizations of f.
The sheaf $\mathcal{E}$ is then the sheaf of Hochschild homology of $\mathcal{M}$.

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- Notion of geometric quantization in the derived setting (Wallbridge).
- Ultimately $\rightsquigarrow$ fully extended TQFT's.
- Motivic aspects : objects in the quantized $D_{qcoh}(Bun_G(X))$ can be thought as "non-commutative motives" over $Bun_G(X)$ (at least their "de Rham" part).

Thank you