Derived symplectic stacks over a space

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Septembre 2019

The setting: Let X be a locally compact Hausdorff topological space and k an algebraically closed field of characteristic zero (but the whole story works over any commutative base ring). Denote by $Sh(X) = Sh(X,k)$ the (∞-)category of sheaves of complexes of k-modules on X. Similarly, $CoSh(X) = CoSh(X,k)$ is the (∞-)category of cosheaves of complexes of k-modules on X.

Verdier duality can be expressed by stating the the functor

$$V : Sh(X) \longrightarrow CoSh(X)$$

sending a sheaf E to the cosheaf $U \mapsto \mathbb{H}_c(U,E)$ of cohomology with compact supports, is an equivalence.

The image of the constant sheaf k by V is denoted by K_c. It is the dualizing cosheaf of X, and it is related to the usual dualizing sheaf K by

$$K \simeq K_c^\vee$$

where $(-)^\vee : CoSh(X) \longrightarrow Sh(X)$ is given by pointwise k-linear duals.

We let $dSt(X)$ be the (∞-)category of sheaves on X with values in derived stacks over k. An object in $dSt(X)$ will be denoted by $\mathcal{F}$, and the value of $\mathcal{F}$ at an open $U \subset X$ will be denoted by $\mathcal{F}_U$. The object $\mathcal{F}_U$ is itself a functor $\mathbf{cdga}_k \longrightarrow \mathbb{T}$, from commutative dg k-algebras to homotopy types. For a continuous map $f : X \longrightarrow Y$ we have the usual push-foward pull-back an adjunction

$$p^* : dSt(Y) \rightleftarrows dSt(X) : p_*.$$

Note that we also have adjunctions on sheaves and cosheaves

$$p^* : Sh(Y) \rightleftarrows Sh(X) : p_* \qquad p_+ : CoSh(Y) \rightleftarrows CoSh(X) : p^+.$$

Using the Verdier equivalence V above, p_+ becomes $p_! : Sh(Y) \longrightarrow Sh(X)$ and p^+ becomes $p^! : Sh(Y) \longrightarrow Sh(X)$. When p is proper $p_! \simeq p_*$.

1 Closed forms

Let $\mathcal{F} \in dSt(X)$. We consider $U \mapsto \mathcal{A}^{p,cl}(\mathcal{F}_U)$, which is a copresheaf of complexes of k-modules on U denote by $\mathcal{A}^{p,cl}(\mathcal{F})$. The associated cosheaf is denoted by $\underline{\mathcal{A}}^{p,cl}(\mathcal{F})$. There is a canonical morphism

$$\underline{\mathcal{A}}^{p,cl}(\mathcal{F}) \longrightarrow \mathcal{A}^{p,cl}(\mathcal{F}).$$

Definition 1.0.1 *The complex of closed p-forms on F is defined by*

$$\mathbb{A}^{p,cl}(\mathcal{F}) := \mathbb{R}\underline{Hom}_{CoSh(X)}(K_c, \underline{\mathcal{A}}^{p,cl}(\mathcal{F}).$$

There is also a notion of closed p-forms with coefficients in a any cosheaf $E \in CoSh(X)$, by considering $\mathbb{R}\underline{Hom}_{CoSh(X)}(E, \underline{\mathcal{A}}^{p,cl}(\mathcal{F}))$.

Let $f : X \longrightarrow Y$ be a proper map between locally compact and Hausdorff spaces. We denote by $K_c^X \in CoSh(X)$ and $K_c^Y \in CoSh(Y)$ the corresponding dualizing cosheaves. Because f is proper, there is a pull-back map on cohomology with compact supports, and thus a natural morphism of cosheaves on Y

$$f^* : K_c^Y \longrightarrow f_+(K_c^X).$$

For $\mathcal{F} \in dSt(X)$, and a closed p-form ω of degree n on $\mathcal{F}$, we represent ω by a map of cosheaves

$$K_c^X \longrightarrow \underline{\mathcal{A}}^{p,cl}(\mathcal{F})[n].$$

Applying p_+ and composing with f^* we get a morphism of cosheaves on Y

$$K_c^Y \longrightarrow p_+(\underline{\mathcal{A}}^{p,cl}(\mathcal{F}))[n].$$

There exists moreover a canonical map of cosheaves

$$p_+(\underline{\mathcal{A}}^{p,cl}(\mathcal{F})) \longrightarrow \underline{\mathcal{A}}^{p,cl}(p_*(\mathcal{F}))$$

obtained as follows. On the level of copreshaves, there is a canonical isomorphism

$$p_+(\mathcal{A}^{p,cl}(\mathcal{F})) \simeq \mathcal{A}^{p,cl}(p_*(\mathcal{F})).$$

We can compose with the associated cosheaf functor to get

$$p_+(\underline{\mathcal{A}}^{p,cl}(\mathcal{F})) \longrightarrow p_+(\mathcal{A}^{p,cl}(\mathcal{F})) \simeq \mathcal{A}^{p,cl}(p_*(\mathcal{F})).$$

As $p_+(\underline{\mathcal{A}}^{p,cl}(\mathcal{F}))$ is a cosheaf, this map factors uniquely through the associated cosheaf $\underline{\mathcal{A}}^{p,cl}(p_*(\mathcal{F}))$ of $\mathcal{A}^{p,cl}(p_*(\mathcal{F}))$.

To conclude, for any closed p-form ω of degree n on $\mathcal{F}$, we get an induced closed p-form $f_*(\omega)$ of degree n on $f_*(\mathcal{F})$. In a more functorial manner: there is a natural map of complexes

$$f_* : \mathbb{A}^{p,cl}(\mathcal{F}) \longrightarrow \mathbb{A}^{p,cl}(f_*(\mathcal{F}))$$

2 Non-degeneracy

We now express the notion of being non-degenerate. At the moment we havent put conditions on $\mathcal{F}$ at all. We need some extra conditions in order to be able to spell out the non-degeneracy conditions. The strongest conditions require representability, that will be discussed in the next section. Here we put a very weak condition: that all the stacks $\mathcal{F}_x$ are formally representable at each closed point.

Let $\mathcal{F} \in dSt(X)$. We can consider $\mathcal{F}$ as a functor

$$\mathcal{F}(-) : \mathbf{cdga}_k \longrightarrow St(X)$$

from commutative cdga's to stacks (of homotopy types) over X.

Definition 2.0.1 *The derived stack $\mathcal{F}$ over X is* locally formally representable *if for all cartesian diagram of local artinian cdga's*

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & A \end{tikzcd}$$

where $H^0(A') \to H^0(A)$ is surjective, the corresponding diagram

$$\begin{tikzcd} \mathcal{F}(B') \arrow[r] \arrow[d] & \mathcal{F}(B) \arrow[d] \\ \mathcal{F}(A') \arrow[r] & \mathcal{F}(A) \end{tikzcd}$$

is cartesian in $St(X)$.

The above notion is equivalent to the fact that for all point $x \in X$, the derived stack $\mathcal{F}_x \in dSt$ is an unpointed formal moduli problem over k, or equivalently that for any open $U \subset X$ the derived stack $\mathcal{F}_U$ is an unpointed formal moduli problem. Well, this is true if $St(X)$ is defined as being the hypercomplete topose associated to X, unless it is wrong. We will be working in situations where X is nice enough so that $St(X)$ is hypercomplete anyway.

Recall also that a functor $F : \mathbf{cdga}_k \longrightarrow \mathbb{T}$ is an unpointed formal moduli problem if for all k-point $z \in F(k)$, the restriction of (F, z) to augmented artinian dg-algebras is a formal moduli problem (i.e. is associated to a dg-lie). Therefore, for such F and any point $z \in F(k)$ we have a well defined dg-lie algebra $\mathcal{L}_{F,z} \in dg - lie_k$.

For any unpointed formal moduli problem F, and any point $z \in F(k)$, there is a canonical morphism of complexes

$$\mathcal{A}^{p,cl}(F) \longrightarrow \wedge_k^p(\mathcal{L}_{F,z}^\vee)[-p],$$

which restricts the forms at the closed point z.

We can now define non-degeneracy, for any closed 2-form ω of degree n on any derived stack $\mathcal{F}$ over X which is moreover locally formally representable in the sense of definition 2.0.1. For this, we first notice that for any open $U \subset X$ and any k-point $z \in \mathcal{F}_U(k)$, we have the corresponding dg-lie algebra $\mathcal{L}_{\mathcal{F}_U,z}$. This construction, for small opens $U' \subset U$ defines a sheaf of dg-lie on U denoted by $\mathcal{L}_{\mathcal{F},z} \in dglie(Sh(U))$. By duality this defines a copresheaf on U, whose associated cosheaf will be denoted by $\mathcal{L}_{\mathcal{F},z}^\vee$.

There is a canonical morphism of cosheaves on U (we still have $z \in \mathcal{F}_U(k)$ fixed here)

$$\underline{\mathcal{A}}^{2,cl}(\mathcal{F}) \longrightarrow \wedge^2(\mathcal{L}_{\mathcal{F},z}^\vee)[-2].$$

The form $\omega : K_c \longrightarrow \underline{\mathcal{A}}^{2,cl}(\mathcal{F})[n]$ then produce a natural morphism of cosheaves on U

$$K_c \longrightarrow \wedge^2(\mathcal{L}_{\mathcal{F},z}^\vee)[n-2].$$

By dualizing again, we get a morphism of sheaves on U

$$(\wedge^2(\mathcal{L}_{\mathcal{F},z}^\vee))^\vee[2-n] \longrightarrow K.$$

There is a canonical map $\mathcal{L}_{\mathcal{F},z} \to (\mathcal{L}_{\mathcal{F},z}^{\vee})^{\vee}$, as well as

$$\wedge^2(\mathcal{L}_{\mathcal{F},z}) \longrightarrow (\wedge^2(\mathcal{L}_{\mathcal{F},z}^{\vee}))^{\vee}.$$

We thus get this way a well defined map of sheaves on U

$$\Theta_{\omega,z} : \wedge^2(\mathcal{L}_{\mathcal{F},z})[2] \longrightarrow K[n].$$

Definition 2.0.2 *With the notations and conditions above the form ω is non-degenerate if for all open $U \in X$ and all $z \in \mathcal{F}_U(k)$, the above map of sheaves on U*

$$\Theta_{\omega,z} : \wedge^2(\mathcal{L}_{\mathcal{F},z})[2] \longrightarrow K[n]$$

induces an equivalence of sheaves on U

$$\mathcal{L}_{\mathcal{F},z}[1] \simeq \mathbb{R}\underline{\mathcal{H}om}(\mathcal{L}_{\mathcal{F},z}, K[n-1]).$$

Here is the proposed statement for preservation of non-degeneracy conditions.

Proposition 2.0.3 *Let $f : X \longrightarrow Y$ be a proper map between locally compact and Hausdorff spaces. Let $\mathcal{F}$ be a derived stack over X which is locally formally representable and ω a closed 2-form on $\mathcal{F}$ which is non-degenerate. Then $f_*(\omega)$ is non-degenerate on $f_*(\mathcal{F})$.*

Not sure if there are extra conditions to put in the statement (e.g. finiteness statements on $\mathcal{L}_{\mathcal{F},z}$). This seems to follow from the general fact that the image by f_* of a perfect pairing of sheaves $E \otimes F \to K^X$ remains a perfect pairing $f_*(E) \otimes f_*(F) \to K^Y$.

3 More on representability

We can go further on representability of $\mathcal{F}$ by internalizing Artin's axiom inside the topos of stacks over X. I havent looked at it carefully, and what follows does not seem the most general notion. Still, I think it is useful for us.

We start by $\mathcal{F} \in dSt(X)$, considered as a functor

$$\mathcal{F}(-) : \mathbf{cdga}_k \longrightarrow St(X).$$

We state Artin's conditions for the functor $\mathcal{F}$ as follows.

1. (local presentability) $\mathcal{F}$ commutes with filtered colimits.
2. (Algebraization) For a noetherian complete (discrete) local k-algebra (A, m), the natural map

$$\mathcal{F}(A) \longrightarrow lim_i \mathcal{F}(A/m^i)$$

is an equivalence of stacks over X.

3. (Nil-completeness) For any cdga A with postnikov tower $A_{\leq n}$, the natural map

$$\mathcal{F}(A) \longrightarrow lim_n \mathcal{F}(A_{\leq n})$$

is an equivalence of stacks over X.

4. (Inf-cartesian) For cdga A, A-module M and derivation $d : A \to M$, $\mathcal{F}$ sends the cartesian square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \oplus M[1] \end{tikzcd}$$

to a cartesian square of stacks over X.

5. (Cotangent complexes) For any open $U \subset X$, any cdga A and any point $z \in \mathcal{F}_U(A)$, the cotangent complex $\mathbb{L}_{\mathcal{F},z}$ exists as a cosheaf of complexes of A-modules over U. Moreover, it is stable by base change over A.

A word of explanations for the last condition. We state here that there exists a cosheaf of A-modules $\mathbb{L}_{\mathcal{F},z}$ on U, such that for any A-module we have a functorial equivalence

$$Map_{CoSh(U,A)}(\mathbb{L}_{\mathcal{F},z}, M) \simeq Fib(\mathcal{F}_U(A \oplus M) \to \mathcal{F}_U(A)),$$

where the fiber is taken at z.

It seems to me that the following result is true.

Proposition 3.0.1 *For a proper map $f : X \longrightarrow Y$, and $\mathcal{F} \in dSt(X)$ satisfying Artin's condition, then $f_*(\mathcal{F})$ satisfies Artin conditions on Y.*

The importance of Artin's conditions is that we can state a more global non-degeneracy condition for closed 2-forms. For $\mathcal{F}$ satisfying Artin's condition, we can form two global derived categories $QCoh(\mathcal{F})$ and $CoQCoh(\mathcal{F})$ of quasi-coherent sheaves of cosheaves over $\mathcal{F}$.

An object in $QCoh(\mathcal{F})$ is given by a family of quasi-coherent sheaves $E_U \in QCoh(\mathcal{F}_U)$, together with transition morphisms $E_U \to u^*E_V$ for any inclusion $u : V \subset U$, satisfying the usual cocycle conditions. We ask moreover that $U \mapsto E_U$ satisfies the sheaf condition : for any U and a covering sieve B on U

$$E_U \simeq lim_{V \in B} u^*(E_V).$$

In a similar manner we define quasi-coherent cosheaves, by reversing the arrows: they consist of families of objects $E_U \in QCoh(\mathcal{F}_U)$ with transition maps $u^*(E_V) \to E_U$, plus the cosheaf condition

$$colim_{V \in B} u^*(E_V) \simeq E_U.$$

The cotangent complex now exists as a global object $\mathbb{L}_{\mathcal{F}} \in CoQCoh(\mathcal{F})$. Its $\mathcal{O}$-linear dual defines a quasi-coherent sheaf $\mathbb{T}_{\mathcal{F}} \in QCoh(\mathcal{F})$. Now, a closed 2-form on $\mathcal{F}$ defines a well defined pairing of cosheaves

$$K_c \otimes \mathcal{O} \longrightarrow \wedge^2 \mathbb{L}_{\mathcal{F}},$$

which by duality provides a pairing of sheaves

$$\wedge^2\mathbb{T}_{\mathcal{F}} \longrightarrow K \otimes \mathcal{O}.$$

Here we need finiteness conditions on the dualizing cosheaf K_c in order to state that the dual of $K_c \otimes \mathcal{O}$ is $K \otimes \mathcal{O}$, unless we have to replace the right hand side by $\mathcal{O}^{K_c}$, the sheaf of maps $K_c \longrightarrow \mathcal{O}$, which we can call the *dualizing complex of* $\mathcal{F}$ *over* X.

By definition, a closed 2-form will be globally non-degenerate if the corresponding pairing induces an equivalence of quasi-coherent sheaves

$$\mathbb{T}_{\mathcal{F}} \longrightarrow \mathbb{R}\underline{\mathcal{H}om}(\mathbb{T}_{\mathcal{F}}, K \otimes \mathcal{O}).$$

When all the object $\mathbb{T}_{\mathcal{F}}$ and $\mathbb{L}_{\mathcal{F}}$ are perfect over $\mathcal{F}$, this last condition can be checked pointwise, and we recover the condition of definition 2.0.2.