RANDOM POPULATION DYNAMICS UNDER CATASTROPHIC EVENTS.

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Abstract. In this paper we introduce new Birth-and-Death processes with partial catastrophe and study some of their properties. In particular we obtain some estimates for the mean catastrophe time, and the first and second moments of the distribution of the process at a fixed time t. This is completed by some asymptotic results.

Key words : Birth-and-Death process; Population Dynamics; Extinction Time; Birth, Death and Catastrophe Process

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1. Introduction

The aim of this work is to propose a model for the evolution of the size of a population submitted to exceptional conditions, like a genocide, see [Sind16]. To this end, we introduce a new Birth-and-Death type process with *partial catastrophe*. Indeed, Birth-and-Death processes (BD-processes for short) are the more standard stochastic models for the description of the evolution of a population's size.

A BD-process assigns arbitrary non-negative Birth-and-Death rate pairs to a birth or a death of an individual in the population. Hence, whenever the population size changes, it grows or decreases exactly by one individual. The BD-process is under suitable assumptions a continuous time Markov chain (CTMC) on the discrete state space $\mathbb{N}$ and jump size ± 1. For a more in depth discussion on continuous time Markov chains and BD-processes see the Textbooks [And91] or [Nor97].

BD-processes have a long history and were first discussed, amongst others, for arbitrary Birth-and-Death rate by Feller [Fel39] and Kendall [Ken48]. Note that BD-processes can also modelize immigration and emigration of individuals. Nonetheless, changes in the population size which concern not only individuals but groups cannot be considered. Brockwell and his coauthors pioneered in [BGR82], [Bro85] and [Bro86] an extension of the BD-process including the possibility of a catastrophe captured by a sudden and exceptionally large decrease in the population size. They were in particular concerned with the probability of

extinction as well as the time to extinction in such population models.
Mathematically one can capture catastrophes by allowing the process to make larger jumps downwards rather than the jump size of 1 in both directions in the classical framework of a BD-process. The classic Birth-and-Death rate pair is denoted by (λ_i, μ_i) for a population of size $i \in \mathbb{N}$. These are complemented by catastrophe rates $(\gamma_i)_{i\in\mathbb{N}}$ as well as corresponding law of the catastrophe sizes $(d_i(j))_{j\leq i}$ where $d_i(j)$ is the probability that a catastrophe in a population of size i leads to j deaths. The infinitesimal generator of the process has the following form.

Definition 1.1 (BD-process with catastrophes, [Bro85])**.** *A BD-process $X = (X_t)_{t\geq 0}$ with general catastrophes is a Continuous Time Markov Chain with values in $\mathbb{N}$ associated to an infinitesimal generator $Q = (q_{ij})_{i,j\in\mathbb{N}}$, of the form*

$$\begin{cases} q_{ij} = \gamma_i\, d_i(i-j)\mathbb{1}_{[0,i)}(j) + \mu_i \mathbb{1}_{i-1}(j) + \lambda_i \mathbb{1}_{i+1}(j), \quad j \neq i, \\ q_{ii} = -(\lambda_i + \mu_i + \gamma_i) + \gamma_i d_i(0), \end{cases}$$

with for any $i \in \mathbb{N}$, $\lambda_i, \mu_i, \gamma_i, d_i(k) \in \mathbb{R}_+$ and $\sum_{k=0}^{i} d_i(k) = 1$. Moreover $\lambda_0 = \mu_0 = \gamma_0 = 0$ and $\displaystyle\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = +\infty$.

An important class of the BD-processes with general catastrophes considers exclusively *total catastrophes* by setting $d_i(j) = \mathbb{1}_{\{j=i\}}$, i.e., in case of a catastrophe, the process jumps from its current state i to the state 0. Note that in Definition 1.1, since $q_{00} = 0$, the state 0 is absorbing. Therefore a total catastrophe may happen at most once before the population dies out. Moreover, without immigration ($\lambda_0 = 0$), a catastrophe leads to the extinction of the population. Note that the infinitesimal generator Q retains a tridiagonal form, if one only considers the states $i \geq 1$. Van Doorn and Zeifmann use this fact in [vDZ04] and [vDZ05] to investigate the transition probabilities at any time t and to extend the classical representation result of the transition probabilities of a BD-process in terms of associated orthogonal polynomials by Karlin and McGregor in [KM58]. Assuming constant catastrophe rates $\gamma_i \equiv \gamma$, Swift obtains in [Swi01] explicit expressions for the transition probabilities in terms of their generating function.

In [BGR82] the authors introduce BD-processes with different types of catastrophes: Geometric catastrophes, Uniform catastrophes and Binomial catastrophes; see also [Dicetal08]. The binomial model, later studied in [Kapetal16], considers a binomial redistribution of the population on the set of integers up to the current one. This induces a new expected population size concentrated around a fixed proportion $p \in [0, 1]$ of the previous one, which does not seem to correspond to the data we want to fit. The practical and statistical aspects of the problem will however not be discussed here.

In the present paper, we extend the study of populations under total catastrophes by considering populations which are subject to *partial catastrophes*. This means that, with positive rate γ_i, the population size i can be drastically reduced to a distinguished state $\mathfrak{n} \geq 1$ - as soon as i exceeds $\mathfrak{n}$. Remark that, in contrast to a total catastrophe, the population can not die out as a consequence of a partial catastrophe. Of course this model is the simplest one in this spirit, and instead of only one catastrophic new state $\mathfrak{n}$, we could consider a new distribution concentrated around $\mathfrak{n}$. Nevertheless, the study of this simple mathematical model is the first necessary step.

We define in Section 2 the exact class of processes we will study and present our main results. We are interested in the first hitting time $T_X^{\mathfrak{n}}$ of the (catastrophic) population size $\mathfrak{n}$. Under the assumption of linear birth, death and catastrophe rates, we will use the tools introduced by Brockwell in [Bro86], in order to obtain explicit expressions for the expected catastrophe time $\mathbb{E}[T_X^{\mathfrak{n}}|X_0 = n_0]$, $n_0 \geq \mathfrak{n}$. We also identify its limiting behavior for large initial population i.e. when $n_0 \to \infty$. Proofs are presented in Section 3. We then study the first two moments of the population size at a fixed time t. After establishing positive recurrence of the BD-process with partial catastrophe in section 4 we compute and discuss explicit upper bounds for the process' first and second moment in section 5.

2. Birth-and-Death Process with Partial Catastrophe: our main results

We fix a *catastrophic state* $\mathfrak{n} \in \mathbb{N}^*$ and set $d_i(j) = \mathbb{1}_{\{j=i-\mathfrak{n}\}}, i \geq \mathfrak{n}$. We introduce a rate $\nu > 0$ to model the immigration if the population vanishes. $(\gamma_i)_{i\in\mathbb{N}}$, $(\lambda_i)_{i\in\mathbb{N}}$, $(\mu_i)_{i\in\mathbb{N}}$ are respectively the catastrophe, the birth and the death rates, where the index i represents the size of the population. All rates are assumed to be linear in the population size with proportionality coefficients respectively $\gamma, \lambda, \mu > 0$, see (2.1). Finally, throughout all this paper we assume that $\lambda > \mu$, that is the individual birth rate exceeds the individual death rate. Hence, in the absence of catastrophic event, the basic Birth-and-Death process with immigration would model a growing population.

We consider the CTMC on the state space $\mathbb{N}$ denoted by $X = (X_t)_{t\geq 0}$ whose infinitesimal generator $Q = (q_{ij})_{i,j\in\mathbb{N}}$ is given by

$$q_{ij} = \begin{cases} \lambda_0 = \nu, & \\ \lambda_i = \lambda \cdot i, & j = i+1, \\ \mu_i = \mu \cdot i, & j = i-1,\ i \geq 1, i \neq \mathfrak{n}+1, \\ \gamma_i = \gamma \cdot i, & j = \mathfrak{n},\ i > \mathfrak{n}+1, \\ \mu_i + \gamma_i, & j = \mathfrak{n},\ i = \mathfrak{n}+1, \\ -\sum_{j\neq i} q_{ij}, & j = i. \end{cases} \tag{2.1}$$

The choice of linear dependence with respect to the size of the population for the birth and death rates is very natural.

Its transition graph is depicted in Figure 1.

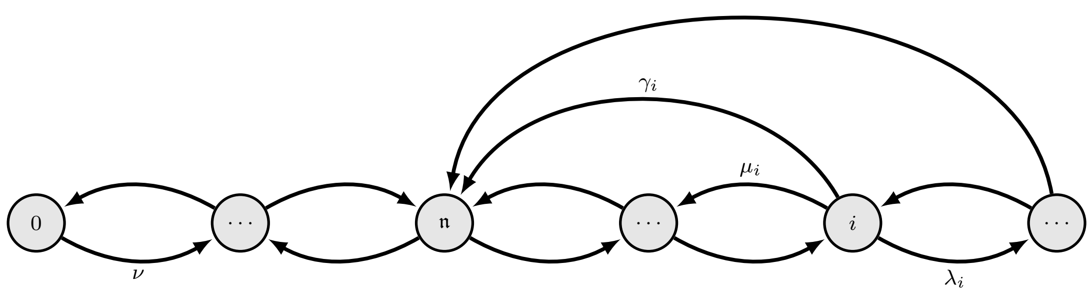

Figure 1. Transition graph of X.

Definition 2.1. *A Birth-and-Death process with partial catastrophe (BD+$C_{\mathfrak{n}}$ process) X is a CTMC with infinitesimal generator Q defined by the equations (2.1) and $X_0 = n_0 > \mathfrak{n}$. Its so-called* catastrophe time $T_X^{\mathfrak{n}}$ *is defined as its hitting time of the catastrophic state $\mathfrak{n}$:*

$$T_X^{\mathfrak{n}} := \inf\{t \geq 0 | X_t = \mathfrak{n}\}. \tag{2.2}$$

Our results on the catastrophe time $T_X^{\mathfrak{n}}$ are presented in the next three theorems. We state in Theorem 2.2 its almost sure finiteness and go on to study in Theorems 2.3 and 2.4 its expectation and its asymptotic behavior as the initial size of the population tends to ∞.

Theorem 2.2 (Finiteness of catastrophe time)**.** *Let X be a BD+$C_{\mathfrak{n}}$ process whose infinitesimal generator Q is given by (2.1). Then its catastrophe time $T_X^{\mathfrak{n}}$ is almost surely finite.*

Better, we can compute the first moment of $T_X^{\mathfrak{n}}$. Using the following notation

$$\mathbb{E}_i[T_X^{\mathfrak{n}}] := \mathbb{E}[T_X^{\mathfrak{n}} | X_0 = i + \mathfrak{n}], \tag{2.3}$$

we find an explicit expression for $\mathbb{E}_i[T_X^{\mathfrak{n}}]$ and infer its asymptotic behavior for $i \to \infty$.

Theorem 2.3 (Explicit computation of the mean catastrophe time)**.** *Let X be a BD+$C_{\mathfrak{n}}$ process whose infinitesimal generator Q is given by (2.1). Denote by $\underline{a} < \bar{a}$ the distinct real zeros of the polynomial $\mathbf{x} \mapsto \mu\mathbf{x}^2 - (\lambda + \mu + \gamma)\mathbf{x} + \lambda$. The mean catastrophe time - defined in (2.3) - is given by*

$$\mathbb{E}_i[T_X^{\mathfrak{n}}] = c\left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) \sum_{k=1}^{\infty} \frac{\underline{a}^k}{k + \mathfrak{n}} + c \sum_{k=1}^{i-1} \frac{1}{k + \mathfrak{n}} \left(\frac{1}{\bar{a}^{i-k}} - \frac{1}{\underline{a}^{i-k}}\right), \quad i \geq 1 \tag{2.4}$$

with $c = (\sqrt{(\lambda + \mu + \gamma)^2 - 4\lambda\mu})^{-1}$.

Moreover, we obtain an explicit decreasing rate of the mean catastrophe time for large initial populations, which is in some sense counterintuitive.

Corollary 2.4. *Let X be the above BD+$C_{\mathfrak{n}}$ process. The asymptotic behavior of its mean catastrophe time for large initial populations is:*

$$\mathbb{E}_i[T_X^{\mathfrak{n}}] = \mathcal{O}(i^{-1}).$$

Proofs of Theorems 2.2 - 2.3 and Corollary 2.4 are postponed to Section 3.

After establishing positive recurrence of the BD+$C_{\mathfrak{n}}$ process in Section 4 we will present in Section 5 the proofs of the following properties of - upper bounds for - the process' first and second moment.

Theorem 2.5 (Upper bound for the mean)**.** *Consider $(X_t)_{t\geq 0}$ the BD+$C_{\mathfrak{n}}$ process whose infinitesimal generator Q is given by (2.1). Then, the following upper bound holds:*

$$\mathbb{E}[X_t] \leq \bar{m}(t), \quad t \geq 0, \tag{2.5}$$

where the function $\bar{m}$ is the solution to the differential equation

$$\begin{cases} \bar{m}(0) = n_0, \\ \bar{m}'(t) = -\gamma\, \bar{m}(t)^2 + (\lambda - \mu + \gamma\, \mathfrak{n})\, \bar{m}(t) + \nu, \quad t > 0. \end{cases} \tag{2.6}$$

We also obtain a similar result for the second moment.

Theorem 2.6 (Upper bound for the second moment)**.** *Let $(X_t)_{t\geq 0}$ be the BD+C$_{\mathfrak{n}}$ process whose infinitesimal generator Q is given by (2.1). Consider the function $\bar{m}$ solution to (2.6) and assume that the initial size of the population is larger than a constant $\bar{m}_e$ computed in (5.4). Then, the second moment admits the following upper bound:*

$$\mathbb{E}[X_t^2] \leq \bar{v}(t), \quad t \geq 0, \tag{2.7}$$

the function $\bar{v}(t)$ being solution to the differential equation

$$\begin{cases} \bar{v}(0) = n_0^2, \\ \bar{v}'(t) = 2(\lambda - \mu)\bar{v}(t) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t) - \gamma\bar{v}(t)^{\frac{3}{2}} + \nu, \quad t > 0. \end{cases}$$

Such a $\bar{v}$ is bounded uniformly in time and so is, thus, $\mathbb{E}[X_t^2]$.

We close Section 5 with a discussion of the quality of the bounds $\bar{m}$ and $\bar{v}$ as defined in the previous theorems with a focus on their long time behavior.

3. Expected Catastrophe Time

This section consists of three subsections which lead through the proofs of Theorem 2.2, Theorem 2.3 and Corollary 2.4.

The proofs are based on various lemmas and propositions, which shed light on recurrence relations of order 2. In particular, we examine the limit behavior of their solutions in Lemma 3.3, the speed of divergence in Lemma 3.4, the dependence on the initial values in Lemma 3.5, Lemma 3.6 as well as possible explicit expressions for the minimal solution in Lemma 3.7 and Lemma 3.9.

3.1. Finiteness of Time of Catastrophe.

We analyze the catastrophe time $T_X^{\mathfrak{n}}$ using the auxiliary process $Y = (Y_t)_{t\geq 0}$ defined by shifting the process X by $\mathfrak{n}$ and stopping it at catastrophe time $T_X^{\mathfrak{n}}$:

$$Y_t := X_{t\wedge T_X^{\mathfrak{n}}} - \mathfrak{n}, \qquad t \geq 0. \tag{3.1}$$

We discuss its properties and their implications for $T_X^{\mathfrak{n}}$ in the following straightforward lemma.

Lemma 3.1. *Let X be the BD+C$_{\mathfrak{n}}$ process whose generator Q satisfies (2.1) with initial condition $X_0 = n_0 > \mathfrak{n}$. Then Y is a BD-process with total catastrophe whose birth, death and catastrophe rates are affine and given respectively by*

$$\tilde{\lambda}_i := \lambda\,(i + \mathfrak{n}), \quad \tilde{\mu}_i := \mu\,(i + \mathfrak{n}) \text{ and } \tilde{\gamma}_i := \gamma\,(i + \mathfrak{n}), \quad i \in \mathbb{N}. \tag{3.2}$$

The catastrophe time of X corresponds to the extinction time for the process Y.

While the proof of this lemma is evident, it yields a helpful tool for the further analysis. That way we switch from characterizing the first hitting time of the state $\mathfrak{n}$ for the process X to the more classical study of the extinction time of the process Y. In particular, the state 0 is *absorbing* for Y and is a *boundary state.* This gives us an advantage compared to analyzing $T_X^{\mathfrak{n}}$ directly, since X may leave the state $\mathfrak{n}$ again both to $\mathfrak{n}+1$ or $\mathfrak{n}-1$. We may thus directly use the tools developed by Brockwell to analyze extinction times for Birth-and-Death processes with general catastrophes, see [Bro86] Lemma 3.1. We recall this result in the following proposition.

Proposition 3.2. *For fixed $u \geq 0$ consider the sequence $(a_i(u))_{i\in\mathbb{N}}$ defined per iteration by*

$$a_0(u) = 0, a_1(u) = 1, \quad \sum_{j=0}^{i+1} q_{ij}\, a_j(u) = u\, a_i(u), \quad i \in \mathbb{N}^*.$$

Let $a_\infty(0) := \lim_{i\to\infty} a_i(0)$. Let $(Z_t)_{t\geq 0}$ be a BD-process with general catastrophes as defined in Definition 1.1. Its time to extinction $T_Z^0 := \inf\{t \geq 0 | Z_t = 0\}$ verifies

(1) $\mathbb{P}[T_Z^0 < \infty | Z_0 = i] = 1 - \frac{a_i(0)}{a_\infty(0)}$, $i \in \mathbb{N}$,
(2) $\mathbb{P}[T_Z^0 < \infty | Z_0 = i] = 1\ \forall i \in \mathbb{N}^* \Leftrightarrow \mathbb{P}[T_Z^0 < \infty | Z_0 = 1] = 1 \Leftrightarrow a_\infty(0) = \infty$.

It is worth noticing that, if Brockwell studied in details the linear case $\tilde{\lambda}_i = \lambda\, i$ in [Bro85], we have to consider here the shifted (affine) case $\tilde{\lambda}_i = \lambda i + \lambda\mathfrak{n}$, so that we have to perform all calculations.
Using Proposition 3.2 to study T_Y^0 we only have to analyze the behavior of the associated sequence $(a_i(0))_{i\in\mathbb{N}}$. It is the aim of the following lemma.

Lemma 3.3. *Using the notations (3.2) consider for $u \geq 0$ and $a > 0$ the sequence $(a_i(u))_{i\in\mathbb{N}}$ defined by the recurrence relation*

$$\begin{cases} (a_0(u),\, a_1(u)) = (0, a) \\ \tilde{\lambda}_i a_{i+1}(u) = (u + \tilde{\lambda}_i + \tilde{\mu}_i + \tilde{\gamma}_i)\, a_i(u) - \tilde{\mu}_i\, a_{i-1}(u), \quad i \geq 1. \end{cases} \tag{3.3}$$

Then $(a_i(u))_{i\in\mathbb{N}}$ is non-decreasing and $a_\infty(0) = \infty$.

Proof. Fix $u \geq 0$ and set $a_i := a_i(u)$ to improve readability. Note that (3.3) is equivalent to

$$\tilde{\lambda}_i(a_{i+1} - a_i) = ua_i + \tilde{\mu}_i(a_i - a_{i-1}) + \tilde{\gamma}_i(a_i - a_0).$$

First, $a_1 - a_0 = a > 0$ by assumption. Fix $i \geq 1$ and suppose that for $j \leq i-1$ the inequality $a_{j+1} - a_j \geq 0$ holds. Hence, in particular, $a_i \geq 0$. Moreover

$$\tilde{\lambda}_i(a_{i+1} - a_i) = ua_i + \tilde{\mu}_i(a_i - a_{i-1}) + \tilde{\gamma}_i \sum_{j=1}^{i}(a_j - a_{j-1}) \geq ua_i \geq 0.$$

By induction $(a_i)_{i\in\mathbb{N}}$ is non-decreasing and, thus, $a_i \geq 0$ for all $i \in \mathbb{N}$. Moreover, we obtain $\tilde{\lambda}_i a_{i+1} \geq (\tilde{\lambda}_i + \tilde{\gamma}_i)a_i$. Hence, for all $i \in \mathbb{N}$,

$$a_{i+1} \quad \geq \quad \frac{\tilde{\lambda}_i + \tilde{\gamma}_i}{\tilde{\lambda}_i} a_i = \left(1 + \frac{\gamma}{\lambda}\right) a_i \geq \ldots \geq \left(1 + \frac{\gamma}{\lambda}\right)^i a$$

and, thus, $a_\infty(0) = \lim_{i\to\infty} a_i = \infty$ with at least a geometric rate. □

Indeed, in the following lemma, we quantify the exact speed of divergence for the sequence $(a_i(0))_{i\in\mathbb{N}}$.

Lemma 3.4. *Take $u = 0$ in (3.3). Then the sequence $\left(\frac{a_i(0)}{a_{i+1}(0)}\right)_{i\in\mathbb{N}}$ converges to the smaller real zero $\underline{a} \in (0, 1)$ of the polynomial*

$$\mathbf{x} \mapsto \mu\mathbf{x}^2 - (\lambda + \mu + \gamma)\mathbf{x} + \lambda.$$

Proof. Set $a_i := a_i(0)$ to improve readability. It holds for $i \geq 2$

$$
\begin{aligned}
a_{i-1}a_i^{-1} &= \frac{\tilde{\lambda}_{i-1}a_{i-1}}{(\tilde{\lambda}_{i-1}+\tilde{\gamma}_{i-1}+\tilde{\mu}_{i-1})a_{i-1}-\tilde{\mu}_{i-1}a_{i-2}} \\
&= \frac{\lambda a_{i-1}}{(\lambda+\gamma+\mu)a_{i-1}-\mu\, a_{i-2}}.
\end{aligned}
$$

Set $z_i = a_i a_{i+1}^{-1}$ for $i \geq 1$ we therefore have

$$
z_1 = \frac{\lambda}{\lambda+\gamma+\mu}, \qquad z_i = \frac{\lambda}{\lambda+\gamma+\mu-\mu z_{i-1}}.
$$

Consider the map $\phi : [0,1] \to [0,1]$ defined by $\phi(x) = \dfrac{\lambda}{\lambda+\gamma+\mu-\mu x}$. Since $0 < \phi(0) < \phi(1) < 1$ and ϕ is strictly increasing, ϕ has a unique fixed point $\underset{\sim}{a} \in (0,1)$ given by

$$
\underset{\sim}{a} = \frac{\lambda+\mu+\gamma-\sqrt{(\lambda+\mu+\gamma)^2-4\lambda\mu}}{2\mu}.
$$

Moreover $\lim_{i\to\infty} z_i = \underset{\sim}{a}$. □

Applying Proposition 3.2 and Lemma 3.3 to the shifted process Y defined in Lemma 3.1 we obtain that the extinction time T_Y^0 is almost surely finite. Hence, the catastrophe time $T_X^{\mathfrak{n}}$ associated to $(X_t)_{t\geq 0}$ is almost surely finite. This completes the proof of Theorem 2.2.

3.2. **Explicit Computation of Mean Catastrophe Time.** In this subsection we prove Theorem 2.3. To that aim we first recall well a known result about the mean of hitting times, see for example [Nor97]. If Y is an irreducible CTMC with infinitesimal generator $Q = (q_{ij})_{ij}$, the sequence $(\mathbb{E}[T_Y^0 | Y_0 = i])_i$ is the minimal positive solution of the linear system

$$
\begin{cases} x_i = 0, & i = 0 \\ -\sum_{j\in\mathbb{N}} q_{ij}x_j = 1, & i \neq 0 \cdot \end{cases} \tag{3.4}
$$

Hence, since $T_Y^0 = T_X^{\mathfrak{n}}$ a.s. the sequence $(\mathbb{E}_i[T_X^{\mathfrak{n}}])_{i\geq 0}$ satisfies the recurrence relation

$$
\tilde{\lambda}_i x_{i+1} = -1 + (\tilde{\gamma}_i + \tilde{\lambda}_i + \tilde{\mu}_i)x_i - \tilde{\mu}_i x_{i-1},\ i \geq 1, \tag{3.5}
$$

where $x_0 = 0$ and the value of x_1 has to be determined.
In what follows, we use Lemma 3.3 extensively, always fixing $u = 0$ and abbreviating $a_i := a_i(0)$. We focus firstly on the dependence of the solution of (3.5) with respect to the value of $x_1 \in \mathbb{R}$.

Lemma 3.5. *Let $(x_i)_{i\in\mathbb{N}}$ be a solution to (3.5) where the coefficients are given by (3.2). Then there is at most one possible value for x_1 such that the sequence $(x_i)_{i\in\mathbb{N}}$ is bounded.*

Proof. Consider two solutions $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ of the recurrence relation (3.5) satisfying $x_1 = x$ resp. $x'_1 = x' < x$. The sequence $(\Delta_i := x_i - x'_i, i \in \mathbb{N})$ satisfies (3.3) with $\Delta_1 = x - x'$ and $u = 0$. By Lemma 3.3, $(\Delta_i)_{i\in\mathbb{N}}$ is a non-decreasing sequence tending to ∞ as $i \to \infty$. If there were two bounded sequences with different values x and x' for $i = 1$ also their difference would be bounded which is a contradiction. □

The following lemmas yield step by step the value of x_1 for which the solution of (3.5) is uniformly bounded.

Lemma 3.6. *There exists a unique value $\hat{x} > 0$ such that,*

(1) if $x_1 < \hat{x}$, $\lim_{i\to\infty} x_i = -\infty$,
(2) if $x_1 > \hat{x}$, $\lim_{i\to\infty} x_i = +\infty$.

Proof. Consider unbounded solutions $(x_i)_{i\in\mathbb{N}}$ to (3.5). As in the proof of Lemma 3.5 take two solutions $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ of (3.5) with $x'_1 = x' < x = x_1$. Since the sequence $(\Delta_i = x_i - x'_i,\ i \in \mathbb{N})$ is non-decreasing by Lemma 3.3 we obtain that $x_i > x'_i$ for all $i \geq 1$. Therefore solutions preserve for any i the order of their values for $i = 1$.
Consider the case where there is a $i_0 \in \mathbb{N}$ such that $x_{i_0-1} \geq 0$ and $x_{i_0} < 0$. Note that if there is a $k \in \mathbb{N}$ such that $x_k < 0$ such a pair (x_{i_0-1}, x_{i_0}) may always be found since $x_0 = 0$. Then, because $\tilde{\gamma}_i > 0$ for all $i \in \mathbb{N}$, we have

$$\tilde{\lambda}_{i_0}(x_{i_0+1} - x_{i_0}) = -1 + \tilde{\gamma}_{i_0} x_{i_0} + \tilde{\mu}_{i_0}(x_{i_0} - x_{i_0-1}) < 0. \tag{3.6}$$

Thus, $x_{i_0+1} < x_{i_0} < 0$ and

$$0 > \tilde{\gamma}_{i_0} x_{i_0} > \tilde{\gamma}_{i_0} x_{i_0+1} > \tilde{\gamma}_{i_0+1} x_{i_0+1}.$$

Hence, inductively we see that starting from i_0 the sequence $(x_i)_{i\geq i_0}$ is decreasing and negative. In particular, multiplying the recurrence relation (3.6) with -1 and using Lemma 3.3 with $u = 1$, we obtain that $(x_i)_{i\geq i_0}$ diverges to $-\infty$ as $i \to \infty$. In particular, once the sequence $(x_i)_{i\in\mathbb{N}}$ becomes negative, it stays negative.
Secondly, consider a positive unbounded solution $(x'_i)_{i\geq 0}$ of (3.5). Then, for any level $C > 0$, there is an index $I \in \mathbb{N}$ such that $x'_i < C$ for all $i < I$ and $x'_I \geq C$. Let $C > 0$ be sufficiently large such that $C\tilde{\gamma}_I > 1$ with $I = \inf\{i : x'_I \geq C\}$. Then, by the recurrence relation (3.5) we obtain

$$\tilde{\lambda}_I(x'_{I+1} - x'_I) \;=\; \tilde{\mu}_I(x'_I - x'_{I-1}) + \tilde{\gamma}_I x'_I - 1 > C\tilde{\gamma}_I - 1 > 0.$$

Hence, $x'_{I+1} > x'_I$ and, thus, $\tilde{\gamma}_{I+1} x'_{I+1} > \tilde{\gamma}_I x'_I > 1$. Furthermore, there is a $\varepsilon > 0$ such that $x'_{I+1} \geq C + \varepsilon$ and $x'_I < C + \varepsilon$. Applying the same arguments to x'_{I+1} with a new constant C' set to $C + \varepsilon$ yields inductively that $(x'_i)_{i\geq N}$ is strictly increasing, by assumption unbounded and, therefore, divergent to $+\infty$.
Therefore, since two sequences solution of (3.5) preserve the order of their initial values, using Lemma 3.5, there exists a critical value $\hat{x} > 0$ such that for $x_1 > \hat{x}$, the solution to (3.5) tends to ∞, while for $x_1 < \hat{x}$ it tends to $-\infty$. ☐

In fact, we will be able to compute the critical value $\hat{x}$ employing generating functions as tool.

Lemma 3.7. *Denote by $(x_i)_{i\in\mathbb{N}}$ the solution to the recurrence relation (3.5) with $x_1 = x$. Its generating function $\mathcal{E}(z) := \sum_{i=0}^{\infty} x_i z^i$ satisfies*

$$\mathcal{E}(z) = z \frac{\lambda x - \sum_{k=1}^{\infty} \frac{z^k}{k+\mathfrak{n}}}{\mu z^2 - qz + \lambda} \tag{3.7}$$

within its radius of convergence $R \in [0, \infty)$, where $q := \lambda + \mu + \gamma$.

Proof. Set $\tilde{q}_i := q(i+\mathfrak{n})$ such that (3.5) becomes

$$\tilde{q}_i x_i = \tilde{\lambda}_i x_{i+1} + \tilde{\mu}_i x_{i-1} + 1.$$

By exploiting this formulation we obtain

$$\begin{aligned}
\mathcal{E}(z) &= \sum_{i=1}^{\infty} x_i z^i = \sum_{i=1}^{\infty} \left(\tilde{\lambda}_i x_{i+1} + \tilde{\mu}_i x_{i-1} + 1\right) \frac{z^i}{\tilde{q}_i} \\
&= \frac{\lambda}{q} \sum_{i=1}^{\infty} x_{i+1} z^i + \frac{\mu}{q} \sum_{i=1}^{\infty} x_{i-1} z^i + \sum_{i=1}^{\infty} \frac{z^i}{\tilde{q}_i} \\
&= \frac{\lambda}{qz} \sum_{i=1}^{\infty} x_{i+1} z^{i+1} + \frac{\mu}{q} z \sum_{i=1}^{\infty} x_i z^i + \sum_{i=1}^{\infty} \frac{z^i}{\tilde{q}_i} \\
&= \frac{\lambda}{qz} \left(\mathcal{E}(z) - xz\right) + \frac{\mu}{q} z \mathcal{E}(z) + \sum_{i=1}^{\infty} \frac{z^i}{\tilde{q}_i}
\end{aligned}$$

which leads to the result (3.7).

□

Based on the explicit generating function we derive an expression of the coefficients by differentiating in 0. To justify this approach we have to establish that $\mathcal{E}$ converges within a positive radius of convergence.

Lemma 3.8. *The generating function $\mathcal{E}$ given by (3.7) has a positive radius of convergence.*

Proof. Let $\hat{x}$ as in Lemma 3.6. Consider solutions $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ to (3.5) with $x'_1 = x' < \hat{x} < x = x_1$. Since there is a $i_0 \in \mathbb{N}$ such that $x'_i < 0$ for all $i \geq i_0$ it follows that

$$\Delta_i := x_i - x'_i \geq \max\{x_i, |x'_i|\}, \quad i \geq i_0.$$

Additionally $(\Delta_i)_{i\in\mathbb{N}}$ satisfies (3.3) with $u = 0$; so according to Lemma 3.4 it grows like $\underline{a}^{-i}$ as $i \to \infty$. We obtain that $(x_i)_{i\in\mathbb{N}}$ and $(x'_i)_{i\in\mathbb{N}}$ grow asymptotically at most like $\underline{a}^{-i}$. Note that the solution $(\hat{x}_i)_{i\in\mathbb{N}}$ of (3.5) with $\hat{x}_1 = \hat{x}$ cannot grow faster than $\underline{a}^{-i}$ as $i \to \infty$ by positiveness of Δ_i for all $i \geq 1$. Thus, the series

$$\mathcal{E}(z) = \sum_{i=1}^{\infty} x_i z^i$$

converges for any initial value at least within the radius of convergence $R := \underline{a} > 0$. □

We proceed with a direct calculation of the coefficients. Recall that $\underline{a} < \bar{a}$ are the distinct real zeros of the polynomial $\mathbf{x} \mapsto \mu\mathbf{x}^2 - (\lambda+\mu+\gamma)\mathbf{x} + \lambda$.

Lemma 3.9. *Any solution $(x_i)_{i\in\mathbb{N}}$ to equation (3.5) with $x_1 = x > 0$ has the form*

$$x_i = \lambda\, c\, x \left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c \sum_{k=1}^{i-1} \frac{1}{k+\mathfrak{n}} \left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right), \tag{3.8}$$

with $c = \left(\sqrt{(\lambda+\mu+\gamma)^2 - 4\lambda\mu}\right)^{-1}$.

Proof. According to Lemma 3.7 and Lemma 3.8 the introduced generating function $\mathcal{E}$ has a positive radius of convergence and the particular form

$$\mathcal{E}(z) = z\frac{f(z)}{g(z)} \text{ with } f(z) := \lambda x - \sum_{k=1}^{\infty} \frac{z^k}{k+\mathfrak{n}} \text{ and } g(z) := \mu z^2 - qz + \lambda.$$

Note first, that by that form we have

$$\partial_z^i \mathcal{E}(z)\Big|_{z=0} = i\,\partial_z^{i-1}\left(\frac{f(z)}{g(z)}\right)\Big|_{z=0} = i\sum_{k=0}^{i-1}\binom{i-1}{k}\left(\partial_z^k f(z)\partial_z^{i-1-k}(\frac{1}{g})(z)\right)\Big|_{z=0}.$$

Due to the form of g we have

$$\frac{1}{g(z)} = \frac{c}{z-\bar{a}} - \frac{c}{z-\underline{a}},$$

and therefore any derivative of order n, namely,

$$\partial_z^i \frac{1}{g(z)}\Big|_{z=0} = i!\,c\left(\frac{1}{\underline{a}^{i+1}} - \frac{1}{\bar{a}^{i+1}}\right). \tag{3.9}$$

Considering the numerator $f(z)$, we have

$$\partial_z^i f(z)\Big|_{z=0} = -\sum_{k=i}^{\infty}\frac{z^{k-i}}{k+\mathfrak{n}}k\cdot(k-1)\cdot\ldots\cdot(k-i+1)\Big|_{z=0} = -\frac{i!}{i+\mathfrak{n}}.$$

Hence,

$$\begin{aligned}
\partial_z^i \mathcal{E}(z)\Big|_{z=0} &= i\,\partial_z^{i-1}\left(\frac{f(z)}{g(z)}\right)\Big|_{z=0} \\
&= i\left(\lambda\,x\,(i-1)!c\left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right)\right. \\
&\qquad \left. - \sum_{k=1}^{i-1}\binom{i-1}{k}\frac{k!}{k+\mathfrak{n}}c(i-1-k)!\left(-\frac{1}{\bar{a}^{i-k}} + \frac{1}{\underline{a}^{i-k}}\right)\right) \\
&= i!\left(\lambda\,x\,c\left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - \sum_{k=1}^{i-1}\frac{c}{k+\mathfrak{n}}\left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right)\right)
\end{aligned}$$

Since $x_i = i!^{-1}\partial_z^i \mathcal{E}(z)\Big|_{z=0}$, we find that for any $i \geq 1$

$$x_i = \lambda\,c\,x\left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c\sum_{k=1}^{i-1}\frac{1}{k+\mathfrak{n}}\left(-\frac{1}{\bar{a}^{i-k}} + \frac{1}{\underline{a}^{i-k}}\right).$$

□

We can now complete the proof of Theorem 2.3 by combining as follows the previously presented results. The sequence of expected hitting times $(\mathbb{E}_i[T_X^{\mathfrak{n}}])_{i\in\mathbb{N}}$ is a solution to (3.5).

Lemma 3.9 shows that the expected value for $i \geq 0$ is in fact given by

$$\mathbb{E}_i[T_X^{\mathfrak{n}}] = \lambda\, \mathbb{E}_1[T_X^{\mathfrak{n}}]\, c \left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i} \right) - c \sum_{k=1}^{i-1} \frac{1}{k+\mathfrak{n}} \left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}} \right),$$

with $c = \left(\sqrt{(\lambda+\mu+\gamma)^2 - 4\lambda\mu} \right)^{-1}$ where we followed the usual convention that an empty sum equals 0.

3.3. **Asymptotic behavior of mean catastrophe time for large initial population.** To improve the readability of this subsection, we introduce the following notation:

$$\Phi_n(z) := \sum_{k=0}^{\infty} \frac{z^k}{k+n}, \quad |z| < 1,\ n \in \mathbb{N}^*. \tag{3.10}$$

Therefore

$$\mathcal{E}(z) = z \frac{\lambda x_1 - \Phi_{\mathfrak{n}}(z) + \mathfrak{n}^{-1}}{\mu z^2 - qz + \lambda}$$

Lemma 3.10. *Any sequence $(x_i)_{i\in\mathbb{N}}$ satisfying (3.5) with $x_1 < \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right)$ diverges to $-\infty$ as $i \to \infty$.*

Proof. There exists $\delta > 0$ such that

$$x_1 = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right) - \delta. \tag{3.11}$$

Since the function $\Phi_{\mathfrak{n}}$ is continuous at $\underline{a} < 1$

$$\forall \varepsilon > 0\, \exists\, \bar{\delta} > 0 :\ z > \underline{a} - \bar{\delta} \Rightarrow \Phi_{\mathfrak{n}}(z) > \Phi_{\mathfrak{n}}(\underline{a}) - \varepsilon.$$

Choose $\varepsilon = \frac{\delta\lambda}{2}$. For all $z \in (\underline{a} - \bar{\delta}, \underline{a})$ we obtain by (3.11)

$$\begin{aligned} \mathcal{E}(z) &= z \frac{\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1} - \lambda\delta - \Phi_{\mathfrak{n}}(z) + \mathfrak{n}^{-1}\right)}{\mu z^2 - qz + \lambda} \\ &< \frac{z\,(\varepsilon - \lambda\delta)}{\mu z^2 - qz + \lambda} = -\frac{z\lambda\delta}{2(\mu z^2 - qz + \lambda)} < 0. \end{aligned}$$

since $\mu z^2 - qz + \lambda > 0$ for all $z < \underline{a}$.

Since positivity of the sequence $(x_i)_{i\in\mathbb{N}}$ would imply that $\mathcal{E}(z) > 0$ for all $z > 0$ it proves that there is an index i_0 for which $x_{i_0} < 0$. Consequently, by Lemma 3.6, the sequence $(x_i)_i$ tends to $-\infty$ as $i \to \infty$.

□

In fact, it turns out that the value $x_1 = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right)$ yields the minimal positive solution of (3.5) which does not tend to $+\infty$, as proved in the following lemma.

Lemma 3.11. *The sequence $(x_i)_{i\in\mathbb{N}}$ satisfying (3.5) with $x_1 = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right)$ is the minimal positive solution to (3.5). Moreover $x_i = \mathcal{O}(i^{-1})$ as $i \to \infty$. Therefore*

$$\hat{x} = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right).$$

Proof. By Lemma 3.9,

$$\begin{aligned} x_i &= \lambda\, c\, x_1 \left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c\sum_{k=1}^{i-1}\frac{1}{k+\mathfrak{n}}\left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right) \\ &= c\sum_{k=1}^{\infty}\frac{\underline{a}^k}{k+\mathfrak{n}}\left(\frac{1}{\underline{a}^i} - \frac{1}{\bar{a}^i}\right) - c\sum_{k=1}^{i-1}\frac{1}{k+\mathfrak{n}}\left(\frac{1}{\underline{a}^{i-k}} - \frac{1}{\bar{a}^{i-k}}\right). \end{aligned} \tag{3.12}$$

Using the notation $b_i := \sum_{k=1}^{i-1}\frac{\bar{a}^k}{k+\mathfrak{n}}$, then $\sum_{k=1}^{i-1}\frac{1}{k+\mathfrak{n}}\frac{1}{\bar{a}^{i-k}} = \frac{b_i}{\bar{a}^i}$ and

$$\frac{b_{i+1} - b_i}{\bar{a}^{i+1} - \bar{a}^i} = \frac{1}{(\bar{a}-1)(i+\mathfrak{n})} \to 0, \qquad i \to \infty.$$

Using the fact that $\bar{a} > 1$ and applying Stolz-Césaro Theorem we obtain that

$$\frac{b_i}{\bar{a}^i} = \frac{1}{\bar{a}^i}\sum_{k=1}^{i-1}\frac{\bar{a}^k}{k+\mathfrak{n}} \to 0, \text{ as } i \to \infty.$$

Thus, the asymptotic behavior of $(x_i)_{i\geq 0}$ is the same as of

$$\frac{c}{\underline{a}^i}\sum_{k=i}^{\infty}\frac{\underline{a}^k}{k+\mathfrak{n}} \geq \frac{c}{i+\mathfrak{n}}. \tag{3.13}$$

Let us check an upper bound. Using the integral comparison,

$$\begin{aligned} \sum_{k=i}^{\infty}\frac{\underline{a}^k}{k+\mathfrak{n}} &\leq \frac{\underline{a}^i}{i+\mathfrak{n}} + \int_i^{\infty}\frac{\underline{a}^s}{s+\mathfrak{n}}ds \\ &\leq \frac{\underline{a}^i}{i+\mathfrak{n}} + \frac{\underline{a}^{-\mathfrak{n}}}{i+\mathfrak{n}}\int_{i+\mathfrak{n}}^{\infty}\exp(\log \underline{a} s)ds \\ &= \frac{\underline{a}^i}{i+\mathfrak{n}} + \frac{\underline{a}^i}{(i+\mathfrak{n})(-\log \underline{a})} \end{aligned}$$

which implies that $\frac{1}{\underline{a}^i}\sum_{k=i}^{\infty}\frac{\underline{a}^k}{k+\mathfrak{n}}$ vanishes as $i \to \infty$. Hence $x_i \to 0$ as $i \to \infty$ and the sequence remains positive since otherwise it would diverge by Lemma 3.10. Moreover, the lower bound (3.13) and the upper bound of order $\mathcal{O}(i^{-1})$ implies the decreasing rate $x_i = \mathcal{O}(i^{-1})$ as $i \to \infty$. Finally, by Lemma 3.6, the sequence $(x_i)_{i\geq 0}$ with $x_1 = \frac{1}{\lambda}\left(\Phi_{\mathfrak{n}}(\underline{a}) - \mathfrak{n}^{-1}\right)$ is indeed the minimal positive solution to (3.5), which determines uniquely the value of $\hat{x}$. □

Let us now complete the proof of Corollary 2.4. The sequence of expected hitting times $(\mathbb{E}_i[T_X^{\mathfrak{n}}])_{i\in\mathbb{N}}$ is the solution of

$$\begin{cases} x_0 = 0,\ x_1 = \hat{x}, \\ \lambda_i x_{i+1} = -1 + (\gamma_i + \lambda_i + \mu_i)x_i - \mu_i x_{i-1}, \quad i \geq 1, \end{cases} \tag{3.14}$$

with $\hat{x} = \frac{1}{\lambda}\sum_{k=1}^{\infty} \frac{\underline{a}^k}{k+\mathfrak{n}}$ where Lemma 3.11 yields the minimality property. From Lemma 3.11 also follows the rate of convergence of $\mathbb{E}_i[T_X^{\mathfrak{n}}]$ to 0 as $i \to \infty$.

4. Recurrence Property and Stationary Distribution

Based on the information we obtained on the expected hitting times of the catastrophe state, we now prove the following theorem.

Theorem 4.1 (Positive Recurrence)**.** *Let $X = (X_t)_{t\geq 0}$ be the BD+C$_{\mathfrak{n}}$ process whose infinitesimal generator Q is given by (2.1). Then X is positive recurrent, it exhibits a unique non degenerated stationary distribution π and, for any initial distribution and for large time, X_t converges in distribution to π.*

Note that this is radically different from the behavior of a BD-process with immigration satisfying $\lambda > \mu$. In that case the population grows in expectation over time and no stationary distribution exists, see e.g. [And91]. Hence, by introducing a partial catastrophe to the model its behavior changes drastically, ensuring for any catastrophe rate $\gamma > 0$ the existence of a unique stationary distribution.
To prove the theorem, we apply the existence of Lyapunov functions.

Proof of Theorem 4.1. Recall that a function $V : \mathbb{N} \to \mathbb{R}^+ \cup \{+\infty\}$ is a Lyapunov function associated to the CTMC X if V satisfies

$$QV(n) \leq -1 + \mathbb{1}_A(n)$$

where Q is the generator of X and A is some *petite set*, see e.g. [MLU93]. Take here $A = \{0, \dots, \mathfrak{n}\}$. By the previous section, $\sup_{i\in\mathbb{N}} \mathbb{E}[T_X^A | X_0 = i] < \infty$ where T_X^A is the first time the process X hits the set A. Hence, $i \mapsto 1 + \mathbb{E}[T^A | X_0 = i]$ yields a Lyapunov function. It follows directly that the process X admits a stationary distribution π and that it is positive recurrent. The uniqueness of π follows from the irreducibility of X and the limiting behavior follows from classical results on non-explosive continuous time Markov chains, see e.g. [Nor97]. □

5. Expected Population Size at Fixed Times

In this last section we analyse the first two moments of the BD+C$_{\mathfrak{n}}$ process at a fixed time $t \geq 0$ and prove Theorem 2.5 and Theorem 2.6. Due to the asymmetric form of the generator (2.1) we focus on upper bounds of the first and second moment of $(X_t)_{t\geq 0}$.
In contraposition to the last section, the parameter ν representing the immigration rate in case of extinction of the population now plays an important role, making the state 0 non-absorbing.

5.1. **Associated Kolmogorov Equations.** Let us consider the Kolmogorov equations associated to the BD+C$_{\mathfrak{n}}$ process $(X_t)_{t\geq 0}$ to analyze its behavior at fixed time. With $P_n(t) := \mathbb{P}[X_t = n | X_0 = n_0], n_0 > \mathfrak{n}$, the following identities hold:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} P_0(t) &= \mu P_1(t) - \nu P_0(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} P_n(t) &= \lambda_{n-1} P_{n-1}(t) - n(\mu+\lambda) P_n(t) + (n+1)\mu P_{n+1}(t), \qquad 0 < n < \mathfrak{n}, \\ \frac{\mathrm{d}}{\mathrm{d}t} P_{\mathfrak{n}}(t) &= (\mathfrak{n}-1)\lambda P_{\mathfrak{n}-1}(t) - \mathfrak{n}(\mu+\lambda) P_{\mathfrak{n}}(t) + (\mathfrak{n}+1)\mu P_{\mathfrak{n}+1}(t) + \gamma \sum_{i=\mathfrak{n}+1}^{\infty} i P_i(t), \\ \frac{\mathrm{d}}{\mathrm{d}t} P_n(t) &= (n-1)\lambda P_{n-1}(t) - n(\gamma+\mu+\lambda) P_n(t) + (n+1)\mu P_{n+1}(t), \quad n > \mathfrak{n}, \end{cases} \tag{5.1}$$

with initial condition $P_n(0) = \delta_{n_0,n}$. Recall that $\lambda_n = n\,\lambda$ if $n \geq 1$ but $\lambda_0 = \nu$.

5.2. **Upper bounds for First and Second Moment.** We approach the first and second moment by analyzing their corresponding ODEs.

Proof of Theorem 2.5. Using (5.1), we obtain

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[X_t] &= \lambda \sum_{n=1}^{\infty} n(n-1) P_{n-1}(t) - (\mu+\lambda) \sum_{n=1}^{\infty} n^2 P_n(t) \\ &\quad + \mu \sum_{n=1}^{\infty} n(n+1) P_{n+1}(t) - \gamma \sum_{n=\mathfrak{n}+1}^{\infty} (n-\mathfrak{n}) n P_n(t) + \nu P_0(t) \\ &= (\lambda-\mu)\mathbb{E}[X_t] + \gamma \sum_{n=\mathfrak{n}+1}^{\infty} (\mathfrak{n}-n) n P_n(t) + \nu P_0(t). \end{aligned}$$

If we denote the first moment by $m(t) := \mathbb{E}[X_t]$ then it solves the ODE

$$m'(t) = (\lambda-\mu)\, m(t) - \gamma \sum_{n=\mathfrak{n}+1}^{\infty} (n-\mathfrak{n}) n P_n(t) + \nu P_0(t), \quad m(0) = n_0 \tag{5.2}$$

In particular,

$$\begin{aligned} m'(t) &\leq (\lambda-\mu)\, m(t) + \gamma \sum_{n=0}^{\infty} (\mathfrak{n}-n) n P_n(t) + \nu P_0(t) \\ &\leq (\lambda-\mu)\, m(t) + \gamma(\mathfrak{n} - m(t))\, m(t) + \nu. \end{aligned}$$

The solution $\bar{m}$ to the ODE

$$\bar{m}'(t) = (\lambda-\mu)\bar{m}(t) + \gamma(\mathfrak{n} - \bar{m}(t))\,\bar{m}(t) + \nu, \quad \bar{m}(0) = n_0, \tag{5.3}$$

is, therefore, a candidate for an upper bound of m. To that aim, we will show that the difference $\bar{m} - m$ is a non-negative function on $[0, \infty)$.

We first prove that $\bar{m} - m$ has a strict local minimum at $t = 0$. Note that solutions of both ODEs (5.2) and (5.3) exist globally on $(0, \infty)$. Since both right hand sides are continuous in t for $t \geq 0$ we can extend definition of the derivative to the set $[0, \infty)$. Note that if $n_0 > \mathfrak{n}$ then

$$\lim_{t\to 0^+} m'(t) = (\lambda-\mu)n_0 + \gamma(\mathfrak{n} - n_0)n_0 = \lim_{t\to 0^+} \bar{m}'(t) - \nu < \lim_{t\to 0^+} \bar{m}'(t)$$

and $\bar{m}(0) = m(0)$. Thus, during some time, $\bar{m}$ dominates m:

$$\exists t_0 > 0 : \quad m(t) \leq \bar{m}(t) \quad \forall t \in [0, t_0).$$

Assume that this domination only holds locally, or equivalently, that there is a $t_2 > t_0$ such that $\bar{m}(t_2) < m(t_2)$. By continuity there exists an interval $I_\delta := (t_2 - \delta, t_2 + \delta)$ such that $\bar{m}(t) < m(t)$ for all $t \in I_\delta$. Without loss of generality we may assume that $\bar{m}(t) \geq m(t)$ for all $t \in [0, t_2 - \delta)$. Set $t_1 := t_2 - \delta$. By continuity of $\bar{m}$ and m, $\bar{m}(t_1) = m(t_1)$. Additionally, since $\bar{m}$ and m are C^1-functions, $\bar{m}'(t_1) < m'(t_1)$. But, on the other hand

$$\bar{m}'(t_1) = g(\bar{m}(t_1)) = g(m(t_1)) \geq m'(t_1),$$

where $g(x) := (\lambda - \mu)x + \gamma(\mathfrak{n} - x)x + \nu$, which leads to a contradiction. Thus, $\bar{m}$ dominates m globally.

□

Note that the ODE (5.3) is the same than (2.6), whose solution is a logistic growth function. In particular $\bar{m}(t)$ converges for t large to its equilibrium value $\bar{m}_e$ given by the largest zero of the polynomial $\mathbf{x} \mapsto \gamma\,\mathbf{x}^2 - (\lambda - \mu + \gamma\,\mathfrak{n})\,\mathbf{x} - \nu$,

$$\bar{m}_e := \frac{(\lambda - \mu + \gamma\,\mathfrak{n}) + \sqrt{(\lambda - \mu + \gamma\,\mathfrak{n})^2 + 4\gamma\nu}}{2\gamma}. \tag{5.4}$$

Moreover $t \mapsto \bar{m}(t)$ is decreasing whenever it is larger than its equilibrium $\bar{m}_e$.

Proof of Theorem 2.6. The dynamics of the process' second moment $v(t) := \mathbb{E}[X_t^2]$ can be derived in the same way as the one for the first moment and it reads

$$\begin{aligned} v'(t) &= 2(\lambda - \mu)v(t) + (\lambda + \mu)m(t) - \gamma \sum_{n=\mathfrak{n}+1}^{\infty} (n^3 - \mathfrak{n}^2 n)P_n(t) \\ &\leq 2(\lambda - \mu)v(t) + (\lambda + \mu)m(t) - \gamma \sum_{n=0}^{\infty} (n^3 - \mathfrak{n}^2 n)P_n(t) + \nu P_0(t) \\ &\leq 2(\lambda - \mu)v(t) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t) - \gamma v(t)^{\frac{3}{2}} + \nu \end{aligned}$$

The last inequality, due to Jensen's inequality, is even strict for $t > 0$:

$$v'(t) < 2(\lambda - \mu)v(t) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t) - \gamma v(t)^{\frac{3}{2}} + \nu, \quad t > 0.$$

Consider now the function $\bar{v}$ being solution of the ODE

$$\bar{v}'(t) = 2(\lambda - \mu)\,\bar{v}(t) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\,\bar{m}(t) - \gamma\,\bar{v}(t)^{\frac{3}{2}} + \nu, \quad \bar{v}(0) = n_0^2. \tag{5.5}$$

It is a candidate for an upper bound of v. Compare the right limits in 0 of the first derivatives:

$$\lim_{t \to 0^+} v'(t) < 2(\lambda - \mu)n_0^2 + (\lambda + \mu + \gamma\,\mathfrak{n}^2)n_0 - \gamma n_0^3 + \nu = \lim_{t \to 0^+} \bar{v}'(t). \tag{5.6}$$

By the same argumentation as for the m, we obtain the domination of v by the function $\bar{v}$. The positive function $\bar{v}$ is indeed bounded, as we will prove now.
Assume conversely, that any large value can be taken by $\bar{v}$. In particular choose any c_0 large enough such that

$$2(\lambda - \mu)c_0 + (\lambda + \mu + \gamma\,\mathfrak{n}^2)n_0 - \gamma c_0^{\frac{3}{2}} + \nu < 0 \tag{5.7}$$

and suppose that there exists a time t_0 such that $\bar{v}(t_0) = c_0$. Since by assumption $n_0 > \bar{m}_e$, the function $\bar{m}$ decreases from $\bar{m}(0) = n_0$. It follows that

$$\begin{aligned} \bar{v}'(t_0) &= 2(\lambda-\mu)\bar{v}(t_0) + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}(t_0) - \gamma\bar{v}(t_0)^{\frac{3}{2}} + \nu \\ &\leq 2(\lambda-\mu)c_0 + (\lambda+\mu+\gamma\,\mathfrak{n}^2)n_0 - \gamma c_0^{\frac{3}{2}} + \nu < 0. \end{aligned}$$

Hence, whenever $\bar{v}$ takes the value c_0, it has a negative derivative. Thus, for $t > t_0$, $\bar{v}(t)$ is uniformly bounded by c_0 which is contradictory. □

We also compare $\bar{v}$ and $\bar{m}^2$ and obtain informations on the long time behavior of $\bar{v}$.

Proposition 5.1. *Let $\bar{v}$ be solution of the ODE (5.5) with $n_0 > \bar{m}_e$ and $\lambda+\mu \geq 2\nu$. Then $\bar{v}(t) \geq \bar{m}^2(t)$ for all $t \geq 0$, where $\bar{m}$ is solution of the ODE (5.3). Moreover, as $t \to \infty$, $\bar{v}(t)$ tends to the solution $\bar{v}_\infty$ of the following equation:*

$$0 = 2(\lambda-\mu)\bar{v}_\infty + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}_e - \gamma\bar{v}_\infty^{\frac{3}{2}} + \nu. \tag{5.8}$$

To prove these properties, we analyze the nullcline, i.e., the function $\mathbf{v}(t)$ solution of the limit equation obtained from (5.5) with vanishing l.h.s.:

$$0 = 2(\lambda-\mu)\mathbf{v}(t) + (\lambda+\mu+\gamma\,\mathfrak{n}^2)\bar{m}(t) - \gamma(\mathbf{v}(t))^{\frac{3}{2}} + \nu. \tag{5.9}$$

To prove the uniqueness of the solution of the equation (5.8) we use Descarts' rule of signs, which we now recall (see [HK91]):

Lemma 5.2. *The number of positive roots counted with multiplicities of a polynomial with real coefficients is equal to the number of changes of sign in the list of coefficients, or is less than this number by a multiple of 2.*

We also obtain the C^1-regularity of the function $\mathbf{v}$ by the following lemma.

Lemma 5.3. *Let $t \geq 0$ and $\mathbf{P}^t : \mathbf{x} \mapsto a_0(t) + \sum_{k=1}^n a_k \mathbf{x}^k$ be a real polynomial with $a_0 \in C^1([0,\infty),\mathbb{R})$ and $a_0'(t) < 0$ for all $t > 0$. If for all $t \geq 0$ the polynomial $\mathbf{P}^t$ has a unique positive simple zero denoted by x_0^t then $t \mapsto x_0^t \in C^1((0,\infty),\mathbb{R})$.*

Proof of Lemma 5.3. For any $t > 0$ and sufficiently small $h > 0$,

$$\begin{aligned} 0 &= \sum_{k=1}^n a_k \Big((x_0^{t+h})^k - (x_0^t)^k \Big) + a_0(t+h) - a_0(t) \\ &= \left(x_0^{t+h} - x_0^t \right) \sum_{k=1}^n a_k \sum_{l=1}^{k-1} \left(x_0^{t+h} \right)^l \left(x_0^t \right)^{k-1-l} + a_0(t+h) - a_0(t). \end{aligned}$$

Note that $\sum_{k=1}^n a_k \sum_{l=1}^{k-1} \left(x_0^{t+h} \right)^l \left(x_0^t \right)^{k-1-l} \neq 0$ and $x_0^{t+h} - x_0^t \neq 0$, because a_0 is decreasing by assumption. Thus,

$$\frac{x_0^{t+h} - x_0^t}{h} = \frac{a_0(t) - a_0(t+h)}{h} \left(\sum_{k=1}^n a_k \sum_{l=1}^{k-1} \left(x_0^{t+h} \right)^l \left(x_0^t \right)^{k-1-l} \right)^{-1} \tag{5.10}$$

and, hence, $t \mapsto x_0^t \in C^1((0,\infty),\mathbb{R})$. □

Proof of Proposition 5.1. Define a positive function ψ by $\psi(t)^2 := \mathbf{v}(t)$ where $\mathbf{v}(t)$ solves the equation (5.9). Then $\psi(t)$ is a positive zero of the polynomial

$$\mathbf{x} \mapsto -\gamma \mathbf{x}^3 + 2(\lambda - \mu)\mathbf{x}^2 + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t) + \nu.$$

By Lemma 5.2, for fixed t it exists, is simple and unique. Therefore the nullcline is defined pointwise by $\mathbf{v}(t) := \sqrt{\psi(t)}, t \geq 0$. Since $t \mapsto \bar{m}(t)$ is continuous, $\mathbf{v}$ is also continuous. The function $\mathbf{v}$ is even continuously differentiable by positivity of ψ, smoothness of $\bar{m}$ and Lemma 5.3. Moreover, since $\bar{m}$ converges to m_e as $t \to \infty$, $\mathbf{v}$ converges as $t \to \infty$ to the solution of the equation (5.8). Furthermore, differentiating (5.9), one obtains

$$0 = 2(\lambda - \mu)\mathbf{v}'(t) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}'(t) - \frac{3}{2}\gamma \mathbf{v}'(t)\mathbf{v}(t)^{\frac{1}{2}}. \tag{5.11}$$

Since $\bar{m}'$ tends to 0 as $t \to \infty$, therefore $\mathbf{v}'$ converges to some value denoted by $\mathbf{v}'(\infty)$, which solves the equation

$$0 = 2(\lambda - \mu)\mathbf{v}'(\infty) - \frac{3}{2}\gamma \mathbf{v}'(\infty)\bar{v}_\infty^{\frac{1}{2}}. \tag{5.12}$$

Thus, either $\mathbf{v}'(\infty) = 0$ or $\bar{v}_\infty = \left(\frac{4(\lambda-\mu)}{3\gamma}\right)^2$. This later value cannot solve (5.8). Hence, $\mathbf{v}'(\infty) = 0$. Note that if $x > x_0^t$, then

$$-\gamma x^3 + 2(\lambda - \mu)x^2 + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t) + \nu < 0$$

and vice versa. Thus, $\bar{v}$ always tends towards the nullcline $\mathbf{v}$. Since $\bar{v}$ always tends towards the nullcline $\mathbf{v}$ and $\mathbf{v}'$ tends to 0, also $\bar{v}$ converges to $\bar{v}_\infty$, which proves the claim. □

Note by the way that one deduces from equation (5.8) that $\bar{v}_\infty \sim \mathcal{O}(\mathfrak{n}^2)$ as $\mathfrak{n} \to \infty$.

About the comparison between $\bar{v}(t)$ and $\bar{m}^2(t)$:
By assumption $\bar{v}(0) = n_0^2 = \bar{m}(0)^2$. Note that

$$\bar{m}(t) \geq \bar{m}_e > \mathfrak{n} \Rightarrow (\mathfrak{n} - \bar{m}(t))^2 > 0 \Rightarrow 2\bar{m}(t)(\mathfrak{n} - \bar{m}(t)) < \mathfrak{n}^2 - \bar{m}^2(t).$$

Therefore

$$\begin{aligned}
(\bar{m}^2)'(t) &= 2\bar{m}(t)\bar{m}'(t) = -2\gamma\bar{m}(t)^3 + 2(\lambda - \mu + \gamma\,\mathfrak{n})\bar{m}^2(t) + 2\nu\bar{m}(t) && (5.13)\\
&< \gamma(\mathfrak{n}^2 - \bar{m}^2(t))\bar{m}(t) + 2(\lambda - \mu)\bar{m}^2(t) + 2\nu\bar{m}(t)\\
&= -\gamma\left(\bar{m}^2(t)\right)^{\frac{3}{2}} + 2(\lambda - \mu)\bar{m}^2(t) + (2\nu + \gamma\,\mathfrak{n}^2)\bar{m}(t)\\
&\leq -\gamma\left(\bar{m}^2(t)\right)^{\frac{3}{2}} + 2(\lambda - \mu)\bar{m}^2(t) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\bar{m}(t), && (5.14)
\end{aligned}$$

where the assumption $2\nu \leq \lambda + \mu$ is used for the last inequality. Now by (5.5) and (5.13),

$$\begin{aligned}
\bar{v}'(0) - (\bar{m}^2)'(0) &= 2(\lambda - \mu)\,\bar{v}(0) + (\lambda + \mu + \gamma\,\mathfrak{n}^2)\,\bar{m}(0) - \gamma\,\bar{v}(0)^{\frac{3}{2}} + \nu\\
&\quad -\Big(- 2\gamma\bar{m}(0)^3 + 2(\lambda - \mu + \gamma\,\mathfrak{n})\bar{m}^2(0) + 2\nu\bar{m}(0)\Big)\\
&= (\lambda + \mu - 2\nu)n_0 + \gamma n_0(n_0 - \mathfrak{n})^2 + \nu > 0.
\end{aligned}$$

This inequality propagates for all $t \geq 0$. This can be proved by the same argumentation as in the proof of Theorem 2.5, using (5.14).

5.3. **Accuracy of these upper bounds.** Previously, we found an upper bound $\bar{m}$ to the first moment m. We would like to quantify the sharpness of this bound by estimating the non negative difference function D defined by

$$D(t) := \bar{m}(t) - m(t) \ \geq 0, \quad D(0) = 0. \tag{5.15}$$

Moreover, $\limsup_{t\to\infty} D(t) \leq \bar{m}_e$ which implies that D is a bounded function.

Proposition 5.4 (Upper Bound for the difference function D)**.** *Let m, $\bar{m}$, v and $\bar{v}$ defined as above under the same assumptions. Then, the difference function D defined in equation (5.15) is pointwise bounded from above by the solution $\bar{D}$ of the non linear ODE*

$$\begin{cases} \bar{D}'(t) = -\gamma \bar{D}(t)^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}(t))\big)\bar{D}(t) \\ \qquad\qquad\qquad +\gamma\big(\bar{v}(t) - \bar{m}(t)^2\big) + \gamma\frac{\mathfrak{n}^2}{4} + \nu, \quad t > 0, \\ \bar{D}(0) = 0. \end{cases} \tag{5.16}$$

Moreover, the function $\bar{D}$ tends for large time to the positive solution $\bar{D}_\infty$ of the equation

$$0 = -\gamma \bar{D}_\infty^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}_e)\big)\bar{D}_\infty + \gamma(\bar{v}_\infty - \bar{m}_e^2) + \gamma\frac{\mathfrak{n}^2}{4} + \nu. \tag{5.17}$$

Proof. By differentiating the function D defined in (5.15) one obtains

$$\begin{aligned} D'(t) \ &\leq \ (\lambda - \mu)D(t) + \mathfrak{n}\gamma\bar{m}(t) - \gamma\bar{m}^2(t) + \gamma \sum_{n \geq \mathfrak{n}+1} (n - \mathfrak{n}) n P_n(t) + \nu \\ &\leq \ -\gamma D(t)^2 + (\lambda - \mu + \gamma\mathfrak{n})D(t) - 2\gamma D(t)m(t) + 2\gamma D(t)\bar{m}(t) \\ &\qquad +\gamma(\bar{v}(t) - \bar{m}^2(t)) + \gamma\sum_{n=0}^{\mathfrak{n}} (\mathfrak{n} - n) n P_n(t) + \nu \\ &\leq \ -\gamma D(t)^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}(t))\big)D(t) + \gamma(\bar{v}(t) - \bar{m}^2(t)) + \gamma\frac{\mathfrak{n}^2}{4} + \nu. \end{aligned} \tag{5.18}$$

The function $\bar{D}$, solving the ODE (5.16), is an upper bound for D since $D(0) = \bar{D}(0)$ and

$$\lim_{t\to 0^+} D'(t) \ = \ 0, \qquad \lim_{t\to 0^+} \bar{D}'(t) = \gamma\frac{\mathfrak{n}^2}{4} + \nu > 0.$$

Thus, by similar arguments as for $\bar{m}$ and $\bar{v}$, it follows that $\bar{D}$ is a global upper bound of D. To investigate the asymptotic behavior of $\bar{D}(t)$ we use the similar techniques as we applied in the proof of Proposition 5.1. This time, the equation (5.16) induces the positive nullcline $\mathbf{D}$ satisfying

$$0 \ = \ -\gamma\mathbf{D}(t)^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}(t))\big)\mathbf{D}(t) + \gamma\big(\bar{v}(t) - \bar{m}^2(t)\big) + \gamma\frac{\mathfrak{n}^2}{4} + \nu.$$

Applying again Lemma 5.2, one proves its existence and also the convergence of $\bar{D}$ towards this nullcline. As in Proposition 5.1, since $\bar{v}$ and $\bar{m}$ converges for t large, $\mathbf{D}(t)$ converges to $\bar{D}_\infty$, which is the unique positive solution to

$$0 = -\gamma \bar{D}_\infty^2 + \big(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}_e)\big)\bar{D}_\infty + \gamma(\bar{v}_\infty - \bar{m}_e^2) + \gamma\frac{\mathfrak{n}^2}{4} + \nu.$$

Therefore, analogously as precedently, the function $\bar{D}$ converges to $\bar{D}_\infty$ thanks the positivity of $\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}(t))$ and the C^1-regularity of $\mathbf{D}$.

Solving (5.17) explicitly, we obtain

$$\bar{D}_\infty = \frac{(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}_e)) + \sqrt{(\lambda - \mu + \gamma(\mathfrak{n} + 2\bar{m}_e))^2 + \gamma^2(4(\bar{v}_\infty - \bar{m}_e^2) + \mathfrak{n}^2) + 4\gamma\nu}}{2\gamma}$$

and consequently $\bar{D}_\infty \sim \mathcal{O}(\mathfrak{n})$ as $\mathfrak{n} \to \infty$. Hence, for small $\mathfrak{n}$, the small size of $\bar{D}$ leads to sufficiently good estimates by considering $\bar{m}$ instead of m. Instead, for large $\mathfrak{n}$, the large values of $\bar{D}$ do not allow to conclude if $\bar{m}$ and m are close. Finer estimates would be needed to paint a clearer picture of the sharpness of the upper bounds but they seem to be currently out of reach.

□

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