ENTROPIC MULTIPLIERS METHOD FOR LANGEVIN DIFFUSION AND WEIGHTED LOG SOBOLEV INEQUALITIES.

PATRICK CATTIAUX ♠ , ARNAUD GUILLIN ◇ , PIERRE MONMARCHÉ ♣ ,
AND CHAOEN ZHANG ◇

♠ Université de Toulouse

◇ Université Blaise Pascal

♣ Sorbonne Université

Abstract. In his work about hypocercivity, Villani [20] considers in particular convergence to equilibrium for the kinetic Langevin process. While his convergence results in L^2 are given in a quite general setting, convergence in entropy requires some boundedness condition on the Hessian of the Hamiltonian. We will show here how to get rid of this assumption in the study of the hypocoercive entropic relaxation to equilibrium for the Langevin diffusion. Our method relies on a generalization to entropy of the multipliers method and an adequate functional inequality. As a byproduct, we also give tractable conditions for this functional inequality, which is a particular instance of a weighted logarithmic Sobolev inequality, to hold.

Key words : Langevin diffusion, entropic convergence, hypocercivity, weighted logarithmic Sobolev inequality, Lyapunov conditions.

1. Settings and main results.

Let $U : \mathbb{R}^d \to \mathbb{R}$ be a smooth (C^∞) function such that $U \geq 1$, $U(x) \to +\infty$ as $|x| \to +\infty$ and $\int e^{-U(x)} dx$ is finite. U will represent the confinement potential for the Hamiltonian $H(x, y) = U(x) + \frac{1}{2}|y|^2$ defined on $\mathbb{R}^{2d}$. The associated Boltzmann-Gibbs (probability) measure is given by

$$\mathrm{d}\mu = \frac{1}{Z} e^{-H(x,y)} \mathrm{d}x \mathrm{d}y$$

where Z is the normalizing constant $\int e^{-H(x,y)} \mathrm{d}x \mathrm{d}y$.

The Langevin dynamics associated to this measure is a flow of probability measures $\mathrm{d}\mu_t = f_t \, \mathrm{d}\mu$ for $t \geq 0$, where f_t solves (at least in a weak sense) the Langevin equation

$$\partial_t f_t = L f_t \,,$$

L being given by

$$L = -y.\nabla_x + (\nabla U(x) - y).\nabla_y + \Delta_y \,. \tag{1}$$

We are thus interested in solutions belonging to $\mathbb{L}^1(\mu)$. Of course, the Hörmander's sum of squares hypoelliptic theorem ensures that $(t, x, y) \mapsto f_t(x, y)$ is smooth on $\mathbb{R}_+^* \otimes \mathbb{R}^{2d}$, whatever

---

Date: August 16, 2022.

the regularity of f_0. It is then easy to see that mass and positivity are preserved so that if $f_0 d\mu$ is a probability measure so is $f_t d\mu$ for any $t \geq 0$. We shall discuss below existence and uniqueness for (1).

The corresponding stochastic process is given by the S.D.E.

$$\begin{cases} dx_t = y_t dt \\ dy_t = -y_t dt - \nabla U(x_t) dt + \sqrt{2} dW_t \end{cases}$$

where (W_t) is an usual $d-$dimensional Wiener process. The infinitesimal generator of the process is thus $L^* = y.\nabla_x - (\nabla U(x) + y).\nabla_y + \Delta_y$. Since all coefficients are local Lipschitz, existence and strong uniqueness for the S.D.E. is ensured up to the explosion time τ. But thanks to our assumptions, $H(x, y) \to +\infty$ as $|x| + |y| \to +\infty$ and it is easily seen that $L^* H \leq M < +\infty$ for some constant M. It is then well known that τ is almost surely infinite, whatever the starting point (x_0, y_0), i.e. the diffusion process is conservative. According to what precedes, for any initial distribution, the distribution μ_t at time $t > 0$ of (x_t, y_t) admits a smooth density w.r.t. Lebesgue measure and since μ is equivalent to Lebesgue measure with a smooth density too, $d\mu_t = f_t d\mu$ where f_t is smooth (C^∞) and solves (1). It is easy to see that μ is the unique invariant but not reversible probability measure for the process (steady state).

We denote by $P_t = e^{tL}$ the semi-group on $\mathbb{L}^1(\mu)$ with generator $(L, D(L))$, i.e. $f_t = P_t f_0$. It is easy to see that for any solution g_t of (1) belonging to $\mathbb{L}^1(\mu)$, $Q_t f = \int f g_t d\mu$ is a Markov continuous semi-group on $L^\infty(\mu)$ whose generator coincides with L^* on the set C_0^∞ of smooth and compactly supported functions (just using integration by parts). The uniqueness of this semi-group implies that $g_t = f_t$ i.e. the uniqueness of the solutions of (1) in $\mathbb{L}^1(\mu)$.

We are interested in the long time behavior of the Langevin diffusion. The usual ergodic theorem tells us that $\frac{1}{t} \int_0^t \mu_s \, ds$ weakly converges to μ as t grows to infinity. One can thus ask for the convergence of f_t towards 1 as t goes to infinity.

This question has been investigated by many authors in recent years both in the P.D.E. community and the probability community. One of the main difference is of course the way to look at this convergence: total variation distance, $\mathbb{L}^2(\mu)$ norm, $\mathbb{H}^1(\mu)$ semi-norm, relative entropy, Wasserstein distance. Another associated problem is to get some bounds on the rate of convergence, once convergence holds true. Let's review some results in this direction.

More or less at the same time, both probabilists and PDE specialists have considered the problem of the speed of convergence to equilibrium. Talay [19] and Wu [22] have built Lyapunov functions and using Meyn-Tweedie's approach have established (non quantitative) exponential convergence to equilibrium (see also [3] for this approach for kinetic models) under quite general assumptions. Desvillettes and Villani [12] used an heavy Fourier machinery to established sub-exponential entropic convergence. Then Hérau and Nier [17] have carried out the spectral analysis of this equation and thus obtained a $\mathbb{L}^2$ exponential decay with quite sharp constants under general conditions. It has settled the bases for the theory of hypocercivity of Villani [20] for the $\mathbb{L}^2$ and the entropic convergence to equilibrium, when $\text{Hess}(U)$ is bounded in the entropic case, see also [13] for a version without regularity issues. Let us also mention [1] where an unified approach dealing with various entropies (as we shall do) is performed, still for bounded Hessians for which explicit rates are given. Finally, and quite recently, coupling approaches, using synchronous coupling or coupling by reflection (see [7]

or [14, 15]) have established exponential convergence to equilibrium in Wasserstein distance with sharp constants, once again when $\text{Hess}(U)$ is bounded.

As we will adopt the terminology and adapt the methodology of hypocoercivity as in Villani [20], let us describe a little bit further the formalism of this setting. Recall that the variance of a squared integrable function g with respect to μ is defined by

$$\text{Var}_\mu(g) := \int g^2 \mathrm{d}\mu - \left(\int g \mathrm{d}\mu \right)^2 = \int \left(g - \int g \mathrm{d}\mu \right)^2 \mathrm{d}\mu$$

while the entropy is defined for positive functions by

$$\text{Ent}_\mu(f) := \int f \ln f \mathrm{d}\mu - \int f \mathrm{d}\mu \ln \int f \mathrm{d}\mu \,.$$

The law μ is said to satisfy a Poincaré inequality if there exists a positive constant C_P such that for all smooth functions g

$$\text{Var}_\mu(g) \leq C_P \int |\nabla g|^2 \mathrm{d}\mu \,.$$

Similarly, μ satisfies a logarithmic Sobolev (or log-Sobolev in short) inequality if there exists a constant C_{LS} such that for all smooth functions g,

$$\text{Ent}_\mu(g^2) \leq C_{LS} \int |\nabla g|^2 \mathrm{d}\mu \,.$$

The natural $\mathbb{H}^1_\mu$ semi-norm is defined as $||g||_{H^1_\mu} := ||\nabla g||_{\mathbb{L}^2_\mu}$. Exponential convergence of $P_t f_0$ to 1 in $\mathbb{H}^1_\mu$ and variance was proved by Villani [20] under two conditions:

(1-var) $|\nabla^2 U| \leq c \left(1 + |\nabla U|\right)$;
(2-var) $e^{-U(x)}\mathrm{d}x$ satisfies a Poincaré inequality.

Remark that (2-var) is equivalent to the fact that μ satisfies a Poincaré inequality, thanks to the tensorization property of the latter, since the gaussian measure satisfies a Poincaré inequality.

For convergence in entropy, the assumptions made by Villani are much stronger:

(1-ent) $\nabla^2 U$ is bounded;
(2-ent) $e^{-U(x)}\mathrm{d}x$ satisfies a log-Sobolev inequality.

Again, (2-ent) is equivalent to the fact that μ satisfies a log-Sobolev inequality, thanks to a similar argument of tensorization.

When both these assumptions are satisfied, Villani showed that, for any initial probability density f_0 with finite moments of order 2, the entropy of $P_t f_0$ converges to 0 exponentially fast (see Villani [20] Theorem 39).

Our main goal in this paper is to get rid of the boundedness assumption (1-ent) for $\nabla^2 U$, replacing it by

Assumption 1. *In addition to our assumptions on U, we assume that there exists $\eta \geq 0$ such that $U^{-2\eta}\nabla^2 U$ is bounded.*

A typical situation where Assumption 1 is satisfied is when both U and $\nabla^2 U$ have polynomial growth at infinity, i.e. $U(x) \geq c_1 (1 + |x|)^l$ and $|\nabla^2 U| \leq c_2 (1 + |x|)^j$ so that we may choose $\eta \geq \frac{j}{2l}$. In particular if $j = l - 2 \geq 0$ as it is the case for true polynomials of degree at least 2, we may choose $\eta = \frac{1}{2} - \frac{1}{l}$.

The counterpart is that we have to reinforce (2-ent) replacing it by the stronger

Assumption 2. μ satisfies the following weighted log-Sobolev inequality: there exists $\rho > 0$ s.t. for all smooth enough g with $\int g^2 \, d\mu = 1$:

$$\mathrm{Ent}_\mu(g^2) \leq \rho \int (H^{-2\eta} |\nabla_x g|^2 + |\nabla_y g|^2) d\mu. \tag{2}$$

Once both Assumptions 1 and 2 are satisfied, we can prove exponential decay in entropy for the Langevin diffusion. Our approach is based on the multiplier method. More precisely we will prove the following:

Theorem 1. *Under Assumptions 1 and 2, let*

$$\lambda = \left(\|H^{-2\eta} \nabla^2 U\|_\infty + 3 \right)^2 ,$$

$$\kappa = \frac{1}{16 \left(d + 1 + 5\eta^2 + 6\eta \right)^2} .$$

Then for all initial probability density f,

$$\mathrm{Ent}_\mu(P_t f) \leq \exp \left(-\frac{\kappa}{1 + 8\lambda\rho} \int_0^t (1 - e^{-s})^2 ds \right) \mathrm{Ent}_\mu(f) .$$

Section 2 is devoted to the proof of this theorem which contains Villani's result in the case $\eta = 0$.
Actually as in [1] (also see [8] in the non degenerate case) we shall prove a more general statement including both the variance and the entropic case. To this end introduce an admissible function Ψ, that is

$$\Psi \geq 0 , \ \Psi \in C^4 \text{ and } \frac{1}{\Psi''} \text{ is positive and concave,} \tag{3}$$

as in [18, 1]. Theorem 1 corresponds to

$$\Psi : \mathbb{R}^+ \to \mathbb{R}, u \mapsto u \ln u + 1 - u ,$$

while the $\mathbb{L}^2_\mu$ case corresponds to $\Psi(u) = (u - 1)^2$. We also denote $\psi = \Psi''$. The general statement is the following

Theorem 2. *Let Ψ be an admissible function. Suppose that Assumption 1 is satisfied. If for any bounded density of probability f, the following inequality is satisfied*

$$\int \Psi(f) \, d\mu \leq \rho \int \psi(f) \left(H^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2 \right) d\mu , \tag{4}$$

then

$$\Psi(P_t f) \leq \exp \left(-\frac{\kappa}{1 + 8\lambda\rho} \int_0^t (1 - e^{-s})^2 ds \right) \Psi(f) . \tag{5}$$

Remark 3. For (4) to be satisfied it is immediate that $\Psi(1)$ has to be equal to 0.
A natural family of admissible functions, namely $\Psi_p(u) = u^p - 1 - p(u - 1)$ defined for $1 < p \leq 2$, is introduced in [1]. Remark that, up to the constants, the change $v = c_p u^{p/2}$ shows that (4) is satisfied for Ψ_p as soon as it is satisfied for Ψ_2. This can be viewed as some kind of interpolation result between $\mathbb{L}^1$ and $\mathbb{L}^2$.
Also notice that, if $1/\psi$ is concave at infinity (i.e. outside some compact interval) one can modify it and introduce some $\tilde{\psi}$ which satisfies all the required properties and coincides with ψ outside some larger compact interval. The corresponding $\tilde{\Psi}$ will behave like Ψ at infinity, which is the interesting property for controlling the convergence. $\diamond$

The key idea for proving Theorem 2 is to use a twisted gradient depending on time, see lemma 6. An important aspect of our result is that the bounded Hessian condition in Villani's approach is relaxed as Assumption 1. In fact it was a major issue raised by Villani [20] concerning the entropic convergence. Indeed, his L^2 multiplier method, at the basis of the entropic hypocercivity, does not rely on a Poincaré inequality but on Brascamp-Lieb inequality. It was thus thought that for the multiplier method to hold for entropy, an entropic Brascamp-Lieb inequality was needed. However Bobkov-Ledoux [6] proved that this inequality is false in general, and true in very particular setting. Our strategy is then to show that it is not an entropic Brascamp-Lieb inequality that we need but a particular weighted logarithmic Sobolev inequality. Note also that a first attempt to skip the boundedness assumption for the Hessian is contained in [3] Theorem 6.10, but the statement therein is much weaker than the one of the present theorem and most importantly not at all quantitative. One can also look at [1] for a quantitative result in the bounded Hessian case.

Next we shall show that, similarly to the non weighted case studied in [9] (see also [2, 10]), the weighted log Sobolev inequality in Assumption 2 is equivalent to some Lyapunov type condition.
To this end we introduce the natural second order operator

$$L_\eta := H^{-2\eta}\Delta_x + \Delta_y - H^{-2\eta}\left(2\eta\frac{\nabla_x H}{H} + \nabla_x H\right).\nabla_x - \nabla_y H.\nabla_y\,,$$

which is symmetric in $\mathbb{L}^2_\mu$ and satisfies

$$\int f\,L_\eta g\,\mathrm{d}\mu = -\int (H^{-2\eta}\,\nabla_x f.\nabla_x g + \nabla_y f.\nabla_y g)\,\mathrm{d}\mu\,. \tag{6}$$

Theorem 4. *Recall that U goes to infinity at infinity. Assume that $|\nabla H| \geq h > 0$ outside some large ball. Denote $A_r := \{(x,y) : H(x,y) \leq r\}$, and*

$$\theta(r) = \sup_{z\in\partial A_r}\ \max_{i,j=1,\ldots,2d}|\frac{\partial^2 H}{\partial z_i \partial z_j}|$$

Assume that $\theta(r) \leq ce^{C_0 r}$ with some positive constants C_0 and c for r sufficiently large. Assume that there exists a Lyapunov function W with $W(x) \geq w > 0$ for all (x,y) and some $\lambda, b > 0$ satisfying

$$L_\eta W(x,y) \leq -\lambda H(x,y)\,W(x,y) + b\,.$$

Then μ verifies a weighted logarithmic Sobolev inequality (2).

Remark that the condition $\theta(r) \leq ce^{C_0 r}$ is trivially verified when both U and $\mathrm{Hess}(U)$ have a polynomial growth. Also, a Lyapunov function exists if U satisfies the conditions in the following corollary:

Corollary 5. *Assume that the following conditions hold outside a compact domain:*

(1) $\Delta_x U \leq \kappa |\nabla_x U|^2$ *for some* $\kappa \in (0, 1)$;
(2) *a growth condition:* $|\nabla_x U|^2 \geq cU^{2\eta+1}$ *for some positive constant c.*

Then $d\mu = \frac{1}{Z} e^{-H(x,y)} dx dy$ *satisfies a weighted logarithmic Sobolev inequality.*

Moreover, if we assume that $U^{-2\eta} \nabla^2 U$ *is bounded, then we may apply Theorem 1.*

The next section will present the proof of Theorem 2, where the entropic multipliers method is presented. In Section 3, the treatment via Lyapunov condition of weigthed log-Sobolev inequality, i.e. Theorem 2 and Corollary 3, is done.

The final section discusses some additional points on weighted inequalities. Indeed, the proof of weighted Poincaré inequality used by Villani relies solely on some Poincaré inequality for each measure and adapt the usual argument of tensorization, using heavily the orthogonality inherited from the $\mathbb{L}^2_\mu$ structure. However, in the entropic case, starting with a log-Sobolev for each marginal, we are only able to recover a weaker (but interesting) inequality for the product measure.

2. PROOF OF THEOREM 2.

This section is devoted to the proof of Theorem 2.

We only consider the case where f_0 is bounded away from zero. Indeed, if it is not the case, writing $g_0 = (1-\delta)f_0 + \delta$ for some $\delta > 0$, then we may prove the theorem for $g_t = (1-\delta)f_t + \delta$ and let δ go to zero to recover the result for f_t.

The key point of the proof is to introduce a time and space-dependent twisted gradient. Consider $r \in \mathbb{N}$ and for $0 \leq i \leq r$, $x \mapsto b_i(x) \in \mathbb{R}^d$ a smooth vector field, $C_i = b_i.\nabla$, $Cf = (C_0 f, \ldots, C_r f)$, $t, x \mapsto M_t(x)$ a smooth function from $\mathbb{R}_+ \times \mathbb{R}^d$ to $\mathcal{M}^{sym+}_{r \times r}(\mathbb{R})$ the set of positive semi-definite symmetric real matrices of size r, and

$$F(t) = \int \psi(P_t f) (CP_t f)^T M_t CP_t f d\mu,$$

where A^T stands for the transpose of the matrix A and vectors are seen as 1-column matrices. The coefficients of M_t are the so-called multipliers in the eponymous method introduced in [20, Section I.8].

The following results holds for any diffusion operator:

Lemma 6. *Let* $L = L_s + L_a$, *where* $L_s = \frac{1}{2}(L + L^*)$ *and* $L_a = \frac{1}{2}(L - L^*)$ *stand for the symmetric and antisymmetric part of L in* $\mathbb{L}^2_\mu$. *Then*

$$F'(t) \leq \int \psi(P_t f) (CP_t f)^T \left(2M_t [C, L] + ((2L_s - L)M_t + \partial_t M_t) C \right) P_t f d\mu,$$

where $[C_i, L] = C_i L - LC_i$ *is the (generalized) Lie bracket of* C_i *and L and* $[C, L] = ([C_0, L], \ldots, [C_r, L])$.

Proof. In the following we write f for $P_t f$ and $M_t(x) = (m_{i,j}(t, x))_{0 \leq i,j \leq r}$. First it holds

$$\partial_t \left(\int \psi(f) m_{i,j} C_i f C_j f \mathrm{d}\mu \right) = \int \psi(f) \partial_t(m_{i,j}) C_i f C_j f + m_{i,j} \partial_t \left(\psi(f) C_i f C_j f \right) \mathrm{d}\mu.$$

This derivation is justified by the fact that f_0 is uniformly strictly positive and so is f_t, by hypoellipticity and the control of the growth of the derivative of f_t, using Villani [20, Sect. A.21] or [16]. Denote as usual the Carré-du-Champ operator $2\Gamma(g, h) = L(gh) - gLh - hLg$. Next, μ being invariant for L, and using the diffusion property, i.e. that the chain rule property $L\Psi(f_1, ..., f_d) = \sum_1^d \partial_i \Psi(f) L f_i + \sum_{i,j} \partial_{i,j} \Psi(f) \Gamma(f_i, f_j)$ holds for all nice Ψ and f,

$$\begin{aligned}
0 &= \int L \left(m_{i,j} \psi(f) C_i f C_j f \right) \mathrm{d}\mu \\
&= \int L \left(m_{i,j} \right) \psi(f) C_i f C_j f \mathrm{d}\mu + \int m_{i,j} L \left(\psi(f) C_i f C_j f \right) \mathrm{d}\mu \\
&\quad + 2 \int \Gamma \left(m_{i,j}, \psi(f) C_i f C_j f \right) \mathrm{d}\mu \\
&= \int (L - 2L_s) \left(m_{i,j} \right) \psi(f) C_i f C_j f \mathrm{d}\mu + \int m_{i,j} L \left(\psi(f) C_i f C_j f \right) \mathrm{d}\mu.
\end{aligned}$$

The case where M is constant (and symmetric semi-definite positive) is already treated in [18, Lemma 8] where it is shown that

$$\sum_{i,j} m_{i,j} \Big(L \left(\psi(f) C_i f C_j f \right) - \partial_t \left(\psi(f) C_i f C_j f \right) \Big) \geq 2\psi(f) \sum_{i,j} m_{i,j} \left(C_i f \right) [L, C_j] f.$$

The proof follows by taking the integral of both sides. $\qquad \square$

Proof of Theorem 2. Now consider the case of the Langevin diffusion, namely L is given by (1). Note that

$$[L, \nabla_y] = \nabla_x + \nabla_y \qquad [L, \nabla_x] = -\nabla^2 U(x).\nabla_y.$$

The operator L is decomposed as $L = L_s + L_a$ where

$$L_s = -y.\nabla_y + \Delta_y \qquad L_a = -y.\nabla_x + \nabla U(x).\nabla_y.$$

Recalling $H(x, y) = U(x) + \frac{1}{2}|y|^2$, then $L_a H = 0$ and more generally $L_a(g \circ H) = 0$ for any smooth $g : \mathbb{R} \to \mathbb{R}$. In particular for $\eta > 0$,

$$\begin{aligned}
\left| (2L_s - L) \left(H^{-\eta} \right) \right| &= \left| L_s \left(H^{-\eta} \right) \right| \\
&= \left| \eta(|y|^2 - d) H^{-\eta - 1} + \eta(\eta + 1)|y|^2 H^{-\eta - 2} \right| \\
&\leq (d + \eta^2 + 2\eta) H^{-\eta}.
\end{aligned}$$

Let a, b, c depend on t and $H(x, y)$, and let $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $C = \nabla$, so that Lemma 6 reads

$$F'(t) \leq -2 \int \psi(P_t f) \left(\nabla P_t f \right)^T N \nabla P_t f \mathrm{d}\mu$$

with

$$N = \begin{pmatrix} b - \frac{1}{2}(L_s + \partial_t)a & -a\nabla^2 U + b - \frac{1}{2}(L_s + \partial_t)b \\ c - \frac{1}{2}(L_s + \partial_t)b & -b\nabla^2 U + c - \frac{1}{2}(L_s + \partial_t)c \end{pmatrix}.$$

In the top left corner b is good news since it gives some coercivity in the x variable. Nevertheless as soon as $b \neq 0$, $b\nabla^2 U$ in the bottom right corner is an annoying term that can only be controlled by the entropy production if it is bounded (which is where, in the previous studies, the assumption that $\nabla^2 U$ is bounded barged in).

Writing $\alpha(t) = (1 - e^{-t})$, set

$$c = 2\varepsilon\alpha H^{-\eta} \qquad b = \varepsilon^2\alpha^2 H^{-2\eta} \qquad a = \varepsilon^3\alpha^3 H^{-3\eta}$$

for some $\varepsilon \in (0, 1)$. In other words,

$$(\nabla f)^T M \nabla f \ = \ \varepsilon\alpha H^{-\eta}|\nabla_y f|^2 + \varepsilon\alpha H^{-\eta}|\nabla_y f + \varepsilon\alpha H^{-\eta}\nabla_x f|^2,$$

so that, in particular, M is positive definite. In that case we bound

$$\begin{aligned}
b - \frac{1}{2}(L_s + \partial_t)a \ &\geq \ \varepsilon^2\alpha^2 H^{-2\eta} - \frac{1}{2}(d + 9\eta^2 + 6\eta)\varepsilon^3\alpha^3 H^{-3\eta} - \frac{3}{2}\varepsilon^3\alpha^2 e^{-t}H^{-3\eta} \\
&\geq \ \varepsilon^2\alpha^2 H^{-2\eta}\left(1 - (d + 1 + 5\eta^2 + 6\eta)\varepsilon\right),
\end{aligned}$$

$$\begin{aligned}
-b\nabla^2 U + c - \frac{1}{2}(L_s + \partial_t)c \ &\geq \ -\varepsilon^2\alpha^2\|H^{-2\eta}\nabla^2 U\|_\infty + \left(2\alpha - \frac{1}{2}(d + \eta^2 + 2\eta)\alpha - e^{-t}\right)\varepsilon H^{-\eta} \\
&\geq \ -\varepsilon^2\|H^{-2\eta}\nabla^2 U\|_\infty - \varepsilon\left(d + \eta^2 + \eta\right),
\end{aligned}$$

$$\begin{aligned}
|b + c - a\nabla^2 U - (L_s + \partial_t)b| \ &\leq \ |\varepsilon^2\alpha^2 H^{-2\eta} + 2\varepsilon\alpha H^{-\eta} - 2e^{-t}\varepsilon^2\alpha H^{-2\eta}| \\
&\quad + \ |\varepsilon^3\alpha^3 H^{-3\eta}\nabla^2 U| + (d + 4\eta^2 + 4\eta)\varepsilon^2\alpha^2 H^{-2\eta} \\
&\leq \ \varepsilon\alpha H^{-\eta}\left(\varepsilon^2\|H^{-2\eta}\nabla^2 U\|_\infty + 2 + \varepsilon(d + 4\eta^2 + 4\eta)\right),
\end{aligned}$$

which implies for $\varepsilon = (d + 1 + 5\eta^2 + 6\eta)^{-1}/4$ that

$$(\nabla f)^T N \nabla f \ \geq \ \frac{1}{4}\varepsilon^2\alpha^2 H^{-2\eta}|\nabla_x f|^2 - A|\nabla_y f|^2$$

with

$$\begin{aligned}
A \ &= \ \left(\varepsilon^2\|H^{-2\eta}\nabla^2 U\|_\infty + 2 + \varepsilon(d + 4\eta^2 + 4\eta)\right)^2 + \varepsilon^2\|H^{-2\eta}\nabla^2 U\|_\infty + \varepsilon\left(d + \eta^2 + \eta\right) \\
&\leq \ \frac{1}{2}\left(\|H^{-2\eta}\nabla^2 U\|_\infty + 3\right)^2 \ := \ \frac{1}{2}\lambda.
\end{aligned}$$

Writing

$$G(t) \ = \ \frac{1}{2\lambda}F(t) + \int \Psi(P_t f)\mathrm{d}\mu,$$

we have obtained

$$\begin{aligned}
G'(t) \ &\leq \ -\int \psi(P_t f)\left(\frac{\alpha^2\varepsilon^2}{4\lambda}H^{-2\eta}|\nabla_x P_t f|^2 + \left(1 - \frac{A}{\lambda}\right)|\nabla_y P_t f|^2\right)\mathrm{d}\mu \\
&\leq \ -\frac{\alpha^2\varepsilon^2}{4\lambda}\int \psi(P_t f)\left(H^{-2\eta}|\nabla_x P_t f|^2 + |\nabla_y P_t f|^2\right)\mathrm{d}\mu.
\end{aligned}$$

On the one hand,

$$F(t) \ \leq \ 3\varepsilon\alpha\int \psi(P_t f)\left(H^{-2\eta}|\nabla_x P_t f|^2 + |\nabla_y P_t f|^2\right)\mathrm{d}\mu,$$

and on the other hand, using the inequality (4),

$$\int \Psi(P_t f) d\mu \;\leq\; \rho \int \psi(P_t f) \left(H^{-2\eta} |\nabla_x P_t f|^2 + |\nabla_y P_t f|^2 \right) d\mu,$$

which implies

$$G'(t) \;\leq\; -\frac{\alpha^2 \varepsilon^2}{1 + 4\lambda\rho} G(t),$$

where we have used that $6\varepsilon\alpha \leq 1$ for simplicity. Hence,

$$\Psi(P_t f) \;\leq\; G(t) \;\leq\; G(0) \exp\left(-\frac{\varepsilon^2}{1 + 4\lambda\rho} \int_0^t \alpha^2(s) ds \right),$$

and $G(0) = \Psi(f)$. The proof is complete. $\qquad\square$

3. Weighted Functional Inequalities with $\eta \geq 0$.

We turn to the study of the functional inequality (4). For simplicity we shall only consider the cases $\Psi(u) = (u-1)^2$ (Variance) and $\Psi(u) = u\ln u - u + 1$ (Entropy).
Recall the definition of L_η,

$$L_\eta := H^{-2\eta}\Delta_x + \Delta_y - H^{-2\eta}\left(2\eta\frac{\nabla_x H}{H} + \nabla_x H \right).\nabla_x - \nabla_y H.\nabla_y \,,$$

which satisfies

$$-\int f\, L_\eta f\, d\mu \;=\; \int \left(H^{-2\eta}\,|\nabla_x f|^2 + |\nabla_y f|^2 \right) d\mu \;:=\; \mathcal{E}_\eta(f). \tag{7}$$

Let us state our first main results

Theorem 7. *The weighted Poincaré inequality*

$$Var_\mu(g) \leq \rho \int \left(H^{-2\eta}|\nabla_x g|^2 + |\nabla_y g|^2 \right)\, d\mu$$

is satisfied if and only if there exists a Lyapunov function, i.e. a smooth function W such that $W(x,y) \geq w > 0$ for all (x,y), a constant $\lambda > 0$ and a bounded open set A such that

$$L_\eta W \leq -\lambda W + \mathbf{1}_{\bar{A}}\,.$$

We provide then the equivalent result for the logarithmic Sobolev inequality.

Theorem 8. *Assume that H goes to infinity at infinity and that there exists $a > 0$ such that $e^{aH} \in \mathbb{L}^1(\mu)$.*

(1) *If μ satisfies the weighted log-Sobolev inequality (2), then, there exists a Lyapunov function, i.e. a smooth function W such that $W(x,y) \geq w > 0$ for all (x,y), two positive constants λ and b such that*

$$L_\eta W \leq -\lambda H W + b\,. \tag{8}$$

(2) *Conversely, assume that there exists a Lyapunov function satisfying* (8) *and that* $|\nabla H|(x, y) \geq c > 0$ *for* $|(x, y)|$ *large enough. Define*

$$\theta(r) = \sup_{z \in \partial A_r} \max_{i,j=1,\dots,2d} |\frac{\partial^2 H}{\partial z_i \partial z_j}|$$

and assume that $\theta(r) \leq c e^{C_0 r}$ *with some positive constants* C_0 *and* c *for* r *sufficiently large. Then* μ *satisfies the weighted log-Sobolev inequality* (2).

These theorems are the analogues, in the weighted situation we are looking at, of (part of) Theorem 1.1 and Theorem 1.2 in [9]. Their proofs are very similar concerning the part 1) of the previous theorem and we shall only give some details in the entropic case. Let us begin by a simple and crucial Lemma, at the basis of the use of Lyapunov type condition. Note that it can also be proved via large deviations argument.

Lemma 9. *For every continuous function* $W \geq 1$ *in the domain of* L_η *such that* $-L_\eta W/W$ *is* μ-*a.e. lower bounded, for all* g *in the domain of* L_η,

$$\int -\frac{L_\eta W}{W} g^2 \, d\mu \leq \int \left(H^{-2\eta} |\nabla_x g|^2 + |\nabla_y g|^2 \right) \, d\mu. \tag{9}$$

Proof. This follows from integration by parts and Cauchy-Schwarz inequality. Indeed,

$$
\begin{aligned}
\int -\frac{L_\eta W}{W} g^2 \, \mathrm{d}\mu &= \int H^{-2\eta} \langle \nabla_x W, \nabla_x \frac{g^2}{W} \rangle + \langle \nabla_y W, \nabla_y \frac{g^2}{W} \rangle \mathrm{d}\mu \\
&= \int H^{-2\eta} \left(-\frac{g^2}{W^2} |\nabla_x W|^2 + 2\frac{g}{W} \langle \nabla_x W, \nabla_x g \rangle \right) \\
&\qquad\qquad + \left(-\frac{g^2}{W^2} |\nabla_y W|^2 + 2\frac{g}{W} \langle \nabla_y W, \nabla_y g \rangle \right) \mathrm{d}\mu \\
&\leq \int \left(H^{-2\eta} |\nabla_x g|^2 + |\nabla_y g|^2 \right) \, \mathrm{d}\mu.
\end{aligned}
$$

$\square$

Let us now prove Theorem 8.

Proof. For a given function ϕ, introduce the operator G_η via $G_\eta h = -L_\eta h + \phi h$. For any h in the domain of L_η, $\int h \, G_\eta h \mathrm{d}\mu = \mathcal{E}_\eta(h) + \int h^2 \phi \, \mathrm{d}\mu$. Choosing $\phi = -c + \mathbf{1}_A$, for some set A to be defined, in the variance case and $\phi = \rho(b - H)$ in the entropic case, one deduces that G_η is continuous for the norms whose square are respectively $\mathcal{E}_\eta(h) + \int_A h^2 \, \mathrm{d}\mu$ and $\mathcal{E}_\eta(h) + \int h^2 \, \mathrm{d}\mu$. If a weighted Poincaré inequality (resp. weighted log-Sobolev inequality) is satisfied, following the proof of Theorem 2.1 (resp. Proposition 3.1) in [9], we get that the form $\int h \, G_\eta h \, \mathrm{d}\mu$ is also coercive so that the Lax-Milgram theorem gives a solution to $G_\eta h = 1$, which furnishes the desired Lyapunov function (see [9] for the details).

For the converse, we revisit the proof of [9] Proposition 3.5 in order to adapt it to our case. As usual, we will rather prove the (weighted) log-Sobolev inequality in its equivalent (weighted) Super Poincaré inequality form, i.e. there exist $c, \beta > 0$ such that for all smooth f and $s > 0$,

$$\int f^2 \mathrm{d}\mu \leq s \int (H^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2) \mathrm{d}\mu + c \, e^{\beta/s} \left(\int |f| \mathrm{d}\mu \right)^2.$$

Indeed, the latter implies a defective (weighted) log-Sobolev inequality and a weighted Poincaré inequality (choosing s such that $ce^{\beta/s} = 1$) and we obtain a tight (weighted) log-Sobolev inequality by using Rothaus lemma (see [4] p.239), which states that

$$\mathrm{Ent}_\mu(f^2) \leq \mathrm{Ent}_\mu(\tilde{f}^2) + 2\mathrm{Var}_\mu(f), \tag{10}$$

where $\tilde{f} = f - \int f \, \mathrm{d}\mu$. For all this we refer to [10, 11, 21].

Recall $A_r = \{H \leq r\}$. For r_0 large enough and some $\lambda' < \lambda$ we have

$$L_\eta W \leq -\lambda' \, H \, W \, + \, b \, \mathbf{1}_{A_{r_0}},$$

so that we may assume that

$$\frac{L_\eta W}{W}(x,y) \leq -\lambda H(x,y) + \frac{b}{w} \, \mathbf{1}_{A_{r_0}}.$$

For $r > r_0$,

$$
\begin{aligned}
\int f^2 \, \mathrm{d}\mu \ &\leq \ \int_{A_r} f^2 \, \mathrm{d}\mu \, + \, \int_{A_r^c} \frac{\lambda H}{\lambda r} f^2 \, \mathrm{d}\mu \\
&\leq \ \int_{A_r} f^2 \, \mathrm{d}\mu \, + \, \int \frac{\lambda H}{\lambda r} f^2 \, \mathrm{d}\mu \\
&\leq \ \int_{A_r} f^2 \, \mathrm{d}\mu \, + \, \frac{1}{\lambda r} \int \left(\frac{-L_\eta W}{W} + \frac{b \, \mathbf{1}_{A_{r_0}}}{w} \right) f^2 \, \mathrm{d}\mu \\
&\leq \ \left(1 + \frac{b}{\lambda r w} \right) \int_{A_r} f^2 \, \mathrm{d}\mu \, + \, \frac{1}{\lambda r} \int \left(H^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2 \right) \, \mathrm{d}\mu.
\end{aligned}
$$

It remains to control the integral in A_r. It is in fact a simple consequence of Nash inequalities for the Lebesgue measure rewritten in its Super Poincaré form (c.f. [10, Prop 3.8]): there exists c_d such that for all r large enough, all smooth f and $s > 0$

$$
\begin{aligned}
\int_{A_r} f^2 \mathrm{d}x \mathrm{d}y \ &\leq \ s \int_{A_r} |\nabla f|^2 \mathrm{d}x \mathrm{d}y + c_d \theta^d(r)(1 + s^{-2d}) \left(\int |f| \mathrm{d}x \mathrm{d}y \right)^2 \\
&\leq \ s \int_{A_r} |\nabla f|^2 \mathrm{d}x \mathrm{d}y + c_d c e^{2dC_0 r}(1 + s^{-2d}) \left(\int |f| \mathrm{d}x \mathrm{d}y \right)^2.
\end{aligned}
$$

Recall that $H \geq 1$. We thus have

$$
\begin{aligned}
\int_{A_r} f^2 \, \mathrm{d}\mu \ &\leq \ \frac{1}{eZ} \int_{A_r} f^2 \mathrm{d}x \mathrm{d}y \\
&\leq \ \frac{r^{2\eta} e^r}{e} s \int \left(H^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2 \right) \, \mathrm{d}\mu \, + \, Z c_d c e^{2dC_0 r}(1 + s^{-2d}) e^{2r} \left(\int_{A_r} |f| d\mu \right)^2.
\end{aligned}
$$

Letting $u = s e^{r-1} r^{2\eta}$ and $C' = Zcc_d$, and considering integral on the whole space in the right hand side, we have thus obtained (for r large enough)

$$\int_{A_r} f^2 \, \mathrm{d}\mu \ \leq \ u \int \left(H^{-2\eta} |\nabla_x f|^2 + |\nabla_y f|^2 \right) \, \mathrm{d}\mu \, + \, C' \, r^{4d\eta} \, (1 + u^{-2d}) e^{2(1+dC_0+d)r} \left(\int |f| d\mu \right)^2.$$

Denoting $c = 1 + \frac{b}{\lambda r_0 w}$, and $\beta_d = 2 + d + 2dC_0$, we thus have, for all $u > 0$ and r large enough,

$$\int f^2 d\mu \leq \left(uc + \frac{1}{\lambda r}\right) \int \left(H^{-2\eta}|\nabla_x f|^2 + |\nabla_y f|^2\right) d\mu + C'\left(1 + u^{-2d}\right) r^{2d\eta} c\, e^{\beta_d r} \left(\int |f| \, d\mu\right)^2.$$
(11)

Choosing $r\lambda = (uc)^{-1}$ and $s = 2uc$, we have thus proved the existence of some β_d' such that

$$\int f^2 d\mu \leq s \int \left(H^{-2\eta}|\nabla_x f|^2 + |\nabla_y f|^2\right) d\mu + C'' e^{\beta_d'/s} \left(\int |f| \, d\mu\right)^2,$$

and the proof is complete. $\qquad\square$

Remark 10. For a general weighted logarithmic Sobolev inequality with the weighted energy

$$\int \left(w_1|\nabla_x f|^2 + w_2|\nabla_y f|^2\right) d\mu,$$

we can introduce the symmetric generator

$$L_{w_1,w_2} := w_1 \Delta_x + w_2 \Delta_y - w_1 \left(-\frac{\nabla_x w_1}{w_1} + \nabla_x H\right).\nabla_x - w_2 \left(-\frac{\nabla_y w_2}{w_2} + \nabla_y H\right).\nabla_y.$$

If a Lyapunov function (as in Theorem 4 but for L_{w_1,w_2}) exists, then following the same line, we can obtain (with the required additional assumptions on the weights) a weighted logarithmic Sobolev inequality. $\qquad\diamondsuit$

We now proceed to the

Proof of Corollary 5. Consider a smooth function $W(x, y) = e^{\alpha U(x) + \frac{\beta}{2}|y|^2}$ with two constants $\alpha, \beta \in (0, 1)$ to be determined. Then for $|(x, y)| \geq R$,

$$\begin{aligned} \frac{L_\eta W}{W} &= \alpha H^{-2\eta}\left[\Delta_x U + \left(\alpha - \frac{2\eta}{H} - 1\right)|\nabla_x U|^2\right] + \beta(d - (1-\beta)|y|^2) \\ &\leq \beta d - \alpha\left(1 - \alpha - \kappa\right)|\nabla_x U|^2 H^{-2\eta} - \beta(1-\beta)|y|^2, \end{aligned}$$

where we used the first condition in the assumption of the corollary.

To bound the last term by some $C - \lambda H$, we consider $\alpha \in (0, 1 - \kappa), \beta \in (0, 1)$, and divide it into two cases. If $\frac{|y|^2}{2} \geq \frac{H}{2}$, then

$$-\alpha\left(1 - \alpha - \kappa\right)|\nabla_x U|^2 H^{-2\eta} - \beta(1-\beta)|y|^2 \leq -\beta(1-\beta)H$$

Otherwise, we have $U \geq \frac{H}{2}$. Combined with the second condition, it follows

$$-\frac{|\nabla_x U|^2}{H^{2\eta}} \leq -\frac{cU^{2\eta+1}}{2^{2\eta}U^{2\eta}} \leq -\frac{c}{2^{2\eta+1}}H,$$

which completes the proof of the Lyapunov condition. Since the second condition implies that U goes to infinity at infinity and $|\nabla_x U| \geq u \geq 0$, we get a weighted logarithmic Sobolev inequality for μ by the previous theorem. $\qquad\square$

The next example, which is the simple polynomial case, will show the adequacy of our conditions on weighted log-Sobolev inequality with the Assumption 1.

Example 1. Let us consider the example where $U(x) = |x|^l$ with $l > 2$ for $|x|$ large enough, that is, $H(x,y) = |x|^l + \frac{|y|^2}{2}$. Then $\Delta_x U = (dl + l^2 - 2l))|x|^{l-2}$ and $|\nabla_x U|^2 = l^2 |x|^{2l-2}$. The first condition is satisfied since $l > 2$, while the second condition requires

$$\eta \leq \frac{1}{2} - \frac{1}{l}.$$

Note that $||U^{-2\eta}\nabla^2 U||_\infty \sim |x|^{l-2-2l\eta}$ so that, to ensure that $U^{-2\eta}\nabla^2 U$ is bounded, we have to choose $\eta = \frac{1}{2} - \frac{1}{l}$. With the case $l = 2$ we recover Villani's result.

Let us give another example which will show that our limit growth for the potential U is below the exponential growth

Example 2. Choose now $U(x) = e^{a|x|^b}$ for $a, b > 0$ for $|x|$ large enough. Then $\Delta_x U \sim a^2 b^2 |x|^{2(b-1)} e^{a|x|^b}$ and $|\nabla_x U|^2 \sim a^2 b^2 e^{2a|x|^b}$. The first condition is thus satisfied , while the second one imposes once again that $2\eta + 1 \leq 2$. Now, Assumption 1 imposes that $2\eta > 1$ if $b \geq 1$, leading to an impossible adequacy of the two sets of conditions, and to $2\eta \geq 1$ if $b < 1$, in which case the choice of $\eta = 1/2$ is admissible.

Let us end this section by a remark

Remark 11. For the multipliers method in the variance case, Villani does not use $H^{-2\eta}$ in the energy to get his inequality but, as will be seen in the next section, proves a rather stronger inequality with weight $U(x)^{-2\eta}(1 + |y|^2)^{-2\eta}$ in the derivative in x. The fact that he deals with the variance helps him enough to prove such a weighted Poincaré inequality. We may also consider a weighted logarithmic Sobolev inequality with such a weight. However, via the Lyapunov condition approach, the condition on η is then too strong to match with Assumption 1. It is thus crucial to have a weighted inequality with weight $H^{-2\eta}$ for Theorem 1. $\Diamond$

The next section presents an alternative approach, trying to provide an answer to the problem alluded in the previous remark. Is it possible to provide a 'tensorization-like" approach to provide a weighted logarithmic Sobolev inequality as in Villani's paper, thus giving an alternative to Lyapunov conditions?

4. Some further remarks on weighted inequalities.

In this final section we shall try to understand whether it is possible to impose conditions on U solely in order to get weighted inequalities. We shall use several times the following elementary inequalities, true for all $\eta \geq 0$, all x and y (recall that $U \geq 1$)

$$U^{-\eta}(x) \left(1 + \frac{1}{2}|y|^2\right)^{-\eta} \leq H^{-\eta}(x,y) \leq \min\left(U^{-\eta}(x), \left(1 + \frac{1}{2}|y|^2\right)^{-\eta}\right). \tag{12}$$

We shall use in the sequel the notations $U^{-2\eta}(x) = \phi_1(x)$ and $\left(1 + \frac{1}{2}|y|^2\right)^{-2\eta} = \phi_2(y)$.

4.1. The case of weighted Poincaré inequalities.

Assume that μ satisfies a weighted Poincaré inequality. If we choose an f that only depends on x and use that $H^{-2\eta}(x,y) \le U^{-2\eta}(x)$ for all y, we immediately see that the first marginal of μ, i.e. $d\mu_1(x) := \frac{1}{Z_1} e^{-U(x)} dx$ also satisfies the weighted Poincaré inequality

$$\text{Var}_{\mu_1}(f) \le C \int U^{-2\eta} |\nabla f|^2 d\mu_1. \tag{13}$$

Conversely we have,

Theorem 12. *Write $\mu(dx,dy) = \mu_1(dx) \otimes \mu_2(dy)$. If $\mu_1(dx) = \frac{1}{Z_1} e^{-U(x)} dx$ satisfies the weighted Poincaré inequality (13) with constant C_1, then μ satisfies the following weighted Poincaré inequality*

$$Var_\mu(h) \le C' \int (H^{-2\eta} |\nabla_x h|^2 + |\nabla_y h|^2) d\mu$$

with

$$C' \le \max\left(\left(2 + \frac{4}{M_2}\right), \frac{4C_1}{M_2}\right) \quad \text{where } M_2 = \int \left(1 + \frac{1}{2}|y|^2\right)^{-2\eta} \mu_2(dy).$$

Proof. A proof is given in Villani [20] Theorem A.3. It uses extensively the spectral theory of the sum of operators. We shall give a more pedestrian (similar) proof.

The first point is that, since we assumed that $U \ge 1$,

$$H^{-2\eta}(x,y) \ge \phi_1(x)\phi_2(y) := U^{-2\eta}(x) \left(1 + \frac{1}{2}|y|^2\right)^{-2\eta}. \tag{14}$$

Thus, if we decompose $\mu(dx,dy) = \mu_1(dx) \otimes \mu_2(dy)$ we have

$$
\begin{aligned}
\int H^{-2\eta} |\nabla_x h|^2 \mu(dx,dy) &\ge \int \phi_1(x)\phi_2(y) |\nabla_x h|^2 \mu_1(dx) \otimes \mu_2(dy) \\
&\ge \frac{1}{C_1} \int \phi_2(y) \left(h(x,y) - \int h(u,y)\mu_1(du)\right)^2 \mu(dx,dy).
\end{aligned}
$$

Now, write

$$
\begin{aligned}
h(x,y) - \int h(u,y)\mu_1(du) &= \left(h(x,y) - \int h(u,y)\mu_1(du) - \int h(x,v)\mu_2(dv) + \int\int h d\mu_1 d\mu_2\right) \\
&\quad + \left(\int h(x,v)\mu_2(dv) - \int\int h d\mu_1 d\mu_2\right) \\
&= g_1(x,y) + g_2(x)
\end{aligned}
$$

and use

$$(a+b)^2 \ge \frac{1}{2} b^2 - a^2.$$

This yields, since $\phi_2(y) \le 1$,

$$\int H^{-2\eta} |\nabla_x h|^2 \mu(dx,dy) \ge \frac{1}{2C_1} \left(\int \phi_2 \, d\mu_2\right)\left(\int g_2^2(x)\mu_1(dx)\right) - \frac{1}{C_1} \int\int g_1^2(x,y)\, \mu_1(dx)\mu_2(dy).$$

Notice that for all y,

$$\int g_1^2(x,y)\,\mu_1(dx) = \mathrm{Var}_{\mu_1}\left(h(\cdot,y) - \int h(\cdot,v)\,\mu_2(dv)\right),$$

so that

$$\int g_1^2(x,y)\,\mu_1(dx) \leq \int \left(h(x,y) - \int h(x,v)\,\mu_2(dv)\right)^2 \mu_1(dx).$$

We can thus integrate this inequality w.r.t. μ_2, use Fubini's theorem, then for each fixed x use the usual Poincaré inequality for the standard gaussian measure μ_2 and finally integrate with respect to μ_1. This yields

$$
\begin{aligned}
\int\int g_1^2(x,y)\,\mu_1(dx)\mu_2(dy) &\leq \int\int \left(h(x,y) - \int h(x,v)\,\mu_2(dv)\right)^2 \mu(dx,dy)\\
&\leq \int\int |\nabla_y h|^2(x,y)\,\mu(dx,dy).
\end{aligned}
$$

Gathering all this we have obtained

$$\int g_2^2(x)\mu_1(dx) \leq \frac{2\,C_1}{M_2}\int H^{-2\eta}|\nabla_x h|^2\,\mathrm{d}\mu + \frac{2}{M_2}\int |\nabla_y h|^2\,\mathrm{d}\mu. \tag{15}$$

Finally,

$$
\begin{aligned}
\mathrm{Var}_\mu(h) &= \int \left(h(x,y) - \int h(x,v)\mu_2(dv) + \int h(x,v)\mu_2(dv) - \int\int h\mathrm{d}\mu\right)^2 \mu(dx,dy)\\
&\leq 2\int\int \left(h(x,y) - \int h(x,v)\,\mu_2(dv)\right)^2 \mu(dx,dy) + 2\int g_2^2(x)\mu_1(dx)\\
&\leq 2\int\int |\nabla_y h|^2(x,y)\,\mu(dx,dy) + 2\int g_2^2(x)\mu_1(dx),
\end{aligned}
$$

and the result follows from (15). □

As a conclusion the weighted Poincaré inequality on $\mathbb{R}^{2d}$ reduces to a weighted Poincaré inequality on $\mathbb{R}^d$ (up to some constant). One should think that the previous result is a kind of weighted tensorization property. This is not the case due to the fact that the weight in front of ∇_x depends on both variables x and y.

There are many ways to obtain such an inequality. Of course since it is stronger than the usual Poincaré inequality, our result is weaker than the one of Villani (but with a simpler proof and explicit bounds for the constants), and we will only describe a typical situation where this equality can be obtained.

As we have seen in the previous section, this weighted Poincaré inequality is equivalent to the existence of some Lyapunov function for $L_{1,\eta}$ which is built similarly to L_η replacing H by U. We can also obtain a slightly different condition. Introduce the probability measure $\mu_1^\phi(dy) = \frac{\phi_1(y)}{M_1}\,\mu_1(dy)$ and the μ_1^ϕ symmetric operator

$$G_1^\phi = \Delta_x - \left(1 + \frac{2\eta}{U}\right)\nabla U.\nabla.$$

Assume that we can find a Lyapunov function $W \geq 1$ such that

$$\frac{G_1^\phi W(x)}{W(x)} \leq -a\, U^{2\eta}(x)$$

for $|x|$ larger than some $R > 0$. If h is compactly supported in $|x| > R$, we may write

$$\int h^2 \, d\mu_1 \leq -\frac{M_1}{a} \int \frac{G_1^\phi W}{W} \, h^2 \, d\mu_1^\phi \leq \frac{M_1}{a} \int |\nabla h|^2 \, d\mu_1^\phi = \frac{M_1}{a} \int |\nabla h|^2 \, U^{-2\eta} \, d\mu_1$$

according to the computations in [2] p.64. Following the method introduced in [2] we then obtain that μ_1 satisfies the desired weighted Poincaré inequality. According to [9] Theorem 4.4, the existence of such a Lyapunov function is linked to the fact that μ_1 satisfies some F-Sobolev inequality, with $F = \ln_+^{2\eta}$. This is for instance the case when $U(x) = 1 + |x|^\alpha$ and $\eta = 1 - \alpha^{-1}$.

4.2. The case of weighted log-Sobolev inequalities.

We look now at the similar weighted logarithmic Sobolev inequality, namely,

$$\mathrm{Ent}_\mu(f^2) \leq \rho \int (H^{-2\eta}|\nabla_x f|^2 + |\nabla_y f|^2) d\mu.$$

As in the L^2 setting, it implies a weighted log Sobolev inequality for μ_1 on $\mathbb{R}^d$ i.e.

$$\mathrm{Ent}_{\mu_1}(f^2) \leq C \int U^{-2\eta}|\nabla_x f|^2 d\mu_1. \tag{16}$$

Since the standard gaussian measure μ_2 satisfies a log-Sobolev inequality too (with optimal constant 2), one should expect to obtain the analogue of theorem 12. This is not so easy (actually we did not succeed in proving such a result and believe that it is wrong) and certainly explains the limitation of Villani's approach, since this property reduces to the well known tensorization property of the logarithmic Sobolev inequality only in the case $\eta = 0$. The best we are able to do is to prove that, in this situation

Theorem 13. *Write* $\mu(dx, dy) = \mu_1(dx) \otimes \mu_2(dy)$. *If* $\mu_1(dx) = \frac{1}{Z_1} e^{-U(x)} dx$ *satisfies the weighted log-Sobolev inequality* (16), *then* μ *satisfies* (4) *with an admissible function* $u \mapsto \Psi(u)$ *behaving like* $u \ln^{\frac{1+4\eta}{1+2\eta}}(u)$ *at infinity.*

Combined with theorem 2 which deals with a decay for more general functionals than the variance or entropy, we are thus able to prove under such conditions an exponential decay for Ψ behaving like $u \ln^{\frac{1+4\eta}{1+2\eta}}(u)$ at infinity.

Notice that for a bounded Hessian we recover the weighted log-Sobolev inequality, i.e. we recover Villani's result.

Proof. The first step of the proof is the following

Lemma 14. *Define the probability measure* $\mu_2^\phi(dy) = \frac{\phi_2(y)}{M_2} \mu_2(dy)$. *Then* μ_2^ϕ *satisfies a log-Sobolev inequality.*

An immediate consequence is the following inequality for $\mu^\phi(dx, dy) = \mu_1(dx) \otimes \mu_2^\phi(dy)$,

$$\text{Ent}_{\mu^\phi}(h^2) \leq C \int (\phi_1 |\nabla_x h|^2 + |\nabla_y h|^2) \mathrm{d}\mu^\phi, \tag{17}$$

which follows from the tensorization property of the log-Sobolev inequality.

Proof of Lemma 14. Write

$$\mu_2^\phi(dy) = Z^\phi e^{-\left(\frac{|y|^2}{2} + 2\eta \ln(1+|y|^2/2)\right)} \mathrm{d}y = Z^\phi e^{-V_2(y)} \mathrm{d}y.$$

A simple calculation shows that

$$HessV_2(y) = \left(1 + \frac{2\eta}{1 + |y|^2/2}\right) Id - \frac{2\eta}{(1 + |y|^2/2)^2} M(y)$$

where $M_{i,j}(y) = y_i y_j$. Hence,

$$HessV_2(y) \geq \left(1 + \frac{2\eta}{1 + |y|^2/2} - \frac{2\eta d |y|^2}{(1 + |y|^2/2)^2}\right) Id$$

in the sense of quadratic forms. Hence for $|y|$ large enough (of order $c\sqrt{d}$), the potential $V_2(y)$ is uniformly convex, uniformly in y. This proves (combining Bakry-Emery criterion and Holley-Stroock perturbation argument) the Lemma. $\qquad\square$

As we recalled, the weighted log-Sobolev inequality is equivalent to a (weighted) super Poincaré inequality, for all smooth h and all $s > 0$,

$$\int h^2 \mathrm{d}\mu^\phi \leq s \int (\phi_1 |\nabla_x h|^2 + |\nabla_y h|^2) \mathrm{d}\mu^\phi + c\, e^{\beta/s} \left(\int |h| \, \mathrm{d}\mu^\phi\right)^2. \tag{18}$$

Since $\phi_2 \leq 1$, it follows

$$\int h^2 \mathrm{d}\mu^\phi \leq \frac{s}{M_2} \int (H^{-2\eta} |\nabla_x h|^2 + |\nabla_y h|^2) \mathrm{d}\mu + \frac{c}{M_2} e^{\beta/s} \left(\int |h| \, \mathrm{d}\mu\right)^2. \tag{19}$$

For $R > 1$, introduce the 1-Lipschitz function

$$\varphi(r) = (r - R)\, \mathbf{1}_{R < r < R+1} + \mathbf{1}_{R+1 \leq r}.$$

One can write

$$
\begin{aligned}
\int h^2 \mathrm{d}\mu &\leq \int_{|y| \leq R+1} h^2 \, \mathrm{d}\mu + \int h^2 \, \varphi^2(|y|) \, \mathrm{d}\mu \\
&\leq \frac{M_2}{\phi_2(R+1)} \int_{|y| \leq R+1} h^2 \mathrm{d}\mu^\phi + \int h^2 \, \varphi^2(|y|) \, \mathrm{d}\mu \\
&\leq \frac{M_2}{\phi_2(R+1)} \int h^2 \mathrm{d}\mu^\phi + \int h^2 \, \varphi^2(|y|) \, \mathrm{d}\mu.
\end{aligned}
$$

The first term in the sum will be controlled thanks to (19). In order to control the second term, we introduce, once again, some Lyapunov function.

Denote by G the Ornstein-Uhlenbeck operator $G = \Delta_y - y.\nabla_y$ and consider $W(y) = e^{|y|^2/4}$. A simple calculation shows that

$$\frac{GW}{W} \leq \frac{1}{4}(2d - |y|^2)$$

for $|y| > \sqrt{2d}$. Hence if $R > \sqrt{2d}$, we get for $|y| > R$,

$$1 \le 4 \left(\frac{-GW}{W} \right) \frac{1}{|y|^2 - 2d} \le 4 \left(\frac{-GW}{W} \right) \frac{1}{R^2 - 2d}$$

and finally

$$\int h^2 \, \varphi^2(|y|) \, \mathrm{d}\mu \le \frac{4}{R^2 - 2d} \int \left(\frac{-GW}{W} \right) h^2 \, \varphi^2(|y|) \, \mathrm{d}\mu. \tag{20}$$

Integrating by parts, and after some easy manipulations (see [2] for the details), we will thus obtain for well chosen constants C, C' all $s > 0$ and large enough R (only depending on d),

$$\int h^2 \mathrm{d}\mu \le C \, (sR^{4\eta} + R^{-2}) \int (\phi_1 \, |\nabla_x h|^2 + |\nabla_y h|^2) \mathrm{d}\mu^\phi \, + \, C' \, R^{4\eta} \, e^{\beta/s} \left(\int |h| \, \mathrm{d}\mu \right)^2. \tag{21}$$

Choosing $u = s^{\frac{1}{1+2\eta}} = R^{-2}$, we obtain a super Poincaré inequality

$$\int h^2 \mathrm{d}\mu \le C \, u \int (\phi_1 \, |\nabla_x h|^2 + |\nabla_y h|^2) \mathrm{d}\mu^\phi \, + \, C' \, e^{\beta'/u^{1+2\eta}} \left(\int |h| \, \mathrm{d}\mu \right)^2. \tag{22}$$

which furnishes a $F = \ln_+^{\frac{1}{1+2\eta}}$-Sobolev inequality, i.e. if $\int h^2 \, \mathrm{d}\mu = 1$,

$$\int h^2 \, \ln_+^{\frac{1}{1+2\eta}} h^2 \, \mathrm{d}\mu \le C \, \int (\phi_1 \, |\nabla_x h|^2 + |\nabla_y h|^2) \mathrm{d}\mu^\phi.$$

Notice that, since $\phi_2 \le 1$, the previous inequality is stronger than

$$\int h^2 \, \ln_+^{\frac{1}{1+2\eta}} h^2 \, \mathrm{d}\mu \le C \, \int (H^{-2\eta} \, |\nabla_x h|^2 + |\nabla_y h|^2) \mathrm{d}\mu. \tag{23}$$

It remains to link (23) to (4). Actually, as explained in [5] section 7, one can replace $\ln_+$ by smooth functions F with a similar behaviour at infinity (and satisfying $F(1) = 0$).
Let

$$\psi(u) = \frac{\ln^{\frac{2\eta}{1+2\eta}}(1+u)}{1+u}$$

defined for $u \ge e$. It is easily seen that $1/\psi$ is concave at infinity. Hence as explained in Remark 3, we may modify ψ and consider some admissible Ψ such that $\Psi(u)$ behaves like $(1+u) \ln^{\frac{1+4\eta}{1+2\eta}}(1+u)$ at infinity. If we define $h^2 = f \ln^{\frac{2\eta}{1+2\eta}}(1+f)$, we get from (23) that μ satisfies (4) with this Ψ and some ρ, completing the proof. $\qquad\square$

For a discussion about the connections between F-Sobolev inequalities and the Ψ entropic inequalities one can look at [8] subsection 3.2.

References

[1] F. Achleitner, A. Arnold, and D. Stürzer. Large-time behavior in non-symmetric Fokker-Planck equations. *Riv. Math. Univ. Parma (N.S.)*, 6(1):1–68, 2015.

[2] D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin. A simple proof of the Poincaré inequality for a large class of probability measures. *Electronic Comm. in Probab.*, 13:60–66, 2008.

[3] D. Bakry, P. Cattiaux, and A. Guillin. Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré. *J. Funct. Anal.*, 254(3):727–759, 2008.

[4] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and Geometry of Markov Diffusion Operators*, volume 348 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2014.

[5] F. Barthe, P. Cattiaux, and C. Roberto. Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry. *Rev. Mat. Iberoam.*, 22(3):993–1067, 2006.

[6] S. Bobkov and M. Ledoux. From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities. *Geom. Funct. Anal.*, 10:1028–1052, 2000.

[7] F. Bolley, A. Guillin, and F. Malrieu. Trend to equilibrium and particle approximation for a weakly sel-fconsistent Vlasov-Fokker-Planck equation. *M2AN Math. Model. Numer. Anal.*, 44(5):867–884, 2010.

[8] P. Cattiaux and A. Guillin. Trends to equilibrium in total variation distance. *Ann. Inst. Henri Poincar Probab. Stat.*, 45(1):117–145, 2009.

[9] P. Cattiaux and A. Guillin. Hitting times, functional inequalities, Lyapunov conditions and uniform ergodicity. *J. Funct. Anal.*, 256:1821–1841, 2016.

[10] P. Cattiaux, A. Guillin, F.Y. Wang, and L. Wu. Lyapunov conditions for super Poincaré inequalities. *J. Funct. Anal.*, 256(6):1821–1841, 2009.

[11] P. Cattiaux, A. Guillin, and L. Wu. Some remarks on weighted logarithmic Sobolev inequality. *Indiana Univ. Math. J.*, 60(6):1885–1904, 2011.

[12] L. Desvillettes and C. Villani. On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation. *Comm. Pure Appl. Math.*, 54(1):1–42, 2001.

[13] J. Dolbeault, C. Mouhot, and C. Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. *Trans. Amer. Math. Soc.*, 367(6):3807–3828, 2015.

[14] A. Eberle, A. Guillin, and R. Zimmer. Quantitative Harris type theorems for diffusions and McKean-Vlasov processes. Math. ArXiv:1606.06012, [math.PR], 2016.

[15] A. Eberle, A. Guillin, and R. Zimmer. Couplings and quantitative contraction rates for Langevin dynamics. Math. ArXiv:1703.01617, [math.PR], 2017.

[16] A. Guillin and F-Y. Wang. Degenerate Fokker-Planck equations: Bismut formula, gradient estimate and Harnack inequality. *J. Differential Equations*, 253(1):20–40, 2012.

[17] F. Hérau and F. Nier. Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. *Arch. Ration. Mech. Anal.*, 171(2):151–218, 2004.

[18] P. Monmarché. Generalized Γ calculus and application to interacting particles on a graph. to appeaur in Potential Analysis, 2018.

[19] D. Talay. Stochastic Hamiltonian systems: exponential convergence to the invariant measure, and discretization by the implicit Euler scheme. *Markov Process. Related Fields*, 8(2):163–198, 2002. Inhomogeneous random systems.

[20] C. Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202(950):iv+141, 2009.

[21] F. Y. Wang. *Functional inequalities, Markov processes and Spectral theory*. Science Press, Beijing, 2005.

[22] L. Wu. Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. *Stochastic Process. Appl.*, 91(2):205–238, 2001.

Patrick CATTIAUX, Institut de Mathématiques de Toulouse, Université de Toulouse, CNRS UMR 5219,, 118 route de Narbonne, F-31062 Toulouse cedex 09.

Email address: `patrick.cattiaux@math.univ-toulouse.fr`

Arnaud GUILLIN, Laboratoire de Mathématiques Blaise Pascal, CNRS UMR 6620, Université Clermont-Auvergne, avenue des Landais, F-63177 Aubière.

Email address: `arnaud.guillin@uca.fr`

Pierre MONMARCHE, Sorbonne Universitaboratoire Jacques-Louis Lions, LJLL, 5 place Jussieu, F-75005 Paris.

Email address: `pierre.monmarche@ens-cachan.org`

Chaoen ZHANG, Laboratoire de Mathématiques Blaise Pascal, CNRS UMR 6620, Université Clermont-Auvergne, avenue des Landais, F-63177 Aubière.

Email address: `chaoen.zhang@uca.fr`