POINCARÉ AND LOGARITHMIC SOBOLEV INEQUALITIES FOR NEARLY RADIAL MEASURES.

PATRICK CATTIAUX ♠ , **ARNAUD GUILLIN** ♢, AND **LIMING WU** ♢

♠ Université de Toulouse

♢ Université Clermont-Auvergne

Abstract. Poincaré inequality has been studied by Bobkov for radial measures, but few is known about the logarithmic Sobolev inequality in the radial case. We try to fill this gap here using different methods: Bobkov's argument and super-Poincaré inequalities, direct approach via L_1-logarithmic Sobolev inequalities. We also give various examples where the obtained bounds are quite sharp. Recent bounds obtained by Lee-Vempala in the log-concave bounded case are refined for radial measures.

Key words : radial measure, log-concave measure, Poincaré inequality, logarithmic Sobolev inequality, Super-Poincaré inequality.

MSC 2010 : 26D10, 39B62, 47D07, 60G10, 60J60.

1. Introduction

Let $\mu(dx) = Z^{-1}\, e^{-V(x)}\, dx$ be a probability measure defined on $\mathbb{R}^n$ ($n \geq 2$). We do not require regularity for V and allow it to take values in $\mathbb{R} \cup \{-\infty, +\infty\}$. We only require that $\int e^{-V} dx = 1$, or more generally that the previous integral is finite. We denote by $\mu(f)$ the integral of f w.r.t. μ.

In this note we will be interested in functional inequalities verified by the measure μ. Recall that μ satisfies a Poincaré inequality if for all smooth f,

$$\text{Var}_\mu(f) := \mu(f^2) - \mu^2(f) \leq C_P(\mu)\, \mu(|\nabla f|^2)\,, \tag{1.1}$$

and that it satisfies a log-Sobolev inequality if for all smooth f

$$\text{Ent}_\mu(f^2) := \mu(f^2\, \ln(f^2)) - \mu(f^2)\, \ln(\mu(f^2)) \leq C_{LS}(\mu)\, \mu(|\nabla f|^2)\,. \tag{1.2}$$

C_P and C_{LS} are understood as the best constants for the previous inequalities to hold. We refer to [1, 2, 29] among many others, for a comprehensive introduction to some of the useful consequences of these inequalities and their most important properties, such as convergence to equilibrium (in L^2 or in entropy) or concentration of measure.
If μ is not normalized as a probability measure, (1.1) reads as

$$\mu(f^2) \,-\, (1/\mu(\mathbb{R}^n))\; \mu^2(f) \;\leq\; C_P(\mu)\, \mu(|\nabla f|^2)\,. \tag{1.3}$$

Date: March 4, 2021.

One key feature of these inequalities is their tensorization property namely

$$C_P(\mu \otimes \nu) = \max(C_P(\mu), C_P(\nu))$$

(the same for C_{LS}) giving a natural way to control these constants for product measures.

Another particular family of measures is the set of *nearly radial* defined as follows:

Definition 1.1. A measure μ is called *nearly radial* if it admits a decomposition

$$\mu(dx) = \mu_r(d\rho)\, \mu_a(d\theta) \tag{1.4}$$

with $x = \rho\, \theta$, $\rho \in \mathbb{R}^+$ and $\theta \in \mathbb{S}^{n-1}$. This amounts to $V(x) = V_r(\rho) + V_a(\theta)$ and

$$\mu(dx) \;=\; n\, \omega_n\, \rho^{n-1}\, e^{-V_r(\rho)}\; e^{-V_a(\theta)}\, d\rho\, \sigma_n(d\theta)$$

where σ_n denotes the uniform distribution on $\mathbb{S}^{n-1}$ and ω_n denotes the volume of the unit euclidean ball.
If in addition V_a is bounded below and above, we shall call these measures *almost radial*.
When $\mu_a = \sigma_n$, μ is said to be *radial* (or *spherically symmetric*) as in the usual literature.

Note also that as our main purpose is functional inequalities such as Poincaré inequality or logarithmic Sobolev inequality which are translation invariant, one may always consider probability measures which are nearly radial measures up to a preliminary translation.

It is natural to ask how to control $C_P(\mu)$ and $C_{LS}(\mu)$ in terms of constants related to μ_r and μ_a. Since μ_a is supported by the sphere we will use the natural riemanian gradient, in other words, for $\theta \in \mathbb{S}^{n-1}$, we will decompose

$$\nabla f = (\nabla_\theta f, \nabla_{\theta^\perp} f) := (\langle \nabla f, \theta\rangle, \Pi_{\theta^\perp} \nabla f)$$

where $\Pi_{\theta^\perp}$ denotes the orthogonal projection onto $\theta^\perp$.

Though natural it seems that the previous question was not often addressed in the literature with the notable exception of radial log-concave measures for which S. Bobkov (see [8]) studied the Poincaré constant (his result is improved in [13]) and for which Huet (see [22]) studied isoperimetric property.

Our main results in the radial (or almost radial) case say that both the Poincaré and the log-Sobolev constant are controlled up to universal constants, by the corresponding constants for the radial part μ_r and $\mu(\rho^2)/(n-1)$ for Poincaré and some slightly more intricate combinaition for log-Sobolev, i.e.

Theorem 1.2. *Let μ be a radial measure.*

1. *(Bobkov's result)*
$$C_P(\mu) \leq \max\left(C_P(\mu_r), \frac{\mu(\rho^2)}{n-1}\right).$$
2. *(Th.4.5) there exists a universal constant c such that*
$$C_{LS}(\mu) \leq c\left(C_{LS}(\mu_r) + \mu(\rho)\max\left(C_P(\mu_r), \frac{\mu(\rho^2)}{n-1}\right)^{1/2}\right).$$

Other results and some consequences are also described. More precisely, the "tensorization" part of Bobkov's proof is elementary and will be explained in Section 2, where we will also show how it applies to other types of Poincaré inequalities (weak or super). As a byproduct we will obtain a first (bad) bound for the log-Sobolev constant.

In section 4 we propose a direct approach of the logarithmic Sobolev inequality for (almost) radial measures. This approach uses in particular $\mathbb{L}^q$ $(1 \leq q \leq 2)$ log-Sobolev inequalities for the uniform measure on the sphere we establish in Section 3, based on the study made in [10] and results in [26]. In the framework of general log-concave measures, similar ideas already appear in [7].

All these results are applied in section 5 to some examples. In particular, in the radial case, we improve upon the bound recently obtained by Lee and Vempala ([25]) for compactly supported (isotropic) log-concave measures. It reads

Theorem 1.3 (Th.5.9). *For any radial log-concave probability measure μ with support contained in the ball centered at* 0 *of radius R, then*

$$C_{LS}(\mu) \leq C \, \frac{R^2}{n-1}$$

for some universal constant C.

It is not difficult to see that, according to Hahn-Banach theorem, for a compactly supported and centered log-concave measure such a R always exists and is less than the diameter of the support. The previous result can thus be immediately compared with the one in [25], in the radial case.

2. Poincaré inequality and variants for nearly radial measures.

2.1. **Bobkov's tensorization method revisited.** Let us begin by explaining Bobkov's tensorization method.

For $\mu(dx) = \mu_r(d\rho)\, \mu_a(d\theta)$ and a smooth bounded function f one has firstly

$$\begin{aligned}\int f^2(\rho\,\theta)\, \mu_r(d\rho) \quad &\leq \quad C_P(\mu_r) \int \langle \nabla f(\rho\theta), \theta\rangle^2 \,\mu_r(d\rho) + \left(\int f(\rho\theta)\, \mu_r(d\rho)\right)^2 \\ &= \quad C_P(\mu_r) \int |\nabla_\theta f(\rho\theta)|^2 \,\mu_r(d\rho) + \left(\int f(\rho\theta)\, \mu_r(d\rho)\right)^2 . \qquad (2.1)\end{aligned}$$

(**Be careful:** therefore ∇_θ is NOT the gradient w.r.t. θ, but the derivation in direction θ, used throughout the paper.) Integrating with respect to μ_a, we obtain

$$\mu(f^2) \;\leq\; C_P(\mu_r)\, \mu(|\nabla_\theta f|^2) + \int \left(\int f(\rho\theta)\, \mu_r(d\rho)\right)^2 \mu_a(d\theta)\,. \qquad (2.2)$$

But if we define $w(\theta) = \int f(\rho\theta)\, \mu_r(d\rho)$ it holds,

$$\begin{aligned} \int \left(\int f(\rho\theta)\, \mu_r(d\rho)\right)^2 \mu_a(d\theta) &\le C_P(\mu_a) \int |\nabla w(\theta)|^2 \, \mu_a(d\theta) + \left(\int w(\theta)\, \mu_a(d\theta)\right)^2 \\ &\le C_P(\mu_a) \int \left| \int \rho\, \nabla_{\theta^\perp} f(\rho\theta) \mu_r(d\rho)\right|^2 \mu_a(d\theta) \, + \, \mu^2(f) \\ &\le C_P(\mu_a)\, \mu_r(\rho^2)\, \mu(|\nabla_{\theta^\perp} f|^2) \, + \, \mu^2(f)\,, \end{aligned} \tag{2.3}$$

where we have used the Cauchy-Schwarz inequality in the last inequality. Here we assume that the Poincaré constant of μ_a on the sphere S^{n-1} is w.r.t. the usual gradient and not the gradient on the sphere. Using that

$$|\nabla_\theta f|^2 + |\nabla_{\theta^\perp} f|^2 = |\nabla f|^2\,,$$

we have thus obtained

Theorem 2.1. *(essentially due to Bobkov) For a nearly radial probability measure $\mu(dx) = \mu_r(d\rho)\, \mu_a(d\theta)$ then*

$$C_P(\mu) \, \le \, \max(C_P(\mu_r)\,,\, \mu_r(\rho^2)\, C_P(\mu_a))\,.$$

Recall that if $C_P(\mu_r) < +\infty$ then $\mu_r(e^{\lambda\rho}) < +\infty$ for $\lambda < 2/\sqrt{C_P(\mu_r)}$ (see e.g. [1]) so that $\mu_r(\rho^2)$ is finite too. See the sharp estimate (5.1) due to Bonnefont-Joulin-Ma [13] for log-concave radial measures.

2.2. **Weak and super Poincaré inequality.** A weak version of the Poincaré inequality has been introduced in [28] (also see the related papers [3, 9, 16]). It has, as the usual Poincaré inequality, interesting properties such as polynomial (or stretched exponential) convergence to equilibrium or concentration property. A weak Poincaré inequality is a family of inequalities taking the form: for any $t > 0$ and all smooth f,

$$\mathrm{Var}_\mu(f) \, \le \, \beta_\mu^{WP}(t)\, \mu(|\nabla f|^2) + \, t\, \mathrm{Osc}^2(f) \tag{2.4}$$

where β_μ^{WP} is a non increasing function that can explode at $t = 0$ (otherwise the classical Poincaré inequality is satisfied) and $\mathrm{Osc}(f) := \sup_{x,y} |f(x) - f(y)|$ denotes the Oscillation of f. The previous proof yields (the detail is left to the reader)

Theorem 2.2. *If $\mu(dx) = \mu_r(d\rho)\, \mu_a(d\theta)$ then*

$$\beta_\mu^{WP}(t) \, \le \, \max(\beta_{\mu_r}^{WP}(t/2)\,,\, \mu_r(\rho^2)\, \beta_{\mu_a}^{WP}(t/2))\,.$$

The integrability of ρ^2 is ensured as soon as β_μ^{WP} does not explode too quickly at the origin (see e.g. [3]).

Similarly one can reinforce the Poincaré inequality introducing super Poincaré inequalities ([29, 30, 4, 5, 18, 2]): for any $t \ge 1$ and all smooth f

$$\mu(f^2) \, \le \, \delta_\mu(t)\, \mu(|\nabla f|^2) + \, t\, \mu^2(|f|)\,, \tag{2.5}$$

which is called a generalized Nash inequality in [2] Chapter 8.4. Once again they imply super-exponential concentration of measure and depending on the coefficients ultracontractivity. Here δ_μ is assumed to be a non increasing function. It immediately follows $C_P(\mu) \le \delta_\mu(1)$. If $\delta_\mu(t) \to 0$ as $t \to +\infty$ one may consider the inverse function $\beta_\mu^{SP}(t) = \delta_\mu^{-1}(t)$ defined for

$t \in]0, \delta_\mu(1)]$ and which is non increasing with values in $[1, +\infty[$. We can thus rewrite (2.5) as: for $t \in]0, \delta_\mu(1)]$,

$$\mu(f^2) \leq t\,\mu(|\nabla f|^2) + \beta_\mu^{SP}(t)\,\mu^2(|f|)\,. \tag{2.6}$$

Conversely, assume that there exists a function β (that can always be chosen non increasing) defined for $t > 0$ and such that

$$\mu(f^2) \leq t\,\mu(|\nabla f|^2) + \beta(t)\,\mu^2(|f|)$$

for all $t > 0$ and nice function f. Applying this inequality to constant functions shows that $\beta(t) \geq 1$ for all t. But we may replace $\beta(t)$ by 1 as soon as $t \geq C_P(\mu)$.

It is known that if $\beta_\mu^{SP}(t)$ behaves like $ce^{c'/t}$ as $t \to 0$, (2.6) together with a Poincaré inequality is equivalent to the logarithmic Sobolev inequality (we will provide precise estimates below).

Following the same route we immediately get that for any positive t and s,

$$\mu(f^2) \leq (s + \beta_{\mu_r}^{SP}(s)\,t\,\mu_r(\rho^2))\,\mu(|\nabla f|^2) + \beta_{\mu_r}^{SP}(s)\,\beta_{\mu_a}^{SP}(t)\,\mu^2(|f|)\,. \tag{2.7}$$

Of course this is a new super Poincaré inequality or more precisely a new family of super Poincaré inequalities. The most natural choice (not necessarily the best one) is

$$t = \frac{s}{\mu(\rho^2)\,\beta_{\mu_r}^{SP}(s)}$$

yielding the following

Theorem 2.3. *If $\mu(dx) = \mu_r(d\rho)\,\mu_a(d\theta)$, then*

$$\beta_\mu^{SP}(t) \leq \beta_{\mu_r}^{SP}(t/2)\,\beta_{\mu_a}^{SP}\left(\frac{t}{2\,\mu(\rho^2)\,\beta_{\mu_r}^{SP}(t/2)}\right)\,.$$

If, for example, we suppose that a logarithmic Sobolev inequality holds for μ_r, it is not hard to remark that the angular part has to satisfy a much stronger inequality than a logarithmic Sobolev inequality for such an inequality to hold for μ.

2.3. **The radial case.**

If μ is radial i.e. $\mu_a = \sigma_n$ the uniform measure on $\mathbb{S}^{n-1}$, we deduce from the famous Lichnerowitz's estimate $C_P(\sigma_n) = \frac{1}{n-1}$ and Theorem 2.1, that

$$C_P(\mu) \leq \max\left(C_P(\mu_r)\,,\, \frac{\mu_r(\rho^2)}{n-1}\right)\,. \tag{2.8}$$

Actually σ_n satisfies the much stronger Sobolev inequality (for $n \geq 4$ see [2] p.308 written for spherical gradient but recall the introduction)

$$\parallel g \parallel^2_{\frac{2n-2}{n-3}} \leq \sigma_n(g^2) + \frac{4}{(n-1)(n-3)}\,\sigma_n(|\nabla g|^2)\,. \tag{2.9}$$

We deduce from this and the Poincaré inequality, the following inequality

$$\parallel g \parallel^2_{\frac{2n-2}{n-3}} \leq \sigma_n^2(g) + \frac{1}{n-3}\,\sigma_n(|\nabla g|^2)\,. \tag{2.10}$$

One can thus derive the corresponding $\beta^{SP}_{\sigma_n}$. If one wants to see the dimension dependence one has to be a little bit careful.

First we apply Hölder's inequality for $p>2$,

$$\sigma_n(g^2) \;\leq\; \sigma_n^{\frac{1}{p-1}}(|g|^p)\,\sigma_n^{\frac{p-2}{p-1}}(|g|)\,,$$

then choose $p=\frac{2n-2}{n-3}$, yielding according to what precedes

$$\begin{aligned}\sigma_n(g^2) \quad &\leq \quad \left(\sigma_n^2(g)+\frac{1}{n-3}\sigma_n(|\nabla g|^2)\right)^{\frac{n-1}{n+1}}\;\sigma_n^{\frac{4}{n+1}}(|g|)\\ &\leq \quad \sigma_n^2(|g|)\;+\;\left(\frac{1}{n-3}\sigma_n(|\nabla g|^2)\right)^{\frac{n-1}{n+1}}\;\sigma_n^{\frac{4}{n+1}}(|g|)\,.\end{aligned}$$

Recall Young's inequality: for all $p>1$ and $a,b,t>0$,

$$ab\leq t\,\frac{a^p}{p}+t^{-(q-1)}\,\frac{b^q}{q}$$

with $1/p+1/q=1$. We deduce from what precedes, this time with $p=(n+1)/(n-1)$,

$$\sigma_n(g^2)\;\leq\;t\,\sigma_n(|\nabla g|^2)\;+\;\left(1\;+\;\frac{1}{4(n-3)^{(n-1)/2}}\,t^{-(n-1)/2}\right)\,\sigma_n^2(|g|)$$

and finally that

$$\beta^{SP}_{\sigma_n}(t)\;\leq\;1+\frac{1}{4(n-3)^{(n-1)/2}}\;t^{-(n-1)/2}\,. \tag{2.11}$$

We can thus plug (2.11) in Theorem 2.3 and get

Corollary 2.4. *If $\mu(dx)=\mu_r(d\rho)\,\sigma_n(d\theta)$ (radial) and $n\geq 4$, then*

$$\beta^{SP}_{\mu}(t)\;\leq\;\beta^{SP}_{\mu_r}(t/2)\,\left(1+\frac{1}{4}\,\left(\frac{\mu(\rho^2)\,\beta^{SP}_{\mu_r}(t/2)}{(n-3)}\right)^{(n-1)/2}\right)\,.$$

The cases $n=2$ and $n=3$ can be studied separately.

2.4. Application to the log-Sobolev inequality.

The equivalence between a log-Sobolev inequality and a super-Poincaré inequality is well known (see e.g. [29] Theorems 3.3.1 and 3.3.3, despite some points we do not understand in the proofs). But here we need precise estimates on the constants. One way to get these estimates is to use the capacity-measure description of these inequalities following the ideas in [5] (also see [31] for some additional comments).

Lemma 2.5. *[see* [15] *Proposition 3.4]*
If μ satisfies a logarithmic-Sobolev inequality with constant $C_{LS}(\mu)$ then $\beta^{SP}_{\mu}(t)\leq\;2\,e^{2\,C_{LS}(\mu)/t}$.

A simplified version of the results of [5, 4] is contained in [2], see in particular Proposition 8.3.2 and Proposition 8.4.1, from which one can deduce the (slightly worse bound) $\beta^{SP}_{\mu}(t)\leq e^{96\,C_{LS}(\mu)-2/t}$.

The converse part is a consequence of [2].

Lemma 2.6. *[see [2] Proposition 8.3.2 and Proposition 8.4.1]*
Conversely, if $\beta_\mu^{SP}(t) \leq C_1\, e^{C_2/t}$ and μ satisfies a Poincaré inequality with constant $C_P(\mu)$, then

$$C_{LS}(\mu) \leq 64 \left(C_2 \,+\, \ln\left(1 \vee \frac{(1+2e^2)C_1}{4}\right)\, C_P(\mu)\right)\,.$$

Proof. Recall that it is enough to look at $t \leq C_P(\mu)$ and then take $\beta_\mu^{SP}(t) = 1$ for $t \geq C_P(\mu)$. So we may replace $C_1\, e^{C_2/t}$ by the larger $\frac{4}{1+2e^2}\, e^{C_2'/t}$ with

$$C_2' = C_2 + C_P(\mu)\, \ln\left(1 \vee \frac{(1+2e^2)C_1}{4}\right)\,.$$

In [2] terminology we thus have $\delta(s) = C_2'/\ln(s(1+2e^2)/4)$, that satisfies the assumptions of Proposition 8.4.1 in [2] with $q=4$. We thus get for $\mu(A) \leq \frac{1}{2}$,

$$Cap_\mu(A) \geq \frac{\mu(A)\, \ln((1+2e^2)/2\mu(A))}{8C_2'}\,.$$

But for $\mu(A) \leq 1/2$,

$$\ln((1+2e^2)/2\mu(A)) \geq \ln\left(1 + \frac{e^2}{\mu(A)}\right)\,.$$

We may thus apply Proposition 8.3.2 in [2] yielding $C_{LS}(\mu) \leq 64C_2'$. □

Now assume that μ is radial. If μ_r satisfies a log-Sobolev inequality, $\beta_{\mu_r}^{SP}(t) \leq 2e^{2C_{LS}(\mu_r)/t}$ so that using Corollary 2.4,

$$\beta_\mu^{SP}(t) \;\leq\; 2\, e^{4C_{LS}(\mu_r)/t} \;+\; \frac{1}{2}\left(\frac{2\,\mu(\rho^2)}{n-3}\right)^{(n-1)/2} \; e^{2(n+1)\, C_{LS}(\mu_r)/t}\,. \tag{2.12}$$

Once again since we only have to look at $t < C_P(\mu)$, and using (2.8), we obtain the following worse bound

$$\beta_\mu^{SP}(t) \;\leq\; 2\, \left(e^{-(n-1)C_{LS}(\mu_r)/\max(C_P(\mu_r), \frac{\mu(\rho^2)}{n-1})} \;+\; \frac{1}{4}\left(\frac{2\mu(\rho^2)}{n-3}\right)^{(n-1)/2}\right)\; e^{2(n+1)\, C_{LS}(\mu_r)/t}\,. \tag{2.13}$$

But we can use the following homogeneity property of the log-Sobolev and the Poincaré inequalities: defining μ_λ for $\lambda > 0$, by,

$$\int f(z)\, \mu_\lambda(dz) \;=\; \int f(\lambda z)\, \mu(dz)\,,$$

it holds $C_{LS}(\mu_\lambda) = \lambda^2\, C_{LS}(\mu)$ (the same for $C_P(\mu)$).
Looking at the pre-factor in (2.13), we see that making λ go to 0, the second term goes to 0 while the first one is unchanged. Using the homogeneity property for both μ and μ_r, and using lemma 2.6 again, we have thus obtained

$$C_{LS}(\mu) \;\leq\; 64\, \left((n+1)\, C_{LS}(\mu_r) \;+\; C\, \max\left(C_P(\mu_r), \frac{\mu(\rho^2)}{n-1}\right)\right)\,, \tag{2.14}$$

where

$$C = \ln\left(1 \vee \frac{1+2e^2}{2}\, e^{-(n-1)\, C_{LS}(\mu_r)/\max(C_P(\mu_r), \frac{\mu(\rho^2)}{n-1})}\right)\,.$$

Of course in many (almost all) situations $C = 0$. Using $C_P(\mu_r) \leq C_{LS}(\mu_r)/2$ it is not very difficult to show that $C \neq 0$ if and only if

$$\mu(\rho^2) \geq \frac{2(n-1)^2\, C_{LS}(\mu_r)}{1+e^2}\,,$$

in which case $C \leq \ln(1+e^2)$.
These results extend to the "almost" radial situation, using the standard perturbation result for a (super)-Poincaré inequality or a log-Sobolev inequality, as explained in the next Corollary

Corollary 2.7. *Assume that $\mu(dx) = \mu_r(d\rho)\,\mu_a(d\theta)$ is almost radial (recall definition 1.1) with*

$$m \leq \left|\left|\frac{d\mu_a}{d\sigma_n}\right|\right|_\infty \leq M$$

where σ_n is the uniform probability measure on $\mathbb{S}^{n-1}$. Then

$$C_P(\mu) \;\leq\; C_P(\mu_r) \,+\, \frac{M}{m}\,\frac{\mu_r(\rho^2)}{n-1}\,.$$

If $n \geq 4$ and μ_r satisfies a log-Sobolev inequality then so does μ and

$$C_{LS}(\mu) \leq\; 64\,\frac{M}{m}\,(n+1)\,C_{LS}(\mu_r)\,,$$

except if $\mu(\rho^2) \geq \frac{2(n-1)^2\,C_{LS}(\mu_r)}{1+e^2}$ in which case

$$C_{LS}(\mu) \leq\; 64\,\frac{M}{m}\,\left((n+1)\,C_{LS}(\mu_r) \,+\, \ln(1+e^2)\,\frac{\mu(\rho^2)}{n-1}\right)\,.$$

The previous two bounds amounts to the existence of a universal constant C such that

$$C_{LS}(\mu) \leq\; C\,\frac{M}{m}\,\max\left(n\,C_{LS}(\mu_r)\,,\;\frac{\mu(\rho^2)}{n-1}\right)\,.$$

We may of course adapt the above proof to characterize the logarithmic Sobolev inequality starting from Theorem 2.3, i.e:

Theorem 2.8. *Assume that $n \geq 4$, $\mu(dx) = \mu_r(d\rho)\,\mu_a(d\theta)$ and that μ_a satisfies a super Poincaré inequality with $\beta^{SP}_{\mu_a}(t) = C_a\,t^{-\kappa}$. Then*

$$C_{LS}(\mu) \leq\; 64\,\left(c(\kappa)\,C_{LS}(\mu_r) \,+\, c\,\mu_r(\rho^2)\,C_P(\mu_a)\right)\,.$$

3. $\mathbb{L}^q$ log-Sobolev inequality for the uniform measure on the sphere.

In this section we shall recall some isoperimetric property of the uniform measure σ_n on the unit sphere $\mathbb{S}^{n-1}$, in the purpose of obtaining the L^q-log-Sobolev inequality of σ_n. This will be crucial for our results in the next section.

3.1. **Recalls.** Let us recall the $\mathbb{L}^1$ inequalities we are interested in, namely Cheeger (or $\mathbb{L}^1$ Poincaré) inequality

$$\mu(|f - m_\mu f|) \leq C \int |\nabla f| \, d\mu \tag{3.1}$$

where $m_\mu f$ denotes a μ-median of f and similarly the $\mathbb{L}^q$ log-Sobolev inequality ($1 \leq q \leq 2$)

$$\mu(|f|^q \ln(|f|^q)) - \mu(|f|^q) \ln(\mu(|f|^q)) \leq C \int |\nabla f|^q \, d\mu \,. \tag{3.2}$$

As usual we denote by $C_C(\mu)$ and $C_{LSq}(\mu)$ the optimal constants in the previous inequalities. It is known (see e.g. [7]) that $C_P(\mu) \leq 4C_C^2(\mu)$. One can also show that there exists a universal constant D such that $C_{LS}(\mu) \leq D\, C_{LS1}^2(\mu)$ (see below).
These inequalities are strongly related to the isoperimetric profile of μ. Recall that the isoperimetric profile I_μ of μ is defined for $p \in [0,1]$ as

$$I_\mu(p) = \inf_{A \text{ s.t. } \mu(A)=p} \mu_n^+(\partial A)$$

where

$$\mu_n^+(\partial A) = \liminf_{h \to 0} \frac{\mu(A^h) - \mu(A)}{h}$$

A^h being the geodesic enlargement of A of size h. Of course in "smooth" situations, as absolutely continuous measures w.r.t. the natural riemanian measure on a riemanian manifold, $I_\mu(p) = I_\mu(1-p)$ so that it is enough to consider $p \in [0, \frac{1}{2}]$.
The following results are then well known (see e.g. [7] for the first one and [10] Theorem 1.1 for the second one)

Proposition 3.1. *There is an equivalence between the following two statements:*

1)

$$I_\mu(p) \geq C \, \min(p, 1-p)$$

2) *and* (3.1) *holds with constant* $1/C$.

There is an equivalence between the following two statements:

3)

$$\text{for } p \in [0, 1/2], \quad I_\mu(p) \geq C \, p \, \ln(1/p)$$

4) *and* (3.2) *holds for* $q = 1$ *and with constant* $1/C$.

According to the previous proposition an $\mathbb{L}^1$ log-Sobolev inequality implies

$$I_\mu(p) \geq (1/C_{LS1}(\mu)) \, p \, \ln(1/p) \geq \sqrt{\ln(2)} \, (1/C_{LS1}(\mu)) \, p \, \ln^{1/2}(1/p)$$

for $p \in [0, 1/2]$. According to the results in [12] the latter implies that $C_{LS}(\mu) \leq D\, C_{LS1}^2(\mu)$ for some new universal constant D.

3.2. **The uniform measure on the sphere.** We turn to the uniform distribution σ_n on the sphere $\mathbb{S}^{n-1}$. We shall first gather known results that appear in several places or immediate consequences of these results. In what follows s_n denotes the area of $\mathbb{S}^{n-1}$ which is equal to $\frac{2\pi^{n/2}}{\Gamma(n/2)}$.

Many properties rely on the fact that the sphere $\mathbb{S}^{n-1}$ satisfies the curvature-dimension condition $CD(n-2, n-1)$ (see [2] p.87). It follows from Proposition 4.8.4 and Theorem 5.7.4 in [2] that for $n \geq 3$,

$$C_P(\sigma_n) \leq \frac{1}{n-1} \quad \text{and} \quad C_{LS}(\sigma_n) \leq \frac{2}{n-1}\,.$$

These bounds are also true for $n = 2$. It is easy to check for the Poincaré constant using e.g. [2] Proposition 4.5.5 iii). For the logarithmic Sobolev inequality see [21]. Actually these bounds are optimal.

The isoperimetric profile I_{σ_n} is described in by Bobkov and Houdré in [10] Lemma 9.1 and Lemma 9.2

Proposition 3.2. *Let $n \geq 3$. Define*

$$f_n(t) = \frac{s_{n-1}}{s_n}\,(1-t^2)^{\frac{n-3}{2}}\ ;\ -1 \leq t \leq 1\,.$$

Let F_n be the distribution function on $[-1,1]$ whose probability density is f_n, $G_n = F_n^{-1}$ be the inverse function of F_n. Then

$$I_{\sigma_n}(p) = \frac{s_{n-1}}{s_n}\,(1-G_n^2(p))^{\frac{n-2}{2}}\,.$$

Notice in particular that

$$I_{\sigma_n}(1/2) = \frac{s_{n-1}}{s_n} = \frac{1}{\sqrt{\pi}}\,\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\,.$$

Using the extension of Stirling's formula to the Γ function, one sees that

$$\lim_{n\to+\infty} \sqrt{2\,\pi/(n-1)}\ \frac{s_{n-1}}{s_n} = 1\,, \tag{3.3}$$

so that one can find universal constants c and C such that

$$c\sqrt{n-1} \leq \frac{s_{n-1}}{s_n} \leq C\sqrt{n-1}\,.$$

Using their Lemma 8.2, Bobkov and Houdré ([10]) also show in their section 9 that the minimum of $I_{\sigma_n}(p)/p(1-p)$ is attained for $p = 1/2$ so that for all $p \in [0,1/2]$

$$I_{\sigma_n}(p) \geq 2\,\frac{s_{n-1}}{s_n}\,p \tag{3.4}$$

and

$$C_C(\sigma_n) = \frac{1}{2}\,\frac{s_n}{s_{n-1}} \leq \frac{C}{\sqrt{n-1}} \tag{3.5}$$

for some universal constant C. Actually, the application of Lemma 8.2 in [10] to $I^{\alpha}_{\sigma_n}$ for $\alpha \in [1, n/(n-1)]$ furnishes some Sobolev inequality (see Proposition 8.1 in [10]).

Remark 3.3. Since the curvature dimension condition $CD(n-2,n-1)$ implies $CD(0,+\infty)$ for $n\geq 2$ (i.e. σ_n is log-concave), it is known (see e.g. [27] for numerous references) that I_{σ_n} is concave on $[0,1/2]$. This furnishes another proof of (3.4).

Actually for any log-concave measure μ it was shown by Ledoux ([23] formula (5.8)) that $C_C(\mu)\leq 6\sqrt{C_P(\mu)}$. The constant 6 is improved by $16/\pi$ in [14] proposition 2.11. We thus have the precise estimate $C_C(\sigma_n)\leq (16/\pi)\,(1/\sqrt{n-1})$ and a lower bound for the isoperimetric profile linked to the Poincaré constant. $\diamondsuit$

Another remarkable property of log-concave measures identified by Ledoux ([23] Theorem 5.3) for the usual ($\mathbb{L}^2$) log-Sobolev inequality and generalized in Theorem 1.2 of [26] by E. Milman, is that the $\mathbb{L}^q$ log-Sobolev inequality also furnishes such a control for the isoperimetric profile, more precisely for any log-concave probability measure μ and $p\in[0,1/2]$,

$$I_\mu(p)\geq \frac{\sqrt{2}}{34\sqrt{C_{LS}(\mu)}}\,p\,\ln^{\frac12}(1/p)\,,$$

in the case $q=2$, and more generally there exists a universal constant c such that for all $1\leq q\leq 2$,

$$I_\mu(p)\geq \frac{c}{C_{LSq}^{1/q}(\mu)}\,p\,\ln^{\frac1q}(1/p)\,.$$

The converse statement $I_\mu(p)\geq c_\mu\,p\,\ln^{\frac1q}(1/p)$ implies $C_{LSq}(\mu)\leq C\,\frac{1}{c_\mu^q}$ for some universal constant C does not require log-concavity and was shown by Bobkov-Zegarlinski [12].

A precise dimension dependence in the isoperimetric profile.

Our goal is now to determine the best possible constant $C_n(q)$ (best in terms of the dimension) such that

$$I_{\sigma_n}(p)\geq C_n(q)\,p\,\ln^{1/q}(1/p)\,, \tag{3.6}$$

and then to apply the equivalence we explained before to derive the best possible $\mathbb{L}^q$ log-Sobolev inequality for σ_n.

We shall consider $p=F_n(x)$ in order to have a tractable expression

$$I_{\sigma_n}(F_n(x))=\frac{s_{n-1}}{s_n}\,(1-x^2)^{\frac{n-2}{2}}\,.$$

To this end we will use the following elementary lemma

Lemma 3.4. *For all $x\in[-1,0)$ define $A_n(x)=\frac{1}{(n-1)}\,(1-x^2)^{1/2}\,I_{\sigma_n}(F(x))$. It holds*

$$A_n(x)\;\leq\;F_n(x)\;\leq\;\frac{A_n(x)}{-x}\,.$$

Proof. For all $x\in[-1,0)$, we have on one hand,

$$\begin{aligned}
F_n(x)&=\int_{-1}^{x}\frac{s_{n-1}}{s_n}\,(1-u^2)^{(n-3)/2}\,du\\
&\leq\int_{-1}^{x}\frac{s_{n-1}}{s_n}\,\frac{-u}{-x}\,(1-u^2)^{(n-3)/2}\,du\\
&=\frac{s_{n-1}}{(-x)(n-1)s_n}\,(1-x^2)^{(n-1)/2}\;=\;\frac{1}{(-x)(n-1)}\,(1-x^2)^{1/2}\,I_{\sigma_n}(F(x))\,.
\end{aligned}$$

On the other hand

$$\begin{aligned} F_n(x) &= \int_{-1}^x \frac{s_{n-1}}{s_n}\,(1-u^2)^{(n-3)/2}\,du \\ &\geq \int_{-1}^x \frac{s_{n-1}}{s_n}\,(-u)\,(1-u^2)^{(n-3)/2}\,du \\ &= \frac{s_{n-1}}{(n-1)s_n}\,(1-x^2)^{(n-1)/2} \;=\; \frac{1}{(n-1)}\,(1-x^2)^{1/2}\,I_{\sigma_n}(F(x))\,. \end{aligned}$$

□

It follows for $F_n(x) \leq 1/2$ and provided $(-x)/A_n(x) > 1$ (for its logarithm to be positive),

$$F_n(x)\,\ln^{1/q}(1/F_n(x)) \;\geq\; F_n(x)\,\ln^{1/q}((-x)/A_n(x)) \;\geq\; A_n(x)\,\ln^{1/q}((-x)/A_n(x))\,.$$

But, remarking that $\ln(n-1)s_n/s_{n-1}) \geq 0$,

$$\ln((-x)/A_n(x)) \geq \ln(-x) + \frac{n-1}{2}\,\ln(1/(1-x^2)) \;\geq\; \ln(-x) + \frac{1}{4}\,\ln(1/(1-x^2)) \geq 0$$

for all $x \in (-1,-a]$ for some $0 < a < 1$ using continuity, so that for $x \leq -a$,

$$\ln(-x) + \frac{n-1}{2}\,\ln(1/(1-x^2)) \geq \frac{2n-3}{4}\,\ln(1/(1-x^2))\,.$$

This yields for such an x,

$$F_n(x)\,\ln^{1/q}(1/F_n(x)) \;\geq\; \frac{1}{(n-1)}\,(1-x^2)^{1/2}\,I_{\sigma_n}(F(x))\,\left(\frac{2n-3}{4}\,\ln(1/(1-x^2)) + \ln\left(\frac{(n-1)s_n}{s_{n-1}}\right)\right)^{1/q}$$

and finally that there exists a constant $c(q)$ such that for such an x,

$$F_n(x)\,\ln^{1/q}(1/F_n(x)) \;\geq\; c(q)\,(n-1)^{\frac{1-q}{q}}\,I_{\sigma_n}(F_n(x))\,. \tag{3.7}$$

In particular the constant $C_n(q)$ in (3.6) cannot be bigger than a constant times $(n-1)^{\frac{1-q}{q}}$.

For $q = 1$, it is known (see Theorem 2 in [24]) that

$$I_{\sigma_n}(p) \;\geq\; \frac{1}{2\pi}\,p\,\ln(1/p)$$

for $p \in [0,1/2]$, so that if not optimal, this result is optimal up to a constant factor.

Since we cannot hope for a better result, our goal will be now to prove that

$$C_n(q) \geq C(q)\,(n-1)^{\frac{1-q}{q}}\,.$$

To this end we will take advantage of Ledoux's result applied to σ_n, i.e.

$$I_{\sigma_n}(p) \;\geq C(2)\,\sqrt{n-1}\,p\,\ln^{1/2}(1/p)\,. \tag{3.8}$$

Indeed, its is easy to check that

$$\sqrt{n-1}\,p\,\ln^{1/2}(1/p) \;\geq\; (n-1)^{\frac{1-q}{q}}\,p\,\ln^{1/q}(1/p)$$

as soon as $p \geq e^{-(n-1)}$. It is thus enough to consider the remaining $p = F_n(x) \leq e^{-(n-1)}$. Using lemma 3.4 we thus have

$$\frac{s_{n-1}}{(n-1)s_n}\,(1-x^2)^{(n-1)/2} \;\leq\; e^{-(n-1)}$$

so that

$$\begin{aligned} \ln(1-x^2) &\leq 2\left(-1+\frac{\ln((n-1)s_n/s_{n-1})}{n-1}\right) \\ &\leq 2\left(-1+\frac{\ln(n-1)}{n-1}\right) \leq -2\,\frac{e-1}{e}\,, \end{aligned}$$

which implies

$$x \leq -\left(1-e^{-2(e-1)/e}\right)^{\frac{1}{2}} = y\,. \tag{3.9}$$

We can thus deduce, for such an x

$$\begin{aligned} F_n(x)\,\ln^{1/q}(1/F_n(x)) &\leq \frac{1}{(-x)(n-1)}\,(1-x^2)^{1/2}\,I_{\sigma_n}(F(x)) \\ &\qquad \left(\ln((n-1)s_n/s_{n-1})+\frac{n-1}{2}\ln(1/(1-x^2))\right)^{1/q} \\ &\leq \frac{1}{(-y)(n-1)}\,(1-x^2)^{1/2}\,I_{\sigma_n}(F(x)) \\ &\qquad \left(\ln^{1/q}((n-1)s_n/s_{n-1})+\left(\frac{n-1}{2}\right)^{1/q}\ln^{1/q}(1/(1-x^2))\right) \\ &\leq D\,(n-1)^{\frac{1-q}{q}}\,I_{\sigma_n}(F(x))\,, \end{aligned} \tag{3.10}$$

for some constant D. We may thus state

Proposition 3.5. *Let $n \geq 3$. For any $1 \leq q \leq 2$ there exist constants C and $C(q)$ such that for all $p \in [0,1/2]$ one has*

$$I_{\sigma_n}(p) \geq C\,(n-1)^{1-\frac{1}{q}}\,p\,\ln^{1/q}(1/p)\,,$$

yielding

$$C_{LSq}(\sigma_n) \leq C(q)\left(\frac{1}{n-1}\right)^{q-1}.$$

Actually all constants can be chosen independently of $q \in [1,2]$. The bound is of optimal order with respect to the dimension.

4. The Logarithmic-Sobolev inequality in the nearly radial case.

4.1. **An alternate direct approach.**

Instead of using super Poincaré inequalities, let us try to directly mimic Bobkov's tensorization in the case of log-Sobolev. First

$$\begin{aligned} \int (f^2\ln(f^2))(\rho\,\theta)\,\mu_r(d\rho) &\leq C_{LS}(\mu_r)\int |\nabla_\theta f(\rho\theta)|^2\,\mu_r(d\rho)\,+ \\ &\quad + \left(\int f^2(\rho\theta)\,\mu_r(d\rho)\right)\ln\left(\int f^2(\rho\theta)\,\mu_r(d\rho)\right), \end{aligned} \tag{4.1}$$

so that integrating with respect to μ_a we get

$$\text{Ent}_\mu(f^2) \leq C_{LS}(\mu_r)\,\mu(|\nabla_\theta f|^2) \;+ \text{Ent}_{\mu_a}(w^2) \tag{4.2}$$

with

$$w(\theta) = \left(\int f^2(\rho\theta)\,\mu_r(d\rho)\right)^{\frac{1}{2}} ,$$

after having remarked that

$$\mu(f^2)\,\ln(\mu(f^2)) = \mu_a(w^2)\,\ln(\mu_a(w^2))\,.$$

We are thus facing a difficulty. Indeed

$$\nabla w(\theta) = \frac{\int f(\rho\theta)\,\rho\,\nabla_{\theta^\perp} f(\rho\theta)\,\mu_r(d\rho)}{\left(\int f^2(\rho\theta)\,\mu_r(d\rho)\right)^{\frac{1}{2}}}\,.$$

Hence if we use the classical log-Sobolev inequality

$$\begin{aligned} \text{Ent}_{\mu_a}(w^2) &\leq C_{LS}(\mu_a)\,\mu_a(|\nabla w|^2) \\ &\leq C_{LS}(\mu_a) \int \frac{\left(\int f(\rho\theta)\,\rho\,\nabla_{\theta^\perp} f(\rho\theta)\,\mu_r(d\rho)\right)^2}{\int f^2(\rho\theta)\,\mu_r(d\rho)}\,\mu_a(d\theta)\,. \end{aligned} \tag{4.3}$$

Using Cauchy-Schwarz inequality in two different ways we obtain

Proposition 4.1. *If $\mu(dx) = \mu_r(d\rho)\,\mu_a(d\theta)$, then*

(1)

$$Ent_\mu(f^2) \leq \max\left(C_{LS}(\mu_r)\,,\, \parallel \rho \parallel^2_{\mathbb{L}^\infty(\mu_r)}\; C_{LS}(\mu_a)\right)\,\mu(|\nabla f|^2)\,.$$

(2)

$$Ent_\mu(f^2) \leq \max\,(C_{LS}(\mu_r)\,,\,C_{LS}(\mu_a))\;\mu((1\vee\rho)^2\,|\nabla f|^2)\,.$$

This is observed also by Joulin and Ma (in a private communication). The first inequality in Proposition 4.1 implies

Corollary 4.2. *If $\mu(dx) = \mu_r(d\rho)\,\mu_a(d\theta)$ and μ is supported by a bounded ball $B(0,R)$ centered at the origin of radius R, then*

$$C_{LS}(\mu) \leq \max\left(C_{LS}(\mu_r)\,,\,R^2\,C_{LS}(\mu_a)\right)\,.$$

The second inequality in Proposition 4.1 is a *weighted* log-Sobolev inequality, with weight $(1\vee\rho)^2$, which is much weaker than the log-Sobolev inequality. These inequalities have been studied for instance in [30, 20]. Consequences in terms of concentration, rate of convergence or transport are in particular discussed in section 3 of [20]. The weight $(1\vee\rho)^2$ is however too big for being really interesting. In particular this weighted log-Sobolev inequality does not imply a Poincaré inequality in whole generality.

Now consider the almost radial situation. According to section 3, $C_{LS}(\sigma_n) \leq \frac{2}{n-1}$ for $n \geq 2$. It thus follows from corollary 4.2 and perturbation arguments

Corollary 4.3. *For all $\mu(dx) = \mu_r(d\rho)\,\mu_a(d\theta)$ supported by some bounded ball $B(0,R)$ and satisfying*

$$m \leq \left|\left|\frac{d\mu_a}{d\sigma_n}\right|\right|_\infty \leq M\,,$$

it holds

$$C_{LS}(\mu) \leq \max \left(C_{LS}(\mu_r) \, , \, \frac{M}{m} \, \frac{2R^2}{n-1} \right) \, . \tag{4.4}$$

4.2. The (almost) radial case.

Our main new results for the log-Sobolev inequality of almost radial measure are

Theorem 4.4. (bounded support case) *There exists a universal c such that, for every almost radial measure $\mu(dx) = \mu_r(d\rho)\,\mu_a(d\theta)$ with support in $B(0,R)$ and satisfying*

$$m \leq \left|\left| \frac{d\mu_a}{d\sigma_n} \right|\right|_\infty \leq M \, ,$$

$$C_{LS}(\mu) \leq \, 2\, C_P(\mu) \, + \, \max \left(C_{LS}(\mu_r) \, , \, c \, \frac{M}{m} \, \min \left(R\sqrt{C_P(\mu)} \, , \, \frac{R^2}{n-1} \right) \right) . \tag{4.5}$$

Theorem 4.5. (unbounded support case) *There exists a universal constant c such that, for every almost radial measure $\mu(dx) = \mu_r(d\rho)\,\mu_a(d\theta)$ satisfying*

$$m \leq \left|\left| \frac{d\mu_a}{d\sigma_n} \right|\right|_\infty \leq M \, ,$$

it holds

$$C_{LS}(\mu)) \leq \, c \, \frac{M}{m} \, \left(C_{LS}(\mu_r) + \mu(\rho) \max \left(C_P(\mu_r) \, , \, \frac{\mu(\rho^2)}{n-1} \right)^{1/2} \right) \, . \tag{4.6}$$

Notice that if μ is supported by a bounded ball $B(0,R)$, we recover only partially the conclusion of Corollary 4.3.

Remark 4.6. Of course the previous results are much better, in terms of the dimension dependence, than Corollary 2.7 since the pre-factor of $C_{LS}(\mu_r)$ is "dimension free" (more precisely can be bounded from above by a universal constant), while the dimension appears in front of $C_{LS}(\mu_r)$ in Corollary 2.7. Let us remark also that the constant appearing in the second formulation is close from the one obtained by Bobkov in dimension one [7]. It will appear again in the next Section. Remark also that contrary to the corollary, the proof cannot be extended to more general cases, except if the angular part μ_a satisfies a similar $\mathbb{L}^1$ log-Sobolev inequality. ♢

Proof of Theorem 4.4. Rewrite (4.2),

$$\mu(f^2 \, \ln(f^2)) \, \leq \, C_{LS}(\mu_r) \, \mu(|\nabla_\theta f|^2) \, + \int g^q \, \ln^q(g) \, d\mu_a \tag{4.7}$$

with $g(\theta) = \left(\int \, f^2(\rho\,\theta) \, \mu_r(d\rho) \right)^{1/q}$ and $1 \leq q \leq 2$.

Instead of the usual log-Sobolev inequality, we now use the $\mathbb{L}^q$ log-Sobolev inequality for σ_n we have obtained in section 3. By the standard perturbation result, it thus holds

$$\int g^q \, \ln(g^q) \, d\mu_a \, \leq \, c \, \frac{M}{m} \, (n-1)^{1-q} \int \, |\nabla g|^q \, d\mu_a \, + \, \left(\int \, g^q \, d\mu_a \right) \, \ln \left(\int \, g^q \, d\mu_a \right) \, , \tag{4.8}$$

so that

$$\mathrm{Ent}_\mu(f^2) \,\leq\, C_{LS}(\mu_r)\, \mu(|\nabla_\theta f|^2) \,+\, c\,\frac{M}{m}\,(n-1)^{1-q} \int |\nabla g|^q \, d\mu_a \,.$$

Now

$$\begin{aligned} \int |\nabla g|^q \, d\mu_a &= \frac{2^q}{q} \int \left(\int f^2(\rho\,\theta)\,\mu_r(d\rho) \right)^{1-q} \left(|\int f\,\nabla_{\theta^\perp} f\,\rho\,\mu_r(d\rho)| \right)^q d\mu_a \\ &\leq \frac{2^q}{q} \parallel \rho \parallel_\infty^q \int \left(\int f^2 \, d\mu_r \right)^{1-\frac{q}{2}} \left(\int |\nabla_{\theta^\perp} f|^2 \, d\mu_r \right)^{\frac{q}{2}} d\mu_a \\ &\leq \frac{2^q}{q} \parallel \rho \parallel_\infty^q \left(\int f^2 \, d\mu \right)^{1-\frac{q}{2}} \left(\int |\nabla_{\theta^\perp} f|^2 \, d\mu \right)^{\frac{q}{2}} \end{aligned} \tag{4.9}$$

where we have used Hölder's inequality in the last line.

Recall Rothaus lemma

$$\mathrm{Ent}_\mu(f^2) \,\leq\, \mathrm{Ent}_\mu((f-\mu(f))^2) \,+\, 2\,\mathrm{Var}_\mu(f)\,.$$

We may thus replace f by $f-\mu(f)$ (that changes g) and use Poincaré's inequality in order to get

$$\int |\nabla g|^q \, d\mu_a \leq \frac{2^q}{q} \parallel \rho \parallel_\infty^q \, C_P(\mu)^{1-\frac{q}{2}} \left(\int |\nabla_{\theta^\perp} f|^2 \, d\mu \right) . \tag{4.10}$$

Gathering all these results, using again Poincaré inequality for bounding the variance, we thus have if μ is supported by $B(0,R)$,

$$\begin{aligned} \mathrm{Ent}_\mu(f^2) &\leq C_{LS}(\mu_r) \int |\nabla_\theta f(\rho\theta)|^2 \, d\mu + 2C_P(\mu) \left(\int |\nabla f|^2 \, d\mu \right) \\ &\quad + c\,\frac{M}{m}\,(n-1)^{1-q}\,\frac{2^q}{q} R^q \, C_P(\mu)^{1-\frac{q}{2}} \left(\int |\nabla_{\theta^\perp} f|^2 \, d\mu \right) . \end{aligned} \tag{4.11}$$

Taking $q=2$ if $R \leq (n-1)$ and $q=1$ if $R \geq (n-1)$, we obtain the desired estimate (4.5) by Theorem 2.1. □

Proof of Theorem 4.5. we will use the preceding argument with $q=1$ (i.e. the L^1 log-Sobolev inequality for σ_n). Instead of (4.9) we have

$$\int |\nabla g| \, d\sigma_n \leq 2 \int \left(\int f^2 \, \rho^2 \, d\mu_r \right)^{1/2} \left(\int |\nabla_{\theta^\perp} f|^2 \, d\mu_r \right)^{\frac{1}{2}} d\sigma_n \tag{4.12}$$

obtained from the the first equality in (4.9) and the Cauchy-Schwarz inequality.
Let us consider $\int f^2 \, \rho^2 \, d\mu$ integrating first w.r.t. μ_r. By using the variational description of the relative entropy we have for any $t>0$

$$\int f^2(\rho\,\theta)\,\rho^2\,\mu_r(d\rho) \,\leq\, \mathrm{Ent}_{\mu_r}(f^2(.\,\theta)) + \frac{1}{t}\,\ln\left(\int e^{t\rho^2}\,\mu_r(d\rho) \right) \left(\int f^2(\rho\,\theta)\,\mu_r(d\rho) \right) . \tag{4.13}$$

We may of course stop here to get a first control of the logarithmic Sobolev constant of μ by using recentering and Rothaus lemma (see the end of the argument) but let us see how using the same approach will provide us with an easy to apprehend formulation of the logarithmic Sobolev constant. We first use

$$\int f^2 \, \rho^2 \, d\mu_r \,\leq\, 2 \int f^2\,(\rho-\mu(\rho))^2 \, d\mu \,+\, 2\,\mu_r^2(\rho) \int f^2 \, d\mu_r \,,$$

and using again the variational description of the relative entropy we have for any $t > 0$,

$$\int f^2(\rho\,\theta)\,(\rho-\mu(\rho))^2\,\mu_r(d\rho) \,\leq\, \mathrm{Ent}_{\mu_r}(f^2(.\,\theta)) + \frac{1}{t}\,\ln\left(\left(\int e^{t(\rho-\mu(\rho))^2}\,\mu_r(d\rho)\right)\right)\left(\left(\int f^2(\rho\,\theta)\,\mu_r(d\rho)\right)\right). \tag{4.14}$$

But since $\rho \mapsto \rho - \mu_r(\rho)$ is 1-Lipschitz and of μ_r mean equal to 0, it is known (see e.g. [7] formula (4.9)) that for $t < 1/C_{LS}(\mu_r)$

$$\int e^{t(\rho-\mu(\rho))^2}\,\mu(d\rho) \,\leq\, \frac{1}{\sqrt{1-t\,C_{LS}(\mu_r)}}\,.$$

For $t = 1/2\,C_{LS}(\mu_r)$ we thus deduce

$$\int f^2(\rho\,\theta)\,(\rho-\mu(\rho))^2\,\mu_r(d\rho) \,\leq\, C_{LS}(\mu_r)\int |\nabla_\theta f(\rho\,\theta)|^2\,\mu_r(d\rho) + \ln(2)\,C_{LS}(\mu_r)\int f^2(\rho\,\theta)\,\mu_r(d\rho)\,.$$

Finally

$$\int f^2\,\rho^2\,d\mu \,\leq\, 2\,C_{LS}(\mu_r)\int |\nabla_\theta f|^2\,d\mu \,+\, 2(\ln(2)\,C_{LS}(\mu_r) \,+\, \mu_r^2(\rho))\int f^2\,d\mu\,. \tag{4.15}$$

Replacing f by $f - \mu(f)$ and using the Poincaré inequality, we thus deduce

$$\int (f-\mu(f))^2\,\rho^2\,d\mu \,\leq\, 2\,C_{LS}(\mu_r)\int |\nabla f|^2\,d\mu + 2(\ln(2)\,C_{LS}(\mu_r) + \mu_r^2(\rho))\,C_P(\mu)\int |\nabla f|^2\,d\mu\,.$$

Gathering all what precedes we get

$$\mathrm{Ent}_\mu((f-\mu(f))^2) \,\leq\, A\,\mu(|\nabla f|^2)$$

with, for some universal constant c,

$$A \,=\, C_{LS}(\mu_r) + c\,\left(C_{LS}(\mu_r) \,+\, (\,C_{LS}(\mu_r) + \mu^2(\rho))\,C_P(\mu))\right)^{\frac{1}{2}}\,. \tag{4.16}$$

To conclude it remains to use Rothaus lemma again. We obtain (4.6) after some simple manipulations using in particular $2C_P \leq C_{LS}$, the concavity of the square root, and the homogeneity of the inequalities w.r.t. dilations as we did in order to get (2.14) but this time with $\lambda \to +\infty$, and finally (2.8) (replacing for simplicity the max by the sum). □

To finish this section, let us remark that one could also get another way to control (4.12) by using Lyapunov conditions rather than using the Logarithmic Sobolev inequality for the radial part. Of course, by [17], one also knows that in our setting a logarithmic Sobolev inequality is equivalent to some Lyapunov type conditions. However we will see that in order to control (4.12) one needs a slightly weaker inequality. Indeed let us suppose here that there exist some function $W(\rho) \geq 1$ and constants $a, b > 0$ such that

$$\rho^2 \leq -a\frac{L_\rho W}{W} + b,\;\; \rho > 0 \tag{4.17}$$

where $L_\rho f = f'' - (V_r' - \frac{n-1}{\rho})f'$ is the generator corresponding to the radial part of μ. Recall now that we need to control

$$\int f^2\rho^2\mu_r(d\rho).$$

Using (4.17) we get

$$\int f^2\rho^2\mu_r(d\rho) \leq a\int f^2\frac{-L_\rho W}{W}\mu_r(d\rho) + b\int f^2\mu_r(d\rho).$$

The first term is easily dealt with, using integration by parts or a large deviations argument as in [18]:

$$\int f^2 \frac{-L_\rho W}{W} \mu_r(d\rho) \leq \int |\nabla_\theta f(\rho\theta)|^2 \mu_r(d\rho).$$

We then use the same trick as before, i.e centering and Rothaus lemma to get

Proposition 4.7. *Assume* (4.17). *There exists a universal constant c such that, for all $\mu(dx) = \mu_r(d\rho)\, \mu_a(d\theta)$ satisfying*

$$m \leq \left|\left| \frac{d\mu_a}{d\sigma_n} \right|\right|_\infty \leq M \,,$$

it holds

$$C_{LS}(\mu)) \leq \; c\, \frac{M}{m} \left(C_{LS}(\mu_r) + \sqrt{a} + \sqrt{b} \max\left(C_P(\mu_r) \,,\, \frac{\mu(\rho^2)}{n-1} \right)^{1/2} \right) .$$

Remark 4.8. Let us make a few comments about the Lyapunov condition (4.17). It has been shown in [19] that it is a sufficient condition for Talagrand inequality, and that there exists examples satisfying this condition and not a logarithmic Sobolev inequality. Nevertheless, we need for the first part of the proof that the radial part satisfies a logarithmic Sobolev inequality. More crucial are the values of the constants a and b with respect to the dimension. If a can be chosen dimension free in usual cases, say $V_r(\rho) = \rho^k$, b is then of order n and we then get an additional $\sqrt{n}$ factor for the logarithmic Sobolev constant in this case. Of course, it is surely better than the n factor by using Super-Poincaré inequality. ♢

5. Some applications in the (almost) radial case.

The main interest of the previous results is that they reduce the study of functional inequalities for μ to the one of its radial part μ_r which is supported by the half line. For such one dimensional measures explicit criteria of Muckenhoupt (or Hardy) type are well known [1, 2, 6, 5, 4]). Let us recall the case of the Poincaré inequality (see e.g. [2] Theorem 4.5.1) and of the log-Sobolev inequality (see [5] Theorem 7 with $T(u) = 2u$, or [15] Proposition 2.4 for a slightly different version).

Proposition 5.1. *Assume that μ_r is absolutely continuous w.r.t. Lebesgue measure with density ρ_r. Assume in addition that the support of μ_r is an interval I. Let m be a median of μ_r, then*

$$\frac{1}{2}\, \max(b_-, b_+) \;\leq\; C_P(\mu_r) \;\leq\; 4\, \max(b_-, b_+)$$

and

$$\frac{1}{12}\, \max(B_-, B_+) \;\leq\; C_{LS}(\mu_r) \;\leq\; 10\, \max(B_-, B_+)$$

where

$$b_- = \sup_{x \in I, 0 \leq x < m} \mu_r([0, x]) \int_x^m \frac{1}{\rho_r(\rho)}\, d\rho$$

$$b_+ = \sup_{x \in I, x > m} \mu_r([x, +\infty)) \int_m^x \frac{1}{\rho_r(\rho)}\, d\rho$$

while

$$B_- = \sup_{x\in I, 0\leq x<m} \mu_r([0,x]) \, \ln\left(1+\frac{1}{\mu_r([0,x])}\right) \int_x^m \frac{1}{\rho_r(\rho)}\, d\rho$$

$$B_+ = \sup_{x\in I, x>m} \mu_r([x,+\infty)) \, \ln\left(1+\frac{1}{\mu_r([x,+\infty))}\right) \int_m^x \frac{1}{\rho_r(\rho)}\, d\rho \, .$$

Several methods are known to furnish estimates for quantities like $\mu_r([a,+\infty[)$ or $\int(1/\rho_r)$ (see e.g. [1] chapter 6.4).

Remark 5.2. We will not study in more details the $\mathbb{L}^1$ inequalities on the real (half) line. However, because it is immediate, let us only give an upper bound for C_C.

Proposition 5.3. *Let $\nu(dx) = e^{-W(x)}dx$ be a probability measure on $\mathbb{R}$.*
Then $C_C(\nu) \leq \max(b^1_-, b^1_+)$ where

$$b^1_- = \sup_{t\in\mathbb{R}} e^{W(t)}\, \nu(]-\infty,t]) \quad \text{and} \quad b^1_+ = \sup_{t\in\mathbb{R}} e^{W(t)}\, \nu([t,+\infty[)\, .$$

Proof. We simply write for any a,

$$\begin{aligned} \nu(|f-m_\nu(f)|) &\leq \nu(|f-f(a)|) = \int \left| \int_a^x f'(t)\, dt \right| \nu(dx) \\ &\leq \int_{-\infty}^a \int_x^a |f'(t)| dt\, \nu(dx) + \int_a^{+\infty} \int_a^x |f'(t)| dt\, \nu(dx) \\ &\leq \int_{-\infty}^a |f'(t)|\, \nu(]-\infty,t])\, dt \, + \int_a^{+\infty} |f'(t)|\, \nu([t,+\infty[)\, dt\, , \end{aligned}$$

and then write $dt = e^{W(t)}\, \nu(dt)$ to get the result. □

It is possible to get a lower bound, and bounds for C_{LS1} using ad-hoc Orlicz spaces as in [10] and the previous trick in one dimension. ♢

Remark 5.4. Before to study some simple examples, let us say a word about "optimality" in the radial case.
Notice first that if $f(x) = g(|x|)$, $\nabla f(x) = g'(|x|)\, \frac{x}{|x|}$ so that $|\nabla f|^2(x) = (g'(|x|))^2$. It follows that $C_P(\mu) \geq C_P(\mu_r)$ and similarly for the log-Sobolev constant.
If we take $f(x) = \sum_{i=1}^n x_i/|x|$, $\text{Var}_\mu(f) = 1$ while $\mu(|\nabla f|^2) = (n-1)\, \mu(1/\rho^2)$ so that $C_P(\mu) \geq 1/((n-1)\, \mu(1/\rho^2)$. The latter is smaller than $\mu(\rho^2)/(n-1)$ but both quantities are comparable in many situations. We shall see it on the examples below. ♢

In all the examples below, we will pay a particular attention to *dimension dependence*, so that any sentence like "the good order" or "good dependence" has to be understood "with respect to the dimension n.

Example 5.5. **The uniform measure on an euclidean ball.**

Consider the simplest example for radial μ i.e. the uniform measure on the euclidean ball of radius R. The good order for the log-Sobolev constant has been derived in [11] Proposition 5.3 by pushing forward the gaussian distribution onto the uniform one on the ball (also see Proposition 5.4 therein for the LSq inequality).

We shall obtain here similar bounds by using our results. First

$$\mu_r(d\rho) = \frac{n\,\rho^{n-1}}{R^n}\,\mathbf{1}_{0\leq\rho\leq R}\,d\rho$$

so that the mean, the (unique) median, the second moment, the Variance of μ_r are respectively:

$$\mu_n = \frac{n}{n+1}R,\ m_n = (1/2)^{1/n}\,R,\ \mu_r(\rho^2) = \frac{n}{n+2}\,R^2,\ v_n = \frac{n}{(n+2)(n+1)^2}\,R^2\,.$$

It is then easily seen that b_- and b_+ are both less than $\frac{R^2}{n(n-2)}$ provided $n > 2$. The case $n = 2$ can be handled separately. It follows that $C_P(\mu_r) \leq 4\,(R^2/n^2)$. We also have $C_P(\mu_r) \geq v_n$ so that R^2/n^2 is the good order. Finally

$$C_P(\mu) \leq \max\left(\frac{4}{n^2}\,,\,\frac{n}{(n+2)(n-1)}\right)\,R^2\,.$$

This bound is not sharp, but asymptotically sharp. Indeed $\mu(1/\rho^2) = \frac{n}{n-2}\,R^{-2}$, so that according to the discussion in remark , $C_P(\mu) \geq \frac{n-2}{n(n-1)}\,R^2$.

Notice that the upper bound is exactly the one obtained in [13] Theorem 1.2, while the lower bound is better than the one in [13]. Theorem 1.2 in [13] deals with general radial log-concave measures by using dedicated tools for log-concave one dimensional measures, we shall come back to this later.

We turn now to the log-Sobolev constant assuming first that $n \geq 3$.
First

$$B_- = \sup_{0\leq x<m_n}\,\frac{x^2}{n(n-2)}\,\left(1-(x/m_n)^{n-2}\right)\,\ln(1+(R/x)^n)\,,$$

so that using $\ln(1+u^n) \leq \ln 2 + n\ln(u)$ for $u \geq 1$, we deduce

$$B_- \leq \frac{R^2}{n-2}\,\sup_{u\geq 1}\left(\frac{1}{u^2}\,\left(\frac{\ln 2}{n} + \ln u\right)\right)\,\leq\,c_1\,\frac{R^2}{n-2}\,,$$

with $c_1 = \frac{1}{2e} + \frac{\ln 2}{n} \leq 1$. Similarly, using $\ln(1+u^n) \geq n\ln u$ for $u \geq 1$ we have

$$B_-\,\geq\,\frac{R^2}{n-2}\,\sup_{2^{1/n}\leq u}\,\left(\frac{\ln u}{u^2}\,\left(1-2^{(n-2)/n}u^{-(n-2)}\right)\right)\,\geq\,c_2\,\frac{R^2}{n-2}\,,$$

with for instance $c_2 = \ln 2/16$ obtained by choosing $u = 4$.
Next

$$B_+ = \sup_{m_n<x\leq R}(1-(x/R)^n)\,\frac{R^2}{n(n-2)}\,\left(2^{(n-2)/n}-(R/x)^{n-2}\right)\,\ln\left(1+\frac{1}{1-(x/R)^n}\right)\,,$$

so that

$$B_+ \leq \frac{R^2}{n(n-2)}\,\sup_{0<v\leq 1}\,(v\,\ln(1+(1/v)))\,\leq\,\frac{R^2}{n(n-2)}\,.$$

Gathering all this we obtain that

$$\frac{\ln 2}{192}\,\frac{R^2}{n-2}\,\leq\,C_{LS}(\mu_r) \leq 10\,\frac{R^2}{n-2}\,.$$

Note that choosing $f(x) = x$ one gets the better lower bound

$$C_{LS}(\mu_r) \geq \; R^2 \, \frac{n^2}{n+2} \left(\ln(1 + \frac{2}{n}) - \frac{2}{n+2} \right) \geq \; 2 \, R^2 \, \frac{n-2}{(n+2)^2} \, .$$

Using Corollary 4.3 we have thus shown, since $C_{LS}(\mu) \geq C_{LS}(\mu_r)$, and after simple manipulations

Proposition 5.6. *For all $n \geq 3$ the uniform measure μ on the euclidean ball of radius R satisfies* $\frac{2\,(n-2)\,R^2}{(n+2)^2} \; \leq \; C_{LS}(\mu) \; \leq \; 10 \, \frac{R^2}{n-2}$.

Example 5.7. **Spherically symmetric log-concave measures.**

The previous example is a particular example of a radial log-concave measure, i.e. $\mu(dx) = e^{-V(|x|)} \, dx$ where V is convex and non decreasing. Actually for what follows (in the radial situation) we do not need V to be non decreasing, but still convex.

For such measures the Poincaré constant was first studied by Bobkov in [8]. Bobkov's result was improved by Bonnefont-Joulin-Ma in [13] Theorem 1.2 who states that for such measures

$$\frac{\mu(\rho^2)}{n} \; \leq \; C_P(\mu) \; \leq \; \frac{\mu(\rho^2)}{n-1} \, . \tag{5.1}$$

The case of the log-Sobolev constant was not really addressed in the specific radial situation, but rather for general log concave distributions. Define

$$Orl(\mu) \; = \; \inf\{ \, t > 0 \, ; \, \int \, \exp(|x - \mu(x)|^2/t^2) \, \mu(dx) \; \leq \; 2 \, \} \, .$$

Then, according to Bobkov's theorem 1.3 in [7], if μ is log-concave

$$C_{LS}(\mu) \leq C \, Orl^2(\mu) \tag{5.2}$$

the right hand side being finite or infinite. When μ is supported by a bounded set K, this yields the rough bound $C_{LS}(\mu) \leq C \, diam^2(K)$. The latter has been improved in [25] Theorem 8, where it is shown that $C_{LS}(\mu) \leq C \, diam(K)$ provided μ is isotropic (i.e. its covariance matrix equals identity). Notice that Bobkov's Corollary 2.3 in [7] tells that

$$C_{LS}(\mu) \; \leq \; 2 \, (C_P(\mu) + diam(K) \, \sqrt{C_P(\mu)}) \, , \tag{5.3}$$

so that, it also furnishes the diameter bound if the K-L-S conjecture is true.

Actually the specific radial case was only addressed in [22] where the author studies the isoperimetric profile of radial log concave distributions (see theorem 4 and theorem 5 therein). Connections between the log-Sobolev constant and the isoperimetric profile are strong in the log-concave situation (see [26] Theorem 1.2) but the results of [22] are not easy to handle with. We will thus use our previous results to derive explicit bounds.

Writing $\mu_r(d\rho) \; = \; n \, \omega_n \, \rho^{n-1} \, e^{-V(\rho)} \, d\rho$ with ω_n the volume of the unit euclidean ball, we see that μ_r is also log-concave. One can thus apply Bobkov's results in [7], starting with Proposition 4.4 therein

$$\frac{3}{4} \, Orl^2(\mu_r) \; \leq \; C_{LS}(\mu_r) \; \leq \; 48 \, Orl^2(\mu_r) \tag{5.4}$$

where

$$Orl(\mu_r) \;=\; \inf\{\; t>0\;;\; \int \exp((\rho-\mu_r(\rho))^2/t^2)\,\mu_r(d\rho)\;\leq\; 2\;\}\,.$$

As a byproduct we get thanks to Theorem 4.5 and (5.1)

Proposition 5.8. *If μ is a radial log-concave distribution,*

$$C_{LS}(\mu) \leq C\,\left(Orl^2(\mu_r) + \frac{\mu(\rho^2)}{\sqrt{n-1}}\right)\,.$$

Using (5.4) *and remark 5.5, one can also derive some lower bound.*

If we assume in addition that μ is supported by a bounded ball $B(0,R)$, we may obtain other bounds using our Corollary 4.3 (the same with Theorem 4.4). We remark that

$$\mu_r(d\rho) = Z_n^{-1}\, e^{(n-1)\ln\rho - V(\rho)}\, d\rho = Z_n^{-1}\, e^{-W(\rho)}\, d\rho$$

with

$$W''(\rho) = V''(\rho) + \frac{n-1}{\rho^2} \geq \frac{(n-1)}{R^2}\,.$$

According to Bakry-Emery criterion (see e.g. [2] Proposition 5.7.1) we deduce

$$C_{LS}(\mu_r)\;\leq\;\frac{R^2}{n-1}\,. \tag{5.5}$$

Applying Corollary 4.3 we have thus obtained

Theorem 5.9. *For any radial log-concave probability measure μ with support contained in $B(0,R)$,*

$$C_{LS}(\mu)\;\leq\;C\,\frac{R^2}{n-1}\,.$$

This bound is sharp in the sense that a similar lower bound is true for the uniform measure on a ball (recall Proposition 5.6).

When the radius is less than $n/2$ this result is better than [25]. Actually we do not need μ to be isotropic contrary to [25].

Acknowledgements. This work has been (partially) supported by the Project EFI ANR-17-CE40-0030 of the French National Research Agency. We sincerely thank two anonymous referees for their well pointed remarks and suggested corrections.

References

[1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2000.

[2] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and Geometry of Markov diffusion operators.*, volume 348 of *Grundlehren der mathematischen Wissenchaften*. Springer, Berlin, 2014.

[3] F. Barthe, P. Cattiaux, and C. Roberto. Concentration for independent random variables with heavy tails. *AMRX*, 2005(2):39–60, 2005.

[4] F. Barthe, P. Cattiaux, and C. Roberto. Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry. *Rev. Mat. Iber.*, 22(3):993–1066, 2006.

[5] F. Barthe, P. Cattiaux, and C. Roberto. Isoperimetry between exponential and Gaussian. *Electron. J. Probab.*, 12:1212–1237, 2007.

[6] F. Barthe and C. Roberto. Sobolev inequalities for probability measures on the real line. *Studia Math.*, 159(3):481–497, 2003.
[7] S. G. Bobkov. Isoperimetric and analytic inequalities for log-concave probability measures. *Ann. Probab.*, 27(4):1903–1921, 1999.
[8] S. G. Bobkov. Spectral gap and concentration for some spherically symmetric probability measures. In *Geometric aspects of functional analysis, Israel Seminar 2000-2001,*, volume 1807 of *Lecture Notes in Math.*, pages 37–43. Springer, Berlin, 2003.
[9] S. G. Bobkov. Large deviations and isoperimetry over convex probability measures. *Electron. J. Probab.*, 12:1072–1100, 2007.
[10] S. G. Bobkov and C. Houdré. Some connections between isoperimetric and Sobolev-type inequalities. *Mem. Amer. Math. Soc.*, 129(616):viii+111, 1997.
[11] S. G. Bobkov and M. Ledoux. From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities. *Geom. Funct. Anal.*, 10(5):1028–1052, 2000.
[12] S. G. Bobkov and B. Zegarlinski. Entropy bounds and isoperimetry. *Mem. Amer. Math. Soc.*, 176(829):x+69, 2005.
[13] M. Bonnefont, A. Joulin, and Y. Ma. Spectral gap for spherically symmetric log-concave probability measures, and beyond. *J. Funct. Anal.*, 270(7):2456–2482, 2016.
[14] P. Cattiaux and Guillin A. On the Poincaré constant of log-concave measures. To appear in Geometric Aspects of Functional Analysis: Israel Seminar (GAFA)., 2019.
[15] P. Cattiaux, I. Gentil, and A. Guillin. Weak logarithmic Sobolev inequalities and entropic convergence. *Probab. Theory and Rel. Fields*, 139(3-4):563–603, 2007.
[16] P. Cattiaux, N. Gozlan, A. Guillin, and C. Roberto. Functional inequalities for heavy tailed distributions and applications to isoperimetry. *Electron. J. Probab.*, 15:346–385, 2010.
[17] P. Cattiaux and A. Guillin. Hitting times, functional inequalities, Lyapunov conditions and uniform ergodicity. *J. Funct. Anal.*, 272(6):2361–2391, 2017.
[18] P. Cattiaux, A. Guillin, F.Y. Wang, and L. Wu. Lyapunov conditions for super Poincaré inequalities. *J. Funct. Anal.*, 256(6):1821–1841, 2009.
[19] P. Cattiaux, A. Guillin, and L. Wu. A note on Talagrand's transportation inequality and logarithmic Sobolev inequality. *Probab. Theory Related Fields*, 148(1-2):285–304, 2010.
[20] P. Cattiaux, A. Guillin, and L. Wu. Some remarks on weighted logarithmic Sobolev inequalities. *Indiana Univ. Math. J.*, 60(6):1885–1904, 2011.
[21] M. Émery and J. E. Yukich. A simple proof of the logarithmic Sobolev inequality on the circle. In *Séminaire de Probabilités, XXI*, volume 1247 of *Lecture Notes in Math.*, pages 173–175. Springer, Berlin, 1987.
[22] N. Huet. Isoperimetry for spherically symmetric log-concave probability measures. *Rev. Mat. Iberoamericana*, 27(1):93–122, 2011.
[23] M. Ledoux. Spectral gap, logarithmic Sobolev constant, and geometric bounds. In *Surveys in differential geometry.*, volume IX, pages 219–240. Int. Press, Somerville MA, 2004.
[24] M. Ledoux. From concentration to isoperimetry: semigroup proofs. In *Concentration, functional inequalities and isoperimetry*, volume 545 of *Contemp. Math.*, pages 155–166. Amer. Math. Soc., Providence, RI, 2011.
[25] Y. T. Lee and S. S. Vempala. Stochastic localization + Stieltjes barrier = tight bound for Log-Sobolev. Available on Math. ArXiv. 1712.01791 [math.PR], 2017.
[26] E. Milman. On the role of convexity in functional and isoperimetric inequalities. *Proc. Lond. Math. Soc. (3)*, 99(1):32–66, 2009.
[27] E. Milman. On the role of convexity in isoperimetry, spectral-gap and concentration. *Invent. math.*, 177:1–43, 2009.
[28] M. Röckner and F. Y. Wang. Weak Poincaré inequalities and $\mathbb{L}^2$-convergence rates of Markov semigroups. *J. Funct. Anal.*, 185(2):564–603, 2001.
[29] F. Y. Wang. *Functional inequalities, Markov processes and Spectral theory.* Science Press, Beijing, 2005.
[30] F. Y. Wang. From super Poincaré to weighted log-Sobolev and entropy cost inequalities. *J. Math. Pures Appl.*, 90:270–285, 2008.
[31] P. A. Zitt. Super Poincaré inequalities, Orlicz norms and essential spectrum. *Potential Anal.*, 35(1):51–66, 2011.

Patrick CATTIAUX, Institut de Mathématiques de Toulouse. CNRS UMR 5219., Université Paul Sabatier,, 118 route de Narbonne, F-31062 Toulouse cedex 09.

Email address: cattiaux@math.univ-toulouse.fr

Arnaud GUILLIN, Laboratoire de Mathématiques Blaise Pascal, CNRS UMR 6620, Université Clermont-Auvergne, avenue des Landais, F-63177 Aubière.

Email address: arnaud.guillin@uca.fr

Liming WU, Laboratoire de Mathématiques Blaise Pascal, CNRS UMR 6620, Université Clermont-Auvergne, avenue des Landais, F-63177 Aubière.

Email address: li-ming.wu@uca.fr