A JOURNEY WITH THE INTEGRATED Γ_2 CRITERION AND ITS WEAK FORMS.

PATRICK CATTIAUX [♠]   AND   ARNAUD GUILLIN [◇]

♠ Université de Toulouse

◇ Université Clermont-Auvergne

ABSTRACT. As the title indicates this paper will describe several extensions and applications of the Γ_2 integrated criterion introduced by M. Ledoux following ideas of B. Hellffer. We introduce general weak versions and show that they are equivalent to the weak Poincaré inequalities introduced by M. Röckner and F. Y. Wang. We also discuss special weak versions appropriate to the study of log-concave measures and log-concave perturbations of product measures.

Key words : Poincaré inequality, Γ_2 operator, log-concave measures.

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1. INTRODUCTION, FRAMEWORK AND PRESENTATION OF THE RESULTS.

Introduced in [3] the Γ_2 criterion (also called $CD(\rho, \infty)$ curvature condition) is the best known sufficient condition for Poincaré and log-Sobolev inequalities to hold for some probability measure μ. It reads as

$$\Gamma_2(f) \geq \rho \, \Gamma(f)$$

for some $\rho > 0$ (see the definitions in the next subsection), i.e. is a pointwise condition. In [28], M. Ledoux introduced an integrated version

$$\mu(\Gamma_2(f)) \geq \rho \, \mu(\Gamma(f))$$

and proved that this integrated version for some $\rho > 0$ is equivalent to a Poincaré inequality (see Theorem 1.3 below). The Poincaré inequality is thus a mean curvature condition.

As it is well known, Poincaré inequality is related to the "exponential" concentration of measure, to the $\mathbb{L}^2(\mu)$ contraction of some associated Markov semi-group (implying exponential stabilization) and to some isoperimetric questions.

During the last years weaker (and also stronger) forms of the Poincaré inequality have been discussed. They allow us to describe weaker concentration properties (polynomial for instance) and slower rates of convergence to equilibrium (see subsection 1.2). It is natural to ask whether these weak Poincaré inequalities are equivalent to some weak integrated Γ_2 criteria. This was the starting point of this work.

We then describe some applications of weak integrated Γ_2 criteria to log-concave measures, perturbation of product measures or of radial measures.

1.1. Framework (The heart of darkness following [4]).

We will first introduce the objects we are dealing with. The aficionados of [4] will (almost) recognize what is called a full Markov triple therein. Nevertheless in order to understand some of our approaches, one has to understand why this framework is the good one.

Let $\mu(dx) = Z_V^{-1} e^{-V(x)} dx$ be a probability measure defined on an open domain $D \subseteq \mathbb{R}^n$. When needed, we will require some regularity for V and assume that it takes finite values. We denote by $\mu(f)$ the integral of f w.r.t. μ.

If V is in $C^2(D)$, we may introduce the operator

$$A = \Delta - \nabla V.\nabla$$

and the diffusion process

$$X_t^x = x + \sqrt{2}\, B_t - \int_0^t \nabla V(X_s^x) ds$$

living in D up to an explosion time T_∂^x since ∇V is local Lipschitz. Of course here $B_\cdot$ is a standard Brownian motion.

When $D = \mathbb{R}^n$, $T_\partial^x = \sup_{k \in \mathbb{N}*} T_k^x$ where T_k^x denotes the exit time of the euclidean ball of radius k, while if D is a bounded open subset, T_∂^x denotes the hitting time of the boundary ∂D, i.e.

$$T_\partial^x = \sup_k T_k^x \quad \text{where} \quad T_k^x = \inf\{t, d(X_t^x, D^c) \leq 1/k\},$$

where $d(.,.)$ denotes the euclidean distance. In the sequel we will assume that

$$T_\partial^x = +\infty \quad a.s. \text{ for all } x \in D. \tag{1.1}$$

In other words the process $X_\cdot$ is conservative (in D) and we define $P_t f(x) = \mathbb{E}(f(X_t^x))$ for bounded f's, so that P_t is a markovian semi-group of contractions in $\mathbb{L}^\infty(D)$.

Definition 1.1. We shall say that Assumption (H) is satisfied if (1.1) holds true and if in addition

$$\mu \text{ is a reversible (symmetric) measure for the process.} \tag{1.2}$$

We will denote $\langle u, v \rangle$ the usual scalar product and introduce

$$\Gamma(f, g) = \langle \nabla f, \nabla g \rangle \quad , \quad \mathcal{E}(f, g) = \mu(\Gamma(f, g))$$

the associated Dirichlet form, with domain $\mathcal{D}(\mathcal{E})$. We will write $\Gamma(f)$ for $\Gamma(f, f)$.
The next result is the key of the construction

Proposition 1.2. *Assume that (H) is satisfied. In the following two cases*

    (1) $D = \mathbb{R}^n$,

    (2) D *is an open bounded domain and* $V \in C^\infty(D)$,

then P_t extends to a μ-symmetric continuous Markov semi-group on $\mathbb{L}^2(\mu)$ with generator $\tilde{A}$ and domain $\mathcal{D}(\tilde{A})$.

In addition the generator $\tilde{A}$ is essentially self-adjoint on $C_0^\infty(D)$ (C^∞ functions with compact support). We shall call ESA this property. In particular $C_0^\infty(D)$ is a core for $\mathcal{D}(\tilde{A})$. The latter is exactly the set of $f \in H^2_{loc}(D)$ such that f and Af are in $\mathbb{L}^2(\mu)$.

We shall give a proof of this Proposition in section 7, where sufficient conditions for (H) are discussed as well as examples. For simplicity we will only use the notation A in the sequel both for A and $\tilde{A}$.

If $g \in \mathcal{D}(A)$ it holds

$$\mathcal{E}(f, g) = -\mu(f\, Ag)\,. \tag{1.3}$$

If $f \in \mathbb{L}^2(\mu)$ it is well known that $P_t f \in \mathcal{D}(A)$ for $t > 0$ and

$$\partial_t P_t f = A P_t f\,. \tag{1.4}$$

If in addition $f \in \mathcal{D}(A)$,

$$\partial_t P_t f = A P_t f \ = \ P_t A f\,. \tag{1.5}$$

In particular if f is in $\mathcal{D}(A)$, for $t > 0$,

$$\partial_t A P_t f = \partial_t P_t A f = A P_t A f\,. \tag{1.6}$$

1.2. Presentation of the main results.

We define the Poincaré constant $C_P(\mu)$ as the smallest constant C satisfying

$$\mathrm{Var}_\mu(f) := \mu(f^2) - \mu^2(f) \ \leq\ C\, \mu(|\nabla f|^2)\,, \tag{1.7}$$

for all $f \in C_b^1(D)$ the set of C^1 functions which are bounded with a bounded derivative. For simplicity we will say that μ satisfies a Poincaré inequality provided $C_P(\mu)$ is finite.

As it is well known, the Poincaré constant is linked to the exponential stabilization of the Markov semi-group P_t.

For a Diffusion Markov Triple, the following is well known (see chapter 4 in [4]), it extends to our situation

Theorem 1.3. *If (H) is satisfied, the following three statements are equivalent*

    (1) μ satisfies a Poincaré inequality,

    (2)   there exists C such that for every $f \in C_0^\infty(D)$ (or $C_b^\infty(D)$ the set of smooth functions with bounded derivatives of any order), it holds

$$\mu(|\nabla f|^2) \ \leq\ C\, \mu((Af)^2)\,, \tag{1.8}$$

    (3)   there exists $C > 0$ such that for every $f \in \mathbb{L}^2(\mu)$,

$$Var_\mu(P_t f) \ \leq\ e^{-2t/C}\, Var_\mu(f)\,. \tag{1.9}$$

In addition the optimal constants in (1.8) and (1.9) are equal to $C_P(\mu)$.

It is important to check that the previous theorem only requires the properties we have recalled before. Actually the proof of (1) $\Leftrightarrow$ (3) ([4] Theorem 4.2.5) only requires (1.4) so that it is always satisfied. The one of (2) $\Leftrightarrow$ (1) ([4] Proposition 4.8.3) requires to use ESA. In addition one has to check that the semi-group is ergodic, i.e. that the only invariant functions ($P_t f = f$ for all t) are the constants. A proof is provided in the Appendix.

Following D. Bakry we may define (provided V is C^2) the Γ_2 operator

$$\Gamma_2(f,g) = \frac{1}{2} \left[A\Gamma(f,g) - \Gamma(f, Ag) - \Gamma(Af, g) \right]. \tag{1.10}$$

for f, g in $C_b^\infty(D)$. A simple calculation yields in this case

$$\Gamma_2(f) := \Gamma_2(f,f) = \| Hess(f) \|_{HS}^2 + \langle \nabla f, Hess(V) \nabla f \rangle, \tag{1.11}$$

where $Hess(f)$ denotes the Hessian matrix of f, and $\| Hess(f) \|_{HS}^2 = \sum_{i,j} |\partial_{i,j}^2 f|^2$.
Using symmetry we get

$$\mu(\Gamma_2(f,g)) = \mu((Af)(Ag)). \tag{1.12}$$

still for C_b^∞ functions since if (H) is satisfied, they belong to $\mathcal{D}(A)$. The latter extends to f, g in $\mathcal{D}(A)$ thanks to ESA.

It is important to see that without (H) this result is wrong in general. To justify (1.12) it is at least necessary to know that $\Gamma(f, g) \in \mathcal{D}(A)$ which is not always the case even for C_b^∞ functions if they are not all in $\mathcal{D}(A)$, as in the case of reflected diffusions for instance. Fortunately if (H) is satisfied it suffices to verify it for C_b^∞ functions.

Assume from now on that (1.12) is satisfied for f and g in the domain of A. It immediately follows that, if the curvature-dimension condition $CD(\rho, N)$ i.e.

$$\Gamma_2(f) \geq \rho \, |\nabla f|^2 + \frac{1}{N} \, (Af)^2$$

is satisfied, then

$$C_P(\mu) \leq \frac{N-1}{\rho \, N}$$

the result being true for $N \in \,]1, +\infty]$. This is the famous Bakry-Emery criterion for the Poincaré inequality. For $N = +\infty$ the criterion is satisfied provided V is strictly convex in which case it is also a consequence of Brascamp-Lieb inequality.

The second statement in Theorem 1.3 is thus sometimes called "the integrated Γ_2 criterion". This statement appears in Proposition 1.3 of M. Ledoux's paper [28] as "*a simple instance of the Witten Laplacian approach of Sjöstrand and Helffer*", but part of the argument goes back to Hörmander (see e.g. [1] p.14). It is worth noticing that, if the semi-group does not appear in the statement, it is an essential tool of Ledoux's proof.

The integrated Γ_2 criterion is used in M. Ledoux's work [28] on Gibbs measures. Under the denomination of "Bochner's method" it appeared more or less at the same time in the statistical mechanics world. More recently it was used in the context of convex geometry in [27, 6] under the denomination of $\mathbb{L}^2$ method. Lemma 1 in [6] contains another proof (without using the semi-group) of (2) $\Rightarrow$ (1) in the previous Theorem.

The third statement in Theorem 1.3 can be improved in the following way

Proposition 1.4. *The third statement (hence the first two too) of Theorem 1.3 is equivalent to the following one: there exists $C > 0$, such that for every f in a dense subset $\mathcal{C}$ of $\mathbb{L}^2(\mu)$ one can find a constant $c(f)$ such that*

$$Var_\mu(P_t f) \leq c(f) e^{-2t/C}$$

and the optimal C is again $C_P(\mu)$.

The proof of this proposition lies on the log-convexity of $t \mapsto \mu(P_t^2 f)$ for which several proofs are available (see the simplest one in [20] lemma 2.11 or in [4]).

A natural subset $\mathcal{C}$ is furnished by $\mathbb{L}^\infty(\mu)$. An exponential decay to 0 of the variance controlled by the initial uniform norm thus implies that the same holds for the $\mathbb{L}^2$ norm and is equivalent to the Poincaré inequality.

The semi-group property shows that $\mathbb{L}^2$ decay to 0 cannot be faster than exponential and the previous result that any uniform decay i.e. $Var_\mu(P_T f) \leq c \, Var_\mu(f)$ for some $T > 0$, $c < 1$ and $f \in \mathbb{L}^2(\mu)$ implies exponential decay. A natural question is then to describe what happens for slower decays. After a pioneering work by T. Liggett ([32]), this question was tackled by M. Röckner and F. Y. Wang in [38]. These authors introduced the notion of weak Poincaré inequalities and relate them to all possible decays of the variance along the semi-group. Let us recall the main result in this direction

Theorem 1.5. *Consider the following two statements*

(1)   *There exists a non-increasing function $\beta_{WP} : (0, +\infty) \to \mathbb{R}^+$, such that for all $s > 0$ and any bounded and Lipschitz function f,*

$$Var_\mu(f) \leq \beta_{WP}(s) \, \mu(|\nabla f|^2) + s \, Osc^2(f), \tag{1.13}$$

   where $Osc(f) = \sup f - \inf f$ denotes the Oscillation of f. (1.13) is called a weak Poincaré inequality (WPI) and it is clear that we may always choose $\beta_{WP}(s) = 1$ for $s \geq 1$.

(2)   *There exists a non-increasing function ξ going to 0 at infinity such that*

$$Var_\mu(P_t f) \leq \xi(t) \, Osc^2(f).$$

The weak Poincaré inequality (1) implies statement (2) with

$$\xi(t) = 2 \inf\{s > 0, \beta_{WP}(s) \ln(1/s) \leq 2t\} = \inf_{s > 0} \left(s + e^{-2t/\beta_{WP}(s)} \right).$$

Conversely statement (2) implies statement (1) with

$$\beta_{WP}(s) = 2s \inf_{r > 0} \left(\frac{1}{r} \xi^{-1}(r \exp(1 - \frac{r}{s})) \right)$$

where ξ^{-1} denotes the converse of ξ, i.e. $\xi^{-1}(r) = \inf\{s > 0, \xi(s) \leq r\}$.

Of course if $\beta_{WP}(0) < +\infty$ one recovers the usual Poincaré inequality.

Remark 1.6. Röckner and Wang (see [38] Corollary 2.4 (2)) introduce a trick that allows to improve ξ in the previous result. The basic idea is to use repeatedly (1.13). We will choose four sequences:

(1)   a decreasing sequence of positive numbers $(\theta_i)_{i \in \mathbb{N}}$ such that $\theta_0 = 1$ and $\theta_i \to 0$ as $i \to +\infty$,

(2)   for $i \geq 1$, $\alpha_i = \theta_{i-1} - \theta_i$ so that $\sum_i \alpha_i = 1$,
(3)   a sequence $(\gamma_i)_{i \geq 0}$ of positive numbers such that $\gamma_0 = 1$ and $\prod_i \gamma_i = 0$,
(4)   for $i \geq 1$, $s_i(t)$ is defined by $e^{-2t\alpha_i/\beta_{WP}(s_i(t))} = \gamma_i$, hence $s_i(t) = \beta_{WP}^{-1}(2t\alpha_i/\ln(1/\gamma_i))$.

Applying (1.13) between $t\theta_i$ and $t\theta_{i-1}$ we thus have

$$Var_\mu(P_{\theta_{i-1}t}f) \leq e^{-2\alpha_i t/\beta_{WP}(s_i(t))} \operatorname{Var}_\mu(P_{\theta_i t}f) + s_i(t)\operatorname{Osc}^2(f) = \gamma_i \operatorname{Var}_\mu(P_{\theta_i t}f) + s_i(t)\operatorname{Osc}^2(f),$$

which yields

$$\operatorname{Var}_\mu(P_t f) \leq \sum_{i \geq 0}(\gamma_i \, s_{i+1}(t))\operatorname{Osc}^2(f). \tag{1.14}$$

So that we may choose $\xi(t) = \sum_{i \geq 0}(\gamma_i \, s_{i+1}(t))$.   ◇

Remark 1.7. In order to prove that statement (2) implies statement (1) we may follow another route. Using

$$\operatorname{Var}_\mu(f) - \operatorname{Var}_\mu(P_t f) = 2 \int_0^t \mu(|\nabla P_u f|^2)\,du \tag{1.15}$$

and the fact that $t \mapsto \mu(|\nabla P_t f|^2)$ is non-increasing (we shall recall a proof in the next section), we have

$$\operatorname{Var}_\mu(f) \leq 2t\,\mu(|\nabla f|^2) + \operatorname{Var}_\mu(P_t f) \leq 2t\,\mu(|\nabla f|^2) + \xi(t)\operatorname{Osc}^2(f)$$

from which we deduce that

$$\beta_{WP}(s) \leq 2\xi^{-1}(s).$$

This expression is simpler than the one in [38] we recalled in Theorem 1.5, but can be slightly worse.   ◇

Example 1.8. Let us give some examples of (non optimal) pairs for (β_{WP}, ξ)

(1)   If for $p > 0$, $\xi(t) = c'\,t^{-p}$ one can take $\beta_{WP}(s) = c\,s^{-1/p}$. Conversely if $\beta_{WP}(s) = c\,s^{-1/p}$ the previous Theorem yields $\xi(t) = c'\,t^{-p}\ln^p(t)$.
      Using the trick in remark 1.6, when $\beta_{WP}(s) = cs^{-1/p}$, and choosing $\gamma_i = 2^{-i}$ and $\alpha_i = \frac{6}{\pi^2}i^{-2}$ we get that $\xi(t) \sim t^{-p}$, for large t's, i.e. the logarithmic term disappeared as expected.
      Examples of measures satisfying such a weak Poincaré inequality are for $V(x) = (n+q)\ln(1+|x|)$ for some $q > 0$, i.e. measures with polynomial tails. For the explicit link between p and q see [38] example 1.4.a).
(2)   For $p > 0$, $\xi(t) = c'\ln^{-p}(1+t)$ and $\beta_{WP}(s) = c\,e^{\delta/s^{1/p}}$.
      This time it corresponds to $\mu(dx) = C\frac{1}{|x|^n\,\ln^q(1+|x|)}\,dx$ for some $q > 1$, see [38] example 1.4.b).
(3)   For $0 < p \leq 1$, $\xi(t) = c\,e^{-c't^p}$ and $\beta_{WP}(s) = d' + d\ln^{(1-p)/p}(1+1/s)$.
      This case covers the Subbotin distributions $\mu(dx) = Z^{-1}\,e^{-|x|^\delta}\,dx$ for $\delta \leq 1$ with $p = \delta/(4-3\delta)$, see [38] example 1.4.c). In particular for $p = 1$ one recovers the radial exponential distribution which satisfies an usual Poincaré inequality.

All the constants depend on p.   ◇

A natural question is thus to understand whether there is an integrated Γ_2 version of these weak inequalities or not. This will be done in the next section where we introduce a first weak version: for some decreasing β for any bounded $g \in D(A)$ and any $s > 0$,

$$\textbf{(WI}\Gamma_2\textbf{Osc)} \quad \mu(|\nabla g|^2) \leq \beta(s)\,\mu((Ag)^2) + s\,\mathrm{Osc}^2(g)\,. \tag{1.16}$$

We shall see that (WIΓ_2Osc) can be compared with the weak Poincaré inequality.

In section 3 we introduce another, perhaps more natural, weak version

$$\textbf{(WI}\Gamma_2 \textbf{ grad)} \quad \mu(|\nabla g|^2) \leq \beta(s)\,\mu((Ag)^2) + s\,|||\nabla g|^2||_\infty\,, \tag{1.17}$$

which is useful in the log-concave situation, i.e. provided V is convex (not necessarily strictly convex). It is known since S. Bobkov's work [9], that a log-concave probability measure always satisfies some Poincaré inequality (see [2] for a direct proof using Lyapunov functions). Recent results by E. Milman ([35]) combined with Brascamp-Lieb inequality allow us to get the following result : if μ is log-concave,

$$C_P(\mu) \leq C_{univ}\,\mu(|||Hess^{-1}V|||)$$

for some universal C_{univ}. Here for a real and non-negative symmetric matrix, we denote by

$$|||M||| := \sup_{|u|=1} \langle u, Mu \rangle = \lambda_{max}(M)$$

the operator norm of M, $\lambda_{max}(M)$ being the largest eigenvalue of M.

We recover this result in corollary 3.2 as a consequence of (WIΓ_2 grad) (and not Brascamp-Lieb) and obtain new explicit bounds in corollary 3.5 involving

$$\mu(\ln^{1+\varepsilon}(1 + |||Hess^{-1}V|||)$$

only.

The next section 4 deals with log-concave perturbations of either log-concave product measures or log-concave radially symmetric measures. Actually M. Ledoux introduced the integrated Γ_2 criterion in order to study the Poincaré inequality of perturbations (non necessarily log-concave but wit a potential whose curvature is bounded from below) of product measures and to obtain results for Gibbs measures on continuous spin systems ([28]). In the same paper he extended his approach to the log-Sobolev constant (see [28] Proposition 1.5 and the comments immediately after its statement). This approach was then developed in [36] and several works.

In their subsection 3.4, Barthe and Klartag [7] indicate that this method should be used in order to get some results on log-concave perturbations of product measures that are uniformly log-concave in the large, but not for heavy tailed product measures. In section 4 we show that the weak integrated Γ_2 criterion allows us to (partly) recover similar but slightly worse results as in [7]. Other results in this direction are shown in [17]. We then extend the method and replace product measures by radial distributions.

In all the paper, unless explicitly stated, we assume for simplicity that assumption (H) is in force.

Dedication. *A tribute to Michel Ledoux.*
The origin of this work was an attempt to convince M. Ledoux of the interest of weak inequalities of Poincaré type. After reading the beautiful wink to Michel's heroes [31], we

understood that the only way to succeed was to introduce some "curvature condition" inside. It was thus natural to weaken the integrated Γ_2 criterion introduced in [28] and to see what happens. The byproduct results in the paper were a nice surprise.

2. WEAK INTEGRATED Γ_2.

Let us start with an obvious remark: since ∇f and Af are unchanged when replacing f by $f - a$ for any constant a, we have

$$\mu(|\nabla f|^2) = \mu(|\nabla (f - a)|^2) = -\mu((f - a) A(f - a)) = -\mu((f - a) Af) \leq \frac{1}{2} \operatorname{Osc}(f) \mu(|Af|) ,$$

by choosing $a = (\sup(f) + \inf(f))/2$. Using for $s > 0$, $2uv \leq \frac{1}{s} u^2 + s v^2$ we thus deduce, using Cauchy-Schwarz inequality, that for all $s > 0$,

$$\mu(|\nabla f|^2) \leq \frac{1}{16s} \mu((Af)^2) + s \operatorname{Osc}^2(f) . \tag{2.1}$$

This is a special instance of (1.16) we recall here: for some decreasing β for any bounded $g \in D(A)$ and any $s > 0$,

$$(\mathbf{WI\Gamma_2Osc}) \quad \mu(|\nabla g|^2) \leq \beta(s) \mu((Ag)^2) + s \operatorname{Osc}^2(g) . \tag{2.2}$$

Hence some (very) weak form of the integrated Γ_2 is always satisfied. The previous inequality is thus certainly insufficient in order to get interesting consequences.

Remark 2.1. Contrary to (WPI), (2.2) is not always satisfied for $s \geq 1$, so that, apriori, β does not necessarily goes to 0 at infinity. However if (2.2) is satisfied with two functions β_1 and β_2, it is also satisfied with $\beta = \min(\beta_1, \beta_2)$. According to (2.1), it is thus always satisfied for $s \mapsto \min(\beta(s), 1/16s)$, so that we may always assume without loss of generality that β goes to 0 at infinity. Again in all what follows we denote $\beta^{-1}(t) = \inf\{s > 0 , \beta(s) \leq t\}$. $\diamond$

To see how to reinforce (2.1) it is enough to look at the proof of (2) implies (1) in Theorem 1.3. We follow the proof in [28].
The starting point is again (1.15),

$$\operatorname{Var}_\mu(f) - \operatorname{Var}_\mu(P_t f) = 2 \int_0^t \mu(|\nabla P_u f|^2) \, du$$

yielding the equality (1.7) in [28],

$$\operatorname{Var}_\mu(f) = 2 \int_0^{+\infty} \mu(|\nabla P_t f|^2) \, d\mu$$

as soon as

$$\operatorname{Var}_\mu(P_t f) \to 0 \quad \text{as } t \to +\infty .$$

Since μ is symmetric, the latter is satisfied as soon as the semi-group is ergodic, i.e. the eigenspace of A associated to the eigenvalue 0 is reduced to the constants. Actually this property is ensured by our assumptions: as shown for instance in [38] Theorem 3.1 and the remark following this theorem, if $\mu(dx) = e^{-V} dx$ is a probability measure with V of C^1 class, hence locally bounded, μ satisfies some weak Poincaré inequality so that the above convergence holds true.

Now defining $F(t) = \mu(|\nabla P_t f|^2)$, one can check (using (1.3) and (1.6)) that

$$F'(t) = -2\mu((AP_t f)^2).$$ (2.3)

Notice that this equality shows that F is non increasing.

Using this property in (1.15) we get

$$\mathrm{Var}_\mu(f) \geq \mathrm{Var}_\mu(P_t f) + 2t\,\mu(|\nabla P_t f|^2) \geq 2t\,\mu(|\nabla P_t f|^2)$$

so that

$$\mu(|\nabla P_t f|^2) \leq \frac{1}{2t}\,\mathrm{Var}_\mu(f) \leq \frac{1}{2t}\,\mathrm{Osc}^2(f).$$ (2.4)

Assuming that a weak integrated Γ_2 inequality (2.2) is satisfied we get, using that $\mathrm{Osc}(P_t f) \leq \mathrm{Osc}(f)$,

$$F'(t) \leq -\frac{2}{\beta(s)}\,F(t) + \frac{2s}{\beta(s)}\,\mathrm{Osc}^2(f).$$

This immediately yields

$$\mu(|\nabla P_t f|^2) = F(t) \leq e^{-2t/\beta(s)}\,\mu(|\nabla f|^2) + s\left(1 - e^{-2t/\beta(s)}\right)\mathrm{Osc}^2(f).$$ (2.5)

We may apply the previous inequality replacing f by $P_a f$, next t by $t - a$ and use again $\mathrm{Osc}(P_a f) \leq \mathrm{Osc}(f)$. Using (2.4) we thus have for $t > a > 0$,

$$\mu(|\nabla P_t f|^2) \leq \inf_{s>0}\left(s + \frac{1}{2a}\,e^{-2(t-a)/\beta(s)}\right)\mathrm{Osc}^2(f) = \eta(t)\,\mathrm{Osc}^2(f).$$ (2.6)

We have thus obtained

Proposition 2.2. *Assume that μ satisfies a weak integrated Γ_2 inequality (WI Γ_2 Osc) (2.2). Define for $t > a > 0$,*

$$\eta(t) = \inf_{s>0}\left(s + \frac{1}{2a}\,e^{-2(t-a)/\beta(s)}\right) = 2\inf\{s > 0\,;\,\beta(s)\ln(1/as) \leq 2(t-a)\}.$$

If η is integrable at infinity, then for $t > a$,

$$Var_\mu(P_t f) \leq 2\left(\int_t^{+\infty} \eta(u)du\right)Osc^2(f).$$

In particular μ satisfies a (WPI) where β_{WP} is given in Theorem 1.5 with

$$\xi(t) = 2\left(\int_t^{+\infty} \eta(u)du\right).$$

Remark 2.3. Notice that if μ satisfies a Poincaré inequality we recover the correct exponential decay thanks to proposition 1.4.

If we come back to (2.1) we may always use $\beta(s) \sim c/s$. Using proposition 2.2 with $a = t/2$ the best possible $\eta(t)$ is of order $1/t$ (for large t's) and thus is not integrable, in accordance with the fact that (2.1) cannot furnish the rate of decay to 0 since it is satisfied for all measures μ.

Notice that, as for the (WPI), if $\beta(s) = cs^{-1/(p+1)}$ we obtain $\eta(t) = c' (\ln(t)/t)^{p+1}$ and finally $\xi(t) \sim c' (\ln(t)/t)^p$. But here again we may apply the trick of remark 1.6, simply replacing (1.13) by (2.5), yielding

$$\mu(|\nabla P_t f|^2) \leq \sum_{i \geq 0} (\gamma_i \, s_{i+1}(t)) \operatorname{Osc}^2(f) , \tag{2.7}$$

with

$$s_i(t) = \beta^{-1}(2t\alpha_i/\ln(1/\gamma_i)) .$$

As for (WPI) this remark allows us to skip the logarithmic term. $\diamond$

Remark 2.4. Taking $a = \mu(f)$ we may replace (2.1) by

$$\mu(|\nabla f|^2) \leq \frac{1}{4s} \, \mu((Af)^2) + s \operatorname{Var}_\mu(f) ,$$

so that we could also consider weak inequalities of the form

$$\mu(|\nabla f|^2) \leq \beta(s) \, \mu((Af)^2) + s \operatorname{Var}_\mu(f) . \tag{2.8}$$

It is immediately seen that the previous derivation is unchanged if we replace $\operatorname{Osc}^2(f)$ by $\operatorname{Var}_\mu(f)$ so that if η is integrable we get

$$\operatorname{Var}_\mu(P_t f) \leq 2 \left(\int_t^{+\infty} \eta(u)du \right) \operatorname{Var}_\mu(f) .$$

But according to what we already said, such a decay implies that μ satisfies a Poincaré inequality, hence thanks to Theorem 1.3 that β is constant equal to $C_P(\mu)$ (or if one prefers that $\beta(0) < +\infty$). Thus, in the other cases, (2.8) furnishes a non-integrable η. $\diamond$

Let us look at the converse statement. According to Theorem 1.5 we may associate some (WPI) inequality to any decay controlled by the Oscillation. Thus for $a = \mu(f)$,

$$\begin{aligned} \mu^2(|\nabla f|^2) &= -\mu^2((f-a)Af) \leq \mu((Af)^2) \operatorname{Var}_\mu(f) \\ &\leq \mu((Af)^2) \left(\beta_{WP}(s) \, \mu(|\nabla f|^2) + s \operatorname{Osc}^2(f) \right) . \end{aligned} \tag{2.9}$$

Since $u^2 \leq Cu + B$ implies that

$$u \leq \frac{1}{2} \left(C + (C^2 + 4B)^{\frac{1}{2}} \right) \leq C + B^{\frac{1}{2}} ,$$

we obtain

$$\begin{aligned} \mu(|\nabla f|^2) &\leq \beta_{WP}(s) \, \mu((Af)^2)) + s^{\frac{1}{2}} \, \mu^{\frac{1}{2}}((Af)^2) \operatorname{Osc}(f) , \\ &\leq (\beta_{WP}(s) + \frac{1}{2}) \, \mu((Af)^2)) + \frac{1}{2} \, s \operatorname{Osc}^2(f) . \end{aligned}$$

We have thus obtained (since we know that μ always satisfies some (WPI) inequality) and according to remark 2.1

Proposition 2.5. *μ always satisfies a weak integrated Γ_2 inequality (WI Γ_2 Osc) (2.2), with*

$$\beta(s) = \min \left(1/2 + \beta_{WP}(2s) , \, \frac{1}{16s} \right) .$$

The previous results need some comments.

In first place, if we cannot assume that $\beta(s) = 1$ for $s \geq 1$ in the weak integrated Γ_2 inequality (2.2), the interesting behaviour of this function is nevertheless as $s \to 0$ for proposition 2.2 to have some interest.

In second place proposition 2.5 is certainly non sharp. In particular we do not recover the same β when β_{WP} is constant, i.e. when μ satisfies a Poincaré inequality, while using (2.9) with $s = 0$ yields the correct value.

A still worse remark is that the previous proposition cannot be always used in conjunction with proposition 2.2. Indeed if $\beta_{WP}(s) \geq c/s$ as it is the case in the second case of example 1.8 the η obtained in proposition 2.2 is not integrable.

Let us look at some other example.

Example 2.6. Assume that for some $p > 0$, $\beta_{WP}(s) = cs^{-1/p}$. In this case one can improve upon the result of proposition 2.5. Indeed we may replace the weak Poincaré inequality by its equivalent Nash type inequality

$$\operatorname{Var}_\mu(f) \leq c \, (p + (1/p)^p)^{\frac{1}{p+1}} \, \mu^{\frac{p}{p+1}}(|\nabla f|^2) \operatorname{Osc}^{\frac{2}{p+1}}(f) \,.$$

We thus deduce

$$\mu^2(|\nabla f|^2) \leq \mu((Af)^2) \operatorname{Var}_\mu(f) \leq c(p) \, \mu((Af)^2) \, \mu^{\frac{p}{p+1}}(|\nabla f|^2) \operatorname{Osc}^{\frac{2}{p+1}}(f)$$

for some $c(p)$ that may change from line to line, so that

$$\mu^2(|\nabla f|^2) \leq c(p) \, \mu^{\frac{p+1}{p+2}}((Af)^2) \operatorname{Osc}^{\frac{2}{p+2}}(f)$$

and finally that μ satisfies a weak integrated Γ_2 inequality with

$$\beta(s) = c(p) \, s^{-1/(p+1)} \,.$$

This result is of course better than the $s^{-1/p}$ obtained by directly using proposition 2.5 and according to remark 1.6 allows to recover the correct decay for $\xi(t)$. $\diamond$

3. The log-concave case.

If one wants to mimic (WPI) it seems more natural to consider another type of weak integrated Γ_2 inequalities, namely

$$\textbf{(WI } \Gamma_2 \textbf{ grad)} \quad \mu(|\nabla g|^2) \leq \beta(s) \, \mu((Ag)^2) + s \, |||\nabla g|^2||_\infty \,. \tag{3.1}$$

But contrary to the previous derivation it is no more true that $|||\nabla P_t f|^2||_\infty \leq |||\nabla f|^2||_\infty$ so that the analogue of (2.4) will involve $\sup_{u \leq t} |||\nabla P_u f|^2||_\infty$ which is not really tractable.

If we want to guarantee $|||\nabla P_t f|^2||_\infty \leq |||\nabla f|^2||_\infty$ a sufficient condition is that μ is log-concave, i.e. V is convex. Indeed in this case on can show (see a stochastic immediate proof in [15]) that

$$|\nabla P_t f|^2 \leq P_t^2(|\nabla f|) \leq P_t(|\nabla f|^2) \leq |||\nabla f|^2||_\infty \,. \tag{3.2}$$

In this case we will thus obtain the analogue of (2.5)

$$\mu(|\nabla P_t f|^2) \leq e^{-2t/\beta(s)} \, \mu(|\nabla f|^2) + s \, |||\nabla f|^2||_\infty \,. \tag{3.3}$$

The difference with the previous section is that (3.1) is satisfied by $\beta(s) = 0$ for $s \geq 1$. The converse function β^{-1} is thus bounded by 1, hence integrable at the origin.

Now we can use the trick described in Remark 1.6 which yields,

$$\mu(|\nabla P_t f|^2) \leq \left(\sum_{i=0}^{+\infty} \gamma_i \, \beta^{-1}(2\alpha_{i+1} \, t / \ln(1/\gamma_{i+1})) \right) \, |||\nabla f|^2||_\infty = \eta(t) \, |||\nabla f|^2||_\infty. \qquad (3.4)$$

Since β^{-1} is integrable at 0, we have thus obtained after a simple change of variable, provided η is integrable at infinity

$$\begin{aligned} \mathrm{Var}_\mu(f) &\leq 2 \left(\int_0^{+\infty} \eta(u)du \right) |||\nabla f|^2||_\infty \\ &\leq \left(\sum_{i=0}^{+\infty} \frac{\gamma_i \, \ln(1/\gamma_{i+1})}{\alpha_{i+1}} \right) \left(\int_0^{+\infty} \beta^{-1}(t)dt \right) |||\nabla f|^2||_\infty. \end{aligned} \qquad (3.5)$$

As before we may choose $\gamma_i = 2^{-i}$ and $\alpha_i = \frac{6}{\pi^2} i^{-2}$ so that

$$\sum_{i=0}^{+\infty} \frac{\gamma_i \, \ln(1/\gamma_{i+1})}{\alpha_{i+1}} = \kappa$$

where κ is thus a universal constant. Hence if β^{-1} is integrable with integral equal to M_β we have obtained

$$\mathrm{Var}_\mu(f) \leq \kappa \, M_\beta \, |||\nabla f|^2||_\infty. \qquad (3.6)$$

As first shown by E. Milman in [35], for log-concave measures (3.6) implies a Poincaré inequality. A semi-group proof of E. Milman's result was then given by M. Ledoux in [30]. Another semi-group proof and various improvements were recently shown in [16]. We shall follow the latter to give a precise result.

Starting with

$$\mu(|f - \mu(f)|) \leq \mathrm{Var}_\mu^{1/2}(f) \leq \kappa^{1/2} M_\beta^{\frac{1}{2}} \, |||\nabla f|||_\infty,$$

we deduce from [16] Theorem 2.7 that the $\mathbb{L}^1$ Poincaré constant $C_C'(\mu)$ is less than $16 \sqrt{\kappa \, M_\beta}/\pi$. Using Cheeger's inequality $C_P(\mu) \leq 4 \, (C_C'(\mu))^2$ we have thus obtained

Proposition 3.1. *Assume that μ is log-concave and satisfies a weak integrated Γ_2 inequality (WI Γ_2 grad) (3.1). Then*

$$C_P(\mu) \leq \frac{1024}{\pi^2} \kappa \, M_\beta,$$

where κ is some (explicit) universal constant and $M_\beta = \int_0^{+\infty} \beta^{-1}(t)dt$.

It turns out that there always exists a (non necessarily optimal) function β such that (3.1) is satisfied for a log-concave measure μ

Indeed recall (1.10) and (1.12). We have

$$\begin{aligned} \mu((Af)^2) &= \mu(||Hessf||_{HS}^2) + \mu(\langle \nabla f, HessV \, \nabla f \rangle) \\ &\geq \frac{1}{u} \mu \left(|\nabla f|^2 \, \mathbf{1}_{\lambda_{min}(HessV) \geq \frac{1}{u}} \right) \\ &\geq \frac{1}{u} \mu(|\nabla f|^2) - \frac{1}{u} \mu \left(\lambda_{min}(HessV) \leq \frac{1}{u} \right) |||\nabla f|^2||_\infty. \end{aligned}$$

It follows
$$\mu(|\nabla f|^2) \leq \beta(s)\,\mu((Af)^2) + s\,|||\nabla f|^2||_\infty$$
where, using $|||Hess^{-1}V||| = 1/\lambda_{min}(HessV)$ one has
$$\beta^{-1}(s) = \mu(|||Hess^{-1}V||| \geq s)\,. \tag{3.7}$$

Since
$$\mu(|||Hess^{-1}V|||) = \int_0^{+\infty} \mu(|||Hess^{-1}V||| \geq s)\,ds$$
we have obtained

Corollary 3.2. *If μ is log-concave and such that $\mu(|||Hess^{-1}V|||) < +\infty$, then*
$$C_P(\mu) \leq C_{univ}\,\mu(|||Hess^{-1}V|||)\,,$$
for some universal constant C_{univ}.

This result is not new and as remarked by E. Milman is an immediate consequence of the fact that (3.6) implies that μ satisfies some Poincaré inequality and of one of the favorite inequality of M. Ledoux, namely the Brascamp-Lieb inequality
$$\mathrm{Var}_\mu(f) \leq \mu(\langle \nabla f, Hess^{-1}V\,\nabla f\rangle) \leq \mu(|||Hess^{-1}V|||)\,|||\nabla f|^2||_\infty\,.$$
Actually this method furnishes a slightly better pre-constant than the one obtained with our method (since our $\kappa \geq 1$).

Still in the log-concave situation, if we assume (2.2) we may derive another control for the Poincaré constant.

Proposition 3.3. *Assume that μ is log-concave and satisfies a weak integrated Γ_2 inequality (WI Γ_2Osc) (2.2). If in addition there exists a function $s(t)$ such that*
$$\int_0^{+\infty} s(t)\,dt = \frac{s_0}{2} < \frac{1}{12} \quad and \quad \int_0^{+\infty} e^{-2t/\beta(s(t))}\,dt = \kappa/2 < +\infty\,,$$
then
$$C_P(\mu) \leq \frac{64\ln(2)\,\kappa}{(1 - 6s_0)^2}\,. \tag{3.8}$$

Proof. Starting with (2.5) in the simplified form
$$\mu(|\nabla P_t f|^2) = F(t) \leq e^{-2t/\beta(s(t))}\,\mu(|\nabla f|^2) + s(t)\,\mathrm{Osc}^2(f)\,,$$
we get
$$\mathrm{Var}_\mu(f) \leq \kappa\,\mu(|\nabla f|^2) + s_0\,\mathrm{Osc}^2 f$$
so that the conclusion follows from [16] Theorem 9.2.14. □

Still in the log-concave case it was shown by M. Ledoux in [29] that
$$|||\nabla P_t f|||_\infty \leq \frac{1}{\sqrt{2t}}\,||f||_\infty$$
so that replacing f by $f - a$ with $a = \frac{1}{2}(\inf f + \sup f)$ we have
$$|||\nabla P_t f|||_\infty \leq \frac{1}{2\sqrt{2t}}\mathrm{Osc}(f)\,.$$

This bound was improved in [15] replacing $\sqrt{2}$ by $\sqrt{\pi}$ and is one of the key element in the proof of Theorem 2.7 in [16].

We may combine this bound with the (WI $\Gamma_2$grad) inequality in order to improve upon the previous result. If a (WI $\Gamma_2$grad) inequality is satisfied we have

$$
\begin{aligned}
\mu(|\nabla P_{2t}f|^2) &\leq e^{-2t/\beta(s)}\mu(|\nabla P_t f|^2) + s\,|||\nabla P_t f|^2||_\infty \\
&\leq e^{-2t/\beta(s)}\mu(|\nabla f|^2) + \frac{s}{4\pi t}\,\mathrm{Osc}^2(f)\,.
\end{aligned}
$$

We have thus obtained

Proposition 3.4. *Assume that μ is log-concave and satisfies a weak integrated Γ_2 inequality (WI Γ_2 grad) (3.1). If in addition there exists a function $s(t)$ such that*

$$
\int_0^{+\infty} \frac{s(t)}{4\pi t}\,dt = \frac{s_0}{4} < \frac{1}{24} \quad \text{and} \quad \int_0^{+\infty} e^{-2t/\beta(s(t))}\,dt = \kappa/4 < +\infty\,,
$$

then

$$
C_P(\mu) \leq \frac{64\ln(2)\,\kappa}{(1-6s_0)^2}\,. \tag{3.9}
$$

In the previous proposition we can choose a generic function $s(t)$ given by

$$
s(t) = \frac{\theta}{16}\left(t\,\mathbf{1}_{t\leq 2} + \ln^{-(1+\theta)}(t)\,\mathbf{1}_{t>2}\right)\,, \tag{3.10}
$$

so that

$$
\int_0^{+\infty} \frac{s(t)}{4\pi t}\,dt = \frac{\theta}{32\pi} + \frac{1}{64\pi\,\ln^\theta(2)} \leq \frac{1}{48}
$$

as soon as $0 < \theta \leq 1$. So we may always choose

$$
\kappa = 4\left(2 + \int_2^{+\infty} e^{-2t/\beta((\theta/16)\,\ln^{-(1+\theta)}(t))}dt\right)\,,\quad s_0 = \frac{1}{12}\,,\quad C_P(\mu) \leq 256\ln(2)\,\kappa\,. \tag{3.11}
$$

As we previously saw, we may also use the previous proposition with

$$
\beta^{-1}(s) = \mu(|||Hess^{-1}V||| \geq s)\,.
$$

This yields

Corollary 3.5. *If μ is log-concave and such that $M_\varepsilon := \mu(\ln^{1+\varepsilon}(1 + |||Hess^{-1}V|||)) < +\infty$ for some $\varepsilon > 0$, then*

$$
C_P(\mu) \leq c + 4\max\left(2, \exp\left(\left[\frac{2^\varepsilon\,64\,M_\varepsilon}{\theta}\right]^{\frac{1}{\varepsilon-\theta}}\right)\right)\,,
$$

with $\theta = 1$ if $\varepsilon \geq 2$ and $\theta = \varepsilon/2$ if $\varepsilon \leq 2$, where c is some universal constant.

Proof. Denote by $M_\varepsilon = \mu(\ln^{1+\varepsilon}(1 + |||Hess^{-1}V|||))$. According to Markov inequality

$$
\beta^{-1}(s) \leq \frac{M_\varepsilon}{\ln^{1+\varepsilon}(1+s)}\,.
$$

It follows

$$
\beta(t) \leq \exp\left[\left(\frac{M_\varepsilon}{t}\right)^{\frac{1}{1+\varepsilon}}\right]
$$

so that for $t \geq 2$,

$$\beta(s(t)) \leq \exp\left[\left(\frac{8M_\varepsilon}{\theta}\right)^{\frac{1}{1+\varepsilon}} \ln^{\frac{1+\theta}{1+\varepsilon}}(1+t)\right].$$

In particular, using $t^2 \geq t + 1$ for $t \geq 2$,

$$\frac{1}{2}\ln(t) \geq \left(\frac{8M_\varepsilon}{\theta}\right)^{\frac{1}{1+\varepsilon}} \ln^{\frac{1+\theta}{1+\varepsilon}}(1+t)$$

as soon as

$$t \geq \max\left(2, \exp\left[\frac{2^\varepsilon\, 64\, M_\varepsilon}{\theta}\right]^{\frac{1}{\varepsilon-\theta}}\right).$$

For such t's we thus have

$$e^{-2t/\beta(s(t))} = e^{-2\exp(\ln(t)-\ln(\beta(s(t))))} \leq e^{-2\sqrt{t}}$$

so that finally

$$\frac{\kappa}{4} \leq \max\left(2, \exp\left[\frac{2^\varepsilon\, 64\, M_\varepsilon}{\theta}\right]^{\frac{1}{\varepsilon-\theta}}\right) + \int_2^{+\infty} e^{-2\sqrt{t}}\, dt\,.$$

Hence the result choosing $\theta = 1$ if $\varepsilon \geq 2$ and $\theta = \varepsilon/2$ otherwise. $\qquad\square$

Of course our bounds are far from being sharp. Notice that the previous corollary allows to look at Subbotin distributions $\mu(dx) = Z^{-1}e^{-|x|^p}\, dx$ for large p's, while Brascamp-Lieb inequality cannot be used. However other known methods (see e.g. S. Bobkov's results on radial measures in [10]) furnish better bounds in this case. Of course the previous corollary covers non radial cases.

Remark 3.6. If $M_\varepsilon := \mu(|||Hess^{-1}V|||^\varepsilon) < +\infty$ for some $\varepsilon > 0$ we can obtain another explicit bound choosing $\theta = 1$ in (3.10). Using again Markov inequality we have $\beta(s) \leq (M_\varepsilon/s)^{1/\varepsilon}$ so that

$$
\begin{aligned}
\int_2^{+\infty} e^{-2t/\beta(s(t))}\, dt &\leq \int_2^{+\infty} e^{-2t/\ln^{2/\varepsilon}(M_\varepsilon^{1/\varepsilon} t)}\, dt \\
&\leq \frac{1}{2} M_\varepsilon^{1/\varepsilon} \ln^{2/\varepsilon}(M_\varepsilon^{2/\varepsilon}) + \int_2^{+\infty} e^{-2t/\ln^{2\varepsilon}(t^2)}\, dt
\end{aligned}
$$

and finally

$$C_P(\mu) \leq c(\varepsilon) + \max\left(2, \frac{1}{2} M_\varepsilon^{1/\varepsilon} \ln^{2/\varepsilon}(M_\varepsilon^{2/\varepsilon})\right).$$

Notice that for $\varepsilon = 1$ we recover a slightly worse result than corollary 3.2 since an extra logarithm appears. Of course choosing $s(t)$ with a slower decay, we may improve upon this result but it seems that in all cases an extra worse term always appears. In addition constants are quite bad. But of course the result is new for $\varepsilon < 1$. $\qquad\diamondsuit$

4. SOME APPLICATIONS: PERTURBATION OF PRODUCT MEASURES AND RADIAL MEASURES.

We will first recall how the (usual) integrated Γ_2 criterion can be used in order to relate the Poincaré constant of μ to the ones of its one dimensional conditional distributions, in some special situations. We copy here Proposition (3.1) in [28] and its proof to see how to potentially extend it. In the sequel we denote

$$SG(\mu) = \frac{1}{C_P(\mu)}$$

the spectral gap of μ.

Proposition 4.1. *(M. Ledoux)*
Let $\mu(dx) = Z^{-1} e^{-W(x)-\sum_{i=1}^n h_i(x_i)} dx = Z^{-1} e^{-V(x)} dx$ be a probability measure on $\mathbb{R}^n$, W and the h_i's being C^2. Introduce the one dimensional conditional distributions

$$\eta_{i,x}(dt) = Z_{i,x}^{-1} e^{-W(x_1,\ldots,x_{i-1},t,x_i,\ldots x_n)-h_i(t)} dt.$$

Let

$$S = \inf_{i,x} SG(\eta_{i,x}).$$

Assume that $Hess W(x) \geq w$ and $\max_i \partial_{ii}^2 W(x) \leq \bar{w}$ for all $x \in \mathbb{R}^n$.
Then

$$SG(\mu) \geq S + w - \bar{w}.$$

Proof. It holds

$$
\begin{aligned}
\Gamma_2 f &= \sum_{i,j}(\partial_{ij}^2 f)^2 + \sum_i h_i''(x_i)(\partial_i f)^2 + \langle \nabla f, Hess W \, \nabla f \rangle \\
&\geq \sum_i (\partial_{ii}^2 f)^2 + \sum_i h_i''(x_i)(\partial_i f)^2 + w|\nabla f|^2 \qquad (4.1) \\
&\geq \sum_i (\partial_{ii}^2 f)^2 + \sum_i (h_i''(x_i) + \partial_{ii}^2 W)\,(\partial_i f)^2 + (w - \bar{w})\,|\nabla f|^2 \\
&\geq \sum_i \Gamma_{2,i} f + (w - \bar{w})\,|\nabla f|^2.
\end{aligned}
$$

It follows

$$
\begin{aligned}
\mu((Af)^2) = \mu(\Gamma_2 f) &\geq \sum_i \mu(SG(\eta_{i,x})\,|\partial_i f|^2) + (w - \bar{w})\,\mu(|\nabla f|^2) \qquad (4.2) \\
&\geq (S + w - \bar{w})\,\mu(|\nabla f|^2),
\end{aligned}
$$

hence the result applying Theorem 1.3.   □

Remark 4.2. Choosing $W = 0$ the previous result contains the renowned tensorization property of Poincaré inequality if $\mu = \otimes_i \mu_i$,

$$C_P(\otimes_i \mu_i) \leq \max_i C_P(\mu_i).$$

Similar results for weak Poincaré inequalities involve a "dimension dependence" (see e.g. [5]).
◇

Remark 4.3. For the proof of proposition 4.1 to be rigorous, it is enough to assume that ESA is satisfied for $C_0^\infty(\mathbb{R}^n)$ (which is implicit in M. Ledoux's work). Indeed in this case one only has to consider such test functions. The delicate point in the previous proof is that one has to check

$$\mu(\Gamma_{2,i}f) = \mu((A_i f)^2)$$

where $A_i f = \partial_{ii}^2 f - (h'(x_i) + \partial_i W)\partial_i f$ in order to use the integrated Γ_2 criterion. If f is compactly supported, this is immediate as we already discussed in the introduction. Hence for $D = \mathbb{R}^n$, (H) ensures that the result holds true.

The case of a bounded domain D will be discussed later.   ◇

In the previous proof, assume that $w = 0$ (W is convex), we thus obtain

$$\Gamma_2 f \geq \sum_i h_i''(x_i)\,(\partial_i f)^2$$

which is interesting only if the right and side is non-negative for any f, i.e. if $h_i'' \geq 0$. Hence as we did for obtaining (3.7) we have for $u > 0$, since we may integrate w.r.t. μ,

$$\mu(|\nabla f|^2) \;\leq\; u\,\mu((Af)^2) \;+\; \mu\left(\min_i(h_i''(x_i) \leq 1/u)\right)\,|||\nabla f|^2||_\infty \tag{4.3}$$

that furnishes a (WI Γ_2 grad) inequality. Of course

$$\mu\left(\min_i(h_i''(x_i) \leq 1/u)\right) \leq n\,\max_i \mu\left(h_i''(x_i) \leq \frac{1}{u}\right).$$

We have seen that such a weak inequality is interesting provided on one hand μ is log-concave and on the other hand $u \mapsto \max_i \mu\left(h_i''(x_i) \leq \frac{1}{u}\right)$ which is clearly non-increasing goes to 0 as $u \to +\infty$. We will thus assume that all h_i are convex, yielding thanks to Proposition 3.4 with the choice (3.10) with $\theta = 1$

Lemma 4.4. *Let $\mu(dx) = Z^{-1}\,e^{-W(x) - \sum_{i=1}^n h_i(x_i)}\,dx = Z^{-1}\,e^{-V(x)}dx$ be a probability measure on $\mathbb{R}^n$, W and the h_i's being convex and C^2. Define*

$$\alpha(v) = \max_i \mu(h_i''(x_i) \leq v)$$

and assume that (the non-decreasing) α goes to 0 as $v \to 0$. Then

$$C_P(\mu) \leq 256\,\ln(2)\,\kappa$$

with

$$\kappa = 4\left(2 + \int_2^{+\infty} e^{-2t/\alpha^{-1}(1/16\,n\,\ln^2(t))}\,dt\right).$$

Let us illustrate this situation in the particular case $h_i(u) = |u|^p$ for $p > 1$. We immediately see that the situation is completely different depending on whether $p < 2$ or $p > 2$. Denote by μ_i the probability distribution of x_i under μ. For $p < 2$ we have to control the tails of μ_i while for $p > 2$ we have to control the mass of small intervals centered at the origin.

Remark 4.5. For $p < 2$, $h_p : u \mapsto |u|^p$ is not C^2. But if $p > 1$, the only problem lies at the origin, and using that h_p'' is integrable at the origin it is not difficult to check (regularizing h_p at the origin for instance) that all what was done above is still true.   ◇

More generally we may consider h_i's who satisfy similar concentration bounds. Let us state a first result

Proposition 4.6. *Let $\mu(dx) = Z^{-1} e^{-W(x) - \sum_{i=1}^{n} h_i(x_i)} dx$ be a probability measure on $\mathbb{R}^n$. We assume that the h_i's are even convex functions. In addition we assume that for all $i = 1, ..., n$,*

$$h_i''(u) \geq \rho(|u|)$$

where ρ is a non-increasing positive function going to 0 at infinity. Then for all even convex function W it holds

$$C_P(\mu) \leq 4 \left(2 + \int_2^{+\infty} e^{-2t\,\rho(\sqrt{2 \max_i C_P(\eta^i)} \, \ln(n \, \ln^2(t)))} \, dt \right),$$

where $\eta^i(du) = Z_i^{-1} e^{-h_i(u)} du$.

Proof. According to Prekopa-Leindler theorem we know that the i-th marginal law μ_i of μ, i.e. the μ distribution of x_i, is a one dimensional distribution, that can be written

$$\mu_i(du) = Z_i^{-1} \, \rho_i(u) \, e^{-h_i(u)} \, du, \tag{4.4}$$

with an even and log-concave (thus non increasing on $\mathbb{R}^+$) function ρ_i. For such one dimensional distributions we may use a remarkable result due to O. Roustant, F. Barthe and B. Ioos (see [39]) recalled in proposition 6 of [7], namely

Lemma 4.7. *(Roustant-Barthe-Ioos)*

Let $\eta(du) = e^{-V(u)} \mathbf{1}_{(-b,b)}(u) \, du$ be a probability measure on $\mathbb{R}$, with V a continuous and even function. For any even function ρ which is non-increasing on $\mathbb{R}^+$ and such that $\nu(du) = \rho(u)\,\eta(du)$ is a probability measure, it holds

$$C_P(\nu) \leq C_P(\eta).$$

Applying the lemma we get

$$C_P(\mu_i) \leq C_P(Z^{-1} e^{-h_i(u)} du) := C_P(\eta^i). \tag{4.5}$$

We can thus use the concentration of measure property obtained via the Poincaré inequality, first shown by S. Bobkov and M. Ledoux ([8]). Here we use an explicit form we found in [4] (4.4.6). Since $u \mapsto u$ is 1-Lipschitz and centered (again thanks to symmetry), it yields

$$\mu(h''(x_i) \leq 1/u) \leq \mu(|x_i| \geq \rho^{-1}(1/u)) \leq 6 \exp\left(-\frac{\rho^{-1}(1/u)}{\sqrt{C_P(\eta^i)}} \right). \tag{4.6}$$

We thus have for $v > 0$

$$v\,\mu((Af)^2) + 6\,n \, \max_i \exp\left(-\frac{\rho^{-1}(1/v)}{\sqrt{C_P(\eta^i)}} \right) ||\nabla f|^2||_\infty \geq \mu(|\nabla f|^2), \tag{4.7}$$

yielding, for $s > 0$ small enough,

$$\beta(s) = \frac{1}{\rho\left(\sqrt{\max_i C_P(\eta^i)} \, \ln(6n/s) \right)}. \tag{4.8}$$

It remains to use the lemma 4.4. $\square$

Remark 4.8. When $h_i(u) = |u|^p$ for some $1 < p \leq 2$, one knows that $C_P(\eta^i) \leq \frac{4}{p^{2(1-1/p)}}$ according to [11] Theorem 2.1. It follows that for some (explicit) constant $c(p)$,

$$C_P(\mu) \leq \frac{c(p)}{p(p-1)} \left(1 + \ln^{2-p}(6n)\right).$$

The study of such μ's is not new. A much better result has been recently shown by F. Barthe and B. Klartag (see Theorem 1 in [7]),

Theorem 4.9. *(Barthe-Klartag) Let $\mu(dx) = Z^{-1} e^{-W(x) - \sum_{i=1}^n |x_i|^p} dx$ be a probability measure. We assume that $1 \leq p \leq 2$ and that W is an even convex function. Then*

$$C_P(\mu) \leq C \ln^{\frac{2-p}{p}} \left(\max(n, 2)\right),$$

where C is some universal constant.

The key point here is naturally that the result holds true for any even and convex W. The proof by Barthe and Klartag lies on a lot of properties of log-concave measures and uses in particular the extension of the gaussian correlation inequality shown by Royen, to mixtures of gaussian measures. We of course refer the reader to [7]. We do not only loose something on the power of the logarithm, but the constant becomes infinite as p goes to 1, which is natural since the Γ_2 requires some strict convexity except at some point. However our result does not require the full machinery of gaussian mixtures, and shows that the result only depends on the behaviour of the second derivative of the h's at infinity. $\diamond$

Remark 4.10. In the previous proof we implicitly have used the fact that (H) is satisfied. We know that it is the case when $W \in C^2(\mathbb{R}^n)$. If W is only continuous, we may replace W by $W_\varepsilon = W * \gamma_\varepsilon$ where γ_ε is a tiny centered gaussian density. W_ε is still even and convex, so that the Theorem applies. Since the bound does not depend on ε we may take limits in the corresponding Poincaré inequalities and get the same bound for W. $\diamond$

Remark 4.11. Let now consider the case $p > 2$. This time we have to control

$$\mu\left(|x_i| \leq (p(p-1)\,u)^{-1/(p-2)}\right),$$

for large u's.
Using (4.4) and since ρ_i is even and log-concave, we see that

$$\mu\left(|x_i| \leq u^{-1/(p-2)}\right) \leq Z_i^{-1}\,\rho_i(0)\,u^{-1/(p-2)}.$$

But

$$Z_i^{-1}\,\rho_i(0) = \frac{\int e^{-W(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n) - \sum_{j \neq i} |x_j|^p} \prod_{i \neq j} dx_j}{\int e^{-W(x) - \sum_j |x_j|^p} \prod_j dx_j}.$$

Denote by

$$\alpha = \max_i Z_i^{-1}\,\rho_i(0). \tag{4.9}$$

Then we get

$$\beta(s) = \frac{1}{p(p-1)} \left(\frac{\alpha n}{s}\right)^{p-2}$$

so that we have to estimate (choosing $\theta = 1$ in (3.10))

$$\int_2^{+\infty} \exp\left(-\frac{2p(p-1)}{(\alpha n)^{p-2}} \frac{t}{\ln^{2(p-2)}(t)}\right) dt.$$

Using that $t/\ln^k(t)$ is bounded below by $c(k,\varepsilon)t^{1-\varepsilon}$ for any $\varepsilon > 0$, we easily obtain

Proposition 4.12. *Let* $\mu(dx) = Z^{-1} e^{-W(x)-\sum_{i=1}^n |x_i|^p} dx$. *We assume that* $p > 2$ *and that* W *is convex and even so that* μ *is log-concave. Then for all* $\varepsilon > 0$, *there exists a constant* $c(p,\varepsilon)$ *such that*

$$C_P(\mu) \leq c(p,\varepsilon)(\alpha n)^{(p-2)(1+\varepsilon)}$$

where

$$\alpha = \max_i \frac{\int e^{-W(x_1,...,x_{i-1},0,x_{i+1},...,x_n)-\sum_{j\neq i}|x_j|^p} \prod_{i\neq j} dx_j}{\int e^{-W(x)-\sum_j |x_j|^p} \prod_j dx_j}.$$

For instance if we assume that $t \mapsto W(x_1,...t,...,x_n)$ is β Hölder continuous, uniformly in x and i, using $|W(x_1,...t,...,x_n) - W(x_1,...0,...,x_n)| \leq L|t|^\beta$, we get

$$\alpha \leq \frac{1}{\int e^{-L|t|^\beta-|t|^p} dt}.$$

The previous result may have some interest only if $2 < p < 3$. This is also quite natural: for large p's, $|x|^p$ becomes flat near the origin so that one cannot expect to use some convexity approach.

The best general control (thus including the case $p > 2$ for Subbotin distributions) is obtained in Theorem 18 of [7], and says that

$$C_P(\mu) \leq c\, n\, \max_i (C_P(\nu_i)).$$

In addition, in subsection 3.4 of [7], it is shown that the factor n is optimal by considering log concave perturbations of Subbotin distributions ν_i with exponent p for large p's for which $C_P(\mu)$ is at least of order $n^{(p-2)/p}$.

However if W is unconditional (i.e. $W(\sigma x) = W(x)$ for all $\sigma \in \{-1,1\}^n$), one can deeply reinforce the previous result and show that $C_P(\mu) \leq \max_i(C_P(\nu_i))$ as shown in [7] Theorem 17. $\diamond$

Remark 4.13. Denote by $Cov(\mu)$ the covariance matrix, i.e. $Cov_{i,j}(\mu) = \mu(x_i x_j) - \mu(x_i)\mu(x_j)$. It is immediate that $\lambda_{max}(Cov(\mu)) \leq C_P(\mu)$. Our proof thus gives an universal bound (that does not depend on W) for the Covariance matrix. The proofs by Barthe and Klartag use first estimates for this covariance matrix. $\diamond$

Looking at log-concave perturbations of log-concave product measures as above, can be partly motivated by statistical issues. We refer to [17] (in particular the final section) for some of them. Of course looking at product measures is interesting thanks to the tensorization property of Poincaré inequality, furnishing dimension free bounds. For log-concave measures, another case is well understood since S. Bobkov's work [10], namely radial measures. The following version is due to M. Bonnefont, A.Joulin and Y. Ma ([12] Theorem 1.2)

Theorem 4.14. *(Bobkov, Bonnefont-Joulin-Ma)*
Let μ be a spherically symmetric (radial) log-concave probability measure on $\mathbb{R}^n$, $n \geq 2$. Then

$$C_P(\mu) \leq \frac{\mu(|x|^2)}{n-1} \,.$$

We can obtain a result similar to proposition 4.6 or proposition 4.12

Theorem 4.15. *Let $\mu(dx) = Z_\mu^{-1} e^{-W(x) - h(|x|^2)} dx$ be a probability measure on $\mathbb{R}^n$. We assume that W is even and convex and that h is convex and non-decreasing on $\mathbb{R}^+$, so that μ is log-concave. W and h are also normalized so that $W(0) = h(0) = 0$ (and consequently W and h are non-negative). Introduce*

$$\nu_h(dx) = e^{-h(|x|^2)} dx \,.$$

There exists an universal constant c such that

$$C_P(\mu) \leq c \left(1 + \int_2^{+\infty} e^{-4t\, h'\left(\frac{1}{(c_n(\mu) \ln^2(t))^{2/n}} \right)} dt \right),$$

with

$$c_n(\mu) = Z_\mu^{-1} \frac{\pi^{n/2}}{n\Gamma(n/2)} \leq \inf_\theta \left\{ \frac{\pi^{n/2}}{n\Gamma(n/2)} \frac{e^{\max_{|x|=\theta} W(x)}}{\nu_h(|x| \leq \theta)} \right\} = \inf_\theta \left\{ \frac{e^{\max_{|x|=\theta} W(x)}}{n \int_0^\theta r^{n-1} e^{-h(r^2)} dr} \right\} \,.$$

Remark 4.16. Let $\mu_\lambda(dx) = Z_\mu^{-1} \lambda^{-n} e^{-W(x/\lambda) - h(|x|^2/\lambda^2)} dx$ a dilation of μ. Notice that $\lambda^2 c_n^{2/n}(\mu_\lambda) = c_n(\mu)$. Since one has a factor $1/\lambda^2$ in front of h', we partly recover the homogeneity of the Poincaré constant under dilations. $\diamondsuit$

Proof. Once again we may assume that W and h are smooth, convolving with a tiny gaussian kernel, that preserves convexity and parity. For simplicity we also assume that h' is (strictly) increasing, so that h' is one to one.

For two vectors x and y we write xy for the vector with coordinates $(xy)_i = x_i y_i$. It holds

$$\begin{aligned}
\Gamma_2 f &= \sum_{i,j} (\partial_{ij}^2 f)^2 + \langle \nabla f, Hess W \nabla f \rangle + 4\, h''(|x|^2)\, |x\nabla f|^2 + 2\, h'(|x|^2)|\nabla f|^2 \\
&\geq 2\, h'(|x|^2)|\nabla f|^2
\end{aligned}$$

so that

$$u\, \mu((Af)^2) + \mu\left(2\, h'(|x|^2) \leq \frac{1}{u} \right) \, |||\nabla f|^2||_\infty \geq \mu(|\nabla f|^2) \,. \tag{4.10}$$

So μ satisfies a (WI Γ_2 grad) inequality, with

$$\beta^{-1}(u) = \mu\left(2\, h'(|x|^2) \leq \frac{1}{u} \right) = \mu_r\left(h'(r^2) \leq \frac{1}{2u} \right) = \mu_r\left(r \leq \sqrt{(h')^{-1}(1/2u)} \right)$$

where μ_r denotes the probability distribution of the radial part of μ. We have

$$\mu_r(dv) = Z_\mu^{-1} n\, \omega_n\, v^{n-1}\, e^{-h(v)} \left(\int_{S^{n-1}} e^{-W(v\theta)} \sigma_n(d\theta) \right) dv$$

where σ_n denotes the uniform measure on the sphere S^{n-1} and $\omega_n = \frac{\pi^{n/2}}{n\Gamma(n/2)}$ denotes the volume of the unit (euclidean) ball. It follows, since W and h are non-negative,

$$\mu_r\left(r \le \sqrt{(h')^{-1}(1/2u)}\right) \le Z_\mu^{-1} \frac{\pi^{n/2}}{n\,\Gamma(n/2)} \left((h')^{-1}(1/2u)\right)^{n/2}$$

from which we deduce that we can choose

$$\beta(t) = \frac{1}{2\,h'((s/c_n)^{2/n})} \quad \text{with} \quad c_n = Z_\mu^{-1} \frac{\pi^{n/2}}{n\Gamma(n/2)}.$$

It remains to apply proposition 3.1.

The next step is thus to get some tractable bound for c_n, i.e a lower bound for Z_μ. The simplest way to do it is to use the fact that W is non-decreasing on each radial direction so that for all $\theta > 0$

$$Z_\mu \ge \int_{|x|\le\theta} e^{-W(x)-h(|x|^2)}\,dx \ge e^{-\max_{|x|=\theta} W(x)}\,\nu_h(|x| \le \theta).$$

$\square$

Corollary 4.17. *In particular if $h(u) = u^p$ with $p \ge 1$, we have*

$$C_P(\mu) \le 12288\,\ln(2)\,\frac{c_n^{\frac{2(p-1)}{n}}(\mu)}{4p}\,(4(p-1))^{\frac{4(p-1)}{n}}.$$

Proof. If $h(u) = u^p$ for $p > 1$, the corresponding dilation μ_λ is given by $h_\lambda(u) = \lambda^{-2p}u^p$. Recall that $c_n(\mu_\lambda) = \lambda^{-n}c_n(\mu)$.
We shall use that

$$\ln(t) \le \frac{1}{\alpha\,2^\alpha}\,t^\alpha + (\ln(2) - (1/\alpha)) \quad \text{for } t \ge 2 \text{ and } \alpha > 0.$$

If $t \ge 2$, we thus have $\ln(t) \le \frac{1}{\alpha\,2^\alpha}\,t^\alpha$ if $\alpha \le 1$ and $\ln(t) \le t^\alpha$ if $\alpha \ge 1$.
It follows

$$\ln^{\frac{4(p-1)}{n}}(t) \le c_\beta\,t^\beta$$

for $t \ge 2$ and $0 < \beta$, with

$$c_\beta = 2^{-\beta}\left(\frac{4(p-1)}{\beta n}\right)^{\frac{4(p-1)}{n}} \quad \text{if } \frac{\beta n}{4(p-1)} \le 1 \quad ; \quad c_\beta = 1 \text{ if } \frac{\beta n}{4(p-1)} \ge 1.$$

This yields

$$e^{-4t\,h'_\lambda\left(\frac{1}{(c_n(\mu_\lambda)\,\ln^2(t))^{2/n}}\right)} \le e^{-\kappa_\beta\,t^{1-\beta}}$$

for

$$\kappa_\beta = \frac{4p}{c_\beta\,\lambda^2\,c_n^{\frac{2(p-1)}{n}}(\mu)}.$$

A simple change of variables $u = \kappa_\beta\,t^{1-\beta}$, together with the positivity of all constants yields

$$\int_2^{+\infty} e^{-4t\,h'_\lambda\left(\frac{1}{(c_n(\mu_\lambda)\,\ln^2(t))^{2/n}}\right)}\,dt \le ((1-\beta)\kappa_\beta)^{-1}\int_0^{+\infty} u^{\frac{\beta}{1-\beta}}\,e^{-u^{1-\beta}}\,du.$$

Choosing for simplicity $\beta = 1/n$, so that $\beta n/4(p-1) \leq 1$, for $n \geq 2$ the final integral is bounded independently of n for instance by $c = \int_0^{+\infty} u\, e^{-\sqrt{u}}\, du = 12$. It follows

$$C_P(\mu_\lambda) \leq 1024 \ln(2) \left(2 + c\, \frac{\lambda^2\, c_n^{\frac{2(p-1)}{n}}(\mu)}{4p} \left(4(p-1)\right)^{\frac{4(p-1)}{n}} \right).$$

Using $C_P(\mu) = \lambda^{-2}\, C_P(\mu_\lambda)$ and letting λ go to infinity furnishes the result.
For $p = 1$ the result follows from strict convexity. □

Remark 4.18. If μ_r denotes the radial distribution of μ,

$$\mu_r(dv) = \rho(v)\, v^{n-1}\, e^{-h(v)}.$$

ρ is clearly an even function. Since for a fixed θ, $v \mapsto W(v\theta)$ is even and convex, it is non-decreasing, so that $v \mapsto \rho(v)$ is non-increasing. ρ is non necessarily log-concave, but we can again apply Proposition 6 in [7] furnishing, with $\bar{\nu}_h = Z^{-1}\, \nu_h$,

$$C_P(\mu_r) \leq C_P(\bar{\nu}_h). \tag{4.11}$$

The measure $\bar{\nu}_h$ being log-concave, we know that

$$C_P(\bar{\nu}_h) \leq 12\, \mathrm{Var}_{\bar{\nu}_h}(v).$$

What is important here is that the Poincaré constant of the radial measure μ_r can be bounded independently of W. ◇

Remark 4.19. Is the bound in Corollary 4.17 of the good order ? To see it look at the particular case $W = 0$. In this case $Z_\mu = \frac{n\,\omega_n}{2p}\, \Gamma(n/2p)$ so that $c_n(\mu) = \frac{2p}{n\,\Gamma(n/2p)}$, and our bound furnishes

$$C_P(\mu) \leq c\, \frac{2^{10(p-1)/n}\, p^{6(p-1)/n}}{4p\, n^{2(p-1)/n}}\, \frac{1}{\Gamma^{2(p-1)/n}(n/2p)},$$

for some universal c. In this case the following very precise bounds were obtained by Bonnefont, Joulin and Ma in [12],

$$\frac{\mu(|x|^2)}{n} \leq C_P(\mu) \leq \frac{\mu(|x|^2)}{n-1}.$$

Since

$$\mu(|x|^2) = \frac{\Gamma((n+2)/2p)}{\Gamma(n/2p)}$$

for $n/p \gg 1$ (even for large p's) we may use

$$\Gamma(z) \sim_{z \to +\infty} \sqrt{2\pi} z^{z-(1/2)}\, e^{-z}$$

so that the Bonnefont, Joulin, Ma theorem furnishes

$$C_P(\mu) \sim (2ep)^{-1/p}\, n^{1-(1/p)}. \tag{4.12}$$

For the same asymptotics our bound furnishes (for some new constant c)

$$C_P(\mu) \leq c\, p^{3(p-1)/n}\, n^{1-(1/p)}. \tag{4.13}$$

Hence provided $p \ln(p) \leq Cn$, we get the good order (but of course not the good constant). This shows that our bound is not so bad. ◇

5. THE CASE OF COMPACTLY SUPPORTED MEASURES.

Let us come back to the proof of Proposition 4.1 starting with

$$\Gamma_2 f = \sum_{i,j} (\partial_{ij}^2 f)^2 + \sum_i h_i''(x_i)(\partial_i f)^2 + \langle \nabla f, Hess W \nabla f \rangle. \qquad (5.1)$$

If W is convex, we thus have

$$\begin{aligned} \mu(\Gamma_2 f) &\geq \sum_i \mu((\partial_{ii}^2 f)^2 + \sum_i h_i''(x_i)(\partial_i f)^2) \\ &= \sum_i \mu(\eta_{i,x}((\partial_{ii}^2 f)^2 + h_i''(x_i)(\partial_i f)^2)) \end{aligned} \qquad (5.2)$$

Instead of adding and substracting $\partial_{ii}^2 W (\partial_i f)^2$, consider $\eta_{i,x}$ as a perturbation of

$$\theta_i(dt) = z_i^{-1} e^{-h_i(t)} dt$$

using the notation

$$\eta_{i,x}(dt) = Z_{i,x}^{-1} e^{-W_{i,x}(t)} \theta_i(dt).$$

Since we integrate a non-negative quantity it holds

$$\begin{aligned} \eta_{i,x}((\partial_{ii}^2 f)^2 + h_i''(x_i)(\partial_i f)^2) &\geq e^{-\sup W_{i,x}} \theta_i((\partial_{ii}^2 f)^2 + h_i''(x_i)(\partial_i f)^2) \\ &\geq e^{-\sup W_{i,x}} SG(\theta_i) \theta_i((\partial_i f)^2) \\ &\geq e^{-Osc W_{i,x}} SG(\theta_i) \eta_{i,x}((\partial_i f)^2), \end{aligned} \qquad (5.3)$$

provided

$$\theta_i(\Gamma_2 g) = \theta_i((L_i g)^2)$$

with $L_i g = g'' - h_i' g'$. Notice since $W_{i,x}$ is convex, its Oscillation cannot be bounded on $\mathbb{R}$, unless $W_{i,x}$ is constant. Hence the previous result has no interest on $\mathbb{R}^n$ and we shall only consider the case where the process lives in a bounded domain D.

We have thus obtained some variation of the renowned Holley-Stroock perturbation result namely

Proposition 5.1. *Let* $\mu(dx) = Z^{-1} e^{-W(x) - \sum_{i=1}^n h_i(x_i)} \mathbf{1}_D(x) \, dx$ *be a probability measure on the hypercube* $D = \prod_i]a_i, b_i[$. *Assume that*

(G1) *For all* i, *the one dimensional diffusion* $dy_t^i = \sqrt{2} \, dB_t^i - h_i'(y_t^i) dt$ *satisfies (H) on* $]a_i, b_i[$ *with reversible measure* $\theta_i(du) = z_i^{-1} e^{-h_i(u)} \mathbf{1}_{u \in]a_i, b_i[} \, du$.
(G2) $W \in C^\infty(\mathbb{R}^n)$ *and is convex.*

Introduce the one dimensional conditional log-density

$$W_{i,x}(t) = W(x_1, ..., x_{i-1}, t, x_i, ..x_n).$$

Then

$$C_P(\mu) \leq \max_i \sup_x e^{Osc(W_{i,x})} \max_i C_P(\theta_i).$$

Since $\max_i \sup_x e^{Osc(W_{i,x})} \leq e^{Osc W}$ we recover (provided W is convex) Holley-Stroock result for a product reference measure on a hypercube. But what is important here is that we only have to consider the Oscillation of W along lines parallel to the axes.

Proof. The only thing remaining to prove is that we can work with $C_b^\infty(D)$ functions f so that $u \mapsto f(x_1, ..., x_{i-1}, u, x_{i+1}, ..., x_n)$ is also $C_b^\infty(]a_i, b_i[)$ and we may use (G1) to justify the calculations we have done before. It is thus enough to show that (H) is satisfied for the full process i.e. with $V = W + \sum_i h_i$.

Since ∇W and ΔW are bounded on $\bar{D}$, the law of $X_\cdot^x$ is absolutely continuous w.r.t. to the one of $(y_\cdot^1, ..., y_\cdot^n)$ thanks to Girsanov theory. It follows that the exit time of D is almost surely infinite since the same holds for $(y_\cdot^1, ..., y_\cdot^n)$ according to (G1). In addition the Feynman-Kac representation of the density F_T (on $C^0([0, T], D)$) is again given by the formula of Example 7.1, so that, as we have seen, (H) is satisfied. $\qquad\square$

Corollary 5.2. *Let $\mu(dx) = Z^{-1} e^{-W(x) - \sum_{i=1}^n h_i(x_i)} \mathbf{1}_D(x) \, dx$ be a probability measure on the hypercube $D = \prod_i]a_i, b_i[$. Assume that the h_i's and W are convex and $C_b^2(\bar{D})$. Then, with the notations of Proposition 5.1 we have*

$$C_P(\mu) \le 12 \max_i \sup_x e^{Osc(W_{i,x})} \max_i C_P(\theta_i) \,.$$

Proof. As usual, using smooth approximations, we may assume that $W \in C^\infty(\mathbb{R}^n)$. We shall perturb μ in order to apply the previous proposition. To this end, on the interval $]a_i, b_i[$ define

$$h_\varepsilon^i(u) = \varepsilon \left(\frac{1}{u - a_i} + \frac{1}{b_i - u} \right) \,.$$

Consider

$$\mu_\varepsilon(dx) = Z^{-1} e^{-W(x) - \sum_i (h_i(x_i) + h_\varepsilon^i(x_i))} \mathbf{1}_D(x) \, dx \,.$$

Denote $g_\varepsilon^i = h_i + h_\varepsilon^i$.

Assumptions (G1) and (G2) of proposition 5.1 are satisfied. We already assumed (G2). In order to show (G1) it is first enough to use Feller test of non explosion for a one dimensional diffusion, i.e. to check that for $c_i = \frac{1}{2}(a_i + b_i)$,

$$\int_{c_i}^{a_i} \exp \left(\int_{c_i}^y (g_\varepsilon^i)'(u) du \right) dy = -\infty$$

(replacing a_i by b_i we similarly get $+\infty$) according for instance to [26] Chapter VI, Theorem 3.1, which is immediate. It follows that (1.1) is satisfied. In addition $(g_\varepsilon^i)' \in \mathbb{L}^2(\theta_\varepsilon^i(du))$ where $\theta_\varepsilon^i(du) = z_\varepsilon^{-1} e^{-g_\varepsilon^i(u)} \mathbf{1}_{]a_i, b_i[}(u) \, du$, so that we are in the situation of Example 7.2 ensuring that the one dimensional $y_\cdot$ in (G1) satisfies (H).

We have thus obtained

$$C_P(\mu_\varepsilon) \le \max_i \sup_x e^{Osc(W_{i,x})} \max_i C_P(\theta_\varepsilon^i(dt)) \,.$$

Using Lebesgue's bounded convergence Theorem, for all $f \in C_b^0(D)$ it holds

$$\lim_{\varepsilon \to 0} \int_D f(x) e^{-W(x) - \sum_{i=1}^n g_\varepsilon^i(x_i)} \, dx = \int_D f(x) e^{-W(x) - \sum_{i=1}^n h_i(x_i)} \, dx$$

so that using this result for f and 1, μ_ε weakly converges to μ. It follows

$$C_P(\mu) \le \liminf_{\varepsilon \to 0} C_P(\mu_\varepsilon) \le \max_i \sup_x e^{Osc(W_{i,x})} \liminf_{\varepsilon \to 0} \max_i C_P(\theta_\varepsilon^i) \,.$$

We may now use the fact that θ_ε^i is log-concave since both h_i and h_ε^i are convex. We thus have

$$C_P(\theta_\varepsilon^i) \leq 12 \, \mathrm{Var}_{\theta_\varepsilon^i}(x_i) \, .$$

Once again θ_ε^i weakly converges to θ_i and since $x_i \mapsto x_i^2$ is continuous and bounded on $[a_i, b_i]$,

$$\mathrm{Var}_{\theta_\varepsilon^i}(x_i) \to \mathrm{Var}_{\theta_i}(x_i)$$

so that the conclusion follows from the immediate $Var_{\theta_i}(x_i) \leq C_P(\theta_i)$. $\qquad\square$

6. Super Γ_2 condition

As there are weak Poincaré inequalities, Super Poincaré inequalities (SPI) have also been introduced by Wang [42] as a concise description of functional inequalities strictly stronger than Poincaré inequalities, in particular logarithmic Sobolev (or more generally F-Sobolev) inequalities.

(SPI) is often written in the following form: $\forall s > 0$, there exists a non-increasing $\beta :]0, \infty[\mapsto [1, +\infty[$ such that

$$\mu(f^2) \leq s\mu(|\nabla f|^2) + \beta(s)\mu(|f|)^2. \tag{6.1}$$

Applying (6.1) to constant functions one sees that $\beta(s) \geq 1$ for all s. Since 1 is assumed to belong to the range of β, the (SPI) inequality implies a Poincaré inequality with $C_P(\mu) \leq \beta^{-1}(1)$, and one has $\beta(s) = 1$ for $s \geq C_P(\mu)$. When $\beta(s) = ae^{b/s}$ for positive a and b, then the Super Poincaré inequality is equivalent to a logarithmic Sobolev inequality (see [19] lemma 2.5 and lemma 2.6 for a precise statement).

It is also possible to consider SPI with a L^p norm rather than the L^1 norm, so that we will introduce general (p-SPI) for $1 \leq p < 2$ and all $s > 0$,

$$\mu(f^2) \leq s\mu(|\nabla f|^2) + \beta(s)\mu(|f|^p)^{2/p}. \tag{6.2}$$

This time, (6.2) does not imply a Poincaré inequality, so that it is natural to assume in addition that $C_P(\mu) < +\infty$. In this case we have the following

Lemma 6.1. *Assume that $C_P(\mu) < +\infty$ and that the following centered (cp-SPI) inequality is satisfied for all $s > 0$,*

$$Var_\mu(f) \leq s\mu(|\nabla f|^2) + \beta_c(s)\mu(|f - \mu(f)|^p)^{2/p} \, , \tag{6.3}$$

where β is non increasing. Then (p-SPI) holds with $\beta(s) = 1 + 4\beta_c(s)$.

Proof. Since $C_P(\mu) < +\infty$ we may choose $\beta_c(s) = 0$ for $s > C_P(\mu)$. Let f be given. It holds

$$\mu(|f - \mu(f)|^p) \leq 2^{p-1} \left(\mu(|f|^p) + \mu^p(|f|)\right) \leq 2^p \, \mu(|f|^p)$$

yielding

$$\begin{aligned}
\mu(|f|^2) &= \mathrm{Var}_\mu(f) + \mu^2(f) \leq s\mu(|\nabla f|^2) + \beta_c(s)\mu^{2/p}(|f - \mu(f)|^p) + \mu^2(|f|) \\
&\leq s\mu(|\nabla f|^2) + (4\beta_c(s) + 1) \, \mu^{2/p}(|f|^p) \, .
\end{aligned}$$

$\qquad\square$

It is then natural to introduce an integrated super Γ_2 condition: for some $1 \leq p < 2$, there exists a positive non-increasing function β such that $\forall s > 0$

$$(pSI - \Gamma_2) \qquad \mu(|\nabla f|^2) \leq s\,\mu((Af)^2) + \beta(s)\mu(|f|^p)^{2/p}.$$

In the sequel we assume that $C_P(\mu) < +\infty$, so that for all $s \geq C_P(\mu)$ one may take $\beta(s) = 0$. Let us begin by this simple proposition

Proposition 6.2. *We have the following*

(1) A $(p - SPI)$ inequality is equivalent to

$$\mu((P_t f)^2) \leq e^{-2t/s}\mu(f^2) + \beta(s)\mu(|f|^p)^{2/p}(1 - e^{-2t/s}), \tag{6.4}$$

for all $s > 0$ and all $t \geq 0$.
(2) A $(pSI - \Gamma_2)$ condition is equivalent to

$$\mu(|\nabla P_t f|^2) \leq e^{-2t/s}\mu(|\nabla f|^2) + \beta(s)\mu(|f|^p)^{2/p}(1 - e^{-2t/s}). \tag{6.5}$$

for all $s > 0$ and all $t \geq 0$.

Proof. The first part is well known and is included in Wang's work [42]. The second point will follow the same line of proof. As already emphasized in the previous sections, denoting

$$F(t) = \mu(|\nabla P_t f|^2)$$

one has

$$F'(t) = -2\mu((AP_t f)^2)$$

so that the $(pSI - \Gamma_2)$ gives directly

$$F'(t) \leq -\frac{2}{s}F(t) + \frac{2\beta(s)}{s}\mu(|P_t f|^p)^{2/p}$$

and since $\mu(|P_t f|^p)^{2/p} \leq \mu(|f|^p)^{2/p}$ we conclude thanks to Gronwall's lemma. The other implication comes from differentiating with respect to time at time 0. $\qquad\square$

6.1. From (p-SPI) to (pSI-Γ_2).

We follow the same proof as in section 2, assuming that a $(p - SPI)$ holds, i.e. we use Cauchy-Schwartz inequality in order to get

$$\begin{aligned}
\mu(|\nabla f|^2) &= \mu(-fAf) \\
&\leq \sqrt{\mu(f^2)\,\mu((Af)^2)} \\
&\leq \left(s\,\mu(|\nabla f|^2)\,\mu((Af)^2) + \beta(s)\mu(|f|^p)^{2/p}\mu((Af)^2)\right)^{1/2}.
\end{aligned}$$

Recall now the already used following fact: if $0 \leq u \leq \sqrt{Au + B}$ then $u \leq A + B^{1/2}$. It yields

$$\begin{aligned}
\mu(|\nabla f|^2) &\leq s\,\mu((Af)^2) + \sqrt{\beta(s)\mu(|f|^p)^{2/p}\mu((Af)^2)} \\
&\leq \frac{3}{2}s\,\mu((Af)^2) + \frac{\beta(s)}{2s}\mu(|f|^p)^{2/p}.
\end{aligned}$$

We thus see that we have "lost" a factor $1/s$ but if we think to the logarithmic Sobolev inequality, it roughly means the loss of a constant.

6.2. **From (pSI-Γ_2) to (p-SPI).**
. Starting with

$$\mu(|\nabla P_t f|^2 \le e^{-2t/s}\mu(|\nabla f|^2) + \beta(s)\mu(|P_t f|^p)^{2/p}(1 - e^{-2t/s})$$

and using

$$\mathrm{Var}_\mu(f) = 2\int_0^\infty \mu(|\nabla P_u f|^2)du$$

we get

$$\mathrm{Var}_\mu(f) \le 2\int_0^\infty e^{-2u/s}\mu(|\nabla f|^2)du + 2\beta(s)\int_0^\infty \mu(|P_u f|^p)^{2/p}(1 - e^{-2u/s})du.$$

Assume first that f is centered. If $p > 1$ then Poincaré inequality implies back an exponential convergence in L^p norm (see [18] Theorem 1.3) so that for all centered f we get

$$\mu(f^2) \le s\mu(|\nabla f|^2) + K_p\beta(s)\mu(|f|^p)^{2/p}$$

where K_p depends on p and is going to infinity as p goes to 1. Applying lemma 6.1 we thus obtain

$$\mu(|f|^2) \le s\mu(|\nabla f|^2) + (1 + 4K_p\beta(s))\mu(|f|^p)^{2/p}.$$

7. APPENDIX: ABOUT THE HEART OF DARKNESS.

Let us come back to the framework of this work especially Proposition 1.2.

First of all, if (H) is satisfied, according to Theorem 2.2.25 and its proof in Royer's book [40] (also see the english version [41]), the following holds

(A1) P_t extends to a μ-symmetric continuous Markov semi-group $e^{-t\tilde{A}}$ on $\mathbb{L}^2(\mu)$. We denote by $\mathcal{D}(\tilde{A})$ the domain of the generator $\tilde{A}$ of this $\mathbb{L}^2(\mu)$ semi-group.

(A2) Any $f \in C^2(D)$ such that $|\nabla f|$ is bounded and $Af \in \mathbb{L}^2(\mu)$ belongs to $\mathcal{D}(\tilde{A})$, and $\tilde{A}f = Af$.

(A3) If $f \in \mathcal{D}(\tilde{A})$ the set of Schwartz distributions on D, then $f \in \mathcal{D}'(D)$ and satisfies $\tilde{A}f = Af$ in $\mathcal{D}'(D)$.

Actually Royer only considers the case $D = \mathbb{R}^n$, but the key point in the proof is that one can apply Ito's formula for such an f up to time t (without any stopping time) which is ensured by the conservativeness in D.

In the case $D = \mathbb{R}^n$ the proof of ESA for C_0^∞ the set of smooth compactly supported functions is contained in [43] using an elliptic regularity result Theorem 2.1 in [24] (actually the latest result certainly appeared in other places). The proof is explained in Theorem 2.2.7 of [40] (also see Proposition 3.2.1 in [4]) when V is C^∞. The structure of $\mathcal{D}(\tilde{A})$ is also proved in the same Theorem.

We shall explain the proof when D is bounded, still assuming for simplicity that $V \in C^\infty(D)$. The same elliptic regularity should be used to extend the result to $V \in C^2(D)$, but will introduce too much intricacies to be explained here.

Proof. First consider the Dirichlet form $\mathcal{E}(f,g) = \mu(\langle\nabla f, \nabla g\rangle)$ whose domain is the closure of $C_0^\infty(D)$ denoted by $H_0^1(\mu, D)$. Since $\mathcal{E}$ is regular, Fukushima's theory (see [25]) allows us to build a symmetric Hunt process associated to $(\mathcal{E}, H_0^1(\mu, D))$. This process is then a solution to the martingale problem associated to A and $C_0^\infty(D)$. Since T_∂^x is almost surely infinite, this martingale problem has an unique solution given by the (distribution) of the stochastic process $X_\cdot^x$.

In order to prove ESA it is enough to show that if $g \in \mathbb{L}^2(\mu)$ satisfies $\mu(g\,(A\varphi - \varphi)) = 0$ for all $\varphi \in C_0^\infty(D)$ then g vanishes (see the beginning of the proof in [40] p.31). According to the proof in [40] p.31, it implies in particular that $g \in \mathcal{D}'(D)$ and satisfies $Ag = g$. Using that A is hypoelliptic since $V \in C^\infty(D)$, we deduce that $g \in C^\infty(D)$.

Using Ito's formula (since the process is conservative) we have

$$\sqrt{2} \int_0^t \langle\nabla g(X_s), dB_s\rangle = g(X_t) - g(X_0) - \int_0^t g(X_s)\,ds \tag{7.1}$$

almost surely. If X_0 is distributed according to μ, the right hand side belongs to $\mathbb{L}^2(\mathbb{P})$ ($\mathbb{P}$ being the underlying probability measure on the path space), so that the left hand side also belongs to $\mathbb{L}^2(\mathbb{P})$. The $\mathbb{L}^2$ norm of this left hand side is equal to $2t\,\mu(|\nabla g|^2)$ so that $\nabla g \in \mathbb{L}^2(\mu)$.

As a consequence

$$t \mapsto \int_0^t \langle\nabla g(X_s), dB_s\rangle$$

is a $\mathbb{P}$ martingale so that for all bounded h,

$$\mathbb{E}(h(X_0)g(X_t)) = \mu(gh) + \int_0^t \mathbb{E}(h(X_0)g(X_s))\,ds\,.$$

Since a regular disintegration of $\mathbb{P}$ is furnished by the distribution of the $X_\cdot^x$'s, it follows

$$P_t g = g + \int_0^t P_s g\,ds$$

μ almost surely, so that $g \in \mathcal{D}(\tilde{A})$ and satisfies $\tilde{A}g = g$. Hence

$$\mu(g^2) = \mu(gAg) = -\mu(|\nabla g|^2)$$

so that $g = 0$.

The proof of the remaining part of the Theorem is the same as in [40] p.42. $\square$

Finally we will indicate how to show that the semi-group is ergodic when D is bounded (we already mention a possible way for $D = \mathbb{R}^n$ in section 2) . If $P_t f = f$ for all $t > 0$ it follows that $f \in \mathcal{D}(A)$ and satisfies $Af = 0$ so that f is smooth thanks to hypoellipticity. Applying Ito's formula we deduce that $f(X_t^x) = f(x)$ a.s. for all $t > 0$. Thanks to the Support Theorem ([26] chapter 6 section 8) we know that the distribution of X_t^x admits a positive density w.r.t. Lebesgue measure, so that if $f(y) \neq f(x)$ for some y, hence all z in a neighborhood N of y by continuity, $\mathbb{P}(X_t^x \in N) > 0$ and thus $f(X_t^x) \neq f(x)$ with positive probability, which is a contradiction.

Let us give now some of the most important examples. In these examples we assume that $V \in C^3$.

Example 7.1. If either

(H1) $V(x) \to +\infty$ as $x \to \partial D$ (i.e. $|x| \to +\infty$ if $D = \mathbb{R}^n$), and $\frac{1}{2}|\nabla V|^2 - \Delta V$ is bounded
        from below, or

(H2) $D = \mathbb{R}^n$ and $\langle x, \nabla V(x) \rangle \geq -a|x|^2 - b$ for some a, b in $\mathbb{R}$,

then (H) is satisfied. If V is convex (H2) is satisfied with $a = b = 0$.

If $D = \mathbb{R}^n$ these two cases are detailed in [40] subsection 2.2.2 (conservativeness is shown in
Theorem 2.2.19 therein). In the (H1) case for a bounded domain the only thing to do is to
replace the exit times of large balls by the T_k's in Lemme 2.2.21 of [40].

In all cases the law of $(X_t^x)_{t \leq T}$ is given by $dQ = F_T dP$ where P is the law of a Wiener process
starting from 0 and

$$F_T = \exp\left(\frac{1}{2}V(x) - \frac{1}{2}V(x + \sqrt{2}\,W_T) + \frac{1}{2}\int_0^T (\frac{1}{2}|\nabla V|^2 - \Delta V)(x + \sqrt{2}\,W_s)ds\right). \quad (7.2)$$

$\diamond$

Example 7.2. Assume now

$$\mu(|\nabla V|^2) < +\infty. \quad (7.3)$$

(7.3) is an entropy condition related to the stationary Nelson processes (see [34, 33, 21,
22, 23]). The stationary (symmetric) conservative diffusion process is built in these papers.
Conservative means here that

$$T_{\partial D} = +\infty \quad \mathbb{P}_\mu \text{ a.s} \quad (7.4)$$

i.e. if X_0 is distributed according to μ.

The proof in the bounded case is a simple modification of the one in [21]. The modification
is as follows (we refer to the notations therein):

(1) First the flow ν_t is stationary with $\nu_t = \mu$.
(2) Next the drift $B = -\nabla V$. Assuming in addition that D has a smooth boundary,
    one can approximate in $\mathbb{L}^2(\mu)$, B by B_k's which are $C_b^\infty(\mathbb{R}^n)$ and coincide with B on
    $\bar{D}_k = \{x\,;, d(x, \partial D) \geq 1/k\}$ (for this we need ∂D_k to be smooth).

One can then follow the "Outline of proof" (4.9 bis) in [21] replacing the T_n therein by the
T_k we have introduced before (the exit times of D_k) so that (4.14) in [21] is trivially satisfied.
(4.16) is then justified when (1.1) is satisfied. The only remaining thing to prove is thus
(4.10) in [21]. For $f \in C_0^\infty(\mathbb{R}^n)$ whose support contains $\bar{D}$ we may then proceed as in the
proof of Theorem (4.18) in [21] in order to prove it.

In order to show the strong existence of the diffusion process starting from x (and not
the stationary measure) it is enough to show (1.1) is satisfied (the strong existence of the
diffusion process up to $T_{\partial D}^x$ is ensured since V is local-Lipschitz). Since the stationary process
is conservative, so is $X_.^x$ for μ, hence Lebesgue, almost all x. Standard results in Dirichlet
forms theory show that this result extends to all x outside some polar set. Actually it is
true for all x using the following (itself more or less standard 40 years ago): choose a small
ball $B(x, \varepsilon)$ with $\varepsilon < d(x, \partial D)/2$ and introduce S^x the exit time of this ball. For $t > 0$ the

distribution of $X_t \, \mathbf{1}_{t<S^x}$ has a density w.r.t. Lebesgue's measure restricted to the ball (using e.g. Malliavin calculus), hence w.r.t μ. It follows from the Markov property and (7.4)

$$\mathbb{P}(T_{\partial D}^x < +\infty) = \lim_{t \to 0} \mathbb{P}(t < S^x \, ; \, T_{\partial D}^x < +\infty) = \lim_{t \to 0} \mathbb{E}(\mathbf{1}_{t<S^x} \, \mathbb{E}(\mathbf{1}_{T_{\partial D}^{X_t^x} < +\infty})) = 0$$

since the second expectation in the last formula vanishes identically.

In all cases the Feynman-Kac representation of F_T in (7.2) is obtained by using Ito's formula with V which is allowed since $V \in C^3(D)$ and (1.1) again. $\diamondsuit$

Example 7.3. If we do no more assume that the hitting time of the boundary is infinite, assumption (H) is not satisfied. The space of interest should be $H_b^1(\mu)$ the closure of $C_b^\infty(D)$ for the Dirichlet form. The corresponding process is the symmetric reflected diffusion process. A good reference is [37] where this normally reflected diffusion process is built (under much more general conditions). Assume that the boundary is smooth.

A little bit more is needed. First if $f \in H_b^1(\mu)$ and $g \in \mathcal{D}(A)$, one has, according to Fukushima's theory (see [25] (1.3.10)),

$$\mu(\langle \nabla f, \nabla g \rangle) = - \mu(f \, Ag) \,. \tag{7.5}$$

If f is smooth ($C^2(\bar{D})$) and belongs to $\mathcal{D}(A)$ it also holds

$$- \mu(f \, Ag) = \mu(\langle \nabla f, \nabla g \rangle) + \int_{\partial D} g \, \langle n_D, \nabla f \rangle \, e^{-V} \, d\sigma_D$$

according to Green's identity. Here n_D denotes the normalized inward normal vector on ∂D and σ_D denotes the surface measure on ∂D. Since the set of the traces on ∂D of bounded functions in $H^1(\mu)$ is dense in $\mathbb{L}^\infty(\sigma_D)$, we deduce that

$$\langle n_D, \nabla f \rangle_{|\partial D} = 0 \,.$$

It is however not clear in general that $P_t f$ is smooth (even if V is). If one assumes that ∂D is C^∞ and $V \in C^\infty(\bar{D})$, $P_t f \in C^\infty(\bar{D})$ is shown in [14] Theorem 2.9 by using the method of [13] (see the proof of Theorem 2.11 therein). Notice that the proof of regularity is using Sobolev imbedding theorem, so that one should relax the C^∞ assumption but with dimension dependent regularity assumptions. Other more important difficulties will be pointed out later. The other major difficulty is that ESA is not satisfied in general.

In comparison with the previous example, the boundary term will disappear if $e^{-V} = 0$ on ∂D. It is what happens if (H) is satisfied, but here again we do not need such a proof which is not useful in the present work. $\diamondsuit$

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Patrick CATTIAUX,, Institut de Mathématiques de Toulouse. CNRS UMR 5219., Université Paul Sabatier,, 118 route de Narbonne, F-31062 Toulouse cedex 09.

Email address: `patrick.cattiaux@math.univ-toulouse.fr`

Arnaud GUILLIN,, Université Clermont Auvergne, CNRS, LMBP, F-63000 CLERMONT-FERRAND, FRANCE.

Email address: `arnaud.guillin@uca.fr`