ON THE POINCARÉ CONSTANT OF LOG-CONCAVE MEASURES.

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Abstract. The goal of this paper is to push forward the study of those properties of log-concave measures that help to estimate their Poincaré constant. First we revisit E. Milman's result [44] on the link between weak (Poincaré or concentration) inequalities and Cheeger's inequality in the log-concave cases, in particular extending localization ideas and a result of Latala, as well as providing a simpler proof of the nice Poincaré (dimensional) bound in the unconditional case. Then we prove alternative transference principle by concentration or using various distances (total variation, Wasserstein). A mollification procedure is also introduced enabling, in the log-concave case, to reduce to the case of the Poincaré inequality for the mollified measure. We finally complete the transference section by the comparison of various probability metrics (Fortet-Mourier, bounded-Lipschitz,...) under a log-concavity assumption.

Key words : Poincaré inequality, Cheeger inequality, log-concave measure, total variation, Wasserstein distance, mollification procedure, transference principle.

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1. **Introduction and overview.**

Let ν be a Probability measure defined on $\mathbb{R}^n$. For a real valued function f, $\nu(f)$ and $m_\nu(f)$ will denote respectively the ν mean and a ν median of f, when these quantities exist. We also denote by

$$\text{Var}_\nu(f) = \nu(f^2) - \nu^2(f)$$

the ν variance of f.
The Poincaré constant $C_P(\nu)$ of ν is defined as the best constant such that

$$\text{Var}_\nu(f) \leq C_P(\nu)\, \nu(|\nabla f|^2)\,.$$

In all the paper, we shall denote equally $C_P(Z)$ or $C_P(\nu)$ the Poincaré constant for a random variable Z with distribution ν. We say that ν satisfies a Poincaré inequality when $C_P(\nu)$ is finite. Note also that in the whole paper, for $x \in \mathbb{R}^n$, $|x| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$ denotes the euclidean norm of x, and a function f is said to be K-Lipschitz if $\sup_{|x-y|>0} \frac{|f(x)-f(y)|}{|x-y|} \leq K$.
It is well known that, as soon as a Poincaré inequality is satisfied, the tails of ν are exponentially small, i.e. $\nu(|x| > R) \leq C\, e^{-cR/\sqrt{C_P(\nu)}}$ for some universal c and C (see [14]), giving a

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very useful necessary condition for this inequality to hold. Conversely, during the last eighty years a lot of sufficient conditions have been given for a Poincaré inequality to hold. In 1976, Brascamp and Lieb ([15]) connected Poincaré inequality to convexity by proving the following: if $\nu(dx) = e^{-V(x)}\,dx$, then

$$\mathrm{Var}_\nu(f) \leq \int {}^t\nabla f\,(\mathrm{Hess}^{-1}(V))\,\nabla f\,d\nu$$

where $\mathrm{Hess}(V)$ denotes the Hessian matrix of V. Consequently, if V is uniformly convex, i.e. $\inf_x {}^t\xi\,\mathrm{Hess}(V)(x)\xi \geq \rho\,|\xi|^2$ for some $\rho > 0$, then $C_P(\nu) \leq 1/\rho$. This result contains in particular the gaussian case, and actually gaussian measures achieve the Brascamp-Lieb bound as it is easily seen looking at linear functions f.
This result was extended to much more general "uniformly convex" situations through the celebrated Γ_2 theory introduced by Bakry and Emery (see the recent monograph [4] for an up to date state of the art of the theory) and the particular uniformly convex situation corresponds to the $\mathrm{CD}(\rho,\infty)$ curvature-dimension property in this theory. This theory has been recently revisited in [18] by using coupling techniques for the underlying stochastic process.

A particular property of the Poincaré inequality is the tensorization property ([4] Proposition 4.3.1)

$$C_P(\nu_1 \otimes ... \otimes \nu_N) \leq \max_{i=1,...,N} C_P(\nu_i)\,.$$

It is of fundamental importance for the concentration of measure and for getting bounds for functionals of independent samples in statistics, due to its "dimension free" character. This "dimension free" character is captured by the Brascamp-Lieb inequality or the Bakry-Emery criterion, even for non-product measures.
Using a simple perturbation of V by adding a bounded (or a Lipschitz) term, one can show that uniform convexity "at infinity" is enough to get a Poincaré inequality ([4] Proposition 4.2.7). This result can also be proved by using reflection coupling (see [26, 27, 18]). However in this situation a "dimension free" bound for the optimal constant is hard to obtain, as it is well known for the double well potential $V(x) = |x|^4 - |x|^2$.

Uniform convexity (even at infinity) is not necessary as shown by the example of the symmetric exponential measure on the line, $\nu(dx) = \frac{1}{2}\,e^{-|x|}$ which satisfies $C_P(\nu) = 4$ (see [4] (4.4.3)). In 1999, Bobkov ([10]) has shown that any log-concave probability measure satisfies the Poincaré inequality. Here log-concave means that $\nu(dx) = e^{-V(x)}\,dx$ where V is a convex function with values in $\mathbb{R} \cup \{+\infty\}$. In particular uniform measures on convex bodies are log-concave. We refer to the recent book [16] for an overview on the topic of convex bodies, and to [49] for a survey of log-concavity in statistics. Another proof, applying to a larger class of measures, was given in [2] using Lyapunov functions as introduced in [3]. If it is now known that a Poincaré inequality is equivalent to the existence of a Lyapunov function (see [21, 19]), this approach is far to give good controls for the Poincaré constant.
Actually Bobkov's result is stronger since it deals with the $\mathbb{L}^1$ version of Poincaré inequality

$$\nu(|f - \nu(f)|) \leq C_C(\nu)\,\nu(|\nabla f|)\,, \tag{1.1}$$

which is often called Cheeger inequality. Another form of Cheeger inequality is

$$\nu(|f - m_\nu(f)|) \leq C'_C(\nu)\,\nu(|\nabla f|)\,. \tag{1.2}$$

Here again, C_C and C'_C have to be understood as the best constants satisfying the related inequalities. Using

$$\frac{1}{2}\,\nu(|f-\nu(f)|)\;\leq\;\nu(|f-m_\nu(f)|)\;\leq\;\nu(|f-\nu(f)|)\,,$$

it immediately follows that $\frac{1}{2}\,C_C \leq C'_C \leq C_C$.
It is well known that the Cheeger constant gives a natural control for the isoperimetric function Is_ν of ν. Recall that for $0 \leq u \leq \frac{1}{2}$,

$$\text{Is}_\nu(u) = \inf_{A,\nu(A)=u}\; \nu_s(\partial A)$$

where

$$\nu_s(\partial A) = \liminf_{h\to 0}\; \frac{\nu(A+hB(0,1))-\nu(A)}{h}$$

denotes the surface measure of the boundary of A. It can be shown (see the introduction of [10]) that

$$\text{Is}_\nu(u) = \frac{u}{C'_C(\nu)}\;\geq\;\frac{u}{C_C(\nu)}\,.$$

The Cheeger inequality is stronger than the Poincaré inequality and (see again [10])

$$C_P(\nu) \leq 4\,(C'_C)^2(\nu)\,. \tag{1.3}$$

The first remarkable fact in the log-concave situation is that a converse inequality holds, namely if ν is log-concave,

$$(C'_C)^2(\nu) \leq 36\,C_P(\nu)\,, \tag{1.4}$$

as shown by Ledoux ([40] formula (5.8)).
Ledoux's approach is using the associated semi-goup with generator $L = \Delta - \nabla V.\nabla$ for which the usual terminology corresponding to the convexity of V is zero curvature. Of course to define L one has to assume some smoothness of V. But if Z is a random variable with a log-concave distribution ν and G is an independent standard gaussian random variable, Prekopa-Leindler theorem ensures that the distribution of $Z+\varepsilon\,G$ is still log concave for any $\varepsilon > 0$ and is associated to a smooth potential V_ε. Hence we may always assume that the potential V is smooth provided we may pass to the limit $\varepsilon \to 0$.

Log-concave measures deserve attention during the last twenty years in particular in Statistics. They are considered close to product measures in high dimension. It is thus important to get some tractable bound for their Poincaré constant, in particular to understand the role of the dimension.
Of course if $Z = (Z_1, ..., Z_n)$ is a random vector of $\mathbb{R}^n$,

$$C_P((\lambda_1\,Z_1, ..., \lambda_n\,Z_n)) \leq \max_i\,\lambda_i^2\,C_P((Z_1, ..., Z_n))\,,$$

and the Poincaré constant is unchanged if we perform a translation or an isometric change of coordinates. It follows that

$$C_P(Z)\;\leq\;\sigma^2(Z)\,C_P(Z')$$

where $\sigma^2(Z)$ denotes the largest eigenvalue of the covariance matrix $Cov_{i,j}(Z) = \text{Cov}(Z_i, Z_j)$, and Z' is an affine transformation of Z which is centered and with Covariance matrix equal to Identity. Such a random vector (or its distribution) is called isotropic (or in isotropic position for convex bodies and their uniform distribution). The reader has to take care about the use of the word isotropic, which has a different meaning in probability theory (for instance in Paul Lévy's work).

Applying the Poincaré inequality to linear functions show that $\sigma^2(Z) \leq C_P(Z)$. In particular, in the uniformly convex situation, $\sigma^2(Z) \leq 1/\rho$ with equality when Z is a gaussian random vector. For the symmetric exponential measure on $\mathbb{R}$, we also have $\sigma^2(Z) = C_P(Z)$ while $\rho = 0$. It thus seems plausible that, even in positive curvature, $\sigma^2(Z)$ is the good parameter to control the Poincaré constant.

The following was conjectured by Kannan, Lovász and Simonovits ([33])

Conjecture 1.1. *(K-L-S conjecture.) There exists a universal constant C such that for any log-concave probability measure ν on $\mathbb{R}^n$,*

$$C_P(\nu) \; \leq \; C \, \sigma^2(\nu) \, ,$$

where $\sigma^2(\nu)$ denotes the largest eigenvalue of the covariance matrix $Cov_{i,j}(\nu) = Cov_\nu(x_i, x_j)$, or if one prefers, there exists an universal constant C such that any isotropic log-concave probability measure ν, in any dimension n, satisfies $C_P(\nu) \leq C$.

During the last years a lot of works have been done on this conjecture. A recent book [1] is totally devoted to the description of the state of the art. We will thus mainly refer to this book for references, but of course apologize to all important contributors. We shall just mention part of these works we shall revisit and extend.

In this note we shall, on one hand investigate properties of log-concave measures that help to evaluate their Poincaré constant, on the other hand obtain explicit constants in many intermediate results. Let us explain on an example: in a remarkable paper ([44]), E. Milman has shown that one obtains an equivalent inequality if one replaces the energy of the gradient in the right hand side of the Poincaré inequality by the square of its Lipschitz norm, furnishing a much less demanding inequality ($(2, +\infty)$ Poincaré inequality). The corresponding constant is sometimes called the spread constant. In other words the Poincaré constant of a log-concave measure ν is controlled by its spread constant. In the next section we shall give another proof of this result. Actually we shall extend it to weak forms of $(1, +\infty)$ inequalities. These weak forms allow us to directly compare the concentration profile of ν with the corresponding weak inequality. We shall also give explicit controls of the constants when one reduces the support of ν to an euclidean ball as in [44] or a l^∞ ball, the latter being an explicit form of a result by Latala [38].

In section 3 we shall describe several transference results using absolute continuity, concentration properties and distances between measures. There is an important overlap between this section and part of the results in [8] or [46]. In addition to different proofs, the main novelty here is that we compare a log-concave measure ν with another non necessarily log-concave measure μ and use the weak form of the $(1, +\infty)$ Poincaré inequality. For instance, we show that if the distance between ν and μ is small enough, then the Poincaré constant of μ controls the one of ν. This is shown for several distances: total variation, Wasserstein, Bounded Lipschitz.

Section 4 is concerned with mollification. The first part is a revisit of results by Klartag [36]. The second part studies convolution with a gaussian kernel. It is shown that if γ is some gaussian measure, $C_P(\nu * \gamma)$ controls $C_P(\nu)$. If the converse is well known, this implication is new and may have some applications elsewhere. The proof is based on stochastic calculus.

Finally in the last section and using what precedes we show that all the previous distances and the Lévy-Prokhorov distance define the same uniform structure on the set of log-concave measures independently of the dimension. We thus complete the transference results using distances. Some dimensional comparisons have been done in [43].

As said by the referee and the editors, this paper collects a (may be too) long list of results, some new, some already proven, presenting new proofs (in general simpler) for these last ones and trying to trace, in general non optimal, constants. We did our best to compare these results with the existing ones, but we apologize in advance for the omissions. We take the opportunity of these few lines to warmly thank an anonymous referee and a less anonymous editor for their suggestions and comments that improve the presentation of the paper at several places.

2. **Revisiting E. Milman's results.**

2.1. (p,q) **Poincaré inequalities.**

Following [44], the usual Poincaré inequality can be generalized in a (p,q) Poincaré inequality, for $1 \leq p \leq q \leq +\infty$,

$$B_{p,q}\, \nu^{1/p}(|f-\nu(f)|^p) \leq \nu^{1/q}(|\nabla f|^q). \tag{2.1}$$

For $p=q=2$ we recognize the Poincaré inequality and $B^2_{2,2}=1/C_P(\nu)$, and for $p=q=1$ the Cheeger inequality with $B_{1,1}=1/C_C(\nu)$.

Among all (p,q) Poincaré inequalities, the weakest one is clearly the $(1,+\infty)$ one, the strongest the $(1,1)$ one, we called Cheeger's inequality previously. Indeed for $1 \leq p \leq p' \leq q \leq +\infty$ except the case $p=q=+\infty$, one has the following schematic array between these Poincaré inequalities (see [44])

$$\begin{array}{ccc} (1,1) & \Rightarrow & (1,q) \\ \Downarrow & & \Uparrow \\ (p,p) & \Rightarrow & (p,q) \\ \Downarrow & & \Uparrow \\ (p',p') & \Rightarrow & (p',q) \end{array}$$

The meaning of all these inequalities is however quite unclear except some cases we shall describe below.

First remark that on $\mathbb{R}^n$,

$$|f(x)-f(a)| \leq \parallel \nabla f \parallel_\infty \, |x-a|$$

yielding

$$\nu(|f-m_\nu(f)|) = \inf_b \, \nu(|f-b|) \leq \inf_a \, \nu^{1/p}(|x-a|^p) \, \parallel |\nabla f| \parallel_\infty ,$$

so that since

$$\frac{1}{2}\,\nu(|f-\nu(f)|) \,\leq\, \nu(|f-m_\nu(f)|) \,\leq\, \nu(|f-\nu(f)|)\,, \tag{2.2}$$

the $(p,+\infty)$ Poincaré inequality is satisfied as soon as ν admits a p-moment. There is thus no hope for this inequality to be helpful unless we make some additional assumption.

Now look at the $(p,2)$ Poincaré inequality ($1 \leq p \leq 2$). We may write, assuming that $\nu(f)=0$,

$$\mathrm{Var}_\nu(f) \quad \leq \quad \| f \|_\infty^{2-p} \; \nu(|f|^p) \leq \frac{1}{B_{p,2}^p} \; \| f \|_\infty^{2-p} \; \nu^{p/2}(|\nabla f|^2)$$

which is equivalent to

$$\mathrm{Var}_\nu(f) \, \leq \, c\, s^{-\frac{2-p}{p}} \; \nu(|\nabla f|^2) + s \; \| f - \nu(f) \|_\infty^2 \qquad \text{for all } s>0\,,$$

with

$$\frac{1}{B_{p,2}^2} = \frac{1}{p} \left(\frac{(c(2-p))^{2/p} + p^{2/p}}{(c(2-p))^{(2-p)/p}} \right) .$$

This kind of inequalities has been studied under the name of weak Poincaré inequalities (see [48, 6, 17]). They can be used to show the $\mathbb{L}^2 - \mathbb{L}^\infty$ convergence of the semi-group with a rate $t^{-p/(2-p)}$.

As shown in [48], any probability measure $\nu(dx) = e^{-V(x)}\, dx$ such that V is locally bounded, satisfies a weak Poincaré inequality. Indeed, using Holley-Stroock perturbation argument ([4] Proposition 4.2.7) w.r.t. the (normalized) uniform measure on the euclidean ball $B(0,R)$, it is easy to see that

$$\mathrm{Var}_\nu(f) \leq \frac{4R^2}{\pi^2}\, e^{Osc_R\, V}\, \nu(|\nabla f|^2) + 2\, \nu(|x|>R) \; \| f - \nu(f) \|_\infty^2$$

where $Osc_R\, V = \sup_{|x|\leq R} V(x) - \inf_{|x|\leq R} V(x)$.

It is thus tempting to introduce weak versions of (p,q) Poincaré inequalities.

Definition 2.1. We shall say that ν satisfies a weak (p,q) Poincaré inequality if there exists some non increasing non-negative function β defined on $]0,+\infty[$ such that for all $s>0$ and all smooth function f,

$$(\nu(|f-\nu(f)|^p))^{\frac{1}{p}} \leq \beta(s) \; \| \, |\nabla f| \, \|_q + s\, \mathrm{Osc}(f)\,,$$

where Osc denotes the oscillation of f. We shall sometimes replace $\nu(f)$ by $m_\nu(f)$. In particular for $p=1$ and $q=\infty$ we have

$$\beta^{med}(s) \leq \beta^{mean}(s) \leq 2\beta^{med}(s/2)\,.$$

Of course if $\beta(0) < +\infty$ we recover the (p,q) Poincaré inequality. Any probability measure satisfies a particular weak $(1,+\infty)$ Poincaré inequality, namely:

Proposition 2.2. *Denote by α_ν the concentration profile of a probability measure ν, i.e.*

$$\alpha_\nu(r) := \sup\{1 - \, \nu(A + B(0,r))\, ;\, \nu(A) \geq \frac{1}{2}\}\,,\, r>0\,,$$

where $B(y,r)$ denotes the euclidean ball centered at y with radius r. Then for any probability measure ν and all $s>0$,

$$\nu(|f - m_\nu(f)|) \leq \; \alpha_\nu^{-1}(s/2) \; \| \, |\nabla f| \, \|_\infty + s\, Osc f\,.$$

and

$$\nu(|f-\nu(f)|) \leq 2\, \alpha_\nu^{-1}(s/4) \; \| \, |\nabla f| \, \|_\infty + s\, Osc(f)\,,$$

where α_ν^{-1} denotes the converse function of α_ν.

Proof. Due to homogeneity we may assume that f is 1-Lipschitz. Hence $\nu(|f - m_\nu(f)| > r) \leq 2\,\alpha_\nu(r)$. Thus

$$\begin{aligned}\nu(|f-\nu(f)|) &\leq 2\,\nu(|f-m_\nu(f)|) \leq 2\,r\,\nu(|f-m_\nu(f)|\leq r) + 2\,\mathrm{Osc}(f)\,\nu(|f-m_\nu(f)|>r)\\ &\leq 2r + 4\,\mathrm{Osc}(f)\,\alpha_\nu(r)\,,\end{aligned}$$

hence the result. □

As asked to us by E. Milman, one should ask about a partial converse to the previous statement. Actually we do not know how to get something convincing.

These results will be surprisingly useful for the family of log-concave measures. Recall the following result is due to E. Milman (see Theorem 2.4 in [44]):

Theorem 2.3. *[E. Milman's theorem.] If $d\nu = e^{-V}dx$ is a log-concave probability measure in $\mathbb{R}^n$, there exists a universal constant C such that for all (p,q) and (p',q') (with $1 \leq p \leq q \leq +\infty$ and $1 \leq p' \leq q' \leq +\infty$)*

$$B_{p,q} \leq C\,p'\,B_{p',q'}\,.$$

Hence in the log-concave situation the $(1,+\infty)$ Poincaré inequality implies Cheeger's inequality, more precisely implies a control on the Cheeger's constant (that any log-concave probability satisfies a Cheeger's inequality is already well known). We shall revisit this last result in the next subsection.

2.2. **Log-concave Probability measures.**

In order to prove Theorem 2.3 it is enough to show that the $(1,+\infty)$ Poincaré inequality implies the $(1,1)$ one, and to use the previous array. We shall below reinforce E. Milman's result. The proof (as in [44]) lies on the concavity of the isoperimetric profile, namely the following proposition which was obtained by several authors (see [44] Theorem 1.8 for a list):

Proposition 2.4. *Let ν be a (smooth) log-concave probability measure on $\mathbb{R}^n$. Then the isoperimetric profile $u \mapsto Is_\nu(u)$ is concave on $[0,\frac{1}{2}]$.*

The previous concavity assumption may be used to get some estimates on Poincaré and Cheeger constants.

Proposition 2.5. *Let ν be a probability measure such that $u \mapsto Is_\nu(u)$ is concave on $[0,\frac{1}{2}]$. Assume that there exist some $0 \leq u \leq 1/2$ and some $C(u)$ such that for any Lipschitz function $f \geq 0$ it holds*

$$\nu(|f-\nu(f)|) \leq C(u)\,\nu(|\nabla f|) + u\,Osc(f)\,. \tag{2.3}$$

Then for all measurable A such that $\nu(A) \leq 1/2$,

$$\nu_s(\partial A) \geq \frac{1-2u}{C(u)}\,\nu(A) \qquad i.e. \quad C'_C(\nu) \leq \frac{C(u)}{1-2u}\,.$$

If we reinforce (2.3) as follows, for some $0 \leq u \leq 1$, $1 < p < \infty$ and some $C_p(u)$

$$\nu(|f-\nu(f)|) \leq C_p(u)\,\nu(|\nabla f|) + u\,\left(\int (f-\nu(f))^p d\nu\right)^{\frac{1}{p}}\,, \tag{2.4}$$

then for all measurable A such that $\nu(A) \leq 1/2$,

$$\nu_s(\partial A) \geq \frac{1\,-\,u}{C_p(u)}\,\nu(A) \quad \textit{i.e.} \quad C'_C(\nu) \leq \frac{C_p(u)}{1\,-\,u}\,.$$

Proof. Let A be some Borel subset with $\nu(A) = \frac{1}{2}$. According to Lemma 3.5 in [13] one can find a sequence f_n of Lipschitz functions with $0 \leq f_n \leq 1$, such that $f_n \to \mathbf{1}_{\bar{A}}$ pointwise ($\bar{A}$ being the closure of A) and $\limsup \nu(|\nabla f_n|) \leq \nu_s(\partial A)$. According to the proof of Lemma 3.5 in [13], we may assume that $\nu(\bar{A}) = \nu(A)$ (otherwise $\nu_s(\partial A) = +\infty$). Taking limits in the left hand side of (2.3) thanks to Lebesgue's bounded convergence theorem, we thus obtain

$$\nu(|\mathbf{1}_A - \nu(A)|) \leq C(u)\,\nu_s(\partial A) + u\,.$$

The left hand side is equal to $2\nu(A)(1-\nu(A)) = \frac{1}{2}$ so that we obtain $\nu_s(\partial A) \geq \frac{\frac{1}{2}-u}{C(u)}$. It remains to use the concavity of Is_ν, which yields $\mathrm{Is}_\nu(u) \geq 2\,\mathrm{Is}_\nu(\frac{1}{2})\,u$.
If we replace (2.3) by (2.4), we similarly obtain, when $\nu(A) = \frac{1}{2}$, $\int (1_A - \nu(A))^p d\nu = (1/2)^p$, so that $\frac{1}{2} \leq C_p(u)\,\nu_s(\partial A) + \frac{1}{2}\,u$ and the result follows similarly. □

Remark 2.6. We may replace $\nu(f)$ by $m_\nu(f)$ in (2.3) without changing the proof, since the explicit form of the approximating f_n in [13] satisfies $m_\nu(f_n) \to \nu(A)$. ♢

According to Proposition 2.4, the previous proposition applies to log-concave measures. But in this case one can weaken the required inequalities.

Theorem 2.7. *Let ν a log-concave probability measure.*
Assume that there exist some $0 \leq s < 1/2$ and some $\beta(s)$ such that for any Lipschitz function f it holds

$$\nu(|f - \nu(f)|) \leq \beta(s)\; \| \, |\nabla f| \, \|_\infty \; + s\, Osc(f)\,, \tag{2.5}$$

respectively, for some $0 \leq s < 1$ and some $\beta(s)$

$$\nu(|f - \nu(f)|) \leq \beta(s)\; \| \, |\nabla f| \, \|_\infty \; + s\; (Var_\nu(f))^{\frac{1}{2}}\;. \tag{2.6}$$

Then

$$C'_C(\nu) \leq \frac{4\beta(s)}{\pi\,(\frac{1}{2}-s)^2} \qquad \textit{resp.} \qquad C'_C(\nu) \leq \frac{16\beta(s)}{\pi\,(1-s)^2}\,.$$

We may replace $\nu(f)$ by $m_\nu(f)$ in both cases.

Proof. In the sequel P_t denotes the symmetric semi-group with infinitesimal generator $L = \Delta - \nabla V.\nabla$. Here we assume for simplicity that V is smooth on the interior of $D = \{V < +\infty\}$ (which is open and convex), so that the generator acts on functions whose normal derivative on ∂D is equal to 0. From the probabilistic point of view, P_t is associated to the diffusion process with generator L normally reflected at the boundary ∂D.
A first application of zero curvature is the following, that holds for all $t > 0$,

$$\| \, |\nabla P_t g| \, \|_\infty \leq \frac{1}{\sqrt{\pi t}} \; \| \, g \, \|_\infty \;. \tag{2.7}$$

This result is proved using reflection coupling in Proposition 17 of [18]. With the slightly worse constant $\sqrt{2t}$, it was previously obtained by Ledoux in [40]. According to Ledoux's duality argument (see (5.5) in [40]), we deduce, with $g = f - \nu(f)$,

$$\nu(|g|) \leq \sqrt{4t/\pi}\,\nu(|\nabla f|) + \nu(|P_t g|)\,. \tag{2.8}$$

Note that $\nu(|P_t g|) = \nu(|P_t f - \nu(P_t f)|)$. Applying (2.5) with $P_t f$, we obtain

$$\nu(|f-\nu(f)|) \leq \sqrt{4t/\pi}\,\nu(|\nabla f|) + \beta(s)\ \parallel |\nabla P_t f| \parallel_\infty\ + s\, Osc(P_t f)\,.$$

Applying (2.7) again, and the contraction property of the semi-group in $\mathbb{L}^\infty$, yielding $Osc(P_t f) \leq Osc(f)$, we get

$$\nu(|f-\nu(f)|) \leq \sqrt{4t/\pi}\,\nu(|\nabla f|) + \left(s\ + \frac{\beta(s)}{\sqrt{\pi t}}\right)\, Osc(f)\,. \tag{2.9}$$

Choose $t = \frac{4\,\beta^2(s)}{\pi\,(\frac{1}{2}-s)^2}$. We may apply proposition 2.3 (and remark 2.6) with $u = (s+\frac{1}{2})/2$ which is less than $\frac{1}{2}$ and

$$C(u) = \frac{4\beta(s)}{\pi\,(\frac{1}{2}-s)}\,.$$

yielding the result.
If we want to deal with the case of $m_\nu(f)$ we have to slightly modify the proof. This time we choose $g = f - m_\nu(P_t f)$ so that, first $\nu(|f - m_\nu(f)|) \leq \nu(|g|)$, second $P_t g = P_t f - m_\nu(P_t f)$, so that we can apply (2.5) with the median. We are done by using remark 2.6.

Next we apply (2.6) and the contraction property of the semi-group in $\mathbb{L}^2$. We get

$$\nu(|f-\nu(f)|) \leq \sqrt{4t/\pi}\,\nu(|\nabla f|) + \beta(s)\ \parallel |\nabla P_t f| \parallel_\infty\ + s\ (\mathrm{Var}_\nu(f))^{\frac{1}{2}}\ .$$

But now, either $\mathrm{Var}_\nu(f) \leq \frac{1}{4}\, Osc(f)$ or $\mathrm{Var}_\nu(f) \geq \frac{1}{4}\, Osc(f)$.
In the first case we get

$$\nu(|f-\nu(f)|) \leq \sqrt{4t/\pi}\,\nu(|\nabla f|) + \beta(s)\ \parallel |\nabla P_t f| \parallel_\infty\ + \frac{s}{2}\, Osc(f)\,,$$

and as we did before we finally get, for

$$\nu(|f-\nu(f)|) \leq\ \frac{8\,\beta(s)}{\pi\,(1-s)}\,\nu(|\nabla f|) +\ \frac{s+1}{4}\, Osc(f)\,. \tag{2.10}$$

One can notice that $\frac{s+1}{4} < \frac{1}{2}$.
In the second case, we first have

$$\parallel |\nabla P_t f| \parallel_\infty \leq\ \frac{2}{\sqrt{\pi t}}\ (\mathrm{Var}_\nu(f))^{\frac{1}{2}}\ ,$$

so that finally

$$\nu(|f-\nu(f)|) \leq\ \frac{8\,\beta(s)}{\pi\,(1-s)}\,\nu(|\nabla f|) +\ \frac{s+1}{2}\ (\mathrm{Var}_\nu(f))^{\frac{1}{2}}\ . \tag{2.11}$$

Looking at the proof of proposition 2.5 we see that both situations yield exactly the same bound for the surface measure of a subset of probability less than or equal to 1/2 i.e the desired result.

If V is not smooth we may approximate ν by convolving with tiny gaussian mollifiers, so that the convolved measures are still log-concave according to Prekopa-Leindler theorem and with smooth potentials. If X has distribution ν and G is a standard gaussian vector independent

of X, ν_ε will denote the distribution of $X+\varepsilon G$.
It is immediate that for a Lipschitz function f,

$$\begin{aligned}\mathbb{E}(|f(X+\varepsilon G)-f(X)|) &\leq \varepsilon \parallel |\nabla f| \parallel_\infty \mathbb{E}(|G|)\\ &\leq \varepsilon\sqrt{n} \parallel |\nabla f| \parallel_\infty ,\end{aligned}$$

so that if ν satisfies (2.5), ν_ε also satisfies (2.5) with $\beta_\varepsilon(s)=\beta(s)+2\varepsilon\sqrt{n}$. We may thus use the result for ν_ε and let ε go to 0.
Assume now that ν satisfies (2.6). It holds

$$\begin{aligned}\nu_\varepsilon(|f-\nu_\varepsilon(f)|) &\leq \nu(|f-\nu(f)|)+2\sqrt{n}\,\varepsilon \parallel |\nabla f| \parallel_\infty\\ &\leq (\beta(s)+2\sqrt{n}\,\varepsilon) \parallel |\nabla f| \parallel_\infty +s\,(\mathrm{Var}_\nu(f))^{\frac{1}{2}} .\end{aligned}$$

But, assuming that $\nu(f)=0$ to simplify the notation,

$$\begin{aligned}\mathrm{Var}_\nu(f) &= \mathbb{E}(f^2(X+\varepsilon G))+\mathbb{E}((f(X+\varepsilon G)-f(X))^2)+2\mathbb{E}(f(X+\varepsilon G)(f(X)-f(X+\varepsilon G)))\\ &\leq \mathrm{Var}_{\nu_\varepsilon}(f)+(\mathbb{E}(f(X+\varepsilon G)))^2+n\,\varepsilon^2 \parallel |\nabla f| \parallel_\infty^2 +2\varepsilon\sqrt{n} \parallel |\nabla f| \parallel_\infty \mathbb{E}(|f(X+\varepsilon G)|)\\ &\leq \mathrm{Var}_{\nu_\varepsilon}(f)+4n\,\varepsilon^2 \parallel |\nabla f| \parallel_\infty^2 +2\varepsilon\sqrt{n} \parallel |\nabla f| \parallel_\infty \mathbb{E}(|f(X)|) .\end{aligned}$$

In particular if f is 1-Lipschitz and bounded by M, we get

$$\nu_\varepsilon(|f-\nu_\varepsilon(f)|)\leq \left(\beta(s)+2\varepsilon\sqrt{n}+s(4n\varepsilon^2+2M\varepsilon\sqrt{n})^{\frac{1}{2}}\right) + s\,\mathrm{Var}_{\nu_\varepsilon}(f)$$

i.e. using homogeneity, ν_ε also satisfies (2.6) with some β_ε, and we may conclude as before. For the median case just remark that $m_{\nu_\varepsilon}(f)$ goes to $m_\nu(f)$ as ε goes to 0. □

Using (1.3) we get similar bounds for $C_P(\nu)$.

Remark 2.8. Of course if ν satisfies a weak $(1,+\infty)$ Poincaré inequality with function $\beta(u)$, we obtain

$$C'_C(\nu)\leq \inf_{0\leq s<\frac{1}{2}} \frac{4\beta(s)}{\pi\,(\frac{1}{2}-s)^2} .$$

Using that β is non increasing, it follows that

$$C'_C(\nu)\leq \frac{4}{\pi\,(\frac{1}{2}-s_\nu)^4} \quad \text{where} \quad \beta(s_\nu)=\frac{1}{(\frac{1}{2}-s_\nu)^2} .$$

We should write similar statements replacing the Oscillation by the Variance. In a sense (2.5) looks more universal since the control quantities in the right hand side do not depend (except the constants of course) of ν. It is thus presumably more robust to perturbations. We shall see this later. Also notice that both (2.5) and (2.6) agree when $s=0$, which corresponds to an explicit bound for the Cheeger constant in E. Milman's theorem.
The advantage of (2.6) is that it looks like a deficit in the Cauchy-Schwarz inequality, since we may take s close to 1. ♢

Notice that in the non weak framework, a not too far proof (i.e. with a functional flavor) is given in [1] theorem 1.10.

Remark 2.9. Notice that (2.5) is unchanged if we replace f by $f+a$ for any constant a, hence we may assume that $\inf f=0$. Similarly it is unchanged if we multiply f by any M, hence we may choose $0\leq f\leq 1$ with $Osc(f)=1$. ♢

2.3. **Some variations and some immediate consequences.**

2.3.1. *Immediate consequences.* Now using our trivial proposition 2.2 (more precisely the version with the median) we immediately deduce:

Corollary 2.10. *For any log-concave probability measure ν,*

$$C'_C(\nu) \leq \inf_{0<s<\frac{1}{4}} \frac{16\,\alpha_\nu^{-1}(s)}{\pi\,(1-4s)^2} \quad \textit{and} \quad C_P(\nu) \leq \inf_{0<s<\frac{1}{4}} \left(\frac{32\,\alpha_\nu^{-1}(s)}{\pi\,(1-4s)^2}\right)^2 .$$

The fact that the concentration profile controls the Poincaré or the Cheeger constant of a log-concave probability measure was also discovered by E. Milman in [44]. Another (simpler) proof, based on the semi-group approach was proposed by Ledoux ([41]). We shall not recall Ledoux's proof, but tuning the constants in this proof furnishes worse constants than ours. The introduction of the weak version of the $(1,+\infty)$ Poincaré inequality is what is important here, in order to deduce such a control without any effort.
A similar and even better result was obtained by E. Milman in Theorem 2.1 of [45], namely

$$C'_C(\nu) \leq \frac{\alpha_\nu^{-1}(s)}{1-2s}$$

that holds for all $s < \frac{1}{2}$. The proof of this result lies on deep geometric results (like the Heintze-Karcher theorem) while ours is elementary. In a sense it is the semi-group approach alternate proof mentioned by E. Milman after the statement of its result.

Also notice that the previous corollary gives a new proof of Ledoux's result (1.4) but with a desperately worse constant. Indeed if we combine the previous bound for C'_C and some explicit estimate in the Gromov-Milman theorem (see respectively [50, 9]) i.e.

$$\alpha_\nu(r) \leq 16\, e^{-r/\sqrt{2C_P}} \quad \text{or} \quad \alpha_\nu(r) \leq e^{-r/3\sqrt{C_P}} ,$$

we obtain $(C'_C)^2(\nu) \leq m\, C_P(\nu)$ for some m. The reader will check that m is much larger than 36.

But actually one can recover Ledoux's result in a much more simple way: indeed

$$\nu(|f-\nu(f)|) \leq \sqrt{\nu(|f-\nu(f)|^2)} \leq \sqrt{C_P(\nu)}\, \nu^{\frac{1}{2}}(|\nabla f|^2) \leq \sqrt{C_P(\nu)}\, \parallel |\nabla f| \parallel_\infty$$

furnishes, thanks to Theorem 2.7,

Proposition 2.11. *If ν is a log-concave probability measure,*

$$C'_C(\nu) \leq C_C(\nu) \leq \frac{16}{\pi}\sqrt{C_P(\nu)} .$$

Since $16/\pi < 6$ this result is slightly better than Ledoux's result recalled in (1.4).

Remark 2.12. Another immediate consequence is the following: since

$$|f(x)-f(a)| \leq \parallel |\nabla f| \parallel_\infty |x-a|$$

we have for all a,

$$\nu(|f-m_\nu(f)|) = \inf_b \int (|f(x)-b|)\nu(dx) \leq \int |f(x)-f(a)|\nu(dx) \leq \parallel |\nabla f| \parallel_\infty \int |x-a|\nu(dx) .$$

Taking the infimum with respect to a in the right hand side, we thus have

$$\nu(|f-m_\nu(f)|) \leq \parallel |\nabla f| \parallel_\infty \int |x-m_\nu(x)|\nu(dx) \ \leq \parallel |\nabla f| \parallel_\infty \int |x-\nu(x)|\nu(dx)\,. \quad (2.12)$$

A stronger similar result (credited to Kannan, Lovász and Simonovits [33]) is mentioned in [1] p.11, namely

$$\mathrm{Var}_\nu(f) \leq 4\,\mathrm{Var}_\nu(x)\,\nu(|\nabla f|^2)\,, \quad (2.13)$$

where $\mathrm{Var}_\nu(x)=\nu(|x-\nu(x)|^2)$.
According to (2.12)

$$C'_C(\nu) \leq \frac{16}{\pi}\int |x-m_\nu(x)|\,d\nu \ \leq \ \frac{16}{\pi}\,\mathrm{Var}_\nu^{1/2}(x)\,.$$

In particular Since $16/\pi < 5,2$, a consequence is the bound $C_P(\nu) \leq 484\,\mathrm{Var}_\nu(x)$.
Notice that this result contains "diameter" bounds, i.e. if the support of ν is compact with diameter D, $C'_C(\nu) \leq \frac{16\,D}{\pi}$. In the isotropic situation one gets $C'_C(\nu) \leq \frac{16\,\sqrt{n}}{\pi}$. ♢

Remark 2.13. Consider an isotropic log-concave random vector X with distribution ν. If f is a Lipschitz function we have for all a,

$$\begin{aligned}\nu(|f-a|) \ &\leq \ \mathbb{E}\left[\left|f(X)-f\left(\sqrt{n}\,\frac{X}{|X|}\right)\right|\right]+\mathbb{E}\left[\left|a-f\left(\sqrt{n}\,\frac{X}{|X|}\right)\right|\right] \\ &\leq \ \parallel |\nabla f| \parallel_\infty \, \mathbb{E}\left[\left||X|-\sqrt{n}\right|\right]+\mathbb{E}\left[\left|a-f\left(\sqrt{n}\,\frac{X}{|X|}\right)\right|\right]. \quad (2.14)\end{aligned}$$

Hence, if we choose $a=\mathbb{E}\left[f\left(\sqrt{n}\,\frac{X}{|X|}\right)\right]$ and if we denote by ν_{angle} the distribution of $X/|X|$ we obtain

$$\nu(|f-m_\nu(f)|) \leq 2\ \parallel |\nabla f| \parallel_\infty \left(\mathbb{E}\left[\left||X|-\sqrt{n}\right|\right]+\sqrt{n}\,\sqrt{C_P(\nu_{angle})}\right). \quad (2.15)$$

This shows that the Cheeger constant of ν is completely determined by the concentration of the radial part of X around $\sqrt{n}$ (which is close to its mean), and the Poincaré constant (we should also use the Cheeger constant) of $X/|X|$.
In particular, if ν is spherically symmetric, the distribution of $X/|X|$ is the uniform measure on the sphere S^{n-1} which is known to satisfy (provided $n \geq 2$) a Poincaré inequality with Poincaré constant equal to $1/n$ for the usual euclidean gradient (not the riemanian gradient on the sphere). In addition, in this situation its known that

$$\mathbb{E}\left[\left||X|-\sqrt{n}\right|\right] \leq 1 \qquad \text{see [11] formula (6)}$$

so that we get that $C'_C(\nu) \leq \frac{64}{\pi}$ for an isotropic radial log-concave probability measure. Since $\frac{16}{\pi}\sqrt{12} < \frac{64}{\pi}$, this result is worse than the one proved in [11] telling that $C_P(\nu) \leq 12$ so that $C'_C(\nu) \leq \frac{16}{\pi}\sqrt{12}$ thanks to proposition 2.11 (12 may replace the original 13 thanks to a remark by N. Huet [32]). Actually, applying (2.13) in dimension 1, it seems that we may replace 12 by 4. ♢

2.3.2. *Variations: the $\mathbb{L}^2$ framework.* In some situations it is easier to deal with variances. We recalled that any (nice) absolutely continuous probability measure satisfies a weak $(2,2)$ Poincaré inequality.

Theorem 2.14. *Let ν a log-concave probability measure satisfying the weak $(2,2)$ Poincaré inequality, for some $s<\frac{1}{6}$,*

$$Var_\nu(f) \leq \beta(s)\nu(|\nabla f|^2) + s\, Osc^2(f)\,.$$

Then

$$C_C'(\nu) \leq \frac{4\sqrt{\beta(s)\ln 2}}{1-6s} \quad \text{and} \quad C_P(\nu) \leq 4\,(C_C'(\nu))^2\,.$$

Proof. We start with the following which is also due to Ledoux ([39]): if ν is log-concave, then for any subset A,

$$\sqrt{t}\,\nu_s(\partial A) \;\geq\; \nu(A) - \;\nu\left((P_t\mathbf{1}_A)^2\right)\,.$$

But

$$\nu\left((P_t\mathbf{1}_A)^2\right) = \mathrm{Var}_\nu(P_t\mathbf{1}_A) + (\nu(P_t\mathbf{1}_A))^2 = \mathrm{Var}_\nu(P_t\mathbf{1}_A) + \nu^2(A)\,.$$

Define $u(t)=\mathrm{Var}_\nu(P_t\mathbf{1}_A)$. Using the semi-group property and the weak Poincaré inequality it holds

$$\frac{d}{dt}u(t) = -2\,\nu(|\nabla P_t\mathbf{1}_A|^2) \leq \frac{-2}{\beta(s)}\,(u(t)-s)$$

since $Osc(P_t\mathbf{1}_A)\leq 1$. Using Gronwall's lemma we thus obtain

$$\mathrm{Var}_\nu(P_t\mathbf{1}_A) \leq e^{-2t/\beta(s)}\,\nu(A) + s\,\left(1-e^{-2t/\beta(s)}\right)$$

so that finally, if $\nu(A)=1/2$ we get

$$\sqrt{t}\,\nu_s(\partial A) \geq \left(1-e^{-2t/\beta(s)}\right)\left(\frac{1}{2}-s\right) - \frac{1}{4}\,.$$

Choose $t=\beta(s)\,\ln 2$. The right hand side in the previous inequality becomes $\frac{1}{8}(1-6s)$, hence

$$\mathrm{Is}_\nu(1/2) \geq \frac{1-6s}{8\sqrt{\beta(s)\,\ln 2}}\,.$$

Hence the result arguing as in the previous proof. □

2.3.3. *Other consequences. Reducing the support.* All the previous consequences are using either the weak or the strong $(1,+\infty)$ Poincaré inequality. The next consequence will use the full strength of what precedes.
Pick some Borel subset A. Let $a\in\mathbb{R}$ and f be a smooth function. Then, provided $\inf f \leq a \leq \sup f$,

$$\nu(|f-a|) \leq \int_A |f-a|d\nu + (1-\nu(A))\,\parallel f-a\parallel_\infty \leq \int_A |f-a|d\nu + (1-\nu(A))\,Osc(f)\,.$$

Denote $d\nu_A = \frac{\mathbf{1}_A}{\nu(A)}\,d\nu$ the restriction of ν to A. Choosing $a=m_{\nu_A}(f)$ we have

$$\begin{aligned}\nu(|f-m_\nu(f)|) &\leq \nu(|f-a|) \leq \nu(A)\,\nu_A(|f-a|) + (1-\nu(A))\,Osc(f)\\ &\leq \nu(A)\,\left(\beta_{\nu_A}(u)\,\parallel |\nabla f|\parallel_\infty + u\,Osc(f)\right) + (1-\nu(A))\,Osc(f)\\ &\leq \nu(A)\,\beta_{\nu_A}(u)\,\parallel|\nabla f|\parallel_\infty + (1-\nu(A)(1-u))\,Osc(f)\,,\end{aligned}$$

provided ν_A satisfies some $(1,+\infty)$ Poincaré inequality. We can improve the previous bound, if ν_A satisfies some Cheeger inequality and get

$$\nu(|f-m_\nu(f)|) \leq \nu(A)\, C'_C(\nu_A)\, \nu(|\nabla f|) + (1-\nu(A))\, Osc(f)\,.$$

Hence, applying theorem 2.7 or proposition 2.5 we have:

Proposition 2.15. *Let ν be a log-concave probability measure, A be any subset with $\nu(A) > \frac{1}{2}$ and $d\nu_A = \frac{\mathbf{1}_A}{\nu(A)}\, d\nu$ be the (normalized) restriction of ν to A. Then*

(1)
$$C'_C(\nu) \leq \frac{\nu(A)\, C'_C(\nu_A)}{2\nu(A)-1}\,,$$

(2) *if ν_A satisfies a $(1,+\infty)$ weak Poincaré inequality with rate β_{ν_A}, then ν satisfies a $(1,+\infty)$ weak Poincaré inequality with rate β_ν satisfying for $u < 1-\frac{1}{2\nu(A)}$,*
$$\beta_\nu(u) \leq \nu(A)\, \beta_{\nu_A}\left(1-\frac{1-u}{\nu(A)}\right)$$
so that
$$C'_C(\nu) \leq \frac{4\nu(A)\, \beta_{\nu_A}(u)}{\pi\,((1-u)\nu(A)-\frac{1}{2})^2}\,.$$

Remark 2.16. A similar result is contained in [44] namely if K is some convex body,

$$C'_C(\nu) \leq \frac{1}{\nu^2(K)}\, C'_C(\nu_K)\,.$$

This result is similar when $\nu(K)$ is close to 1, but it requires the convexity of K. Convexity of K ensures that ν_K is still log-concave. We shall come back to this point later. Of course our result does not cover the situation of sets with small measure.
In addition a careful look at the proof of Lemma 5.2 in [44] indicates that when $\nu(A) > 1/2$, the convexity of A is not required. (5.2) in [44] thus furnishes the same result as (1) in the previous Proposition up to the pre-factor $\nu(A)$. ♢

This result enables us to reduce the study of log-concave probability measures to the study of compactly supported distributions, arguing as follows. Let Z be a random variable (in $\mathbb{R}^n$) with log concave distribution ν. We may assume without loss of generality that Z is centered. Denote by $\sigma^2(Z)$ the largest eigenvalue of the covariance matrix of Z.

- *l^2 truncation.*

  Thanks to Chebyshev inequality, for $a > 1$,
  $$\mathbb{P}(|Z| > a\,\sigma(Z)\,\sqrt{n}) \leq \frac{1}{a^2}\,.$$
  According to Proposition 2.15, if $a > \sqrt{2}$,
  $$C'_C(Z) \leq \frac{a^2}{a^2-2}\, C'_C(Z(a))$$
  where $Z(a)$ is the random variable $\mathbf{1}_{Z\in K_a}\, Z$ supported by $K_a = B(0, a\,\sigma(Z)\,\sqrt{n})$ with distribution $\frac{\mathbf{1}_{K_a}}{\nu(K_a)}\,\nu$. Of course the new variable $Z(a)$ is not necessarily centered, but

we may, without changing the Poincaré constant(s), consider the variable $\bar{Z}(a) = Z(a) - \mathbb{E}(Z(a))$. It is easily seen that

$$|\mathbb{E}(Z_i(a))| \leq |\mathbb{E}(-Z_i \, \mathbf{1}_{K_a^c}(Z))| \leq \frac{\mathrm{Var}^{1/2}(Z_i)}{a} \leq \frac{\sigma(Z)}{a} \, ,$$

i.e.

$$\sum_{i=1}^{n} |\mathbb{E}(Z_i(a))|^2 \leq \frac{n\sigma^2(Z)}{a^2}$$

so that $\bar{Z}(a)$ is centered and supported by $B(0, \sqrt{a^2 + (1/a^2)}\, \sigma(Z)\, \sqrt{n})$.
Notice that for all i,

$$\mathrm{Var}(Z_i) \geq \mathrm{Var}(\bar{Z}_i(a)) \; \geq \; \mathrm{Var}(Z_i) \left(1 - \frac{\kappa}{a} - \frac{1}{a^2}\right) \, , \tag{2.16}$$

where κ is the universal Khinchine constant, i.e. satisfies

$$\mu(z^4) \leq \kappa^2 \, (\mu(z^2))^2$$

for all log concave probability measure on $\mathbb{R}$. According to the one dimensional estimate of Bobkov ([10] corollary 4.3) we already recalled, we know that $\kappa \leq 7$.
The upper bound in (2.16) is immediate, while for the lower bound we use the following reasoning :

$$\begin{aligned} \mathrm{Var}(\bar{Z}_i(a)) &= \mathbb{E}(Z_i^2 \, \mathbf{1}_{K_a}(Z)) - (\mathbb{E}(Z_i(a)))^2 \\ &\geq \mathrm{Var}(Z_i) - \mathbb{E}(Z_i^2 \, \mathbf{1}_{K_a^c}(Z)) - (\mathbb{E}(Z_i(a)))^2 \\ &\geq \mathrm{Var}(Z_i) \; - \frac{1}{a} \, (\mathbb{E}(Z_i^4))^{\frac{1}{2}} \; - \frac{\mathrm{Var}(Z_i)}{a^2} \end{aligned}$$

according to the previous bound on the expectation. We conclude by using Khinchine inequality.
Similar bounds are thus available for $\sigma^2 \bar{Z}(a)$ in terms of $\sigma^2(Z)$.

Remark 2.17. Though we gave explicit forms for all the constants, they are obviously not sharp.
For instance we used the very poor Chebyshev inequality for reducing the support of ν to some euclidean ball, while much more precise concentration estimates are known. For an isotropic log-concave random variable (vector) Z, it is known that its distribution satisfies some "concentration" property around the sphere of radius $\sqrt{n}$.
We shall here recall the best known result, due to Lee and Vempala [42], improving on Guédon and E. Milman ([31]) (also see [16] chapter 13 for an almost complete overview of the state of the art):

Theorem 2.18. *[Lee and Vempala] Let Z be an isotropic log-concave random vector in $\mathbb{R}^n$. Then there exist some universal positive constants C and c such that for all $t > 0$,*

$$\mathbb{P}(|\,|Z| - \sqrt{n}\,| \geq t\,\sqrt{n}) \; \leq \; C \, \exp(-\, c\sqrt{n} \, \min\{t, t^2\}) \, . \qquad \diamondsuit$$

- l^∞ truncation.

Instead of looking at euclidean balls we shall look at hypercubes, i.e. l^∞ balls. We assume that Z is centered.

According to Prekopa -Leindler theorem again, we know that the distribution ν_1 of Z_1 is a log-concave distribution with variance $\lambda_1^2 = \text{Var}(Z_1)$. Hence ν_1 satisfies a Poincaré inequality with $C_P(\nu_1) \leq 12\,\lambda_1^2$ according to [10] proposition 4.1. Of course a worse bound was obtained in remark 2.13.
Using Proposition 4.1 in [14] (see lemma 3.7 in the next section), we have for all $1 \geq \epsilon > 0$

$$\nu_1(z_1 > a_1) \leq \frac{4-\epsilon}{\epsilon}\, e^{-\frac{(2-\epsilon)a_1}{\sqrt{C_P(\nu_1)}}} \leq \frac{4-\epsilon}{\epsilon}\, e^{-\frac{(2-\epsilon)a_1}{2\sqrt{3}\,\lambda_1}}\,. \tag{2.17}$$

Of course changing Z in $-Z$ we get a similar bound for $\nu_1(z_1 < -a_1)$. Choosing $a_1 = \frac{2\sqrt{3}}{2-\epsilon}\,\lambda_1\, a\, \ln n$ for some $a > 0$, we get

$$\nu_1(|z_1| > a_1) \leq 2\frac{4-\epsilon}{\epsilon n^a}\,.$$

Hence if

$$K_a = \left\{ \max_{i=1,\ldots,n} |z_i| < \frac{2\sqrt{3}}{2-\epsilon}\,\sigma(Z)\, a\, \ln n \right\} \tag{2.18}$$

we have

$$\nu(K_a) \geq 1 \,-\, 2\frac{4-\epsilon}{\epsilon n^{a-1}}\,.$$

Using Proposition 2.15 we have

Proposition 2.19. *For $n \geq 2$, K_a being defined by (2.18), we have*

$$C'_C(\nu) \leq \frac{n^{a-1}}{n^{a-1} - (8-2\epsilon)\epsilon^{-1}}\; C'_C(\nu_{K_a})\,.$$

Again we may center ν_{K_a}, introducing a random vector $\bar{Z}$ with distribution equal to the re-centered ν_{K_a}. This time it is easily seen that

$$(1-\varepsilon(n))\,\text{Var}(Z_i) \leq \text{Var}(\bar{Z}_i) \leq\, \text{Var}(Z_i)\,,$$

with $\varepsilon(n) \to 0$ as $n \to +\infty$.

Notice that we have written a more precise version of Latala's deviation result ([38])

Theorem 2.20. *Let Z be an isotropic log-concave random vector in $\mathbb{R}^n$. Then for $n \geq 2$, $1 \geq \epsilon > 0$,*

$$\mathbb{P}\left(\max_{i=1,\ldots,n} |Z_i| \geq t \ln n\right) \leq \frac{(8-2\epsilon)\epsilon^{-1}}{n^{\frac{(2-\epsilon)t}{2\sqrt{3}}-1}} \qquad \textit{for } t \geq \frac{2\sqrt{3}}{2-\epsilon}\,.$$

Let us give another direct application. ν is said to be unconditional if it is invariant under the transformation $(x_1,...,x_n) \mapsto (\varepsilon_1\, x_1,...,\varepsilon_n\, x_n)$ for any n-uple $(\varepsilon_1,...,\varepsilon_n) \in \{-1,+1\}^n$. Defining K_a as in (2.18) (with $\sigma(Z) = 1$ here), the restricted measure ν_{K_a} is still unconditional.

So we may apply Theorem 1.1 in [23] (and its proof in order to find an explicit expression of the constant) saying that ν_{K_a} satisfies some weighted Poincaré inequality

$$\mathrm{Var}_{\nu_{K_a}}(f) \,\leq\, (4\sqrt{3}+1)^2 \sum_{i=1}^n \nu_{K_a}\left(\nu_{K_a}^{i-1}(x_i^2)\,(\partial_i f)^2\right)\,,$$

where $\nu_{K_a}^j$ denotes the conditional distribution of ν_{K_a} w.r.t. the sigma field generated by $(x_1, ..., x_j)$. Actually, up to the pre-constant, we shall replace the weight $\nu_{K_a}^{i-1}(x_i^2)$ by the simpler one $x_i^2 + \nu_{K_a}(x_i^2)$ according to [35].

In all cases, since x_i is ν_{K_a} almost surely bounded by a constant times $\ln(n)$, we have obtained thanks to Proposition 2.19 the following result originally due to Klartag [34]

Proposition 2.21. *There exists an universal constant c such that, if ν is an isotropic and unconditional log-concave probability measure,*

$$C_P(\nu) \leq c\, \max(1, \ln^2 n)\,.$$

Of course what is needed here is the invariance of ν_{K_a} with respect to some symmetries (see [23, 7]). But it is not easy to see how ν_{K_a} inherits such a property satisfied by the original ν, except in the unconditional case.

3. Transference of the Poincaré inequality.

3.1. Transference via absolute continuity.

If μ and ν are probability measures it is well known (Holley-Stroock argument) that

$$C_P(\nu) \,\leq \parallel \frac{d\nu}{d\mu} \parallel_\infty \parallel \frac{d\mu}{d\nu} \parallel_\infty\, C_P(\mu)\,,$$

the same being true with the same pre-factor if we replace C_P by C_C or C'_C. Similar results are known for weak $(2,2)$ Poincaré inequalities too. In this section we shall give several transference principles allowing us to reinforce this result, at least under some curvature assumption.

Such transference results have been obtained in [8] using transference results for the concentration profile.

We first recall the statement of Proposition 2.2 in [8], in a simplified form :

Proposition 3.1. *[Barthe-Milman] Recall that the concentration profile α_ν is defined in Proposition 2.2.*
Assume that for some $1 < p \leq +\infty$,

$$\int \left|\frac{d\nu}{d\mu}\right|^p d\mu \,=\, M_p^p \,<\, +\infty\,.$$

Then if $q = \frac{p}{p-1}$, for all $r > 0$,

$$\alpha_\nu(r) \leq 2M_p\, \alpha_\mu^{1/q}(r/2)\,.$$

We may use this result together with corollary 2.10 to deduce

Corollary 3.2. *Under the assumptions of Proposition 3.1, if ν is log-concave,*

$$C'_C(\nu) \leq \inf_{0<s<\frac{1}{4}} \frac{32\,\alpha_\mu^{-1}((s/2M_p)^q)}{\pi(1-4s)^2}\,.$$

Notice that, using the equivalence results between concentration and Poincaré inequality of [44], Barthe and E. Milman have shown a comparison result in [8] Theorem 2.7, namely:

Corollary 3.3. *[Barthe-Milman] If ν is log-concave and satisfies the assumptions of Proposition 3.1, there exists some universal constant c such that*

$$C_P(\nu) \leq c\,\frac{p-1}{p}\,\frac{1}{1+\ln(M_p)}\,C_P(\mu)\,.$$

The goal of the remaining part of this subsection is to derive similar results using our approach. This allows us to obtain more general results since we shall consider only weak Poincaré inequalities. We can also trace the constants. However, in the log-concave situation, despite the explicit constants, our estimates have a worse dependence in M_p. Notice that we will also consider the relative entropy instead of the $\mathbb{L}^p$ norm. The latter has also been done in [46] Theorem 5.7. Comparison of this Theorem and the results below seems difficult.

Theorem 3.4. *Let ν and μ be two probability measures.*

(1) *If for some $1<p$,*

$$\int \left|\frac{d\nu}{d\mu}\right|^p d\mu \,=\, M_p^p \,<\, +\infty\,,$$

then

$$\beta_\nu(s) \,\leq\, M_p^{p/(p-1)}\,s^{-1/(p-1)}\,C(\mu)\,,$$

where $C(\mu)$ can be chosen as $C'_C(\mu)$, $1/B_{1,\infty}(\mu)$, $C_C(\mu)$ or $\sqrt{C_P(\mu)}$.
In particular if ν is log-concave

$$C'_C(\nu) \,\leq\, D\,C(\mu)\,M_p^{p/(p-1)}$$

where $D = \frac{16\,(p+1)^{1/(p-1)}}{\pi\,(\frac{1}{2}-\frac{1}{p+1})^2}$.

(2) *Let ν be log-concave. If the relative entropy $D(\nu||\mu) := \int \ln(d\nu/d\mu)\,d\nu$ is finite, then with the same $C(\mu)$ and any $u<\frac{1}{2}$,*

$$C'_C(\nu) \,\leq\, \frac{4\,\left(e^{2\max(1,D(\nu||\mu))/u}-1\right)}{\pi\,(\frac{1}{2}-u)^2}\,C(\mu)\,.$$

Proof. Let f be a smooth function. It holds, provided $\inf f \leq a \leq \sup f$,

$$\begin{aligned} \nu(|f-m_\nu(f)|) &\leq \nu(|f-a|) \\ &\leq \int |f-a| \, \frac{d\nu}{d\mu} \, d\mu \\ &\leq K \int \mathbf{1}_{d\nu/d\mu \leq K} \, |f-a| \, d\mu \, + \, Osc(f) \int \mathbf{1}_{d\nu/d\mu > K} \, d\nu \\ &\leq K \int |f-a| \, d\mu \, + \, Osc(f) \int \mathbf{1}_{d\nu/d\mu > K} \, d\nu \, . \end{aligned} \tag{3.1}$$

In order to control the last term we may use Hölder (or Orlicz-Hölder) inequality. In the usual $\mathbb{L}^p$ case we have, using Markov inequality

$$\begin{aligned} \int \mathbf{1}_{d\nu/d\mu > K} \, d\nu &\leq M_p \, \mu^{\frac{1}{q}} \left(\frac{d\nu}{d\mu} > K \right) \\ &\leq M_p \, \frac{M_p^{p/q}}{K^{p/q}} = \frac{M_p^{1+(p/q)}}{K^{p/q}} = \frac{M_p^p}{K^{p-1}} \, , \end{aligned}$$

so that choosing $K = M_p^{p/(p-1)} \, u^{-1/(p-1)}$ we have obtained

$$\nu(|f-m_\nu(f)|) \leq M_p^{p/(p-1)} u^{-1/(p-1)} \int |f-a| \, d\mu \, + \, u \, Osc(f) \, .$$

Then using either $a = m_\mu(f)$ or $a = \mu(f)$ we get the desired result, with $C(\mu) = C'_C(\mu)$ or $1/B_{1,\infty}(\mu)$ or $C_C(\mu)$ or $\sqrt{C_P(\mu)}$. If in addition ν is log-concave it follows

$$C'_C(\nu) \, \leq \, \frac{16}{\pi \, (\frac{1}{2} - u)^2 \, u^{1/(p-1)}} \, M_p^{\frac{p}{p-1}} \, C(\mu) \, .$$

It remains to optimize in u for $0 < u < \frac{1}{2}$ and elementary calculations show that the maximum is attained for $u = 1/(p+1)$.

Starting with (3.1) we may replace Hölder's inequality by Orlicz-Hölder's inequality for a pair of conjugate Young functions θ and θ^*, that is

$$\int \mathbf{1}_{d\nu/d\mu > K} \, d\nu \leq \, 2 \, \| \, \frac{d\nu}{d\mu} \, \|_{\mathbb{L}_\theta(d\mu)} \, \| \, \mathbf{1}_{d\nu/d\mu > K} \, \|_{\mathbb{L}_{\theta^*}(d\mu)} \; .$$

Here the chosen Orlicz norm is the usual gauge (Luxemburg) norm, i.e.

$$\| \, h \, \|_{L_\theta(\mu)} = \inf\{b \geq 0 \quad s.t. \quad \int \theta(|h|/b) \, d\mu \leq 1\} \, ,$$

and recall that for any $\lambda > 0$,

$$\| \, h \, \|_{L_\theta(\mu)} \leq \frac{1}{\lambda} \, \max(1 \, , \, \int \theta(\lambda |h|) d\mu) \, . \tag{3.2}$$

For simplicity we will perform the calculations only for the pair of conjugate Young functions

$$\theta^*(u) = e^{|u|} - 1 \quad , \quad \theta(u) = (|u| \ln(|u|) + 1 - |u|) \, .$$

According to what precedes

$$\| \, \frac{d\nu}{d\mu} \, \|_{\mathbb{L}_\theta(d\mu)} \leq \, \max(1 \, , \, D(\nu||\mu))$$

and

$$\int \theta(\mathbf{1}_{d\nu/d\mu>K}/b)\, d\mu \;=\; (e^{1/b}-1)\, \mu\left(\frac{d\nu}{d\mu}>K\right) \leq (e^{1/b}-1)\, \frac{1}{K}\,,$$

the final bound being non optimal since we only use $\mathbf{1}_{d\nu/d\mu>K} \leq \frac{1}{K}\frac{d\nu}{d\mu}$ and not the better integrability of the density. Using the best integrability does not substantially improve the bound. We thus obtain

$$\| \mathbf{1}_{d\nu/d\mu>K} \|_{\mathbb{L}_{\theta^*(d\mu)}} = \frac{1}{\ln(1+K)}$$

and we may conclude as before. □

The method we used in the previous proof is quite rough. One can expect (in particular in the entropic case) to improve upon the constants, using more sophisticated tools. This is the goal of the next result.

Theorem 3.5. *Let ν and μ be two probability measures.*

(1) *If for some $1<p\leq 2$,*

$$\int \left|\frac{d\nu}{d\mu}\right|^p d\mu \,=\, M_p^p \,<\, +\infty\,,$$

then

$$\frac{1}{B_{1,\infty}(\nu)} \;\leq\; \left(\frac{p}{p-1}\, 8^{\frac{p}{(p-1)}}\, C_P(\mu)\right)^{\frac{1}{2}}\, M_p\,.$$

If in addition ν is log-concave

$$C'_C(\nu) \;\leq\; \frac{16\sqrt{p}}{\pi\,\sqrt{p-1}}\, 8^{\frac{p}{2(p-1)}}\, \sqrt{C_P(\mu)}\, M_p\,.$$

(2) *If the relative entropy $D(\nu||\mu) := \int \ln(d\nu/d\mu)\, d\nu$ is finite, then*

$$\frac{1}{B_{1,\infty}(\nu)} \;\leq 2\,\sqrt{C_P(\mu)}\, \max\left(1\,,\, 3e^{\sqrt{C_P(\mu)}}\right)\, \max\left(1\,,\, D(\nu||\mu)\right)\,.$$

If in addition ν is log-concave

$$C'_C(\nu) \;\leq\; \frac{32}{\pi}\,\sqrt{C_P(\mu)}\, \max\left(1\,,\, 3e^{\sqrt{C_P(\mu)}}\right)\, \max\left(1\,,\, D(\nu||\mu)\right)\,.$$

Proof. Let f be a smooth function. We have

$$\frac{1}{2}\,\nu(|f-\nu(f)|) \;\leq\; \nu(|f-m_\nu(f)|) \leq \nu(|f-\mu(f)|) \leq \left(\int |f-\mu(f)|^q\, d\mu\right)^{1/q} \left(\int \left|\frac{d\nu}{d\mu}\right|^p d\mu\right)^{1/p}. \tag{3.3}$$

Now we can use the results in [20], in particular the proof of Theorem 1.5 (see formulae 2.3, 2.7 and 2.8 therein) where the following is proved:

Lemma 3.6. *For all $q \geq 2$, it holds*

$$\int |f-\mu(f)|^q \, d\mu \; \leq \; q \, 8^q \, C_P(\mu) \, \int |f-\mu(f)|^{q-2} \, |\nabla f|^2 \, d\mu \, .$$

It is at this point that we need $p \leq 2$. Using Hölder inequality we deduce

$$\int |f-\mu(f)|^q \, d\mu \leq \; (q \, 8^q \, C_P(\mu))^{\frac{q}{2}} \, \int |\nabla f|^q \, d\mu \, . \tag{3.4}$$

It follows

$$\nu(|f-m_\nu(f)|) \leq \; (q \, 8^q \, C_P(\mu))^{\frac{1}{2}} \, M_p \; \parallel |\nabla f| \parallel_\infty \; .$$

Since ν is log-concave, we get the desired result, using theorem 2.7.

Now we turn to the second part of the Theorem which is based on Proposition 4.1 in [14] we recall now:

Lemma 3.7. *[Bobkov-Ledoux] If g is Lipschitz,*

$$\mu(e^{\lambda g}) \; \leq \; \left(\frac{2+\lambda \, C_P^{\frac{1}{2}}(\mu) \; \parallel |\nabla g| \parallel_\infty}{2-\lambda \, C_P^{\frac{1}{2}}(\mu) \; \parallel |\nabla g| \parallel_\infty}\right) \; e^{\lambda \mu(g)} \, ,$$

provided $2 > \lambda \, C_P^{\frac{1}{2}}(\mu) \; \parallel |\nabla g| \parallel_\infty > \; 0$.

Hence, in (3.3), we may replace the use of Hölder inequality by the one of the Orlicz-Hölder inequality

$$\nu(|f-m_\nu(f)|) \leq 2 \; \parallel f-\mu(f) \parallel_{L_\theta(\mu)} \parallel \frac{d\nu}{d\mu} \parallel_{L_{\theta^*}(\mu)} \; .$$

Again we are using the pair of conjugate Young functions

$$\theta(u) = e^{|u|}-1 \quad , \quad \theta^*(u) = |u| \ln(|u|) + 1 - |u| \, .$$

Without loss of generality we can first assume that f (hence $f-\mu(f)$) is 1-Lipschitz.
We then apply (3.2) and Lemma 3.7 with $g = |f-\mu(f)|$ and $\lambda = 1/\sqrt{C_P(\mu)}$. Since

$$\mu(|g|) \leq \mu^{\frac{1}{2}}(|g|^2) \leq C_P^{\frac{1}{2}}(\mu) \, \mu^{\frac{1}{2}}(|\nabla g|^2) \leq C_P^{\frac{1}{2}}(\mu) \; \parallel |\nabla g| \parallel_\infty ,$$

we obtain

$$\parallel f-\mu(f) \parallel_{L_\theta(\mu)} \leq \sqrt{C_P(\mu)} \, \max\left(1 \, , \, 3e^{\sqrt{C_P(\mu)}}\right) \; \parallel |\nabla f| \parallel_\infty \; .$$

Similarly

$$\parallel \frac{d\nu}{d\mu} \parallel_{L_{\theta^*}(\mu)} \leq \max\left(1 \, , \, D(\nu||\mu)\right) \; .$$

Again we conclude thanks to theorem 2.7. □

As usual we should try to optimize in p for p going to 1 depending on the rate of convergence of M_p to 1.
We cannot really compare both theorems, since the constant $C(\mu)$ in the first theorem can take various values, while it is the usual Poincaré constant in the second theorem. In the $\mathbb{L}^p$ case, the first theorem seems to be better for large $p's$ and the second one for small $p's$. In the entropic case, the second one looks better.

Finally, let us recall the following beautiful transference result between two log-concave probability measures proved in [44] and which is a partial converse of the previous results

Proposition 3.8. *[E. Milman] Let μ and ν be two log-concave probability measures. Then*

$$C'_C(\nu) \leq \left|\left|\frac{d\mu}{d\nu}\right|\right|_\infty^2 C'_C(\mu)\,.$$

3.2. **Transference using distances.**

As shown in [44] Theorem 5.5, the ratio of the Cheeger constants of two log-concave probability measures is controlled by their total variation distance (which is the half of the W_0 Wasserstein distance). Recall the equivalent definitions of the total variation distance

Definition 3.9. If μ and ν are probability measures the total variation distance $d_{TV}(\mu,\nu)$ is defined by one of the following equivalent expressions

$$\begin{aligned} d_{TV}(\mu,\nu) \quad &:= \quad \frac{1}{2} \sup_{\|f\|_\infty \leq 1} |\mu(f)-\nu(f)| \\ &= \quad \sup_{0\leq f\leq 1} |\mu(f)-\nu(f)| \\ &= \quad \inf\{\mathbb{P}(X\neq Y)\,;\, \mathcal{L}(X)=\mu\,,\, \mathcal{L}(Y)=\nu\,\} \end{aligned}$$

which is still equal to

$$\frac{1}{2}\int \left|\frac{d\mu}{dx} - \frac{d\nu}{dx}\right| dx$$

when μ and ν are absolutely continuous w.r.t. Lebesgue measure.
The second equality is immediate just noticing that for $0\leq f\leq 1$, $\mu(f-\frac{1}{2})-\nu(f-\frac{1}{2}) = \mu(f)-\nu(f)$ and that $\parallel f-\frac{1}{2}\parallel_\infty \leq \frac{1}{2}$.

More precisely one can show the following explicit result, due to E. Milman ([44] Theorem 5.5), except the bound for the constant κ:

Theorem 3.10. *[E. Milman] Let $\mu(dx)=e^{-V(x)}dx$ and $\nu(x)=e^{-W(x)}dx$ be two log-concave probability measures. If $d_{TV}(\mu,\nu)=1-\varepsilon$, for some $\varepsilon>0$, then*

$$C'_C(\nu) \;\leq\; \frac{\kappa}{\varepsilon^2}\left(1 \vee \ln\left(\frac{1}{\varepsilon}\right)\right) C'_C(\mu)\,,$$

for some universal constant κ one can choose equal to $(192\,e/\pi)$.

Proof. We give a short proof (adapted from [44]) but that does not use concentration results. First if Z_μ and Z_ν are random variables with respective distribution μ and ν, and $\lambda>0$ the total variation distance between the distributions of λZ_μ and λZ_ν is unchanged, hence still equal to $1-\varepsilon$. Choosing $\lambda = 1/\sqrt{C_P(\mu)}$ we may thus assume that $C_P(\mu)=1$.
Introduce the probability measure $\theta(dx)=\frac{1}{\varepsilon}\min(e^{-V(x)},e^{-W(x)})\,dx$ which is still log-concave and such that $d\theta/d\mu$ and $d\theta/d\nu$ are bounded by $1/\varepsilon$. Using proposition 3.8 we first have $C'_C(\nu)\leq \frac{1}{\varepsilon^2}\,C'_C(\theta)$. Next $D(\theta||\mu)\leq \ln(1/\varepsilon)$ so that using theorem 3.5 (2) we have $C'_C(\theta)\leq \frac{96\,e}{\pi}\max(1,\ln(1/\varepsilon))$. It follows that

$$C'_C(\nu) \leq \frac{96\,e}{\pi\,\varepsilon^2}\max(1,\ln(1/\varepsilon))$$

provided $C_P(\mu) = 1$. It remains to use $C'_C(\lambda Z_\nu) = \lambda C'_C(\nu) = (1/\sqrt{C_P(\mu)})\, C'_C(\nu)$ and $\sqrt{C_P(\mu)} \leq 2C'_C(\mu)$ to get the result. □

But since the $(1, +\infty)$ Poincaré inequality deals with Lipschitz functions, it is presumably more natural to consider the W_1 Wasserstein distance

$$W_1(\nu, \mu) := \sup_{f\, 1-Lipschitz} \int f\,(d\mu - d\nu) \, = \, \inf_{\mathcal{L}(X)=\mu, \mathcal{L}(Y)=\nu} \mathbb{E}(|X - Y|)\,.$$

Actually we have the following:

Proposition 3.11. *Assume that μ and ν satisfy weak $(1, +\infty)$ Poincaré inequalities with respective rates β_μ and β_ν. Then for all $s > 0$,*

$$\beta_\nu(s) \, \leq \, \beta_\mu(s) \, + \, 2\, W_1(\nu, \mu)\,.$$

Proof. Let f be 1-Lipschitz. We have

$$\begin{aligned} \nu(|f - \nu(f)|) \quad &\leq \quad \nu(|f - \mu(f)|) \, + \, |\nu(f) - \mu(f)| \\ &\leq \quad \mu(|f - \mu(f)|) \, + \, W_1(\nu, \mu) \, + \, |\nu(f) - \mu(f)| \\ &\leq \quad \beta_\mu(s) \, + \, 2\, W_1(\nu, \mu) \, + s\, \mathrm{Osc}(f)\,. \end{aligned}$$

Here we used that $|\mu(|f - \mu(f)|) - \nu(|f - \mu(f)|)| \leq W_1(\mu, \nu)$ since $|f - \mu(f)|$ is still 1-Lipschitz and that $|\nu(f) - \mu(f)| \leq W_1(\nu, \mu)$ too, i.e. the W_1 control for two different functions. □

We immediately deduce:

Corollary 3.12. *Let ν be a log-concave probability measure. Then for all μ,*

$$C'_C(\nu) \, \leq \, \frac{16}{\pi}\, (C_C(\mu) + 2\, W_1(\nu, \mu))\,.$$

Remark 3.13. Comparison of Poincaré constants using Wasserstein distance is not new. In [46] Theorem 5.5, E. Milman already derived a result in this direction. However the formulation of the previous Corollary looks simpler and the constant is explicit. ♢

The proof of proposition 3.11 can be modified in order to give another approach of Theorem 3.10. Consider a Lipschitz function f satisfying $0 \leq f \leq 1 = Osc(f) = \parallel f \parallel_\infty$ (recall remark 2.9), then

$$\nu(|f - \nu(f)|) \leq \mu(|f - \mu(f)|) \, + \, |\mu(|f - \mu(f)|) - \nu(|f - \mu(f)|)| \, + \, |\nu(f) - \mu(f)|\,.$$

Since

$$\mu(|f - \mu(f)|) \leq \beta_\mu(s) \, \parallel |\nabla f| \parallel_\infty \, + s\, Osc(f)\,,$$

while for $0 \leq g \leq 1$,

$$|\nu(g) - \mu(g)| = |\nu(g - \inf(g)) - \mu(g - \inf(g))| \leq d_{TV}(\mu, \nu) \, \parallel g - \inf g \parallel_\infty = d_{TV}(\mu, \nu)\, \mathrm{Osc}(g)\,,$$

we get applying the previous bound with $g = f$ and $g = |f - \mu(f)|$,

$$\nu(|f - \nu(f)|) \leq \beta_\mu(s) \, \parallel |\nabla f| \parallel_\infty + (s + 2\, d_{TV}(\mu, \nu))\, Osc(f)\,.$$

Hence, for any μ and ν, for all s' such that $s' > 2\, d_{TV}(\mu, \nu)$, we have

$$\beta_\nu(s') \leq \beta_\mu\left(s' - 2\, d_{TV}(\nu, \mu)\right)\,. \tag{3.5}$$

We thus have shown:

Proposition 3.14. *Let ν be a log-concave probability measure. Then for all μ such that $d_{TV}(\nu,\mu) \leq 1/4$, we have for all $s < \frac{1}{2} - 2d_{TV}(\mu,\nu)$,*

$$C'_C(\nu) \leq \frac{16\,\beta_\mu(s)}{\pi\,(1-2s-4\,d_{TV}(\mu,\nu))^2}\,.$$

In particular for $s=0$ we get

$$C'_C(\nu) \leq \frac{16}{\pi\,(1-4\,d_{TV}(\nu,\mu))^2}\;C_C(\mu)\,.$$

Of course the disappointing part of the previous result is that, even if the distance between ν and μ goes to 0, we cannot improve on the pre factor. In comparison with Theorem 3.10 we do not require μ to be log-concave, but the previous proposition is restricted to the case of not too big distance between μ and ν while we may take ε close to 0 in Theorem 3.10. Notice that we really need the weak form of the $(1,\infty)$ inequality here (the non weak form of E. Milman is not sufficient), since we only get $\beta_\nu(s')$ for s' large enough.

Remark 3.15. The previous result with $\mu = \nu_A$, see proposition 2.15, furnishes a worse bound than in this proposition.
The fact that we have to use the total variation bounds for two different functions, prevents us to localize the method, i.e. to build an appropriate μ for each f as in Eldan's localization method (see e.g. [28, 1]). Let us explain the previous sentence.
Pick some function β. Let f be a given function satisfying $0 \leq f \leq 1$. Assume that one can find a measure μ_f such that $\beta_{\mu_f} \leq \beta$ and

$$|\mu_f(|f-\mu_f(f)|) - \nu(|f-\mu_f(f)|)| \,+\, |\nu(f)-\mu_f(f)| \leq \varepsilon \leq \frac{1}{2}\,.$$

Then

$$\nu(|f-\nu(f)|) \leq \beta(s)\;\parallel |\nabla f| \parallel_\infty + (s+\varepsilon)\,Osc(f)$$

so that one can conclude as in proposition 3.14. Eldan's localization method is close to this approach, at least by controlling $|\nu(f)-\mu_f(f)|$, but not $|\mu_f(|f-\mu_f(f)|) - \nu(|f-\mu_f(f)|)|$. We shall come back to this approach later. ♢

Remark 3.16. In the proof of Proposition 3.14 we may replace the total variation distance by the Bounded Lipschitz distance

$$d_{BL}(\mu,\nu) = \sup\{\mu(f)-\nu(f)\;\text{ for } f \text{ 1-Lipschitz and bounded by } 1\}\,,$$

or the Dudley distance (also called the Fortet-Mourier distance)

$$d_{Dud}(\mu,\nu) = \sup\{\mu(f)-\nu(f)\;\text{ for } \parallel f \parallel_\infty + \parallel |\nabla f| \parallel_\infty \leq 1\}\,.$$

Recall that

$$d_{Dud}(\nu,\mu) \leq d_{BL}(\nu,\mu) \leq 2\,d_{Dud}(\nu,\mu)\,.$$

Provided one replaces $\parallel g \parallel_\infty$ by $\parallel g \parallel_\infty + \parallel |\nabla g| \parallel_\infty$, (3.5) is replaced by

$$\beta_\nu(s') \leq \beta_\mu(s' - 2d_{Dud}(\nu,\mu)) + 2d_{Dud}(\nu,\mu)\,, \tag{3.6}$$

so that, when ν is log-concave, we get

$$C'_C(\nu) \leq \frac{16\,(\beta_\mu(s) + 2d_{Dud}(\nu,\mu))}{\pi\,(1-2s-4\,d_{Dud}(\mu,\nu))^2}\,. \tag{3.7}$$

One can of course replace d_{Dud} by the larger d_{BL} in these inequalities.
When μ and ν are isotropic log-concave probability measures, it is known that

$$d_{TV}(\mu,\nu) \leq C\,\sqrt{n\,d_{BL}(\mu,\nu)}$$

according to proposition 1 in [43]. Combined with corollary 3.10 this bound is far to furnish the previous result since it gives a dimension dependent result. In addition we do not assume that μ is log concave. $\diamondsuit$

Remark 3.17. Remark that with our definitions $d_{Dud} \leq d_{BL} \leq 2\,d_{TV} \leq 2$. $\diamondsuit$

4. Mollifying the measure.

In this section we shall study inequalities for mollified measures. If Z is a random variable we will call mollified variable the sum $Z+X$ where X is some independent random variable, i.e. the law ν_Z is replaced by the convolution product $\nu_Z * \mu_X$. In this situation it is very well known that

$$C_P(\sqrt{\lambda}\,Z \,+\, \sqrt{1-\lambda}\,X) \;\leq\; \lambda\,C_P(Z) + (1-\lambda)\,C_P(X)$$

for $0 \leq \lambda \leq 1$ (see [5]). Taking $\lambda = 1/2$ it follows that

$$C_P(Z+X) \leq C_P(Z) + C_P(X)\,. \tag{4.1}$$

It is well known that mollifying the measure can improve on functional inequalities. For instance if ν is a compactly supported probability measure, the convolution product of ν with a gaussian measure will satisfy a logarithmic Sobolev inequality as soon as the variance of the gaussian is large enough (see e.g. [51]), even if ν does not satisfy any "interesting" functional inequality (for instance if ν has disconnected support); but the constant is desperately dimension dependent. We shall see that adding the log-concavity assumption for ν will help to improve on similar results.

4.1. **Mollifying using transportation.**

A first attempt to look at mollified measures was done by Klartag who obtained the following transportation inequality on the hypercube in [36]:

Theorem 4.1. *[Klartag] Let $R \geq 1$ and let Q be some cube in $\mathbb{R}^n$ of side length 1 parallel to the axes. Let $\mu = p(x)dx$ be a log-concave probability measure on Q satisfying in addition*

$$p(\lambda x + (1-\lambda)y) \leq R\,(\lambda p(x) + (1-\lambda)p(y)) \tag{4.2}$$

for any $0 \leq \lambda \leq 1$ and any pair $(x,y) \in Q^2$ such that all cartesian coordinates of $x-y$ are vanishing except one ($x-y$ is proportional to some e_j where e_j is the canonical orthonormal basis).
Then, μ satisfies a T_2 Talagrand inequality, i.e. there exists some C (satisfying $C \leq 40/9$) such that for any μ',

$$W_2^2(\mu',\mu) \leq C\,R^2\,D(\mu'||\mu)\,,$$

where W_2 denotes the Wasserstein distance and $D(.||.)$ the relative entropy.
In particular μ satisfies a Poincaré inequality with $C_P(\mu) \leq \frac{C\,R^2}{2}$.

The final statement is an easy and well known consequence of the T_2 transportation inequality, as remarked in [36] corollary 4.6.

In the sequel Q will denote the usual unit cube $[-\frac{1}{2}\,,\,\frac{1}{2}]^n$, and for $\theta > 0$, θQ will denote its homothetic image of ratio θ.
For $\theta > 0$, let Z_θ be a log-concave random vector whose distribution $\mu_\theta = p_\theta(x)\,dx$ is supported by θQ and satisfies (4.2) for any pair $(x,y) \in (\theta Q)^2$ and some given R. Then the distribution μ of $Z = Z_\theta/\theta$ satisfies the assumptions in Theorem 4.1 with the same R since its probability density is given for $x \in Q$ by

$$p(x) = \theta^n\, p_\theta(\theta\, x)\,.$$

In particular $C_P(Z) \leq \frac{1}{2}\, CR^2$ so that $C_P(Z_\theta) \leq \frac{1}{2}\, CR^2\, \theta^2$.
In the sequel we can thus replace Q by θQ. We shall mainly look at two examples of such p's: convolution products with the uniform density and the gaussian density.

4.1.1. *Convolution with the uniform distribution.* Consider U_θ a uniform random variable on θQ, $\theta > 1$. Its density $p(x) = \theta^{-n}\, \mathbf{1}_{\theta Q}(x)$ satisfies (4.2) in θQ with $R = 1$. It is immediate that

$$p(\lambda x + (1-\lambda)x' - y) \leq\ \lambda\, p(x-y) + (1-\lambda)\, p(x'-y)$$

for all x, x', y such that $x - y$ and $x' - y$ belong to θQ.
Let Z be a log-concave random variable whose law μ is supported by Q. The law

$$\nu_\theta(dx) = \left(\int\, p(x-y)\, \nu(dy)\right)\, dx$$

of $Z + U_\theta$ is still log-concave according to Prekopa-Leindler and satisfies (4.2) with $R = 1$ on $(\theta-1)Q$. According to what precedes, its restriction

$$\nu_{\theta,1}(dx) = \frac{\mathbf{1}_{(\theta-1)Q}}{\nu_\theta((\theta-1)Q)}\, \nu_\theta(dx)$$

to $(\theta - 1)Q$ satisfies

$$C_P(\nu_{\theta,1}) \leq \frac{1}{2}\, C\, (\theta-1)^2\,.$$

Thus

$$C'_C(\nu_{\theta,1}) \leq \frac{6}{\sqrt{2}}\, \sqrt{C}\, (\theta-1)\,,$$

according to Ledoux's comparison result.

4.1.2. *Convolution with the gaussian distribution.* Let $\gamma_\beta(x) = (2\pi\beta^2)^{-n/2}\, \exp\left(-\,\frac{|x|^2}{2\beta^2}\right)$ be the density of a centered gaussian variable βG (as before G is the standard gaussian).
It is elementary to show that γ_β satisfies the following convexity type property close to (4.2): for all pair (x, x') and all $\lambda \in [0,1]$,

$$\gamma_\beta(\lambda x + (1-\lambda)x') \leq e^{\frac{|x-x'|^2}{8\beta^2}}\, \left(\lambda\, \gamma_\beta(x) + (1-\lambda)\, \gamma_\beta(x')\right)\ , \tag{4.3}$$

this inequality being optimal and attained for pairs $(x, x') = (x, -x)$.
It immediately follows that

$$\gamma_\beta(\lambda x + (1-\lambda)x' - y) \leq e^{\frac{|x-x'|^2}{8\beta^2}}\, \left(\lambda\, \gamma_\beta(x-y) + (1-\lambda)\, \gamma_\beta(x'-y)\right)\ ,$$

for all (x, x', y). It follows that for all log-concave random vector Z, the distribution of $Z + \beta G$ is still satisfying (4.3). We have thus obtained:

Proposition 4.2. *Let β and θ be positive real numbers. Let Z be some log-concave random vector and denote by ν_β the distribution of $Z + \beta G$. Then the restriction*

$$\nu_{\beta,\theta} = \frac{\mathbb{1}_{\theta Q}}{\nu_\beta(\theta Q)}\, \nu_\beta$$

satisfies

$$C_P(\nu_{\beta,\theta}) \leq \frac{20}{9}\, \theta^2\, e^{\frac{\theta^2}{8\beta^2}}\,.$$

Notice that if we let β go to infinity, $\nu_{\beta,\theta}$ converges to the uniform measure on θQ so that the order θ^2 is the good one (if not the constant $\frac{20}{9}$). Also notice that if ν is supported by αQ, we may replace βG by a random variable whose distribution is γ_β restricted to $(\alpha + \theta)Q$.

The control obtained in proposition 4.2 (or in the uniform case) can be very interesting for practical uses, in particular for applied statistical purposes using censored random variables. But if we want to use it in order to get some information on the original ν, the result will depend dramatically on the dimension, since $\nu_\beta(\theta Q)$ is very small.

4.2. **Mollifying with gaussian convolution. A stochastic approach.**

In this section we shall introduce our approach for controlling the Poincaré constant using some appropriate stochastic process. To this end, we first consider a standard Ornstein-Uhlenbeck process $X_.$, i.e. the solution of

$$dX_t = dB_t - \frac{1}{2}\, X_t\, dt\,. \tag{4.4}$$

The law of X_t^x (i.e. the process starting from point x) will be denoted by $G(t, x, .) = \gamma(t, x, .)\, dx$, $\gamma(t, x, .)$ being thus the density of a gaussian random variable with mean $e^{-t/2}\, x$ and covariance matrix $(1 - e^{-t})\, Id$. The standard gaussian measure γ is thus the unique invariant (reversible) probability measure for the process. The law of the O-U process starting from μ will be denoted by $\mathbb{G}_\mu$.

One way to recover (4.1) is to remark that, if Z denotes the initial random variable X_0,

$$X_t = e^{-(T-s)/2} Z + \sqrt{1 - e^{-(T-s)}} G$$

where G is an independent random variables with distribution γ, and next to apply the "local" Poincaré inequalities for the Ornstein-Uhlenbeck semi-group (see [4] Theorem 4.7.2). An alternate proof using synchronous coupling is given in [18].

One may thus ask whether a converse inequality can be obtained by simply reversing time.

We shall see that in the log-concave situation, it is actually possible.

Let $\nu(dx) = e^{-V(x)}\, dx$ be a probability measure, V being smooth. We assume that ν is log-concave. Let

$$h(x) = (d\nu/d\gamma)(x) = (2\pi)^{n/2}\, e^{((|x|^2/2) - V(x))}\,.$$

The relative entropy $D(\nu||\gamma)$ satisfies

$$D(\nu||\gamma) = \frac{n}{2}\,\log(2\pi) + \int ((|x|^2/2) - V(x))\,\nu(dx) < +\infty\,,$$

since V is non-negative outside some compact subset. We may define for all t (Mehler formula),

$$\mathbb{E}(h(X_t^x)) = G_t h(x) = \int h(e^{-t/2}x + \sqrt{1-e^{-t}}y)\,\gamma(dy)\,,$$

which is well defined, positive, smooth (C^∞), and satisfies for all $t > 0$,

$$\partial_t G_t h(x) = \frac{1}{2}\Delta G_t h(x) - \frac{1}{2}\,\langle x, \nabla G_t h(x)\rangle\,.$$

In particular we may consider the solution of

$$dY_t = dB_t - \frac{1}{2}\,Y_t\,dt\; + \nabla \log G_{T-t}h(Y_t) \;=\; dB_t + b(t, Y_t)\,dt\,, \tag{4.5}$$

for $0 \leq t \leq T$. This stochastic differential equation has a strongly unique solution (since the coefficients are smooth) up to some explosion time $\xi \geq T$.

Since $D(\nu||\gamma) < +\infty$, this explosion time is almost surely infinite when the initial distribution is given by $G_T h\,\gamma$. This well known result follows from the discussion below and Proposition 2.3 in [22].

In addition the law $\mathbb{Q}$ of the solution satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{G}_\gamma}|_{\mathcal{F}_T} = h(\omega_T)\,.$$

Here of course $\mathcal{F}_T$ denotes the natural filtration on the path space. In particular the law $\nu(X_s)$ of the process satisfies,

$$\nu(X_s) = \mathbb{Q} \circ \omega_s^{-1} = G_{T-s}h\,\gamma \quad \text{ for all } 0 \leq s \leq T,$$

thanks to the stationarity of $\mathbb{G}_\gamma$. Of course this is nothing else but the h-process corresponding to h and the O-U process, but with a non bounded h.

If one prefers the solution $\mathbb{Q}$ of (4.5) is defined up to and including time T and is simply the law of the time reversal, at time T, of an Ornstein Uhlenbeck process with initial law ν. For more details see e.g. [18].

A specific feature of the O-U process is that, according to Prekopa-Leindler theorem, $\nu(X_s)$ is still a log-concave measure as the law of $e^{-(T-s)/2}Z + \sqrt{1-e^{-(T-s)}}G$ where Z and G are two independent random variables with respective distribution ν and γ (so that the pair (Z, G) has a log-concave distribution). This property of conservation of log-concave distributions is typical to the "linear" diffusion processes of O-U type (see [37]). Hence $\nu(X_s)(dx) = e^{-V_s(x)}\,dx$ for some potential V_s which is smooth thanks to the Mehler formula we recalled before, and convex.

It follows that $G_{T-s}h\,\gamma$ is log-concave, and finally that $b(t, .)$ (defined in (4.5)) satisfies the curvature condition

$$2\,\langle\, b(t,x) - b(t,y)\,,\, x-y\,\rangle \;\leq\; |x-y|^2\,, \tag{4.6}$$

for all t, x and y. (4.6) is called condition (H.C.-1) in [18]. It follows that for all $T > 0$,

$$C_P(\nu) \;\leq\; e^T\, C_P(G_T h\,\gamma) + (e^T - 1)\,. \tag{4.7}$$

For time homogeneous drifts this is nothing else but the so called "local" Poincaré inequality in [4] (Theorem 4.7.2). The extension to time dependent drift is done in [18] Theorem 5. Actually since the time non-homogeneous semi-group associated to (4.5) is defined on $[0, T-\varepsilon]$ for any $\varepsilon > 0$, we have first to use Theorem 5 of [18] up to time $T-\varepsilon$ and then to pass to the limit $\varepsilon \to 0$.

We have thus obtained:

Theorem 4.3. *Let Z be a random variable with log-concave distribution ν and G be a standard gaussian random variable independent of Z. Then for all $0 < \lambda \leq 1$ it holds*

$$C_P(Z) \leq \frac{1}{\lambda} C_P(\sqrt{\lambda}\, Z + \sqrt{1-\lambda}\, G) + \left(\frac{1}{\lambda} - 1\right) .$$

Of course since $C_P(aY) = a^2\, C_P(Y)$ for any random variable and any $a \in \mathbb{R}$, we obtain that for all real numbers α and β, defining $\lambda = \frac{\alpha^2}{\alpha^2+\beta^2}$,

$$C_P(Z) \leq \frac{1}{\alpha^2}\, C_P(\alpha\, Z \,+\, \beta\, G) + \frac{\beta^2}{\alpha^2} . \tag{4.8}$$

Using all the comparison results we already quoted, it follows

$$C'_C(Z) \leq \frac{12}{\alpha}\, C'_C(\alpha Z + \beta G) \,+\, \frac{6\beta}{\alpha} . \tag{4.9}$$

Remark 4.4. In the proof we may replace the standard Ornstein-Uhlenbeck process by any O-U process

$$dX_t = dB_t - \frac{1}{2}\, A\, X_t\, dt ,$$

where A is some non-negative symmetric matrix. Up to an orthogonal transformation we may assume that A is diagonal with non-negative diagonal terms a_i. Assume that

$$\max_{i=1,\ldots,n} a_i \leq 1 .$$

Then (4.6) is still satisfied, and no better inequality is. Similarly the Poincaré constant of the corresponding gaussian variable G' is unchanged so that (4.7) is still satisfied, and no better inequality is. We may thus in (4.8) replace the standard gaussian vector G by any centered gaussian vector, still called G, with independent entries G_i such that $\text{Var}(G_i) \leq 1$ for all i; in particular a degenerate gaussian vector. ♢

Remark 4.5. (4.8) with $\alpha = 1$ and $\beta = \sqrt{T}$, can also be derived using a similar time reversal argument for the Brownian motion (with reversible measure Lebesgue) instead of the O-U process. The corresponding drift b satisfies $\langle\, b(t,x) - b(t,y)\, ,\, x-y\,\rangle \,\leq\, 0$ which directly yields the result. Unfortunately the invariant measure is no more a probability measure. ♢

Remark 4.6. Theorem 4.3 can be viewed as a complement of previous similar results comparing the behavior of log-concave distributions and their gaussian mollification. Indeed, if Z is an isotropic log-concave probability measure and G a standard gaussian vector, it is elementary to show that for all $t \geq 0$,

$$\text{Var}(|Z + \sqrt{t}\, G|^2) = \text{Var}(|Z|^2) \,+\, 2nt(2+t) , \tag{4.10}$$

so that if for some t_0, $\text{Var}(|Z+\sqrt{t_0}\,G|^2) \leq Cn$ then the same bound is true for $\text{Var}(|Z|^2)$ i.e. the variance conjecture is true. Similarly, it is recalled in [31] (just before formula (4.5) therein), that a Fourier argument due to Klartag furnishes a control for the deviations

$$\mathbb{P}(|Z| > (1+t)\sqrt{n}) \leq C\,\mathbb{P}(|Z+G| > (1+(1+t)^2)\sqrt{n})$$

and

$$\mathbb{P}(|Z| < (1-t)\sqrt{n}) \leq C\,\mathbb{P}(|Z+G| < (1+(1-t)^2)\sqrt{n})$$

for some universal constant C. $\diamondsuit$

5. Probability metrics and log-concavity.

5.1. **Comparing Bounded Lipschitz and Total Variation distances.**

Let μ and ν be two probability measures. It is immediate to show the analogue of (4.1), i.e. if $\mu * \nu$ denotes the convolution product of both measures

$$\beta_{\mu*\nu}(s) \leq \beta_\mu(s/2) \,+\, \beta_\nu(s/2)\,, \tag{5.1}$$

here β corresponds to the usual centering with the mean (not the median).
Let $0 < t$. Denote by γ_t the distribution of tG, that is the gaussian distribution with covariance $t^2 Id$ (whose density is $\tilde{\gamma}_t$). For $0 \leq g \leq 1$, one has

$$(\nu * \gamma_t)(g) = \nu(g * \tilde{\gamma}_t)$$

and $g_t = g * \tilde{\gamma}_t$ is still bounded by 1 and $1/t$-Lipschitz (actually $\sqrt{2}/t\sqrt{\pi}$) according to (2.7) applied to the Brownian motion semi-group at time t^2. It follows that

$$(\nu * \gamma_t)(g) - (\mu * \gamma_t)(g) = \nu(g_t) - \mu(g_t)$$

so that

$$\begin{aligned} d_{TV}(\mu * \gamma_t, \nu * \gamma_t) \quad &\leq \quad \left(1 \,\vee\, \frac{\sqrt{2}}{t\sqrt{\pi}}\right) d_{BL}(\mu,\nu) \\ &\text{and} \\ d_{TV}(\mu * \gamma_t, \nu * \gamma_t) \quad &\leq \quad \left(1 \,+\, \frac{\sqrt{2}}{t\sqrt{\pi}}\right) d_{Dud}(\mu,\nu)\,. \end{aligned} \tag{5.2}$$

We may thus apply (3.5), and the fact that $C'_C(\gamma_t) = t$ in order to get, provided respectively

$$s > 2(1 \vee \sqrt{2}/t\sqrt{\pi})\, d_{BL}(\nu,\mu) \text{ or } s > 2(1 + \sqrt{2}/t\sqrt{\pi})\, d_{Dud}(\mu,\nu)\,,$$

the following

$$\begin{aligned} \beta_{\nu*\gamma_t}(s) \quad &\leq \quad \beta_{\mu*\gamma_t}\left(s - \,2(1 \vee \sqrt{2}/t\sqrt{\pi})\, d_{BL}(\mu,\nu)\right) \\ &\leq \quad \beta_\mu\left(\frac{s}{2} - \,(1 \vee \sqrt{2}/t\sqrt{\pi})\, d_{BL}(\mu,\nu)\right) \,+\, t\,, \end{aligned} \tag{5.3}$$

or

$$\beta_{\nu*\gamma_t}(s) \leq \beta_\mu\left(\frac{s}{2} - \,(1 + \sqrt{2}/t\sqrt{\pi})\, d_{Dud}(\mu,\nu)\right) \,+\, t\,.$$

Gathering the previous inequality, Theorem 2.7 and (4.8), we get new versions of proposition 3.14, slightly different from the one we gave in Remark 3.16.

For example, for a log-concave ν and for all μ such that $d_{BL}(\nu,\mu) \leq 1/4$, if we choose $t=\sqrt{2}/\sqrt{\pi}$ it holds

$$C'_C(\nu * \gamma_t) \leq \frac{16\,(C'_C(\mu)+\sqrt{2/\pi})}{\pi(1-4\,d_{BL}(\mu,\nu))^2}\,,$$

so that

$$C_P(\nu) \leq \frac{2}{\pi} + 4\left(\frac{16\,(C'_C(\mu)+\sqrt{2/\pi})}{\pi(1-4\,d_{BL}(\mu,\nu))^2}\right)^2 .$$

But we may use (5.2) in a potentially more interesting direction. Indeed, using theorem 3.10 and provided ν and μ (hence $\nu * \gamma$ and $\mu * \gamma$) are log-concave

$$C'_C(\nu * \gamma) \leq \frac{\kappa}{(1-d_{BL}(\nu,\mu))^2}\,(1 \vee \ln(1/(1-d_{BL}(\nu,\mu))))\,C'_C(\mu * \gamma)\,, \tag{5.4}$$

for some $\kappa \leq 192e/\pi$, provided $d_{BL}(\nu,\mu) \leq 1$ (we have skipped the $\sqrt{2/\pi}$ for simplicity). Hence using (4.9), the comparison between C_P and C'_C and (4.1) we have obtained a partial analogue of theorem 3.10 with the weaker bounded Lipschitz distance:

Theorem 5.1. *Let ν and μ be two log-concave probability measures on $\mathbb{R}^n$, γ be the standard gaussian distribution on $\mathbb{R}^n$.*

(1) *Then if $d_{BL}(\nu,\gamma)=1-\varepsilon$,*

$$C'_C(\nu) \leq \frac{C}{\varepsilon^2}\,(1 \vee \ln(1/\varepsilon)) + 6\,,$$

where the universal constant C can be chosen less than $\frac{13824\,e}{\pi}$.

(2) *If $d_{BL}(\nu,\mu)=1-\varepsilon$ then*

$$C'_C(\nu) \leq \frac{D}{\varepsilon^2}\,(1 \vee \ln(1/\varepsilon))\,(2C'_C(\mu)+1) + 6\,,$$

where the universal constant D can be chosen less than $6C$.

5.2. **Comparing Total Variation and W_1.**

We still use the notation G_T for the Ornstein-Uhlenbeck semi-group we introduced in (4.4), in particular we write $G_T\nu$ for the law at time T of the O-U process with initial distribution ν.

It is well known that $W_1(G_T\nu, G_T\mu) \leq e^{-T/2}\,W_1(\nu,\mu)$. Recall that this contraction property is an immediate consequence of synchronous coupling, i.e. if we build two solutions $X_.$ and $Y_.$ of (4.4) with the same Brownian motion it holds

$$|X_t - Y_t| = |X_0 - Y_0| - \frac{1}{2}\int_0^t |X_s - Y_s|ds$$

implying the result by choosing an optimal coupling (X_0, Y_0).

If we want to replace the W_1 distance by the total variation distance (or the bounded Lipschitz distance) one has to replace the synchronous coupling by a coupling by reflection following the idea of Eberle ([27]) we already used in [18]. This yields (see [18] subsection 7.4) the following inequality

$$d_{TV}(G_T\nu, G_T\mu) \leq \frac{e^{-T/2}}{\sqrt{2\pi\,(1-e^{-T})}}\, W_1(\nu,\mu)\,. \tag{5.5}$$

What we did before allows us to state a negative result:

Proposition 5.2. *The Ornstein-Uhlenbeck semi-group is not a contraction in total variation distance, nor in bounded Lipschitz distance.*

Proof. Since $d_{TV} \leq 1$ and applying the semi-group property, any uniform decay $d_{TV}(G_T\nu, \gamma) \leq h(T)$ with h going to 0 implies an exponential decay and the contraction property, for some $T > 0$, $d_{TV}(G_T\nu, \gamma) \leq \frac{1}{2}$. If ν_λ is the log-concave distribution of some random vector $\lambda\, X$, it follows that for universal constants C and C',

$$C_P(\nu_\lambda) \leq C\,(C_P(G_T\nu_\lambda) + 1) \leq C'\,.$$

But $C_P(\nu_\lambda) = \lambda^2\, C_P(\nu_1)$ yielding a contradiction.
Similarly, if $d_{BL}(G_T\nu, \gamma) \leq 1/2$ then $d_{TV}(G_{T+1}\nu, \gamma) \leq \frac{1}{2}$ which is impossible. □

5.3. **Comparison with other metrics on probability measures.**

The weakest distance we introduced is the Bounded Lipschitz distance. It is known that this distance metrizes weak convergence. We may thus compare d_{BL} with the Lévy-Prokhorov distance.

Definition 5.3. If μ and ν are two probability measures the Lévy-Prokhorov distance $d_{LP}(\mu,\nu)$ is defined as

$$d_{LP}(\mu,\nu) = \inf\{\varepsilon \geq 0\,;\; \mu(A) \leq \nu(A + B(0,\varepsilon)) + \varepsilon\,;\; \text{for all Borel set } A.\}.$$

It is well known that d_{LP} is a metric (in particular $d_{LP}(\mu,\nu) = d_{LP}(\nu,\mu)$ despite the apparent non symmetric definition, just taking complements), clearly bounded by 1, that actually metrizes the convergence in distribution (weak convergence). We may also replace Borel sets A by closed sets A, hence defining $\rho(\mu,\nu)$ and symmetrizing the definition i.e. $d_{LP}(\mu,\nu) = max(\rho(\mu,\nu), \rho(\nu,\mu))$.
The following properties can be found in [24] Corollaries 2 and 3 (and using that d_{LP} is less than 1), or [25] problem 5 p.398 or in [47] (with a worse constant)

Proposition 5.4. (1) *It holds*

$$\frac{1}{4}\, d_{BL}(\nu,\mu) \leq \frac{1}{2}\, d_{Dud}(\nu,\mu) \leq d_{LP}(\nu,\mu) \leq \sqrt{\frac{3}{2}\, d_{Dud}(\nu,\mu)} \leq \sqrt{\frac{3}{2}\, d_{BL}(\nu,\mu)}\,,$$

(2) $d_{LP}(\nu,\mu) = \inf\{K(X,Y)\,;\; \mathcal{L}(X) = \nu\,,\; \mathcal{L}(Y) = \mu\}$ *where*

$$K(X,Y) = \inf\{\varepsilon \geq 0\,;\; \mathbf{P}(|X-Y| > \varepsilon) \leq \varepsilon\}$$

is the Ky Fan distance between X and Y, and $\mathcal{L}(X)$ denotes the probability distribution of X.

Assume that μ and ν are log-concave. Then $\mu \otimes \nu$ is also log-concave according to Prekopa-Leindler theorem, so that $(x, y) \mapsto x - y$ is a polynomial of degree 1 on $\mathbb{R}^n \otimes \mathbb{R}^n$. For such a polynomial, moment controls have been obtained by several authors. We shall use the version in [29] Corollary 4 of a result by Guédon ([30])

Theorem 5.5. *[Guédon] If η is a log-concave probability measure and P is a polynomial of degree 1, then for all $c > 0$ and all $t \geq 1$,*

$$\eta(x\,;\,|P(x)| > ct) \leq \eta(x\,;\,|P(x)| > c)^{\frac{1+t}{2}} \,,$$

provided the left hand side is (strictly) positive.

If X and Y are independent log-concave random variables, we deduce that for $t \geq 1$,

$$\mathbb{P}(|X - Y| > t\, K(X, Y)) \,\leq\, (K(X, Y))^{\frac{1+t}{2}} \,. \tag{5.6}$$

Using

$$\mathbb{E}(|X - Y|) = \int_0^{+\infty} \mathbb{P}(|X - Y| > t)\, dt$$

we have thus obtained

$$\mathbb{E}(|X - Y|) \,\leq\, K(X, Y) \left(1 \,+\, \frac{2\, K(X, Y)}{\ln(1/K(X, Y))}\right) , \tag{5.7}$$

so that taking an optimal coupling on the right hand side we have obtained

Corollary 5.6. *Let μ and ν be two log-concave probability measures. Then*

$$d^2_{LP}(\mu, \nu) \,\leq W_1(\mu, \nu) \,\leq\, d_{LP}(\mu, \nu) \left(1 \,+\, \frac{2\, d_{LP}(\mu, \nu)}{\ln(1/d_{LP}(\mu, \nu))}\right) ,$$

so that,

$$C'_C(\nu) \leq \frac{16}{\pi} \left(C_C(\mu) + 2\, d_{LP}(\mu, \nu) \left(1 \,+\, \frac{2\, d_{LP}(\mu, \nu)}{\ln(1/d_{LP}(\mu, \nu))}\right)\right) .$$

Recall that the left hand side of the inequality between distances is always true (see e.g. [12] (10.1) p.1045).
Combining all what precedes we have also obtained

Corollary 5.7. *The Ornstein-Uhlenbeck semi-group is not a contraction in Lévy-Prokhorov distance. However if ν is log-concave*

$$d_{LP}(G_T\nu, \gamma) \,\leq\, e^{-T/4} \left[d_{LP}(\gamma, \nu) \left(1 \,+\, \frac{2\, d_{LP}(\gamma, \nu)}{\ln(1/d_{LP}(\gamma, \nu))}\right)\right]^{\frac{1}{2}} .$$

Provided $d_{BL}(\mu, \nu) \leq 2/3$ we also have,

$$W_1(\mu, \nu) \,\leq\, \sqrt{\frac{3}{2}\, d_{BL}(\mu, \nu)} \left(1 \,+\, \frac{\sqrt{6 d_{BL}(\mu, \nu)}}{\ln(\sqrt{2}/\sqrt{3 d_{BL}(\mu, \nu)})}\right) ,$$

so that we get a new bound for two log-concave measures

$$C'_C(\nu) \leq \frac{16}{\pi} \left(C_C(\mu) + \,\sqrt{6 d_{BL}(\mu, \nu)} \left(1 \,+\, \frac{\sqrt{6 d_{BL}(\mu, \nu)}}{\ln(\sqrt{2}/\sqrt{3 d_{BL}(\mu, \nu)})}\right)\right) .$$

For small values of $d_{BL}(\mu,\nu)$, this bound is better than (3.7) (but here we need both measures to be log-concave) and theorem 5.1, which is true for large values of $d_{BL}(\mu,\nu)$.
One can also compare corollary 5.6 with proposition 4 in [43] giving a dimensional inequality

$$W_1(\mu,\nu) \, \leq \, C \, \left(\sqrt{n} \vee \ln\left(\frac{\sqrt{n}}{d_{BL}(\mu,\nu)}\right)\right) \, d_{BL}(\mu,\nu) \, ,$$

for isotropic log-concave probability measures.

Remark 5.8. The previous results give some hints on the (somehow bad) structure of isotropic log-concave measures. Indeed look, on one hand at the uniform measure μ^n on $A = [-\sqrt{3}\, , \, \sqrt{3}]^n$ associated to a random variable U, on the other hand at the standard gaussian distribution γ^n associated to G. $\mu^n(A) = 1$ when

$$\gamma^n(A + B(0,\varepsilon)) \leq \gamma^n([-\sqrt{3} - \varepsilon \, , \, \sqrt{3} + \varepsilon]^n) = (\gamma^1([-\sqrt{3} - \varepsilon \, , \, \sqrt{3} + \varepsilon]))^n \, ,$$

so that

$$d_{LP}(\mu^n,\gamma^n) \, \geq \, 1 - u^n$$

with $u = \gamma^1([-\sqrt{3}\, , \, \sqrt{3}])$, hence, for large n, $d_{LP}(\mu^n,\gamma^n)$ is as close to 1 as we want. Consequently we cannot expect to get a dimension free nice upper bound for the Lévy-Prokhorov distance. The question is then whether such a bound is true if we consider the set of $\nu * \gamma_t$ where ν describes the set of *isotropic* log-concave distributions, or not. We know that such a bound does not exist for all log-concave distributions, according to Corollary 5.7. ♢

If the Lévy-Prokhorov distance seems difficult to estimate, one can relate it to a Wasserstein distance for a new distance. Introduce

$$W_{LP}(\nu,\mu) = \inf \left\{ \int \, \frac{|x-y|}{1+|x-y|} \, \pi(dx,dy) \, ; \, \pi \circ x^{-1} = \nu \, , \, \pi \circ y^{-1} = \mu \right\} \, . \tag{5.8}$$

Proposition 5.9. *Let* $K^*(X,Y) = \mathbb{E}\left(\frac{|X-Y|}{1+|X-Y|}\right)$. *It holds*

$$\frac{1}{2} \, K^*(X,Y) \leq \, K(X,Y) \, \leq \, \sqrt{2K^*(X,Y)} \, .$$

Consequently

$$\frac{1}{2} \, W_{LP}(\mu,\nu) \, \leq \, d_{LP}(\mu,\nu) \, \leq \, \sqrt{2 \, W_{LP}(\mu,\nu)} \, .$$

Proof. Denote $Z = |X-Y|$ and $Z^* = \frac{Z}{1+Z}$, so that $Z = \frac{Z^*}{1-Z^*}$.
On one hand, since $Z^* \leq 1$,

$$\mathbb{E}(Z^*) \leq \mathbb{P}(Z^* > \eta) + \eta \, .$$

But $\mathbb{P}(Z^* > \eta) = \mathbb{P}(Z > \varepsilon)$ for $\eta = \frac{\varepsilon}{1+\varepsilon}$, so that

$$\mathbb{E}(Z^*) \leq K(X,Y) + \frac{K(X,Y)}{1+K(X,Y)} \leq 2 \, K(X,Y) \, .$$

Conversely, using what precedes and Markov inequality,

$$\mathbb{P}(Z > \varepsilon) \leq \frac{\mathbb{E}(Z^*)}{\frac{\varepsilon}{1+\varepsilon}}$$

so that $\mathbb{P}(Z > \varepsilon) \leq \varepsilon$ provided $\mathbb{E}(Z^*) \leq \frac{\varepsilon^2}{1+\varepsilon}$ in particular if $\mathbb{E}(Z^*) \leq \frac{\varepsilon^2}{2}$ (because we only have to consider $\varepsilon \leq 1$) yielding the result. □

Since the cost $c(x,y) = \frac{|x-y|}{1+|x-y|}$ is concave we also have

$$K^*(X,Y) = \mathbb{E}(c(|X-Y|)) \leq c(\mathbb{E}(|X-Y|)) = \frac{E(|X-Y|)}{1+E(|X-Y|)}\,,$$

so that

$$W_{LP}(\nu,\mu) \;\leq\; \frac{W_1(\nu,\mu)}{1+W_1(\nu,\mu)}\,. \tag{5.9}$$

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