Complement

Several statements concerned with the logarithmic Sobolev inequality have to be completed. Namely, we have to complete Assumption 2') in Theorem 1.2 by adding the following assumption :

Assumption Add $\theta(r) = \sup_{\{x; V(x)\leq r\}} \max_{i,j} \left|\partial^2_{i,j} V(x)\right|$ satisfies $\theta(r) \leq m\, e^{Dr}$ for positive m and D, and all r large enough.

Proposition 3.5 and its proof have to be modified in the following way

Proposition 0.1. *Assume that μ is symmetric and that $\sigma.\sigma^* \geq \alpha\, Id$ for some $\alpha > 0$ (i.e. is uniformly elliptic). Assume in addition that*

(1) V goes to infinity at infinity,
(2) $|\nabla V(x)| \geq v > 0$ for $|x|$ large enough,
(3) $e^{aV} \in \mathbb{L}^1(\mu)$ for some $a > 0$,
(4) $\theta(r)$ defined by

$$\theta(r) = \sup_{\{x; V(x)\leq r\}} \max_{i,j} \left|\partial^2_{i,j} V(x)\right|$$

satisfies $\theta(r) \leq m\, e^{Dr}$ for positive m and D, and all r large enough.

If there exists a Lyapunov function W with $W(x) \geq w > 0$ for all $x \in D$, $\frac{\partial W}{\partial n} = 0$ on ∂D and satisfying

$$LW(x) \leq -\lambda\, V(x)\, W(x) \,+\, b\,,$$

for some λ and b strictly positive, then μ satisfies a logarithmic-Sobolev inequality.

Proof. We follow the method in [2] Theorem 2.1 (itself inspired by [1]). Let $A_r = \{V \leq r\}$. For r_0 large enough and some $\lambda' < \lambda$ we have

$$LW(x) \leq -\lambda'\, V(x)\, W(x) \,+\, b\, \mathbf{1}_{A_{r_0}}\,,$$

so that we may assume that

$$\frac{LW}{W}(x) \leq -\,\lambda\, V(x) \quad \text{ for } x \in A_r^c \text{ and all } r \text{ large enough.}$$

Denote by $M = \sup(-V)$. We have for $s \leq s_0$ and $r > r_0$,

$$\begin{aligned}
\int f^2\, d\mu \;&=\; \int_{A_r} f^2\, d\mu + \int_{A_r^c} f^2\, d\mu \\
&\leq\; e^M \int_{A_r} f^2\, dx \,+\, \frac{1}{\lambda r}\int \lambda V(x)\, f^2 d\mu, \\
&\leq\; e^M \left(1 + \frac{b}{\lambda r_0}\right) \int_{A_r} f^2\, dx \,+\, \frac{1}{\lambda\, r}\int f^2 \left(\frac{-LW}{W}\right) d\mu \\
&\leq\; e^M \left(1 + \frac{b}{\lambda r_0}\right) \left(s\int_{A_r} |\nabla f|^2\, dx \,+\, C_d\, \theta^d(r)(1+s^{-d/2}) \left(\int_{A_r} |f|\, dx\right)^2 \right) + \\
&\quad + \frac{1}{\lambda\, r}\int |\sigma.\nabla f|^2\, d\mu\,.
\end{aligned}$$

The first part of the last bound is obtained by using (3.1.4) in [2] (it is here that we are using the assumption on $|\nabla V|$), while the second bound is obtained using integration by parts or the Green-Rieman formula (see [2] (2.2)). Using uniform ellipticity we thus obtain, denoting $c = e^M \left(1 + \frac{b}{\lambda r_0}\right)$

$$\mu(f^2) \,\leq\, \left(\frac{s\,c\,e^r}{\alpha} + \frac{1}{\lambda\,r}\right) \int |\sigma.\nabla f|^2 \, d\mu \,+\, C_d\, c\, m^d \,(1+s^{-d/2})\, e^{2r+dDr} \left(\int |f|\, d\mu\right)^2 . \quad (0.2)$$

Denote $u = sce^r/\alpha$. We thus have for all $0 < u < Cte\, e^r$:

$$\mu(f^2) \,\leq\, \left(u + \frac{1}{\lambda\,r}\right) \int |\sigma.\nabla f|^2 \, d\mu \,+\, C'_d \,(1+u^{-d/2})\, e^{(2+dD+(d/2))r} \left(\int |f|\, d\mu\right)^2 . \quad (0.3)$$

Hence choosing $r = c'/u$, which implies u small enough for $u < Cte\, e^{c'/u}$, we get the following super-Poincaré inequality for small s,

$$\mu(f^2) \,\leq\, s \int |\sigma.\nabla f|^2 \, d\mu \,+\, C' \, e^{\tilde{c}/s} \left(\int |f|\, d\mu\right)^2 ,$$

which is known to be equivalent to a defective logarithmic Sobolev inequality (see the introduction of [2]). But the Lyapunov condition being stronger than $(HP1)$, we know that μ satisfies a Poincaré inequality, hence using Rothaus lemma, that it satisfies a (tight) log-Sobolev inequality.

□

References

[1] D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin. A simple proof of the Poincaré inequality for a large class of probability measures. *Elec. Comm. in Prob.*, 13:60–66, 2008.

[2] P. Cattiaux, A. Guillin, F. Y. Wang and L. Wu. Lyapunov conditions for super Poincaré inequalities. *J. Funct. Anal.*, 256(6):1821–1841, 2009.