About Coulomb gases

Concentration and dynamics

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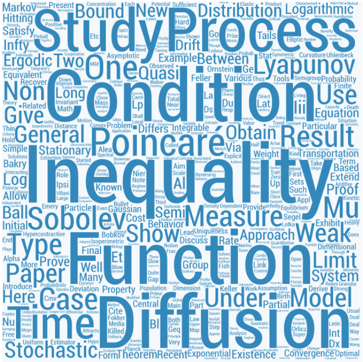
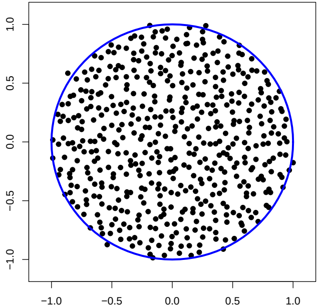


Motivation: Ginibre Ensemble

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plot(eig(randcg(500,500)/sqrt(500)))
```

Outline

Coulomb kernel in mathematical physics

- Coulomb kernel in $\mathbb{R}^d$, $d \geqslant 2$,

$$x \in \mathbb{R}^d \mapsto g(x) = \begin{cases} \log \dfrac{1}{|x|} & \text{if } d = 2, \\ \dfrac{1}{|x|^{d-2}} & \text{if } d \geqslant 3. \end{cases}$$

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- Fundamental solution of Poisson's equation

$$\Delta g = -c_d \, \delta_0 \quad \text{where} \quad c_d = \begin{cases} 2\pi & \text{if } d = 2, \\ (d-2)|\mathbb{S}^{d-1}| & \text{if } d \geqslant 3. \end{cases}$$

Coulomb energy and equilibrium measure

- Coulomb energy of probability measure μ on $\mathbb{R}^d$:

$$\mathscr{E}(\mu) = \iint g(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \in \mathbb{R} \cup \{+\infty\}.$$

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$$\mathscr{E}_V(\mu) = \mathscr{E}(\mu) + \int V(x)\mu(\mathrm{d}x).$$

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- If V is strong then μ_V is compactly supported with density

$$\frac{1}{2c_d}\Delta V$$

Examples of equilibrium measures

d	g	V	μ_V
1	2	$\infty \mathbf{1}_{\text{interval}^c}(x)$	arcsine
1	2	x^2	semicircle
2	2	$\lvert x\rvert^2$	uniform on a disc
$\geqslant 3$	d	$\lVert x\rVert^2$	uniform on a ball
$\geqslant 2$	d	radial	radial in a ring

Outline

Coulomb gas or one component plasma

- Energy of N Coulomb charges in $\mathbb{R}^d$:

$$H_N(x_1,\dots,x_N) = N\sum_{i=1}^{N} V(x_i) + \sum_{i\neq j} g(x_i - x_j)$$

Coulomb gas or one component plasma

- Energy of N Coulomb charges in $\mathbb{R}^d$:

$$H_N(x_1,\ldots,x_N) = N^2\Big(\int V(x)\mu_N(\mathrm{d}x) + \iint_{x\neq y} g(x-y)\mu_N(\mathrm{d}x)\mu_N(\mathrm{d}y)\Big)$$

- Empirical measure: $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$

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- Empirical measure: $\mu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$
- Boltzmann–Gibbs measure $\mathbb{P}_{V,\beta}^N$ on $(\mathbb{R}^d)^N$:

$$\frac{\exp\left(-\frac{\beta}{2} H_N(x_1,\dots,x_N)\right)}{Z_{V,\beta}}$$

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- $\mathbb{P}_{V,\beta}^N$ is neither product nor log-concave

Empirical measure and equilibrium measure

- Random empirical measure under $\mathbb{P}^N_{V,\beta}$:

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- Large Deviation Principle (Gozlan-C.-Zitt)

$$\frac{\log \mathbb{P}^N_{V,\beta}\Big(\mathrm{dist}(\mu_N,\mu_V) \geqslant r\Big)}{N^2} \underset{N\to\infty}{\longrightarrow} -\frac{\beta}{2} \inf_{\mathrm{dist}(\mu,\mu_V)\geqslant r} \big(\mathscr{E}_V(\mu) - \mathscr{E}_V(\mu_V)\big).$$

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- Quantitative estimates? How to relate dist and $\mathscr{E}_V(\cdot)-\mathscr{E}_V(\mu_V)$?

Outline

Electrostatics

Coulomb gas model

Metrics and Coulomb transport inequality

Concentration for Coulomb gases

Dynamics for planar case

Probability metrics and topologies

- Coulomb divergence (Large Deviations rate function)

$$\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)$$

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- Bounded-Lipschitz or Fortet–Mourier distance

$$\mathrm{d}_{\mathrm{BL}}(\mu,\nu) = \sup_{\substack{\|f\|_{\mathrm{Lip}}\leqslant 1\\ \|f\|_\infty\leqslant 1}} \int f(x)(\mu-\nu)(\mathrm{d}x),$$

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- (Monge-Kantorovich-)Wasserstein distance

$$\mathrm{W}_p(\mu, \nu) = \inf_{\substack{(X,Y) \\ X \sim \mu, Y \sim \nu}} \mathbb{E}(|X - Y|^p)^{1/p}.$$

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$$\mathrm{d}_{\mathrm{BL}}(\mu,\nu) \leqslant \mathrm{W}_1(\mu,\nu) = \sup_{\|f\|_{\mathrm{Lip}}\leqslant 1} \int f(x)(\mu-\nu)(\mathrm{d}x).$$

Local Coulomb transport inequality

Theorem (Transport type inequality – C.-Hardy-Maïda)

$$\mathrm{W}_1(\mu,\nu)^2 \leqslant C_D\,\mathscr{E}(\mu-\nu).$$

Local Coulomb transport inequality

Theorem (Transport type inequality – C.-Hardy-Maïda)

$D \subset \mathbb{R}^d$ compact, $\mathrm{supp}(\mu+\nu) \subset D$, $\mathscr{E}(\mu) < \infty$ *and* $\mathscr{E}(\nu) < \infty$,

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- Constant C_D is $\approx \mathrm{Vol}(B_{4\mathrm{Vol}(D)})$

Idea of proof of Coulomb transport inequality

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- Integration by parts & Schwarz’s inequality in $\mathbb{R}^d$ and L^2

$$c_d \int f(x)(\mu - \nu)(\mathrm{d}x) = -\int f(x)\Delta U^{\mu-\nu}(x)\mathrm{d}x$$

$$\leqslant \|f\|_{\mathrm{Lip}} \left(|D_+| \int |\nabla U^{\mu-\nu}(x)|^2 \mathrm{d}x \right)^{1/2}$$

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- Integration by parts again

$$\int |\nabla U^{\mu-\nu}(x)|^2 \mathrm{d}x = -\int U^{\mu-\nu}(x)\Delta U^{\mu-\nu}(x)\mathrm{d}x$$
$$= c_d \mathcal{E}(\mu - \nu).$$

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- Finally $\mathrm{W}_1(\mu,\nu)^2 \leqslant |D_+|c_d\mathscr{E}(\mu-\nu)$.

Coulomb transport inequality for equilibrium measures

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$$\mathrm{d}_{\mathrm{BL}}(\mu,\mu_V)^2 \leqslant C_{\mathrm{BL}}\big(\mathscr{E}_V(\mu)-\mathscr{E}_V(\mu_V)\big).$$

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For any probability measure μ *on* $\mathbb{R}^d$ *with* $\mathcal{E}(\mu) < \infty$

$$\mathrm{d}_{\mathrm{BL}}(\mu,\mu_V)^2 \leqslant C_{\mathrm{BL}}\left(\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)\right).$$

Moreover if V *has at least quadratic growth then*

$$\mathrm{W}_1(\mu,\mu_V)^2 \leqslant C_{\mathrm{W}_1}\left(\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)\right).$$

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- Free transport inequalities ($d = 2$ and $V = +\infty$ on $\mathbb{R}^c$)
- Growth condition is optimal for W_1

Outline

Concentration of measure for Coulomb gases

Theorem (Concentration inequality – C.-Hardy-Maïda)

If V does has reasonable growth then for every β, N, r

$$\mathbb{P}^N_{V,\beta}\Big(\mathrm{d}_{\mathrm{BL}}(\mu_N,\mu_V) \geqslant r\Big) \leqslant \mathrm{e}^{-a\beta N^2 r^2} \quad .$$

Moreover if V has at least quadratic growth then W_1 instead of d_{BL}.

- LDP shows that order in N is optimal

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$$\mathbb{P}^N_{V,\beta}\left(\mathrm{d}_{\mathrm{BL}}(\mu_N,\mu_V)\geqslant r\right)\leqslant \mathrm{e}^{-a\beta N^2 r^2+\mathbf{1}_{d=2}(\frac{\beta}{4}N\log N)+b\beta N^{2-2/d}+c(\beta)N}.$$

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- Explicit constants a, b, c if V is quadratic
- Implies Wasserstein convergence:

$$\mathbb{P}^N_{V,\beta}\left(\mathrm{W}_1(\mu_N,\mu_V)\geqslant r\right)\leqslant \mathrm{e}^{-cN^2r^2},\quad r\geqslant\begin{cases}\sqrt{\frac{\log N}{N}} & \text{if } d=2,\\ N^{-1/d} & \text{if } d\geqslant 3.\end{cases}$$

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- See also Rougerie & Serfaty

Idea of proof of concentration

- Starting point

$$\mathbb{P}^N_{V,\beta}(\mathrm{W}_1(\mu_N,\mu_V)\geqslant r)=\frac{1}{Z^N_{V,\beta}}\int_{\mathrm{W}_1(\mu_N,\mu_V)\geqslant r}\mathrm{e}^{-\frac{\beta}{2}N^2\mathcal{E}^{\neq}_V(\mu_N)}\mathrm{d}x.$$

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- Normalizing constant

$$\frac{1}{Z^N_{V,\beta}} \leqslant \exp\left\{N^2\frac{\beta}{2}\mathscr{E}_V(\mu_V) - N\left(\frac{\beta}{2}\mathscr{E}(\mu_V) - \mathrm{S}(\mu_V)\right)\right\}.$$

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- Regularization: g superharmonic, $\mu_N^{(\varepsilon)} = \mu_N * \lambda_\varepsilon$,

$$-N^2\mathcal{E}^{\neq}_V(\mu_N) \leqslant -N^2\mathcal{E}_V(\mu_N^{(\varepsilon)}) + N\mathcal{E}(\lambda_\varepsilon) + N\sum_{i=1}^N (V*\lambda_\varepsilon - V)(x_i).$$

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- Regularization: g superharmonic, $\mu_N^{(\varepsilon)}=\mu_N*\lambda_\varepsilon$,

$$-N^2\mathcal{E}^{\neq}_V(\mu_N)\leqslant-N^2\mathcal{E}_V(\mu_N^{(\varepsilon)})+N\mathcal{E}(\lambda_\varepsilon)+N\sum_{i=1}^N(V*\lambda_\varepsilon-V)(x_i).$$

- Coulomb transport $-\mathcal{E}_V(\mu_N^{(\varepsilon)})+\mathcal{E}_V(\mu_V)\leqslant-\frac{1}{C}\mathrm{W}_1^2(\mu_N^{(\varepsilon)},\mu_V)$.

Concentration for spectrum of Ginibre matrices

Corollary (Ginibre Random Matrices – C.-Hardy-Maïda)

If M is $N \times N$ with iid Gaussian entries of variance $\frac{1}{N}$ in $\mathbb{C}$

- Eigenvalues of $M \propto \exp(-N\sum_{i=1}^{N} |x_i|^2) \prod_{i<j} |x_i - x_j|^2$

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If M is $N \times N$ with iid Gaussian entries of variance $\frac{1}{N}$ in $\mathbb{C}$

- Eigenvalues of M $\propto \exp(-N\sum_{i=1}^{N}|x_i|^2 - \sum_{i\neq j} g(x_i - x_j))$
- Here $d = 2$, $\beta = 2$, $V = |\cdot|^2$

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If M is $N \times N$ with iid Gaussian entries of variance $\frac{1}{N}$ in $\mathbb{C}$ then

$$\mathbb{P}^N_{|\cdot|^2,2}\Big(\mathrm{W}_1(\mu_N,\mu_\bullet) \geqslant r\Big) \leqslant \mathrm{e}^{-\frac{1}{4C}N^2r^2+\frac{1}{2}N\log N+N[\frac{1}{C}+\frac{3}{2}-\log\pi]}.$$

- Eigenvalues of M $\propto \exp(-N\sum_{i=1}^N |x_i|^2 - \sum_{i\neq j} g(x_i - x_j))$
- Here $d = 2$, $\beta = 2$, $V = |\cdot|^2$
- Provides $\lim_{N\to\infty} \mathrm{W}_1(\mu_N,\mu_\bullet) = 0$ a.s.

Concentration for spectrum of Ginibre matrices

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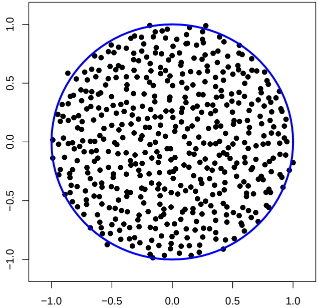
- Open problem: Bernoulli ±1 random matrices (universality)

Concentration for spectrum of Ginibre random matrices




```
plot(eig(randcg(500,500)/sqrt(500)))
```

Outline

Ginibre distribution is not log-concave

- Hermitian (Dyson GUE) 1D

$$e^{-\mathrm{Tr}(H^2)} = \prod_{j,k=1}^{N} e^{-|H_{jk}|^2}$$

Ginibre distribution is not log-concave

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- Ginibre is not log-concave (contrary to Dyson GUE)

Ginibre process

- State space

$$D = \mathbb{C}^N \setminus \cup_{i \neq j} \{(x_1, \ldots, x_N) \in \mathbb{C}^N : x_i = x_j\}$$

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- Ginibre if $\beta_N = N^2$ (determinantal, exactly solvable)

Ginibre process

- Ginibre process on $D \subset \mathbb{C}^N = (\mathbb{R}^2)^N$ reversible for $\mathbb{P}^N$

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- Mean-field interacting particle system $X_t^N = (X_t^{i,N})_{1\leqslant i\leqslant N}$

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- Mean-field interacting particle system $X_t^N = (X_t^{i,N})_{1\leqslant i\leqslant N}$
- 1D log-gases are log-concave:
  - Freeman Dyson
    A Brownian-motion model for the eigenvalues of a random matrix.
    Journal of Mathematical Physics 3 (1962) 1191–1198.
  - László Erdős & Horng-Tzer Yau
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- 2D: no convexity / Brascamp–Lieb / Bakry–Émery / Caffarelli

Well posedness

Theorem (Well posedness – Bolley-C.-Fontbona)

- *D is path-connected in* $(\mathbb{R}^2)^N$ *and H is coercive:*

$$\lim_{x\to\partial D} H(x) = +\infty \quad \textit{and} \quad \lim_{x\to\partial D} |\nabla H(x)| = +\infty.$$

- *X is not explosive: for any* $x \in D$

$$\mathbb{P}(T_{\partial D} = +\infty \mid X_0 = x) = 1.$$

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- D is not convex and $\inf_D \mathrm{Hess}(H) = -\infty$

Poincaré inequality

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For any N, the law $\mathbb{P}^N$ satisfies a Poincaré inequality.

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- Bakry–Barthe–Cattiaux–Guillin Lyapunov approach

$$H(x) = \frac{1}{N}\sum_{i=1}^{N}|x_i|^2 + \frac{1}{N^2}\sum_{j\neq i}\log\frac{1}{|x_j - x_k|}$$

$$Lf = \frac{\alpha_N}{\beta_N}\Delta f - \alpha_N \nabla H \cdot \nabla f$$

$$Le^{\gamma H} \leqslant -ce^{\gamma H} + c'\mathbf{1}_K.$$

Second moment dynamics

Theorem (Second moment dynamics – Bolley-C.-Fontbona)

$(R_t)_{t\geqslant 0} = (\frac{1}{N}\|X_t\|^2)_{t\geqslant 0}$ *is an ergodic Cox–Ingersoll–Ross process:*

$$\mathrm{d}R_t = \sqrt{\frac{8\alpha_N}{N\beta_N}R_t}\,\mathrm{d}B_t + 4\frac{\alpha_N}{N}\left[\frac{N}{\beta_N} + \frac{N-1}{2N} - R_t\right]\mathrm{d}t.$$

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In particular, with $\Gamma_N = \mathrm{Gamma}(N + \frac{N-1}{2N}\beta_N, \beta_N)$, for any $t \geqslant 0$

$$\mathrm{W}_1(\mathrm{Law}(R_t), \Gamma_N) \leqslant \mathrm{e}^{-4\frac{\alpha_N}{N}t}\,\mathrm{W}_1(\mathrm{Law}(R_0), \Gamma_N).$$

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$$\mathrm{W}_1(\mathrm{Law}(R_t), \Gamma_N) \leqslant \mathrm{e}^{-4\frac{\alpha_N}{N}t}\,\mathrm{W}_1(\mathrm{Law}(R_0), \Gamma_N).$$

Furthermore for any $x \in D$ *and* $t \geqslant 0$*, we have*

$$\mathbb{E}(R_t \mid R_0 = r) = r\mathrm{e}^{-\frac{4\alpha_N}{N}t} + \left(\frac{1}{2} + \frac{N}{\beta_N} - \frac{1}{2N}\right)\left(1 - \mathrm{e}^{-\frac{4\alpha_N}{N}t}\right).$$

Uniform Poincaré for single particle

Theorem (Poincaré for one-particle – Bolley-C.-Fontbona)

If $\beta_N = N^2$ then the one-particle marginal $\mathbb{P}^{1,N}$ of $\mathbb{P}^N$ is log-concave and satisfies a Poincaré inequality with a constant uniform in N.

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- If $\beta_N = N^2$ then one particle marginal of $\mathbb{P}^{1,N}$ has density

$$z \in \mathbb{C} \mapsto \varphi_N(z) = \frac{\mathrm{e}^{-N|z|^2}}{\pi} \sum_{\ell=0}^{N-1} \frac{N^\ell |z|^{2\ell}}{\ell!}.$$

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- Circular law

$$\lim_{N\to\infty} \sup_{z\in K} \left| \varphi_N(z) - \frac{\mathbf{1}_{\{|z|\leqslant 1\}}}{\pi} \right| = 0.$$

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- Second moment of φ_N bounded in N then KLS Bobkov's theorem

McKean–Vlasov mean-field limit

$$\mathrm{d}X_t^{i,N} = \sqrt{2\frac{\alpha_N}{\beta_N}}\mathrm{d}B_t^{i,N} - 2\frac{\alpha_N}{N}X_t^{i,N}\mathrm{d}t - 2\frac{\alpha_N}{N}\sum_{j\neq i}\frac{X_t^{i,N} - X_t^{j,N}}{\left|X_t^{i,N} - X_t^{j,N}\right|^2}\mathrm{d}t.$$

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^{i,N}}$$

McKean–Vlasov mean-field limit

Theorem (McKean–Vlasov (in progress) – Bolley-C.-Fontbona)

If $\sigma = \lim_{N\to\infty} \frac{\alpha_N}{\beta_N} \in [0,\infty)$ *then* $\lim_{N\to\infty} \mu_t^N = \mu_t$ *with*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} f(x)\,\mu_t(\mathrm{d}x) = \sigma \int \Delta f(x)\,\mu_t(\mathrm{d}x) - 2\int_{\mathbb{R}^2} x\cdot\nabla f(x)\,\mu_t(\mathrm{d}x) + \int_{\mathbb{R}^4} \frac{(x-y)\cdot(\nabla f(x)-\nabla f(y))}{|x-y|^2}\,\mu_t(\mathrm{d}x)\mu_t(\mathrm{d}y)$$

or more compactly

$$\partial_t \mu_t = \sigma\Delta\mu_t + \nabla\cdot((\nabla V + \nabla W * \mu_t)\mu_t).$$

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- Regimes: $(\alpha_N,\beta_N) = (N,N^2)$ and $(\alpha_N,\beta_N) = (N,N)$

McKean–Vlasov mean-field limit for nonlinear PDE

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  Une loi des grands nombres pour des systèmes de diffusions avec interaction et à coefficients non bornés.
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MIND THE GAP

That's all folks!

Thank you for your attention.