Gaussian kernels also have Gaussian minimizers

F. Barthe & P. Wolff

Institut de Mathématiques de Toulouse

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Outline

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Norm of operators with Gaussian kernels

- $\|f\|_p = \|f\|_{L^p(\mathbb{R}^m, dx)} = \left(\int_{\mathbb{R}^m} |f(x)|^p dx\right)^{\frac{1}{p}}$

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  with $q : \mathbb{R}^n \to \mathbb{R}^+$ (or $\mathbb{C}$), quadratic form with $\mathrm{Re}(q) > 0$
  and $\ell : \mathbb{R}^n \to \mathbb{R}^+$ (or $\mathbb{C}$), linear form

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Theorem (Lieb, 1990)

$$\|\mathcal{K}\|_{p\to q} = \sup\left\{\frac{\|\mathcal{K}g\|_q}{\|g\|_p};\ g \text{ centred Gaussian}\right\}$$ *holds if*

- $1 < p \leq q < \infty$ *in the complex case*
- $1 < p, q < \infty$ *in the real case*

Applications

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- Gaussian hypercontractivity (Nelson) : for $t \geq 0$

$$P_t f(x) := \int_{\mathbb{R}^n} f\left(e^{-t}x + \sqrt{1-e^{-2t}}\,y\right) d\gamma_n(y).$$

If $p > 1$ and $q \leq 1 + e^{2t}(p-1)$ then

$$\|P_t f\|_{L^q(\gamma_n)} \leq \|f\|_{L^p(\gamma_n)}.$$

Multilinear Brascamp-Lieb inequalities

- H and for $1 \leq i \leq m$, H_i Euclidean spaces
- $B_i : H \to H_i$ linear and onto, $f_i : H_i \to \mathbb{R}$
- $p_i \in (1, +\infty)$
- $Q : H \to \mathbb{R}^+$ quadratic form ($Q \geq 0$)

$$I(f_1, \ldots, f_m) := \frac{\int_H e^{-Q(x)} \prod_{i=1}^m f_i(B_i x)\, dx}{\prod_{i=1}^m \|f_i\|_{p_i}}.$$

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$$\sup_{f_1,\ldots,f_m} I(f_1, \ldots, f_m) = \sup\left\{ I(g_1, \ldots, g_m);\ g_i \text{ centred Gaussian} \right\}$$

Multilinear inequalities, with $c_i = 1/p_i$, $h_i = |f_i|^{p_i}$

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$$J(h_1, \dots, h_m) := \frac{\int_H e^{-Q(x)} \prod_{i=1}^m h_i(B_i x)^{c_i} \, dx}{\prod_{i=1}^m \left(\int_{H_i} h_i \right)^{c_i}}.$$

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- **Sharp convolution inequality** on $\mathbb{R}^n$ (Beckner) :
  $p, q, r \geq 1$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,
  $$\|f * g\|_r \leq C\|f\|_p\|g\|_q.$$
  By duality, rewrites as
  $$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)h(x)\, dxdy \leq C\|f\|_p\|g\|_q\|\|h\|_{r'}$$

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- **Geometric Brascamp-Lieb** (K. Ball) :
  Let $u_1, \ldots, u_m \in \mathbb{R}^n$, with $|u_i| = 1$, $c_i \in (0, 1]$ and
  $$\sum_{i=1}^m c_i u_i \otimes u_i = \mathrm{Id}_{\mathbb{R}^n}.$$
  Then for $f_i : \mathbb{R} \to \mathbb{R}^+$,
  $$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, u_i\rangle)^{c_i} dx \leq \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}.$$

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- **Reverse hypercontractivity** (Borell) :
  Let $p, q \in (-\infty, 1)$ with $q \geq 1 + e^{2t}(p-1)$. For $f : \mathbb{R}^n \to \mathbb{R}^+$

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Remark : for some other $q < 1$, it happens

$$\inf \left\{ \frac{\|P_t g\|_{L^q(\gamma_n)}}{\|g\|_{L^p(\gamma_n)}} ; g \text{ centred Gaussian} \right\} = 1,$$

$$\inf \left\{ \frac{\|P_t g\|_{L^q(\gamma_n)}}{\|g\|_{L^p(\gamma_n)}} ; g \text{ non-centred Gaussian} \right\} = 0.$$

- **Sharp inverse Young inequality** (Brascamp-Lieb) : $p, q, r \in (0,1]$ with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and non-negative functions

$$\|f * g\|_r \geq C\|f\|_p\|g\|_q.$$

By "duality", rewrites as

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- **Prékopa-Leindler inequality** (limit case $p, q, r \to 0$) : For $\lambda \in (0, 1)$ and $f, g : \mathbb{R}^n \to \mathbb{R}^+$,

$$\int_{\mathbb{R}^n}^{*} \sup_{\lambda x+(1-\lambda)y=z} f(x)^{\lambda} g(y)^{1-\lambda}\,dz \geq \left(\int_{\mathbb{R}^n} f\right)^{\lambda}\left(\int_{\mathbb{R}^n} g\right)^{1-\lambda}.$$

- **Inverse Brascamp-Lieb inequality** (B.). Geometric form : Let $u_1, \ldots, u_m \in \mathbb{R}^n$, with $|u_i| = 1$, $c_i \in (0,1]$ and

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = \mathrm{Id}_{\mathbb{R}^n}.$$

Then for $f_i : \mathbb{R} \to \mathbb{R}^+$,

$$\int_{\mathbb{R}^n}^{*} \sup_{\sum c_i x_i u_i = z} \prod_{i=1}^{m} f_i(x_i)^{c_i}\, dz \geq \prod_{i=1}^{m} \left(\int_{\mathbb{R}} f_i \right)^{c_i}.$$

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  Remark :
  rather "dual" to Brascamp-Lieb, even in general form :
  $f_i : H_i \to \mathbb{R}^+$, $B_i : H \to H_i$, sup on $\sum_{i=1}^{m} c_i B_i^* x_i = z$.

Theorem of Chen, Dafnis and Paouris (2013)

- $N = n_1 + \cdots + n_m$,
- $X = (X_1, \ldots, X_m)$ random vector with distribution $\mathcal{N}_N(0, \Sigma)$.
- $p_i \in \mathbb{R}^*$ and $P := \mathrm{diag}(p_1 \Sigma_{1,1}, \ldots, p_m \Sigma_{m,m})$,
- $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^+$

Then, if $\Sigma \leq P$,

$$\mathbb{E} \prod_{i=1}^{m} f_i(X_i) \leq \prod_{i=1}^{m} \left(\mathbb{E} f_i(X_i)^{p_i}\right)^{\frac{1}{p_i}},$$

and if $\Sigma \geq P$,

$$\mathbb{E} \prod_{i=1}^{m} f_i(X_i) \geq \prod_{i=1}^{m} \left(\mathbb{E} f_i(X_i)^{p_i}\right)^{\frac{1}{p_i}}.$$

Allows to recover many inverse inequalities.

Outline

Setup for an inverse Lieb principle

- H and for $1 \leq k \leq m$, H_k Euclidean spaces
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- $c_1, \ldots, c_{m_+} > 0 \geq c_{1+m_+}, \ldots, c_m$

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- $\mathcal{B}_+ x := (B_1 x, \ldots, B_{m_+} x)$, $x \in H$.
- $\mathcal{G} := \{(g_1, \ldots, g_m);\ g_i : H_i \to \mathbb{R}_*^+ \text{ Gaussian}\}$
- $\mathcal{CG} := \{(g_1, \ldots, g_m);\ g_i : H_i \to \mathbb{R}_*^+ \text{centred Gaussian}\}$

Theorem (B. and Wolff, 2014)

If $Q_{|\ker \mathcal{B}_+} > 0$ *then*

$$\inf J = \inf_{\mathcal{G}} J < +\infty,$$

else $\inf_{\mathcal{G}} J = +\infty$, *and*

- $\inf J = +\infty$ *when* $\mathcal{B}_+$ *is onto,*
- $\inf J = 0$ *when* $\mathcal{B}_+$ *is not onto.*

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The "gaussian minimizers principle" fails for :
$m = m_+ = 2$, $c_1 = c_2 = 1$ and

$$J(f_1, f_2) := \frac{\int_{\mathbb{R}^2} f_1(x) f_2(x)\, dx\, dy}{\left(\int_{\mathbb{R}} f_1\right)\left(\int_{\mathbb{R}} f_2\right)}.$$

Centred Gaussian minimizers

Theorem (B. and Wolff, 2014)

If $Q_{|\ker \mathcal{B}_+} > 0$ *and*

$$s^+ + \dim(H_1) + \cdots + \dim(H_{m_+}) \leq \dim(H)$$

then

$$\inf J = \inf_{\mathcal{CG}} J.$$

If the above hypothesis is not verified then $\inf J = 0$ *or* $+\infty$*.*

Without the dimension condition, it may happen

$$\inf_{\mathcal{CG}} J > 0 \text{ and } \inf_{\mathcal{G}} J = 0.$$

Outline

Reduction to $\mathcal{B}_+ = B_1 \times \cdots \times B_{m_+}$ bijective

- If $(a_1, \ldots, a_{m_+})$ is far from Range$(B_1 \times \cdots \times B_{m_+})$: choose $f_i := \mathbf{1}_{B(a_i,\varepsilon)}$. Then

$$\prod_{i=1}^{m_+} f_i(B_i x)^{c_i} = 0, \qquad \text{for all } x \in H.$$

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- If $|v| = 1$ and $v \in \bigcap_{i \le m_+} \ker B_i$: set $x = y + tv$ with $y \perp v$

$$\int \prod_k f_k(B_k x)^{c_k} dx = \int_{\{v\}^\perp} \prod_{i \le m_+} f_i(B_i y)^{c_i} \int_{\mathbb{R}} \prod_{j > m_+} f_j(B_j y + tB_j v)^{c_j} dt\, dy.$$

y- a.e. the dt integral is $+\infty$, since

$$\{t;\ f_j(B_j y + tB_j v)^{c_j} \le 1\} = \{t;\ f_j(B_j y + tB_j v) \ge 1\}$$

has finite measure.

Calculating for centred Gaussian functions

$$\int_{\mathbb{R}} e^{-\pi t^2} dt = 1$$

Calculating for centred Gaussian functions

- For $A : H \to H$ linear self-adjoint,

$$\int_H e^{-\pi\langle x, Ax\rangle}\, dx = \begin{cases} \det(A)^{-\frac{1}{2}} & \text{if } A > 0 \\ +\infty & \text{else} \end{cases} .$$

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- For $A_k > 0$ on H_k and $g_k(\cdot) = e^{-\pi\langle\cdot,A_k\cdot\rangle}$,

$$\int_H \prod_k g_k(B_k x)^{c_k}\,dx = \int_H e^{-\pi\langle x,\sum c_k B_k^* A_k B_k x\rangle}\,dx$$

Lemma

$\inf_{\mathcal{CG}} J = D^{-\frac{1}{2}}$, *where*

$$D := \sup\left\{ \frac{\det\left(\sum c_k B_k^* A_k B_k\right)_+}{\prod \det(A_k)^{c_k}};\ A_k > 0 \text{ self-adjoint on } H_k \right\}.$$

Shifting Gaussian functions

- For $A : H \to H$ linear self-adjoint, and $v \in H$

$$\int_H e^{-\pi\langle x, Ax\rangle + 2\pi\langle x, v\rangle}\, dx = \begin{cases} \dfrac{e^{\pi\langle A^{-1}v, v\rangle}}{\det(A)^{-\frac{1}{2}}} & \text{if } A > 0 \\ +\infty & \text{else} \end{cases}.$$

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where

$$A := \sum c_k B_k^* A_k B_k \text{ and } v = \sum_k c_k B_k^* v_k.$$

Thus $J(g_1,\ldots,g_m) =$

$$\begin{cases} \left(\dfrac{\prod \det(A_k)^{c_k}}{\det(A)}\right)^{\frac{1}{2}} e^{\pi\left(\langle A^{-1}v,v\rangle - \sum c_k\langle A_k^{-1}v_k,v_k\rangle\right)} & \text{if } A > 0 \\ +\infty & \text{else} \end{cases}.$$

Shifts don't spoil the constants

Lemma

Assume $\mathcal{B}_+ = B_1 \times \cdots \times B_{m_+} : H \to H_1 \times \cdots \times H_{m_+}$ *is* **onto**.

For $1 \leq k \leq m$, *let* $v_k \in H_k$ *and* $A_k > 0$ *operator on* H_k.
Set

$$v := \sum_k c_k B_k^* v_k \text{ and } A := \sum c_k B_k^* A_k B_k.$$

If $A > 0$ *then*

$$\langle A^{-1} v, v \rangle \geq \sum c_k \langle A_k^{-1} v_k, v_k \rangle.$$

Consequently

$$\inf_{\mathcal{CG}} J = \inf_{\mathcal{G}} J.$$

$$D = \sup \left\{ \frac{\det\left(\sum c_k B_k^* A_k B_k\right)_+}{\prod \det(A_k)^{c_k}};\ A_k > 0 \text{ self-adjoint on } H_k \right\}.$$

Plan of proof : if $D < +\infty$, $\sqrt{D} = D \inf_{\mathcal{CG}} J \geq D \inf J \overset{?}{\geq} \sqrt{D}$

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Lemma

For each $k \leq m$, let $A_k > 0$ on H_k and $g_k : H_k \to \mathbb{R}_*^+$ defined by

$$g_k(\cdot) = e^{-\pi\langle \cdot, A_k^{-1} \cdot \rangle} \det(A_k)^{-\frac{1}{2}},$$

Let $f_k : H_k \to \mathbb{R}^+$ with $\int_H f_k = 1 = \int_{H_k} g_k$. Then

$$D \cdot J(f_1, \ldots, f_m) \geq \left(\frac{\det\left(\sum c_k B_k^* A_k B_k\right)_+}{\prod \det(A_k)^{c_k}} \right)^{1/2}.$$

Optimizing on A_k completes the proof.

Transportation proof

- By Brenier's theorem : $\exists\, \varphi_k : H_k \to \mathbb{R}$ convex s.t. $\nabla\varphi_k$ pushes $f_k(x)dx$ to $g_k(x)dx$, i.e.

$$f_k(\cdot) = g_k(\nabla\varphi_k(\cdot))\ \det(\nabla^2\varphi_k(\cdot)).$$

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- Define $\varphi(x) := \sum_k c_k\, \varphi_k(B_k x), \quad x \in H.$

$$\nabla\varphi(x) = \sum_k c_k B_k^* \nabla\varphi_k(B_k x) \in H,$$

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$$\nabla^2\varphi(x) = \sum_k c_k B_k^* \nabla^2\varphi_k(B_k x) B_k,$$

- On $\Omega := \{x \in H;\ \nabla^2\varphi(x) \geq 0\}$,

$$0 \leq \det(\nabla^2\varphi(x)) \leq D \prod \det\left(\nabla^2\varphi_k(B_k x)\right)^{c_k}.$$

$$
\begin{aligned}
D\,J(f_1,\ldots,f_m) &= D \int \prod f_k(B_k x)^{c_k}\, dx \\
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If $A := \sum c_k B_k^* A_k B_k > 0$, then by former lemma

$$
D\,J(f_1,\dots,f_m) \geq \frac{\int e^{-\pi\langle A^{-1}z,z\rangle}\,dz}{\prod \det(A_k)^{c_k/2}} = \left(\frac{\det\left(\sum c_k B_k^* A_k B_k\right)}{\prod \det(A_k)^{c_k}}\right)^{1/2} \qquad \square
$$

Ensuring surjectivity

Lemma

Let $\psi : \mathbb{R}^n \to \mathbb{R}$*, of class* C^2 *with* $\lim_{|x| \to +\infty} \frac{\psi(x)}{|x|} = +\infty$*. Then*

$$\nabla\psi\left(\{x;\ \nabla^2\psi(x) \geq 0\}\right) = \mathbb{R}^n.$$

Proof. Let $a \in \mathbb{R}^n$.
$x \mapsto \psi(x) - \langle x, a \rangle$ achieves its minimum at some x_0, and

$$\nabla\psi(x_0) = a\,, \qquad \nabla^2\psi(x_0) \geq 0 \quad \square$$

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By approximation, one can ensure that

$$\varphi(x) = \sum_{i\leq m_+} c_i\varphi_i(B_i x) + \sum_{j>m_+} c_j\varphi_j(B_j x)$$

is superlinear.

Outline

General case, à la Bennett-Carbery-Christ-Tao

For $f_k : H_k \to \mathbb{R}^+$, $k \leq m$ and $c \in (0, +\infty)^{m^+} \times (-\infty, 0]^{m-m_+}$

$$J_c(f_1, \ldots, f_m) := \frac{\int_H \prod_{k=1}^m f_k(B_k x)^{c_k}\,dx}{\prod_{k=1}^m \left(\int_{H_k} f_k\right)^{c_k}}.$$

Theorem (B.-Wolff, 2017, $Q = 0$ for shortness)

Assume $\mathcal{B}_+ = B_1 \times \cdots \times B_{m_+} : H \to H_1 \times \cdots \times H_{m_+}$ *is bijective.*

Then $\inf J_c > 0$ *iff :*

- $\dim(H) = \sum c_k \dim(H_k)$, *and*
- *for all subspaces* $V \subset H$ *with* $\mathcal{B}_+ V = B_1 V \times \cdots \times B_{m_+} V$,

$$\dim(V) \geq \sum c_k \dim(B_k V).$$

Sketch for necessity I

- Assume $\forall f_k$,

$$\frac{\int_H \prod_{k=1}^m f_k(B_k x)^{c_k}\,dx}{\prod_{k=1}^m \left(\int_{H_k} f_k(x)\,dx\right)^{c_k}} \geq C > 0.$$

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$$R^{\dim(H)-\sum c_k \dim(H_k)} \times \frac{\int_H \prod_{k=1}^m g_k(B_k x)^{c_k}\, dx}{\prod_{k=1}^m \left(\int_{H_k} g_k(x)\, dx\right)^{c_k}} \geq C > 0.$$

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- Let $V \subset H$ a subspace :

$$H = V \oplus V^{\perp}, \qquad H_k = B_k V \oplus (B_k V)^{\perp}.$$

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$$\frac{\int_{V^{\perp}} \int_V \prod_{k=1}^m f_k(b_k y + \rho_k z, \beta_k z)^{c_k} \, dy \, dz}{\prod_{k=1}^m \left(\int_{(B_k V)^{\perp}} \int_{B_k V} f_k(y, z) \, dy \, dz \right)^{c_k}} \geq C > 0.$$

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- Let $R > 0$. Replace $f_k(y,z)$ with $g_k(y/R, z)$ and y with Ry.

$$R^{\dim(V) - \sum c_k \dim(B_k V)} \times \frac{\int_{V^{\perp}} \int_V \prod_{k=1}^m g_k(b_k y + \frac{\rho_k z}{R}, \beta_k z)^{c_k} dydz}{\prod_{k=1}^m \left(\int_{(B_k V)^{\perp}} \int_{B_k V} g_k(y,z)\, dydz \right)^{c_k}} \geq C$$

Rank 1 case

Setting : $1 \leq n = m_+ \leq m$
$u_1, \ldots, u_m \in \mathbb{R}^n \setminus \{0\}$ with $(u_1, \ldots, u_n)$ basis.
Set $i \sim j$ when u_j has a non-zero i-th coordinate
For $f_k : \mathbb{R} \to \mathbb{R}^+$, $1 \leq k \leq m$ and $c \in (0, +\infty)^n \times (-\infty, 0]^{m-n}$

$$J_c(f_1, \ldots, f_m) := \frac{\int_{\mathbb{R}^n} \prod_{k=1}^m f_k\big(\langle x, u_k\rangle\big)^{c_k} dx}{\prod_{k=1}^m \left(\int_{\mathbb{R}} f_k\right)^{c_k}}.$$

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Theorem (B.-Wolff, 2015)

$$\inf J_c > 0 \iff c \in \mathbf{1}_{[\![1,n]\!]} + \mathrm{Pos}\Big(\big\{\mathbf{1}_{\{i\}} - \mathbf{1}_{\{j\}};\ i \sim j\big\}\Big)$$

$$\iff c \in [1, +\infty)^n \times (-\infty, 0]^{m-n}; \quad \sum_k c_k = n \text{ and}$$

$$\forall S \subset [\![1, n]\!], \quad \sum_{i \in S} (c_i - 1) \leq \sum_{j;\ S \sim j} |c_j|$$

Extreme rays of the positivity domain

Let $\varepsilon \geq 0$, $(u_1, \ldots, u_n)$ be the canonical basis, and $n \sim n+1$: $u_{n+1} = \sum_{i=1}^{n} \alpha_i u_i$ with $\alpha_n \neq 0$.

$$\int_{\mathbb{R}^n} \Big(\prod_{i=1}^{n-1} f_i(\langle x, u_i \rangle) \Big) f_n(\langle x, u_n \rangle)^{1+\varepsilon} f_{n+1}(\langle x, u_{n+1} \rangle)^{-\varepsilon} dx$$

$$= \int \prod_{i=1}^{n-1} f_i(x_i) \left(\int f_n(x_n)^{1+\varepsilon} f_{n+1}\Big(\sum_{i=1}^{n} \alpha_i x_i \Big)^{-\varepsilon} dx_n \right) dx_1 \cdots dx_{n-1}$$

$$\geq \left(\prod_{i=1}^{n-1} \int f_i \right) \left(\int f_n \right)^{1+\varepsilon} \left(\frac{1}{|\alpha_n|} \int f_{n+1} \right)^{-\varepsilon}$$

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$$\geq \left(\prod_{i=1}^{n-1} \int f_i \right) \left(\int f_n \right)^{1+\varepsilon} \left(\frac{1}{|\alpha_n|} \int f_{n+1} \right)^{-\varepsilon}$$

Hence $\inf J_c > 0$ for
$c = (1, \ldots, 1, 1+\varepsilon, -\varepsilon, 0, \ldots 0) = \mathbf{1}_{[\![1,n]\!]} + \varepsilon \left(\mathbf{1}_{\{n\}} - \mathbf{1}_{\{n+1\}} \right)$.

More to do

- complex kernels
- non-linear versions
- geometric applications
- Cauchy maximizers (HLS, Sobolev..)
- ...