Pour les 59 ans de Patrick Cattiaux et Christian Léonard

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La promenade autour des points tardifs

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$\diamondsuit$

Basé sur des travaux avec
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Contents

Souvenirs, souvenirs . . .

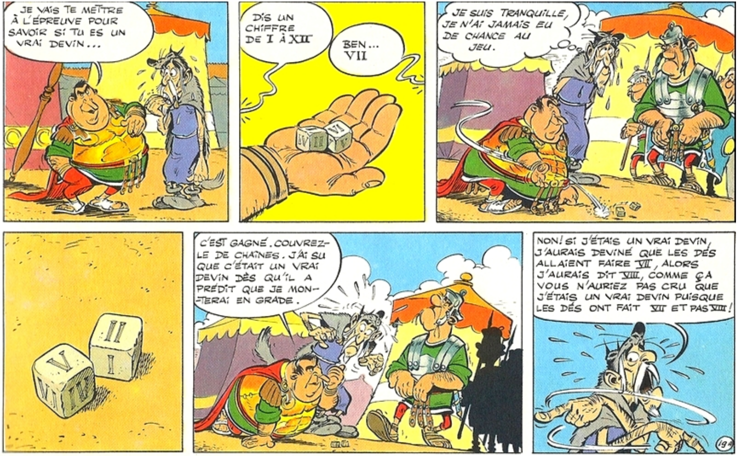

Figure: Notes de cours de probabilité, novembre 1977

Souvenirs, souvenirs . . .

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GEOSTOCHASTIC CALCULUS

by

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ABSTRACT

A stochastic calculus for a family of continuous measure-valued Markov processes is developed. Such processes arise naturally in the construction of stochastic models of spatially distributed populations. The stochastic calculus is a tool whereby a class of density-dependent models can be studied in terms of the multiplicative measure diffusion process. In this paper the stochastic integral is introduced in the space-time setting and a Cameron-Martin-Girsanov theorem is established.

1. INTRODUCTION

In order to model the evolution of a geographically distributed population or a distributed chemical reaction it is useful to introduce the notion of a measure-valued Markov process. A basic model of this type is the multiplicative (branching) measure diffusion process which was introduced by Dawson (1975) and which models a reproducing population in the absence of interaction effects. A number of properties and applications of the multiplicative measure diffusion process were obtained by Dawson (1977), Dawson and Hochberg (1978), Dawson and Ivanoff (1978) and Holley and Stroock (1978). However in order to develop more realistic models it is necessary to introduce density dependent nonlinear interaction effects. The main purpose of this paper is to introduce methods for dealing with the latter which are based on an extension of the Itô stochastic calculus to measure-valued diffusion processes. Starting with the multiplicative measure diffusion process and this geostochastic calculus it is possible to construct stochastic models of complex spatially distributed systems.

In modelling a spatially distributed population there are three main effects which must be incorporated:

(a) the spatial motion and dispersion of the population,
(b) the inherent fluctuation in the population due to demographic and environmental stochasticity,
(c) nonliniear interaction effects such as limitations on the environment carrying capacity.

Measure-valued stochastic processes incorporating these effects can be heuristically associated with a symbolic stochastic evolution equation of the form

$$\partial u/\partial t = Gu + F(u) + W(u) \, , \qquad (1.1)$$

where G is the infinitesimal generator of the spatial motion on R^d, the space on which the system is assumed to live, $F(u)$ represents the nonlinear interaction term and $W(u)$ represents the stochastic fluctuation term. We now consider a few examples which arise in applications.

Example 1.1 Model of Population with Reproduction and Migration. Consider a population of individuals which inhabit Euclidean space R^d. Each individual has a random life span and at the end of its life-time the individual dies and is replaced by a random number of offspring. To model spatial migration each individual is assumed to move in R^d according to a Brownian motion process. Applications of this type of model arise in population genetics to describe the dispersion and mutation of the descendants of a new gene in a high density population or a population of rare mutant genes (cf. Sawyer, 1976). A similar model arises in neutron transport theory to describe the motion and production of neutrons in a nuclear reactor (cf. Bensoussan, Lions and Papanicolaou, 1978). One method of studying this model is to consider a continuous approximation. This is analogous to modelling population dynamics by an ordinary differential equation. In the present example the diffusion approximation leads to the stochastic evolution equation

$$\partial/\partial t(u(t,.)) = \Delta u(t,.) + \alpha u(t,.) + \gamma W(u) \, , \qquad (1.2)$$

where Δ denotes the Laplacian operator, α is the Malthusian parameter and γ is inversely proportional to the mean life span. For each t, $u(t,.)$ is a random measure, that is, $u(t,A)$ denotes the biomass in the region A at time t. The appropriate stochastic fluctuation term, $W(u)$, in this case is described in detail in Section 3. For a mathematical derivation of the diffusion approximation, refer to Dawson and Ivanoff (1978).

Example 1.2. Biogeography: Modelling of a Population with Competition for Resources. Consider a population of individuals subject to reproduction

Contents

Cover time

$(X_t, t = 1, 2, \ldots)$ simple random walk on the torus $\mathbb{Z}_n^2 = \mathbb{Z}^2 / n\mathbb{Z}^2$

$$\Updownarrow$$

$X_t = \Upsilon_n S_t$, $S =$ SRW on $\mathbb{Z}^2$, $\Upsilon_n =$ equivalence mod. n

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Hitting time of x

$$T_n(x) = \min\{t \geq 0 : X_t = x\},$$

and the cover time

$$\boxed{\mathcal{T}_n = \max_{x \in \mathbb{Z}_n^2} T_n(x)}$$

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Hitting time of x

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and the cover time

$$\boxed{\mathcal{T}_n = \max_{x \in \mathbb{Z}_n^2} T_n(x)}$$

Wilf 1989, Aldous-Fill book 1990, Brummelhuis-Hilhorst 1991 . . .

Main questions:

✍ Asymptotics $n \to \infty$ of cover time on a large torus ?
✍ Statistics and Geometry of 2nd, 3rd . . . maxima ? Of nearby maxima ?
✍ Geometric structure of level sets ?

Yet another maximum of r.v.'s more or less dependent

- Maximum of independent identically distributed r.v.'s
- RW on complete graph ($\leftrightsquigarrow$ $d = \infty$). $T_n(x)$'s are independent !

  Cover time = Coupon collector problem with n^d images.

  (sum of independent but not i.d. r.v.'s)

$$\mathcal{T}_n = n^d \ln(n^d) + n^d \text{Gumbel} + \dots \qquad (\textit{Erdos}-\textit{Renyi}\ 1961)$$

- nearest neighbor random walk : ($d < \infty$) Correlations !
- Maxima of correlated fields: huge activity.
  Especially for log-correlated fields: Gaußian Free Field (d=2), Branching Random Walks and BBM, Multiplicative chaos and Liouville quantum gravity, Last Passage Percolation ...

What is special in $d = 2$? Strong correlation !

- Dimension two is critical for the walk (recurrence/transience)
- Hitting time is not much larger than mixing time

Simple Random Walk on $\mathbb{Z}^d$

Pólya's theorem 1921: RW is recurrent for $d = 1, 2$, and transient for $d \geq 3$.

- Green function for $d \geq 3$:

$$G(x, y) = \mathbb{E}_x \sharp\{t : X_t = y\} = g(y - x)$$

solves

$$\Delta_x G(x, y) = -\delta_y(x),$$

with $\Delta f(x) = \frac{1}{2d} \sum_{z \sim x} f(z) - f(x)$ the discrete Laplacian.

$$\boxed{-g = \Delta^{-1}}$$

$$\begin{aligned} g(x) &\leq g(0) < \infty \\ g(x) &= c|x|^{2-d} + O(|x|^{-d}), \qquad |x| \to \infty. \end{aligned}$$

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$$\boxed{-g = \Delta^{-1}}$$

- Green function for $d = 2$: g being infinite is replaced by $-a$,

$$a(x) = \sum_{t \geq 0} [\mathbb{P}_0(X_t = 0) - \mathbb{P}_0(X_t = x)]$$

solves $\Delta a = \delta_0$. Then $a(x) \sim \frac{2}{\pi} \ln |x|$ as $|x| \to \infty$.

Hitting times

Mean value is

$$\mathbb{E} T_n(x) \sim \begin{cases} g(0) n^d, & d \geq 3 \\ \frac{2}{\pi} n^2 \ln n, & d = 2 \end{cases}$$

(start from uniform !) and the law is close to exponential as $n \to \infty$

$$\frac{T_n(x)}{\mathbb{E} T_n(x)} \xrightarrow{\text{law}} \mathcal{E}(1)$$

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- General fact: Matthews' method (1989) shows that

$$\mathcal{T}_n \text{ and } \mathbb{E}\mathcal{T}_n \leq \mathbb{E}T_n(x) \times \ln(n^d),$$

- Sharp at leading order for $d \geq 3$ (Aldous 1990), and $d = 2$ (Dembo-Peres-Rosen-Zeitouni 2004)

Where does dependence show up ?

Question: In which respect does cover time behave like the maximum of n^d independent exponentially distributed r.v.'s ?

$$\mathcal{T}_n \overset{??}{\simeq} \max_{n^d \text{ i.i.d.r.v.}} \mathcal{E}_x(a_n)$$

$\mathcal{E}_x(a_n)$ with the "correct mean" (see last slide) ?

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$\mathcal{E}_x(a_n)$ with the "correct mean" (see last slide) ?

Sloppy answer:

- OK essentially, for large d.
- But when $d = 2$, it is false in some asymptotics.

Contents

Sznitman's Random Interlacements $d \geq 3$

Model of Random Interlacements (RI) of Sznitman (2008):
$\mathrm{RI}(\alpha)$: A stationary point process in $\mathbb{Z}^d$, given by a Poisson process of paths.
It yields "the local picture" left by the trace of a simple random walk in torus.

Theorem (Sznitman'09)

*For $\alpha > 0$, as $n \to \infty$,
the uncovered set at time αn^d converges in law to the vacant set of RI(α).*

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Equilibrium measure of a finite $A \subset\subset \mathbb{Z}^d$ (escape from A)

$$e_A(x) = P_x(S_t \notin A, t \geq 1), \quad x \in \partial A$$

capacity and harmonic measure (from infinity)

$$\mathrm{cap}(A) = \sum_{x \in \partial A} e_A(x) \ , \quad \mathrm{hm}_A(x) = e_A(x)/\,\mathrm{cap}(A).$$

Theorem means

$$\lim_{n \to \infty} \mathbb{P}[\Upsilon_n A \subset U_{\alpha n^d}] = \exp\big(- \alpha\,\mathrm{cap}(A)\big) = \mathbb{P}[A \subset \mathcal{V}^\alpha]$$

with $\mathcal{V}^\alpha$ the vacant set of RI(α).

Times beyond $O(n^d)$:

Belius'13 extends the coupling to larger times, up to the mean Cover time. Also, Miller-Sousi'16.

▷ Roughly, uncovered points are independent in dimension $d \geq 3$.

Asymptotics in dimension $d \geq 3$

- As $n \to \infty$, it holds in probability,

$$\mathcal{T}_n \sim g(0) n^d \ln(n^d)$$

with $g(x)$ the Green function.

- Fluctuations [Belius 2013]:

$$\frac{\mathcal{T}_n}{g(0)n^d} - \ln(n^d) \xrightarrow{\text{law}} \text{Gumbel}$$

as the if $T_n(x)$ were independent for $x \in \mathbb{Z}_n^d$.

$d \geq 3$: LARGE deviations

Recall Law of Large $\sharp$ in green (Aldous 1990)

☛ Lower tail is stretch exponential: For $\gamma \in (0,1)$,

$$\mathbb{P}\Big[\mathcal{T}_n \leq \gamma g(0) n^d \ln n^d\Big] = \exp\big(- n^{d(1-\gamma)+o(1)}\big).$$

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☛ Upper tail has a polynomial decay: For $\gamma > 1$,

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See Goodman-den Hollander 2013 for Brownian motion; with additional (involved) considerations on extremal geometry.

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☛ Compatible with the independent exponential picture.

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Dimension $d = 2$: correlation shows up

☞ For $d = 2$, Dembo-Peres-Rosen-Zeitouni'04

$$\frac{\mathcal{T}_n}{\frac{4}{\pi} n^2 \ln^2 n} \to 1 \qquad \text{in probability.}$$

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☞ Ding'12, Belius-Kistler'17 (exact value of c for Wiener sausage):

$$\sqrt{\mathcal{T}_n / 2n^2} \simeq \sqrt{2/\pi} \ln n - c \ln \ln n.$$

Bramson-Zeitouni's conjecture 2009: $\sqrt{\mathcal{T}_n / 2n^2}$ is tight around its median.

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The 1st limit is as in independent case (as well as large deviations from above). But the red terms are different.

What goes differently in dimension $d=2$?

The set of (n,γ)-late points

$$\mathcal{L}_n(\gamma) = \left\{ x : T_n(x) \geq \gamma \times \frac{4}{\pi} n^2 \ln^2 n \right\} ,$$

with $\gamma \in (0,1)$, is believed to be responsible for the discrepancies.

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☞ Brummelhuis-Hilhorst'91: fractal structure of late points.

☞ Dembo-Peres-Rosen-Zeitouni'06: density of late points is of different order around a fixed point and around a late point.

Clustering instead of a Poisson structure.

Not compatible with weak dependence between hitting times.

☞ Goal: understand the uncovered set at such times

Contents

Sznitman's Random Interlacements (RI) generalized

...by Teixeira'09: Let $\widehat{S}$ be a random walk on a transient weighted graph.

Construction of random interlacements at level α (abbr. RI(α)). To construct its restriction to a finite $\Lambda \subset V$ (vertices; here, $V = \mathbb{Z}^d$) :

- At each x on the boundary of Λ generate a Poisson($\alpha \widehat{e}_\Lambda(x)$)-number of particles; $\widehat{e}_\Lambda =$ equilibrium measure of Λ, has mass $\widehat{\text{cap}}(\Lambda)$;
- Each particle performs an independent $\widehat{S}$-random walks.

"Poisson soup of $\widehat{S}$-spaghettis"

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$$\mathcal{V}^\alpha = V \setminus \{\text{visited vertices}\} .$$

• Consistent definition for $\Lambda \subset V$.
• Characterizing property: $\forall A \subset V$ finite,

$$\boxed{\mathbb{P}[A \subset \mathcal{V}^\alpha] = \exp\big(-\alpha \widehat{\mathrm{cap}}(A)\big).} \qquad \text{(RI)}$$

- In $d \geq 3$, simply take $\hat{S} = S$ usual RW to obtain Sznitman's original RI.

Capacity in the recurrent case

Harmonic measure of a finite $A \subset \mathbb{Z}^2$ = entrance law "starting at infinity",

$$\mathrm{hm}_A(x) = \lim_{\|y\| \to \infty} P_y[S_{\tau(A)} = x].$$

Potential (solution of $\Delta a = \delta_0$):

$$a(x) = \sum_{t \geq 0} [P_0(S_t = 0) - P_x(S_t = 0)].$$

Capacity of a finite $A \subset \mathbb{Z}^2$,

$$\mathrm{cap}(A) = \sum_{x \in A} a(x - x_0)\, \mathrm{hm}_A(x) \qquad (x_0 \in A \text{ arbitrary})$$

The capacity is invariant by translation and the capacity of a singleton is 0.
For the random walk and a finite A in $\mathbb{Z}^2$,

$$P_x\big(\tau(B(0,R)) < \tau(A)\big) = \frac{\ln \|x\|}{\ln R} + \frac{C - \frac{\pi}{2}\mathrm{cap}(A) + \varepsilon(x) + \varepsilon_x(R)}{\ln R},$$

as $R \to \infty$ and then $x \to \infty$.

Contents

Random Interlacements in 2 dimensions

2 Problems:

- (1) What $\widehat{S}$ gives a meaningful process RI when $d = 2$?
- (2) What does the RI describe in terms of RW or BM on torus ?

Random Interlacements in 2 dimensions

• (1) Take $\widehat{S}$ = random walk conditioned to never hit 0 - i.e., Doob's h-transform with transition

$$\widehat{p}(x,y) = \frac{a(y)}{4a(x)}, \qquad y \sim x \neq 0.$$

- $\widehat{S}$ is reversible w.r.t. $\mu_x := a^2(x)$, conductances $a(x)a(y)$, $x \sim y \in \mathbb{Z}^2$,
- $\widehat{S}$ is transient.

Crucial identity: $\forall A \subset \mathbb{Z}^2$ with $0 \in A$,

$$\boxed{\mathrm{cap}(A) = \widehat{\mathrm{cap}}(A)}.$$

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Crucial identity: $\forall A \subset \mathbb{Z}^2$ with $0 \in A$,

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• (2) Finally, for the corresponding interlacement, we have

$$\boxed{\mathbb{P}[A \subset \mathcal{V}^\alpha] = \exp\big(-\alpha\pi\,\mathrm{cap}(A)\big), \quad A \subset\subset \mathbb{Z}^2 \text{ containing } 0} \qquad (*)$$

Contents

V - Late points for RW on the torus and RI ($d = 2$)

Uncovered set by the walk by time

$$U_t^{(n)} = \{x \in \mathbb{Z}_n^2 : T_n(x) > t\}.$$

Taking time of the order of the cover time:

$$t_\alpha := \frac{4\alpha}{\pi} n^2 \ln^2 n,$$

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Theorem (FC+S.Popov+M.Vachkovskaia 2016)

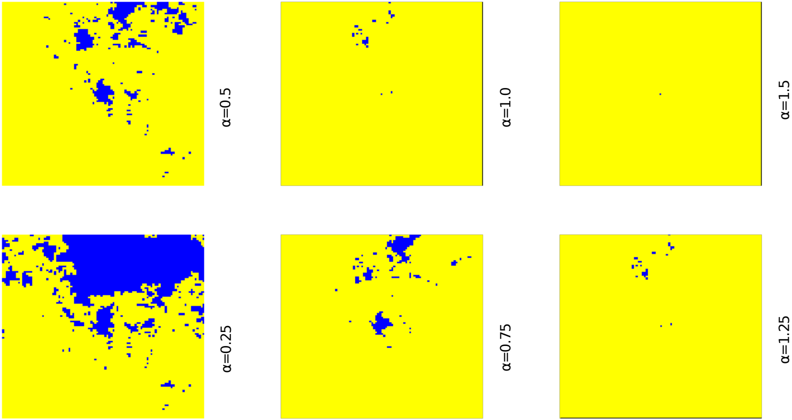
Let $\alpha > 0$ and A is a finite subset of $\mathbb{Z}^2$. We have

$$\lim_{n\to\infty} \mathbb{P}[\Upsilon_n A \subset U_{t_\alpha}^{(n)} \mid 0 \in U_{t_\alpha}^{(n)}] = \exp\big(-\pi\alpha \,\mathrm{cap}(A \cup \{0\})\big).$$

The RI describes the structure of the late points around a randomly picked late point (conditionally that there exist some.)

Vacant set (in blue) of RI(α) in $\mathbb{Z}^2$. Thanks to Darcy Cunha.

For $\alpha = 1.5$ the only vacant site is the origin.



Some properties of RI

- $\lim_{y\to\infty} P_x(\hat{\tau}_y < \infty) = 1/2$
- Density decay

$$\mathbb{P}[x \in \mathcal{V}^\alpha] = \exp\Big(-\pi\alpha \frac{a(x)}{2} \Big) \sim C_\alpha \|x\|^{-\alpha}.$$

- When $s := \|x\| \to \infty$, $\|y\| = s^{1+o(1)}$ and $\|x-y\| = s^{\beta+o(1)}$ with some $\beta \in [0,1]$, the correlation decays like

$$\mathrm{Cor}\big(\mathbf{1}_{\{x\in\mathcal{V}^\alpha\}}, \mathbf{1}_{\{y\in\mathcal{V}^\alpha\}}\big) = s^{-\frac{\alpha\beta}{4-\beta}+o(1)}.$$

Some properties of RI

- If $0 \in A \subset B(r)$,

$$\mathbb{P}[A \subset \mathcal{V}^\alpha \mid x \in \mathcal{V}^\alpha] = \exp\left(-\frac{\pi\alpha}{4}\,\mathrm{cap}(A)\frac{1 + O\big(\frac{r \ln r \ln \|x\|}{\|x\|}\big)}{1 - \frac{\mathrm{cap}(A)}{2a(x)} + O\big(\frac{r \ln r}{\|x\|}\big)}\right).$$

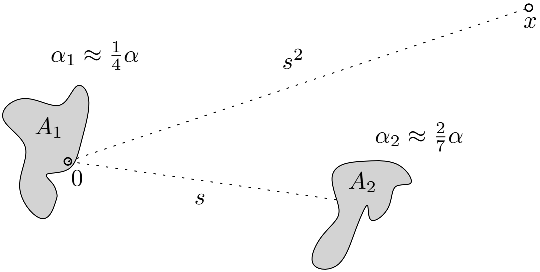


Figure: How the “local rate” looks like if we condition on the event that a “distant” site is vacant.

Contents

Size of the vacant set

Recall

$$\mathcal{V}^\alpha = V \setminus \{\text{visited vertices}\} .$$

Phase transition (2016):

- For $\alpha > 1$, $\quad |\mathcal{V}^\alpha| < \infty \quad$ a.s.,
- For $\alpha \in (0, 1)$, $\quad |\mathcal{V}^\alpha| = \infty \quad$ *a.s.*

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Theorem (FC+S.Popov 2017+)

$$\mathcal{V}^1 \quad \text{is a.s. infinite.}$$

Critical case $\alpha = 1$

Contradicting arguments:

Times corresponds formally to "just after" the actual covering time because of the negative $\log \log n$-correction to the leading order. So around the origin (and assuming it is not visited yet) there should not be much unvisited points:

- ☛ this is in favor of scenario : $\mathcal{V}^1$ a.s. finite ...

On the other hand, conditioning by a rare event (everything has not been visited), we put the walk in a deviating regime, and it may occur that many points around are unvisited, leading to the

- ☛ opposite scenario : $\mathcal{V}^1$ a.s. infinite ...

Why 2nd scenario is correct ???

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Strategy of proof: take a sequence of nested pairs of balls $B_k = B(x_k, b_k)$, with increasing size and increasing distance.

- Local fluctuations of excursions produce too few excursions: for many k's,

$$\sharp(k\text{-excursions } \partial B_{k+1} \to \partial B_k) \ll (\text{mean number})$$

- If too few excursions, $\mathbb{P}(B_k \text{ has an unvisited point}) \geq c > 0$.
- The decorrelation between the different balls is good enough. □

Coming next. . .

More properties of RI. . .

. . . in the continuous case (work in progress FC + S.Popov) . . .

Featuring: The Wiener moustache !