Long time behavior for stochastic switched differential equations

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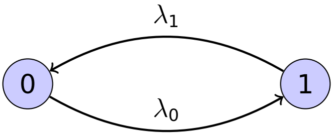
The simplest Markov process

Consider the process $(I_t)_{t \geqslant 0}$ on $\{0, 1\}$ with jump rates



- Jum time starting at i:

$$\mathbb{P}_i(T > t) = e^{-\lambda_i t}$$

- Jump rates: with $p \in (0, 1)$ and $\beta > 0$,

$$\lambda_1 = (1 - p)\beta \quad \text{and} \quad \lambda_0 = p\beta$$

- Invariant measure:

$$\nu = (1 - p)\delta_0 + p\delta_1$$

Switched flows

Framework

- Process $(I_t)_{t \geqslant 0}$ as before
- Two smooth flows F^0 and F^1 on $\mathbb{R}^2$

The full process

The process $Z = (X, I) \in \mathbb{R}^d \times \{0, 1\}$ is defined by

$$\dot{X}_t = F^{I_t}(X_t)$$

Main example today

$$F^i(x) = A_i x$$

where A_0 and A_1 are 2×2 Hurwitz matrices.
$(\mathrm{Spec}(A_i) \subset (-\infty, 0) \times \mathbb{R})$

Lyapunov exponent

Theorem

If $\mathbb{P}(X_0 \neq 0) = 1$ *then*

$$\frac{1}{t} \log \|X_t\| \xrightarrow[t\to\infty]{a.s.} \chi(p, \beta).$$

This deterministic limit does not depend on the initial condition.

Sketch of proof

Introduce the polar coordinates $(R_t, U_t) \in (0, +\infty) \times S^1$ of X_t:

$$\dot{R}_t = R_t \langle U_t, A_{I_t} U_t \rangle$$
$$\dot{U}_t = AU_t - \langle U_t, A_{I_t} U_t \rangle U_t$$

- (U, I) is Markovian
- $\forall t \geqslant 0$

$$R_t = R_0 \exp\left(\int_0^t \langle U_s, A_{I_s} U_s \rangle \, ds \right)$$

Sketch of proof

The invariant measures μ on $S^1 \times \{0,1\}$ are such that

$$\forall f : S^1 \times \{0,1\} \to \mathbb{R}, \quad \int Lf d\mu = 0$$

where L is the generator of (U, I)

$$Lf(u,i) = d_i(u)\partial_u f(u,i) + \lambda_i(f(u,1-i) - f(u,i)).$$

The Ergodic Theorem ensures that

$$\chi(p,\beta) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \langle U_s, A_{I_s} U_s \rangle \, ds = \int_{S^1\times\{0,1\}} \langle u, A_i u \rangle \, d\mu(u,i)$$

First example

Theorem (Benaïm-Le Borgne-M.-Zitt (2014))

Define

$$A_0 = \begin{pmatrix} -1 & 2b \\ -2/b & -1 \end{pmatrix} \quad \text{and} \quad A_1 = A_0^T.$$

If $b - 1/b > 1$ *then* $\beta \mapsto \chi(1/2, \beta)$ *is increasing and*

$$\lim_{\beta \to 0} \chi(1/2, \beta) = -1$$

$$\lim_{\beta \to \infty} \chi(1/2, \beta) = b - \frac{1}{b} - 1 > 0$$

Why may χ be positive for large β?

Even if A_0 and A_1 are Hurwitz matrices, $b - 1/b - b$ is the positive eigenvalue of

$$A_{1/2} = \frac{1}{2}A_0 + \frac{1}{2}A_1$$

For large βs, the inv. meas. concentrates near the instable direction of $A_{1/2}$.

General case. Same situation for $p \neq 1/2$ as soon as $A_p = pA_1 + (1-p)A_0$ has a positive eigenvalue.

A second example

Theorem (Lawley-Mattingly-Reed (2013))

Define

$$A_0 = \begin{pmatrix} -\alpha & 1 \\ 0 & -\alpha \end{pmatrix} \quad \textit{and} \quad A_1 = \begin{pmatrix} -\alpha & 0 \\ -1 & -\alpha \end{pmatrix}$$

- *If β is small or large enough then $\chi(1/2, \beta) < 0$.*
- *If α is small enough, there exists β_0 such that $\chi(1/2, \beta_0) > 0$.*

The Lyapunov exponent $\chi(1/2, \beta)$

$$\chi(1/2, \beta) = -\alpha + G(\beta)$$

Simulations suggest that G is unimodal...

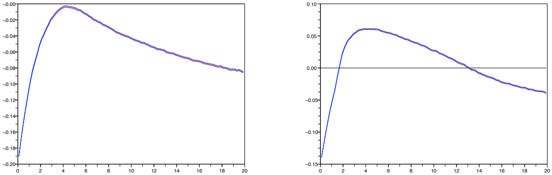


Figure: $\beta \mapsto \chi(1/2, \beta)$ for $\alpha = 0.2$ (on the left) and $\alpha = 0.15$ (in the right).

Link with deterministic control

The system is unbounded if there exists $t \mapsto I_t$ such that $|X_t| \to \infty$.

Theorem (Balde-Boscain-Mason (2009))

Define

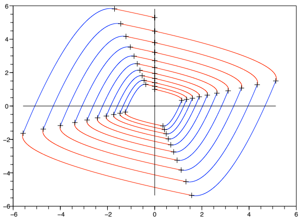
$$A_0 = \begin{pmatrix} -\alpha & 1 \\ 0 & -\alpha \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} -\alpha & 0 \\ -1 & -\alpha \end{pmatrix}$$

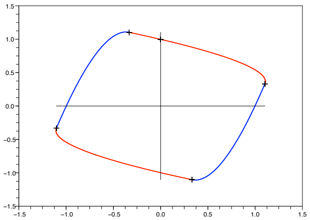
The system is unbounded iff

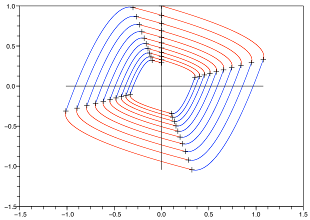
$$T(\alpha^2) := \frac{1 + 2\alpha^2 + \sqrt{1 + 4\alpha^2}}{2\alpha^2} e^{-2\sqrt{1+4\alpha^2}} > 1.$$

The worst trajectory

The *worth trajectory* with $\alpha = 0.32$, $\alpha = 0.3314$ and $\alpha = 0.34$.
The system starts $(0, 1)$ at and evolves clock-wisely.







Lotka-Volterra systems

Evolution of two competitive populations A and B

$$F(x,y) = \begin{cases} \alpha x(1 - ax - by) \\ \beta y(1 - cx - dy) \end{cases}$$

If $a < c$ and $b < d$, then $(1/a, 0)$ is the unique stable point.
The environment is favorable to species A.

Let us switch between two favorable to spec. A environments.

Everything may happen!

Theorem (Benaïm-Lobry (2014))

There exist F^0 and F^1 fav. to spec. A such that the switched process may be fav. to

- *spec. A,*
- *spec. B,*
- *one of the two spec. randomly (bistability),*
- *both spec. (persistence),*

depending on the jump rates.

The sign of a parameter Λ_j encodes the survival of species j.

Everythig may happen!

Theorem (M.-Zitt)

• *There exists an interval* $I \subset (0,1)$ *and* $p \in I \mapsto \beta_c(p)$ *s.t.*

$$\{(p,\beta)\,:\,\Lambda_i(p,\beta) > 0\} = \{(p,\beta)\,:\,\beta > \beta_c(p)\}.$$

• *If persistence occurs, the support of the invariant measure is (almost) known.*

Support of the invariant measure