Schrödinger problem and W_2-geodesics
joint work with Nicola Gigli

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En l'honneur de P. Cattiaux et C. Léonard
Toulouse, le 7 juin 2017

First order differentiation formula

Theorem (Gigli '13)

Let

- $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(K, \infty)$ space;
- (μ_t) a W_2-geodesic with $\mu_0, \mu_1 \leq C\mathfrak{m}$;
- $f \in W^{1,2}(X)$.

Then $t \mapsto \int f \mathrm{d}\mu_t$ is $C^1([0,1])$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \mathrm{d}\mu_t = -\int \langle \nabla f, \nabla \phi_t \rangle \mathrm{d}\mu_t$$

where $(\phi_t) \subset W^{1,2}(X)$ is any continuous choice of locally Lipschitz functions such that

$$\partial_t \mu_t + \mathrm{div}(\nabla \phi_t \mu_t) = 0.$$

Aim

Let

- $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}^*(K, N)$ space;
- (μ_t) a W_2-geodesic with $\mu_0, \mu_1 \leq C\mathfrak{m}$;
- $f \in W^{2,2}(X)$.

Question: can we say that $t \mapsto \int f \mathrm{d}\mu_t$ is $C^2([0,1])$ and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int f \mathrm{d}\mu_t = \int \mathrm{Hess}(f)(\nabla\phi_t, \nabla\phi_t) \mathrm{d}\mu_t$$

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Geodesics in $(\mathscr{P}_2(X), W_2)$

A W_2-geodesic (μ_t) on $\mathscr{P}_2(X)$ solves

$$\partial_t \mu_t + \mathrm{div}(\nabla \phi_t \mu_t) = 0$$

for functions (ϕ_t) such that

$$\partial_t \phi_t + \frac{1}{2} |\nabla \phi_t|^2 = 0$$

Problem: no matter how nice μ_0, μ_1 are, in general the ϕ_t's are only semiconcave.

Question: given a geodesic (μ_t), can we find curves (μ_t^ε) which are smooth and produce a second order approximation of (μ_t)?

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Idea

Let $(\mu_t^\varepsilon), (\phi_t^\varepsilon)$ be smooth and such that

$$\partial_t \mu_t^\varepsilon + \mathrm{div}(\nabla \phi_t^\varepsilon \mu_t^\varepsilon) = 0$$
$$\partial_t \phi_t^\varepsilon + \frac{1}{2}|\nabla \phi_t^\varepsilon|^2 = a_t^\varepsilon$$

Then for every f smooth we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \mathrm{d}\mu_t^\varepsilon = \int \langle \nabla f, \nabla \phi_t^\varepsilon \rangle \mathrm{d}\mu_t^\varepsilon$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int f \mathrm{d}\mu_t^\varepsilon = \int \Big(\mathrm{Hess}(f)(\nabla \phi_t^\varepsilon, \nabla \phi_t^\varepsilon) + \langle \nabla f, \nabla a_t^\varepsilon \rangle \Big) \mathrm{d}\mu_t^\varepsilon$$

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Rigorous statement of the problem

Given X smooth and μ_0, μ_1 with bounded densities and supports, find (μ_t^ε) so that

0th order: (μ_t^ε) uniformly W_2-converges to the only W_2-geodesic (μ_t) from μ_0 to μ_1 with densities uniformly bounded;

1st order: up to subsequences, $\phi_t^{\varepsilon_n} \to \overline{\phi}_t$ in $W^{1,2}(X)$, with $(\overline{\phi}_t)$ a choice of Kantorovich potentials associated to (μ_t);

2nd order: for every $f \in W^{2,2}(X)$ and $\delta \in (0, 1/2)$ it holds

$$\iint_\delta^{1-\delta} \langle \nabla f, \nabla a_t^\varepsilon \rangle \rho_t^\varepsilon \mathrm{d}t\mathrm{d}\mathfrak{m} \to 0$$

The estimates should depend only on:

- the L^∞-norms of the densities of μ_0, μ_1;
- the diameter of their supports;
- the lower bound on the Ricci curvature of X;
- the dimension of X.

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Interpolation between probability densities via the heat flow: the Schrödinger system

Let X be smooth and ρ_0, ρ_1 bounded probability densities with bounded support.

Find functions f, g on X such that

$$\begin{cases} \rho_0 = f\, P_1(g) \\ \rho_1 = P_1(f)\, g \end{cases}$$

The *entropic interpolation* between ρ_0 and ρ_1 is then defined by

$$\rho_t := P_t(f) P_{1-t}(g), \quad t \in [0,1].$$

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Interpolation between probability densities via the heat flow: slowing down

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How to find the functions $f^\varepsilon, g^\varepsilon$: the Schrödinger problem

Let R_ε be the measure on X^2 given by

$$\mathrm{d}R_\varepsilon(x,y) := \frac{\mathrm{d}P_\varepsilon(\delta_x)}{\mathrm{d}\mathfrak{m}}(y)\mathrm{d}(\mathfrak{m}\otimes\mathfrak{m})(x,y)$$

Then $(f^\varepsilon, g^\varepsilon)$ is a solution to the Schrödinger system if and only if

$$f^\varepsilon \otimes g^\varepsilon R_\varepsilon \in \mathrm{Adm}(\rho_0\mathfrak{m}, \rho_1\mathfrak{m})$$

where $f^\varepsilon \otimes g^\varepsilon(x,y) := f^\varepsilon(x)g^\varepsilon(y)$.

How to find the functions $f^\varepsilon, g^\varepsilon$: the Schrödinger problem

Let π^ε be the unique minimum of

$$\inf_{\pi \in \mathrm{Adm}(\rho_0 \mathfrak{m}, \rho_1 \mathfrak{m})} H(\pi \,|\, R_\varepsilon)$$

Its Euler equation is

$$\int \log\left(\frac{\mathrm{d}\pi^\varepsilon}{\mathrm{d}R_\varepsilon}\right) \mathrm{d}\sigma = 0$$

for every σ such that $\pi^1_\# \sigma = \pi^2_\# = 0$. This forces

$$\log\left(\frac{\mathrm{d}\pi^\varepsilon}{\mathrm{d}R_\varepsilon}\right) = a^\varepsilon \oplus b^\varepsilon$$

for some $a^\varepsilon, b^\varepsilon$, where $a^\varepsilon \oplus b^\varepsilon(x,y) := a^\varepsilon(x) + b^\varepsilon(y)$.
Thus for $f^\varepsilon := \exp(a^\varepsilon), g^\varepsilon := \exp(b^\varepsilon)$ we have

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The dual problem

With some manipulations one can show that the dual problem of

$$\inf_{\pi \in \mathrm{Adm}(\rho_0 \mathfrak{m}, \rho_1 \mathfrak{m})} \varepsilon H(\pi \mid R_\varepsilon)$$

is

$$\sup_{\varphi, \psi \in C(X)} \left\{ \int \varphi \rho_0 \mathrm{d}\mathfrak{m} + \int \psi \rho_1 \mathrm{d}\mathfrak{m} - \varepsilon \log \left(\int e^{\frac{\varphi \oplus \psi}{\varepsilon}} \mathrm{d}R_\varepsilon \right) \right\}$$

Moreover, if π^ε is a minimizer and $\varphi^\varepsilon, \psi^\varepsilon$ maximizers (Schrödinger potentials), we have

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Link with optimal transport

Theorem (Mikami-Thieullen '04 and Léonard '12)

As $\varepsilon \downarrow 0$ the curves $t \mapsto \rho_t^\varepsilon$ converge to the (unique) W_2-geodesic between $\rho_0\mathfrak{m}$ and $\rho_1\mathfrak{m}$. Moreover

$$\varepsilon H(\pi^\varepsilon \mid R_\varepsilon) \to \inf_{\pi \in \mathrm{Adm}(\rho_0\mathfrak{m}, \rho_1\mathfrak{m})} \frac{1}{2} \int \mathrm{d}^2(x,y) \mathrm{d}\pi.$$

The precise statement involves:

- abstract spaces;
- Γ-convergence;
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Building the approximation

$$\rho_t^\varepsilon := f_t^\varepsilon g_t^\varepsilon \quad f_t^\varepsilon := P_{\varepsilon t/2} f^\varepsilon \quad g_t^\varepsilon := P_{\varepsilon(1-t)/2} g^\varepsilon$$

$$\partial_t f_t^\varepsilon = \frac{\varepsilon}{2}\Delta f_t^\varepsilon \quad -\partial_t g_t^\varepsilon = \frac{\varepsilon}{2}\Delta g_t^\varepsilon$$

$$\begin{cases} \varphi_t^\varepsilon := \varepsilon \log f_t^\varepsilon \\ \partial_t \varphi_t^\varepsilon = \frac{1}{2}|\nabla \varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta \varphi_t^\varepsilon \end{cases} \quad \begin{cases} \psi_t^\varepsilon := \varepsilon \log g_t^\varepsilon \\ -\partial_t \psi_t^\varepsilon = \frac{1}{2}|\nabla \psi_t^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta \psi_t^\varepsilon \end{cases}$$

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Statement of the convergence results

Given X smooth and μ_0, μ_1 with bounded densities and supports, with the notations previously introduced we have that:

0th order: $(\rho_t^\varepsilon \mathfrak{m})$ uniformly W_2-converges to the only W_2-geodesic (μ_t) from μ_0 to μ_1 with densities uniformly bounded;

1st order: up to subsequences, $\vartheta_t^{\varepsilon_n} \to \overline{\vartheta}_t$ in $W^{1,2}_{loc}(X)$, with $(\overline{\vartheta}_t)$ a choice of Kantorovich potentials associated to (μ_t);

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The estimates only depend on $\|\rho_0\|_{L^\infty}, \|\rho_1\|_{L^\infty}, K, N$ and on the diameter of the supports of ρ_0, ρ_1.

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A couple of estimates

Let (u_t) be a solution of the heat equation with positive compactly supported L^1 initial datum. Then:

- Hamilton's gradient estimate

$$|\nabla \log u_t| \le \frac{C_1}{t}\Big(1 + \mathrm{d}(\cdot, \bar{x})\Big), \quad \forall t \in (0,1];$$

- Li-Yau Laplacian estimate

$$\Delta \log u_t \ge -\frac{C_2}{t}, \quad \forall t \in (0,1].$$

The constants C_1, C_2 are positive and only depend on K, N and on the diameter of $\mathrm{supp}(u_0)$ (on $\bar{x}$ too for C_1).

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Second order approximation

We start from

Theorem (Léonard '13)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} H(\rho_t^\varepsilon \mathfrak{m} \,|\, \mathfrak{m}) = \frac{1}{2} \int \Big(\Gamma_2(\varphi_t^\varepsilon) + \Gamma_2(\psi_t^\varepsilon) \Big) \rho_t^\varepsilon \mathrm{d}\mathfrak{m}$$

$$= \int \Big(\Gamma_2(\vartheta_t^\varepsilon) + \frac{\varepsilon}{2} \Gamma_2(\log \rho_t^\varepsilon) \Big) \rho_t^\varepsilon \mathrm{d}\mathfrak{m}$$

where

$$\Gamma_2(h) := \Delta \frac{|\nabla h|^2}{2} - \langle \nabla h, \nabla \Delta h \rangle$$

Second order approximation

Then from Léonard's formula we deduce that

$$\sup_{\varepsilon \in (0,1)} \iint_{\delta}^{1-\delta} \Big(|\mathrm{Hess}(\vartheta_t^\varepsilon)|^2 + \varepsilon^2 |\mathrm{Hess}(\log \rho_t^\varepsilon)|^2 \Big) \rho_t^\varepsilon \mathrm{d}t \mathrm{d}\mathfrak{m} < +\infty$$

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for every $\delta \in (0, 1/2)$.

Indeed, in the case $K = 0$,

$$\Gamma_2(h) \geq |\mathrm{Hess}(h)|^2$$

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Second order differentiation formula

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Let

- $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(K, N)$ space;
- (μ_t) a W_2-geodesic with $\mu_0, \mu_1 \leq C\mathfrak{m}$ and bounded supports;
- $f \in H^{2,2}(X)$.

Then $t \mapsto \int f \mathrm{d}\mu_t$ is $C^2([0, 1])$ and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int f \mathrm{d}\mu_t = \int \mathrm{Hess}(f)(\nabla\phi_t, \nabla\phi_t)\mathrm{d}\mu_t$$

where $(\phi_t) \subset W^{1,2}(X)$ is any continuous choice of locally Lipschitz functions such that

$$\partial_t \mu_t + \mathrm{div}(\nabla\phi_t \mu_t) = 0.$$

In particular, the choice of evolved Kantorovich potential does the job.

Merci de votre attention!