Generalized flows for the Euler model of incompressible fluids and resolution of the initial value problem by convex minimization

Yann Brenier
CNRS, Centre de mathématiques LAURENT SCHWARTZ,
ECOLE POLYTECHNIQUE, Palaiseau, France

En l'honneur de Patrick CATTIAUX et Christian LEONARD
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OUTLINE OF THE LECTURE:

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Geometric interpretation of the Euler equations for incompressible fluids after V.I. Arnold 1966.

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What about the initial value problem? (NEW!)

Euler, 200 BC (*)

XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations ſuivantes :

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la conſidération de la continuité du fluide :

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Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la conſidération de la continuité du fluide :

(*) Before Christ

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XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations suivantes :

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Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la confidération de la continuité du fluide :

(*) Before Christian...

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d.qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=0.$$

Si le fluide n'étoit pas compreſſible, la denſité q ſeroit la même en Z, & en Z', & pour ce cas on auroit cette équation :

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0.$$

qui eſt auſſi celle ſur laquelle j'ai établi mon Mémoire latin allégué ci-deſſus.

GEOMETRY OF THE EULER MODEL OF INCOMPRESSIBLE FLOWS.

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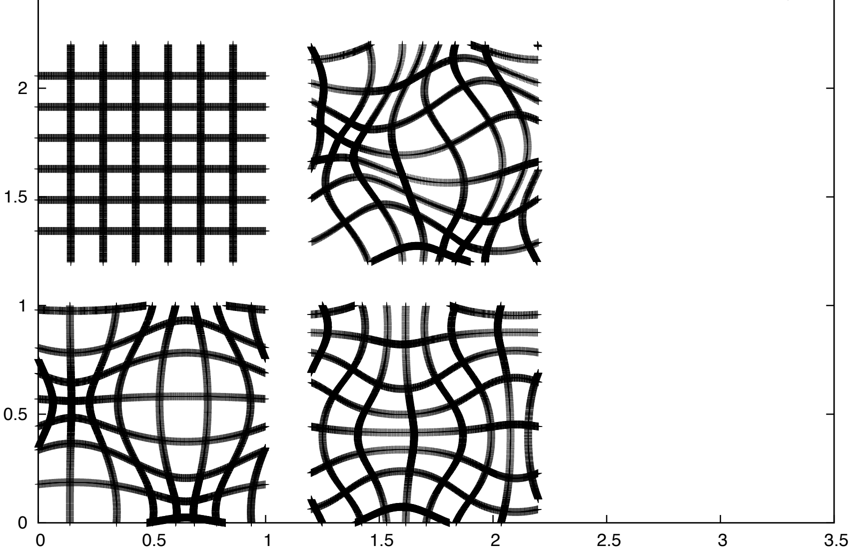
According to V.I. Arnold 1966, an incompressible fluid, confined in a domain D and moving according to the Euler equations, just follows a (constant speed) geodesic curve along the manifold of all possible incompressible maps of D.



Three maps of the (periodized) square: only one is incompressible.

From a more concrete and computational viewpiont, it is worth considering the discrete version of an incompressible motion inside D:

From a more concrete and computational viewpiont, it is worth considering the discrete version of an incompressible motion inside D: the permutation of N sub-cells of equal volume.

From Combinatorics to Fluids (Thanks to Mirko Rokyta, Charles University)!

Example of a discrete incompressible motion with 7 time steps and 12 sub-cells (in line)

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7 time steps have been performed.

Time is on vertical axis and space on horizontal axis.

The trajectories of 2 selected sub-cells (4 and 5) are drawn in red.

"transportation cost" to reach the final permutation

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The "cost" is obtained by adding up the squares of all displacements at all steps. Here: 12+10+12+42+10+12+10=108.

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The "cost" is obtained by adding up the squares of all displacements at all steps. Here: 12+10+12+42+10+12+10=108. This is the "cost" to reach the final permutation in 7 steps. Notice that step 4 costs a lot!

Obviously, there is at least a solution leading to the final permutation at the lowest possible cost, among the... $(12!)^6 \sim 10^{52}$ possible candidates!

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Presumably, passing to the limit (in the number of cubes and steps), we should recover the motion of an incompressible fluid obeying the Euler equations.

Exercise: let us try to find a discrete geodesic leading to permutation 12-11-10-9-8-7-6-5-4-3-2-1 using twelve steps

1	2	3	4	5	6	7	8	9	10	11	12
12	11	10	9	8	7	6	5	4	3	2	1

LET US TRY TO MOVE BY EXCHANGING NEIGHBORS...

1	2	3	4	5	6	7	8	9	10	11	12
2	1	4	3	6	5	8	7	10	9	12	11
2	4	1	6	3	8	5	10	7	12	9	11
12	11	10	9	8	7	6	5	4	3	2	1

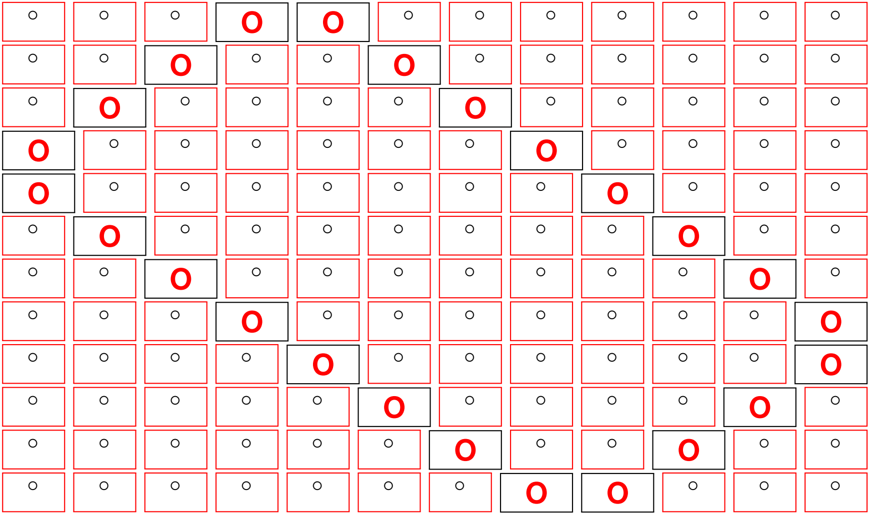
FINALLY ARRIVED...AFTER 12 STEPS.

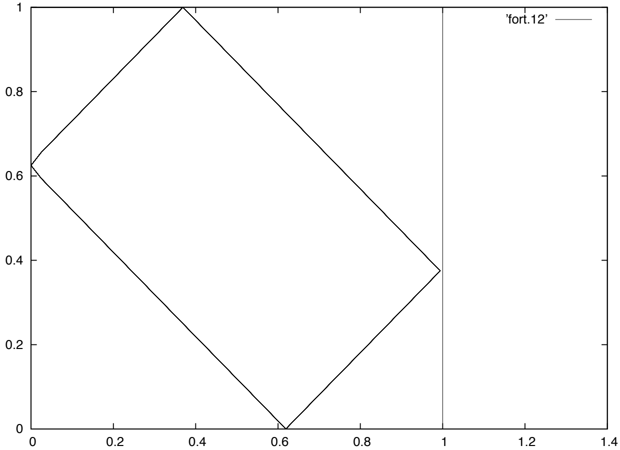
1	2	3	4	5	6	7	8	9	10	11	12
2	1	4	3	6	5	8	7	10	9	12	11
2	4	1	6	3	8	5	10	7	12	9	11
4	2	6	1	8	3	10	5	12	7	11	9
4	6	2	8	1	10	3	12	5	11	7	9
6	4	8	2	10	1	12	3	11	5	9	7
6	8	4	10	2	12	1	11	3	9	5	7
8	6	10	4	12	2	11	1	9	3	7	5
8	10	6	12	4	11	2	9	1	7	3	5
10	8	12	6	11	4	9	2	7	1	5	3
10	12	8	11	6	9	4	7	2	5	1	3
12	10	11	8	9	6	7	4	5	2	3	1
12	11	10	9	8	7	6	5	4	3	2	1

LET US FOLLOW THE TRAJECTORIES OF TWO NEIGHBOURS: 4 AND 5

1	2	3	4	5	6	7	8	9	10	11	12
2	1	4	3	6	5	8	7	10	9	12	11
2	4	1	6	3	8	5	10	7	12	9	11
4	2	6	1	8	3	10	5	12	7	11	9
4	6	2	8	1	10	3	12	5	11	7	9
6	4	8	2	10	1	12	3	11	5	9	7
6	8	4	10	2	12	1	11	3	9	5	7
8	6	10	4	12	2	11	1	9	3	7	5
8	10	6	12	4	11	2	9	1	7	3	5
10	8	12	6	11	4	9	2	7	1	5	3
10	12	8	11	6	9	4	7	2	5	1	3
12	10	11	8	9	6	7	4	5	2	3	1

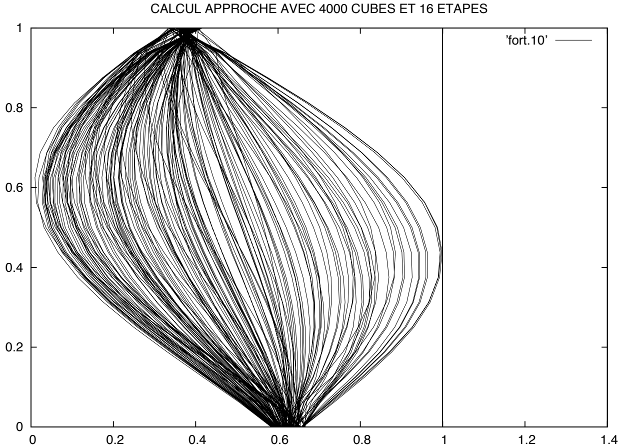
Is it really the lowest possible cost?



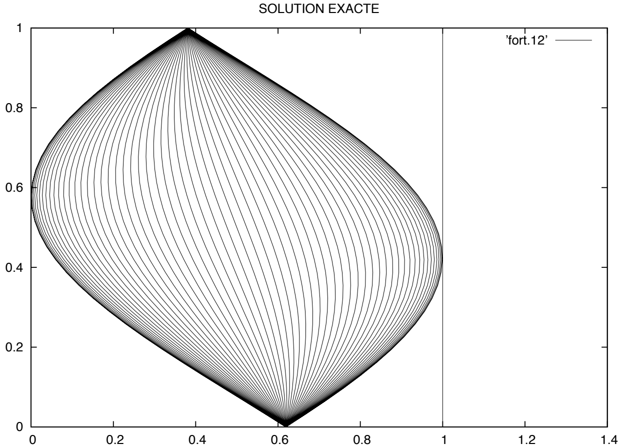


ANYWAY, IT IS EASY TO "PASS TO THE LIMIT"

AS A MATTER OF FACT, THIS IS NOT THE BEST SOLUTION. THE COST CAN BE REDUCED BY FACTOR $\pi^2/12 \sim 0.8225$



NUMERICS WITH 4000 CUBES AND 16 STEPS



EXACT SOLUTION (30 AC)

Incompressible fluids: a probabilist description

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A generalized incompressible flow on a compact domain D is defined as a probability measure μ on paths $t \in [0, T] \to \xi_t \in D$, such that:

i) μ has finite energy (*): $\mathbb{E}_\mu \int_0^T \frac{1}{2}|\frac{d\xi_t}{dt}|^2 dt \ < +\infty$,

Incompressible fluids: a probabilist description

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i) μ has finite energy (*): $\mathbb{E}_\mu \int_0^T \frac{1}{2}|\frac{d\xi_t}{dt}|^2 dt \; < +\infty$,
ii) (incompressibility) for each $t \in [0, T]$, the projection μ_t is the (normalized) Lebesgue measure $\mathcal{L}_D$ on D.

(*) Of course, such measures are very different from Wiener measures.

Generalized solutions to the Euler model

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We say that a generalized incompressible flow (GIF) μ solves the Euler model if there is a scalar field p defined on $]0, T[\times D$, sufficiently smooth, such that, $\mu-$a.s., every path ξ satisfies

$$\frac{d^2\xi_t}{dt^2} = -(\nabla p)(t, \xi_t), \quad \forall t \in]0, T[.$$

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Of course, each classical solution $V(t,x)$ generates a generalized solution μ by

$$\forall \Phi, \quad \mathbb{E}_\mu \Phi[t \to \xi_t] = \int_D \Phi[t \to X_t(a)]da; \quad \partial_t X_t(a) = V(t, X_t(a)), \quad X_0(a) = a, \quad \forall a \in D.$$

The generalized least action principle (31 AC)

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Let D be a convex body and (μ, p) be a generalized solution to the Euler model.
Assume $D_x^2 p(t, x) \leq r\mathbb{I}_d$ uniformly for some finite r.

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Assume $D_x^2 p(t,x) \leq r\mathbb{I}_d$ uniformly for some finite r.

Then, for all $0 \leq t_0 < t_1 \leq T$, μ uniquely minimizes

$$\mathbb{E}_\mu \int_{t_0}^{t_1} \frac{1}{2} \left|\frac{d\xi_t}{dt}\right|^2 dt,$$

among all generalized incompressible flows (GIF) $\tilde{\mu}$ with same end-points: $\tilde{\mu}_{t_0,t_1} = \mu_{t_0,t_1}$, as $(t_1 - t_0)^2 r < \pi^2$.

The converse part of the action principle (35 AC)

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Let $\mu_{0,T}$ be a probability measure on $D \times D$ such that $\mu_0 = \mu_T = \mathcal{L}_D$.

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Let $\mu_{0,T}$ be a probability measure on $D \times D$ such that $\mu_0 = \mu_T = \mathcal{L}_D$. Then, $\mu_{0,T}$ is always achieved by a generalized incompressible flow μ of minimal energy

The converse part of the action principle (35 AC)

Let $\mu_{0,T}$ be a probability measure on $D \times D$ such that $\mu_0 = \mu_T = \mathcal{L}_D$. Then, $\mu_{0,T}$ is always achieved by a generalized incompressible flow μ of minimal energy and there is a *unique* distribution $\nabla p(t,x)$ such that

$$\mathbb{E}_\mu \int_0^T \left(\frac{d\xi_t}{dt} \partial_t A(t,\xi_t) + (\frac{d\xi_t}{dt} \otimes \frac{d\xi_t}{dt}) \cdot \nabla A(t,\xi_t) \right) dt = < \nabla p, A >, \ \forall A \in \mathcal{D}(\mathrm{int}([0,T] \times D)).$$

NB. There is no such result in the classical framework (Shnirelman1985).

Proof: treat the problem as a continuous multi-marginal Monge-Kantorovich problem.

Euler, Monge, Kantorovich

Regularity of the pressure gradient (41-54 AC)

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In the proof, the existence of a unique ∇p requires special efforts (with respect to more standard optimal transport or MFG problems).

Main open question: is p semi-concave? So far, we only know (*) $\nabla p \in L^2_{loc}(\mathcal{M}_{loc})$ and we have example where p is not better than locally semi-concave.

(*) after Y.B. CPAM 1999, square integrability in time due to Ambrosio-Figalli 2008.

SOME REFERENCES...

1) Euler's equations of incompressible fluids

L. Euler, opera omnia, seria secunda 12, p. 274,
Arnold, Ann. Fourier 1966, Ebin-Marsden, Ann. Maths 1970,
Shnirelman Math USSR Sb. 1987. Books:
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3) Entropic (Schrödinger) regularization

Arnaudon, Cruzeiro, Fang, Léonard, Zambrini...—>Ask Christian!

SOME REFERENCES...

1) Euler's equations of incompressible fluids

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2) Generalized incompressible flows

Y.B.: JAMS 1990, ARMA 1993, CPAM 1999, CVPDE 2012,
Shnirelman: GAFA 1994, Ambrosio-Figalli: Arma 2008, CVPDE 2008, Bernot-Figalli-Santambrogio, IHP NL 2009.

3) Entropic (Schrödinger) regularization

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4) Numerics

Y. B.: Comp. Mech. 1989, Physica D 2008 (combinatorics),
Mérigot-Mirebeau arXiv-2015, Gallouët-Mérigot arXiv-2016 (Monge-Ampère solver),
Nenna's PhD 2016 (entropic regularization).

What about the initial value problem?

A priori, convex minimization techniques are hopeless for the initial value problem (IVP).

For a generalized incompressible flow (GIF) μ with finite energy $\mathbb{E}_\mu \int_0^T \frac{1}{2}|\frac{d\xi_t}{dt}|^2 dt < +\infty$,

it does not make sense to prescribe the initial velocity $\frac{d\xi_t}{dt}|_{t=0}$ for $\mu-$a.e. paths ξ.

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However, we are going to see that the initial *bulk* velocity makes sense for a GIF of *minimal* energy.

Initial bulk velocity of a GIF of minimal energy

For a GIF μ of minimal energy, we already know

$$\mathbb{E}_{\mu} \int_0^T \left(\frac{d\xi_t}{dt} \partial_t A(t, \xi_t) + (\frac{d\xi_t}{dt} \otimes \frac{d\xi_t}{dt}) \cdot \nabla A(t, \xi_t) \right) dt =< \nabla p, A > \quad \forall A \in \mathcal{D}(\mathrm{int}([0, T] \times D)).$$

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This suggests to define the initial *bulk* velocity V_0 by

$$< V_0, A(0, \cdot) >= \mathbb{E}_\mu \int_0^T \left(\frac{d\xi_t}{dt} \partial_t A(t, \xi_t) + (\frac{d\xi_t}{dt} \otimes \frac{d\xi_t}{dt}) \cdot \nabla A(t, \xi_t) \right) dt,$$

where A are test fields chosen to be divergence-free (in order to kill the pressure term) without vanishing at $t = 0$ (to be able to "feel" V_0).

A minimization problem for the IVP (59 AC)

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Given a divergence-free velocity field V_0, we look for a GIF μ of minimal energy with initial *bulk* velocity V_0. This reads as the saddle-point problem:

$$\inf_{\mu\geq 0} \sup_{A,\varphi} \quad < V_0, A(0,\cdot) > +$$

$$\mathbb{E}_\mu \int_0^T \left(\frac{1}{2}\left|\frac{d\xi_t}{dt}\right|^2 - \left(\frac{d\xi_t}{dt} \otimes \frac{d\xi_t}{dt}\right) \cdot B(t,\xi_t) - \frac{d\xi_t}{dt} \cdot E(t,\xi_t)\right)dt$$

where $E = \partial_t A + \nabla\varphi$, $B = \frac{1}{2}(\nabla A + \nabla A^T)$, subject to $\nabla \cdot A = 0$, $A(T,\cdot) = 0$ (to take care of V_0), while φ enforces the incompressibility condition.

The dual (convex) minimization problem

$$\inf_{(E,B)} \int_{[0,T]\times D} E \cdot (2\mathbb{I}_d + 4B)^{-1} \cdot E + V_0 \cdot E$$

subject to $B(t=T,\cdot)=0, \quad \partial_t B_{ij} = \frac{1}{2}(\partial_j E_i + \partial_i E_j) + \partial_i \partial_j (-\triangle)^{-1} \partial^k E_k.$

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