Stochastic calculus lecture: discrete martingales

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Chapter 1

Reminder for martingales indexed by $\mathbb{N}$

To integrate processes $H_s(\omega)$ against BM "$dB_s(\omega)$" we will assume that $H : [0, +\infty) \mapsto \mathbb{R}$, depends is "previsible". Roughly it means that $H(t, .)$ is measurable with respect to the sigma-field $\sigma(X_s,\ s < t)$ of the past. Then the time dependence of $t \mapsto \int_0^t H_s dB_s$, will be achieved so that $\int_0^t H_s dB_s$ is a martingale. First we recall results for martingales indexed by $\mathbb{N}$ especially convergence results. Then we will extend these results to martingales indexed by $[0, +\infty)$. The main issue in this case is that $[0, +\infty)$ is not denumerable.

1 Definitions and first examples

Definition 1.1. *A filtration on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a non-decreasing sequence of sub-*σ*-fields of* $\mathcal{F}$*:*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}.$$

One says that $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ *is a filtered probability space.*

Example 1.1. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1[, \mathcal{B}([0, 1[), \lambda)$, where λ is Lebesgue measure. The filtration $(\mathcal{F}_n)_{n\geq 0}$ defined by

$$\mathcal{F}_n = \sigma\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right[, i = 0, \dots, 2^n - 1\right), \quad n \geq 0$$

is called the dyadic filtration.

If the parameter n denotes time, then $\mathcal{F}_n$ is interpreted as available information up to time n.

Example 1.2. For a stochastic process $(X_n)_{n\geq 0}$, we define its natural filtration $\mathcal{F}^X = (\mathcal{F}_n^X)_{n\geq 0}$ by: for all $n \geq 0$,

$$\mathcal{F}_n^X = \sigma(X_0, X_1, \cdots, X_n),$$

which is the smallest σ-field such that $X_0, \ldots, X_n$ are measurable.

Definition 1.2. *We say that a stochastic process $X = (X_n)_{n\geq 0}$ is adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$, if for all $n \geq 0$, X_n is $\mathcal{F}_n$-measurable. We say that a stochastic process $(X_n)_{n\geq 0}$ is adapted if it is adapted to some filtration.*

A stochastic process is obviously adapted to its natural filtration.

Remark 1.1. If $(\mathcal{F}_n)_{n\geq 0}$ and $(\mathcal{G}_n)_{n\geq 0}$ are two filtrations such that $\mathcal{G}_n \subset \mathcal{F}_n$ for all $n \geq 0$, and if $(X_n)_{n\geq 0}$ is adapted to $(\mathcal{G}_n)_{n\geq 0}$, then $(X_n)_{n\geq 0}$ is adapted to $(\mathcal{F}_n)_{n\geq 0}$.

Definition 1.3. *A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a non-decreasing sequence of sub-σ-fields of $\mathcal{F}$:*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}.$$

One says that $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ is a filtered probability space.

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Definition 1.5. *Let $X = (X_n)_n$ be an adapted process on filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n, n \in \mathbb{N}), \mathbb{P})$ such that for all n, X_n is integrable.*

The process X is a **martingale** *if for all n,*

$$\mathbb{E}[X_{n+1}/\mathcal{F}_n] = X_n, \quad \textit{almost surely.}$$

The process $X$is a **sub-martingale** *if for all integer n n,*

$$\mathbb{E}[X_{n+1}/\mathcal{F}_n] \geq X_n, \quad \textit{almost surely.}$$

The process X is a **upper martingale** *if for all integer n,*

$$\mathbb{E}[X_{n+1}/\mathcal{F}_n] \leq X_n, \quad \textit{almost surely.}$$

Examples

(See exercises at the end of the chapter for some proofs of the following properties are left to the reader.)

1. If $X \in L^1(\Omega, \mathcal{A})$, $X_n = \mathbb{E}[X/\mathcal{F}_n]$ is a martingale.This process is also uniformly integrable.

2. (**Fundamental example.**) Let $(Z_n, n \in \mathbb{N}^*)$ be a sequence of independent and integrable random variables and X_0 be an integrable random variable independent of the sequence (Z_n). (Most of the time, X_0 is constant.) Let $X_n := X_0 + \sum_{i=1}^n Z_i$. Then the filtrations $\mathcal{F}_n^X$ and $\mathcal{F}_n = \sigma(X_0, Z_1, \ldots, Z_n)$ are equal and for this filtration :

   (a) if for all integer n, $\mathbb{E}(Z_n) = 0$, X is a martingale;

   (b) if for all integer n, $\mathbb{E}(Z_n) \geq 0$, X is a sub martingale;

   (c) if for all integer n $\mathbb{E}(Z_n) \leq 0$, X is an upper martingale;

   (d) if all r.v. Z_i have same expectation m, $X_n - nm$ is a martingale.

3. A special case of the example 2 comes from the game theory. In this case the distribution of the r.v. Z_n is the Bernoulli distribution with parameter p : $\mathbb{P}(Z_i = 1) = p$, $\mathbb{P}(Z_i = -1) = 1 - p$. with values $+1$ et -1. In this case X_n is the fortune of the player after n bets, when its initial fortune is X_0. The process $(M_n)_n$ where $M_n = X_n - n(2p - 1)$ is a martingale for its natural filtration $\mathcal{F}^X$.

4. In the example 2, if we assume that $\mathbb{E}[\exp(aZ_n)] := \exp(r_n)$ exists and is finite, let $R_n = r_1 + \cdots + r_n$. (Here $R_0 = 0$.)

   Then $M_n = \exp(aX_n - R_n)$ is a martingale for the natural filtration $\mathcal{F}^X$.

A process X can be a martingale (resp. upper, resp sub) with respect to several filtrations.

Proposition 1.1. *If X is a martingale (resp. a upper-martingale, a sub-martingale) with respect to a filtration $(\mathcal{F}_n)$ and the process X is adapted to an other filtration $(\mathcal{G}_n)$ smaller than$(\mathcal{F}_n)$ (that means for all n, $\mathcal{G}_n \subset \mathcal{F}_n$), Then X is a martingale (resp. a upper-martingale, a sub-martingale)with respect to the filtration $\mathcal{G}_n$. A martingale (resp. a upper-martingale, a sub–martingale) is a martingale (resp. a upper-martingale, a sub–martingale)with respect to its natural filtration .*

Proof. Use successive conditioning. . □

We can also increase filtrations by adding to each σ fields $\mathcal{F}_n$ an independent σ field:

Proposition 1.2. *Let (X_n) be a martingale (resp. a sub-martingale, an upper-martingale), with respect to a filtration $\mathcal{F}_n$. Let $\mathcal{B}$ be a σ field independent of $\mathcal{F}_\infty$, and let $\mathcal{G}_n = \mathcal{F}_n \vee \mathcal{B}$. Then (X_n) is a martingale (resp. a sub-martingale, an upper-martingale) with respect to the filtration $\mathcal{G}_n$.*

Proof. Left to the reader. □

Notation 1.1. *In the sequel*

$$(\Delta X)_n := X_n - X_{n-1} \tag{1.1}$$

is the increments process of (X_n).

Proposition 1.3. *Let X be a $\mathcal{F}$-martingale. Then*

1. $\forall n \geq 0$, $\forall k \geq 0$, $\mathbb{E}[X_{n+k}/\mathcal{F}_n] = X_n$; $\mathbb{E}[X_n] = \mathbb{E}[X_0]$.

2. *If the martingale is square integrable the increments $(\Delta X)_n$ of X are orthogonal :*
$$n \neq m \implies \mathbb{E}[(\Delta X)_n(\Delta X)_m] = 0.$$

3. *If X is a upper-martingale, $-X$ is a sub-martingale.*

4. *The set of martingales with respect to a given filtration is a linear space.*

5. *If X is a martingale and ϕ is a convex application such that $Y_n = \phi(X_n)$ is integrable then , Y_n is a sub-martingale.*

6. *If X is a sub-martingale, and if ϕ is increasing and convex, $\phi(X)$ is a sub-martingale if $\phi(X_n)$ is integrable.*

Proof. The proof is left to the reader.

The point 1 relies on successive conditioning and induction.

The point 2 is obtained by conditioning by $\mathcal{F}_{m-1}$ for $n < m$.

The points 3 et 4 are immediate.

The points 5 et 6 rely on Jensen conditionnal inequality.

□

For square integrable martingale, we have

Proposition 1.4. *If M_n is a square integrable martingale*

$$\forall n \leq p,\ \mathbb{E}[(M_p - M_n)^2] = \sum_{k=n+1}^{p} \mathbb{E}[(\Delta M)_k^2].$$

Proof. Apply the property of orthogonal increments

2 of Proposition 1.3. □

Corollary 1.1. *A martingale bounded in L^2 converges in L^2.*

Proof. By definition, since the martingale is bounded in L^2 there exists a constant C such that for all n,

$$\mathbb{E}(X_n^2) \leq C^2.$$

Then,

$$\mathbb{E}(X_n - X_0)^2 \leq 4C^2,$$

and Proposition 1.4 allows to prove that the series

$$\sum_k \mathbb{E}[(\Delta M)_k^2]$$

converges. As a consequence,

$$\lim_{n\to\infty} \sup_{p\geq q\geq n} \sum_q^p \mathbb{E}[(\Delta M)_k^2] = 0.$$

Using the previous proposition again

$$\lim_n \sup_{p,q\geq n} \mathbb{E}(M_p - M_q)^2 = 0,$$

and the sequence is Cauchy in L^2 and converges.

□

2 Doob's decomposition

Definition 1.6. *Let $(A_n)_{n\geq 0}$ be a process indexed by $\mathbb{N}$, A is predictable with respect to the sigma field $\mathcal{F}_n$ if $\forall n$ A_n is $\mathcal{F}_{n-1}$measurable.*

Theorem 1.1. D Doob 's decomposition *: Let X be a sub-martingale ; there exists a martingale M and a predictable increasing process A, null at 0, unique, such that for all integer n, $X_n = M_n + A_n$.*

The process A is called "compensator" of X.

Proof. Let $A_0 = 0$ and $M_0 = X_0$. For $n \geq 1$, define A_n in the following way : let $\Delta_n = \mathbb{E}(X_n/\mathcal{F}_{n-1}) - X_{n-1}$, and

$$A_n = \Delta_1 + \cdots + \Delta_n.$$

Moreover$M_n = X_n - A_n$. By construction A_n is predictable, and since X_n is a sub-martingale, $\Delta_n \geq 0$, and A_n is increasing. Moreover,

$$\mathbb{E}(M_{n+1}/\mathcal{F}_n) = \mathbb{E}(X_{n+1}/\mathcal{F}_n) - A_{n+1} = X_n + \Delta_n - A_{n+1} = M_n.$$

and M_n is a martingale.

Uniqueness comes from the fact that if such a decomposition exists then

$$\mathbb{E}(X_{n+1} - X_n/\mathcal{F}_n) = A_{n+1} - A_n,$$

This characterize A_n if $A_0 = 0$. □

In the particular case of square integrable martingale we obtain the following.

Proposition 1.5. *Let M_n be a square integrable martingale. Recall (notation 1.1) and $(\Delta M)_n = M_n - M_{n-1}$ and let*

$$U_n = \mathbb{E}[(\Delta M)_n^2/\mathcal{F}_{n-1}).$$

Then $M_n^2 - \sum_{k=1}^n U_k$ is a martingale.

Proof. It is the Doob's decomposition applying to the sub-martingale M_n^2, since

$$\mathbb{E}[(\Delta M)_n^2/\mathcal{F}_{n-1}) = \mathbb{E}(M_n^2/\mathcal{F}_{n-1}] - M_{n-1}^2.$$

□

3 Stopping times

3.1 Definition

Definition 1.7. *A random variable $T\colon \Omega \to \mathbb{N} \cup \{+\infty\}$ is called a stopping time (with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$) if for all $n \geq 0$,*

$$\{T \leq n\} \in \mathcal{F}_n.$$

Remark 1.3. Since $\{T = n\} = \{T \leq n\} \setminus \{T \leq n-1\}$, T is a stopping time if and only if for all $n \geq 0$,

$$\{T = n\} \in \mathcal{F}_n.$$

Remark 1.4. A stopping time is thus a random time, which can be interpreted as a stopping rule for deciding whether to continue or stop a process on the basis of the present information and past events, for instance playing until you go broke or you break the bank, etc...

Example 1.4. 1. If $T = n$ a.s., then clearly T is a stopping time.

2. Let $(X_n)_{n\geq 0}$ be an adapted stochastic process, and consider the first time X_n reaches the borel set A:

$$T_A = \inf\{n \geq 0 \,|\, X_n \in A\},$$

with the convention that $\inf \varnothing = +\infty$. It is called the hitting time of A. Then T_A is a stopping time. Indeed,

$$\begin{aligned}\{T_A = n\} &= \{X_0 \notin A, X_1 \notin A, \dots, X_{n-1} \notin A, X_n \in A\} \\ &= \bigcap_{k=0}^{n-1} \{X_k \notin A\} \cap \{X_n \in A\} \in \mathcal{F}_n.\end{aligned}$$

3. Show that $\tau_A = \sup\{n \geq 1 \,|\, X_n \in A\}$ the last passage time in A is not a stopping time in general.

Recall the notations: $x \wedge y = \inf(x, y)$ and $x \vee y = \max(x, y)$.

Proposition 1.6. *If S and T are two stopping times, then $S \wedge T$, $S \vee T$ and $S + T$ are also stopping times.*

Proof. Writing

$$\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\}$$

and

$$\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\}$$

gives the result for $S \wedge T$ and $S \vee T$. For $S + T$, we write:

$$\{S + T \leq n\} = \bigcup_{k\leq n} \{S = k\} \cap \{T \leq n - k\} \in \mathcal{F}_n,$$

since $\mathcal{F}_k \subset \mathcal{F}_n$ for all $k \leq n$. □

Remark 1.5. In particular, if T is a stopping time, then for all $n \geq 0$, $T \wedge n$ is a bounded stopping time.

Proposition 1.7. *If $(T_k)_k$ is a sequence of stopping times, then $\inf_k T_k$, $\sup_k T_k$, $\liminf_k T_k$ and $\limsup_k T_k$ are also stopping times.*

Proof. Exercise. □

Proposition 1.8. *Let T be a stopping time. Then,*

$$\mathcal{F}_T = \{A \in \mathcal{F} \,|\, \forall n \geq 0, A \cap \{T = n\} \in \mathcal{F}_n\}$$

is a σ-field, called the σ-field of T-past.

Remark 1.6. Obviously, T is $\mathcal{F}_T$-measurable.

Proof. It is obvious that $\Omega \in \mathcal{F}_T$. If $A \in \mathcal{F}_T$, then for all n,

$$A^c \cap \{T = n\} = \{T = n\} \setminus A = \{T = n\} \setminus (A \cap \{T = n\}) \in \mathcal{F}_n,$$

hence $A^c \in \mathcal{F}_T$. If $(A_k)_k$ is countable collection of $\mathcal{F}_T$-mesurable set, then

$$\left(\bigcup_k A_k\right) \cap \{T = n\} = \bigcup_k (A_k \cap \{T = n\}) \in \mathcal{F}_n,$$

hence $\bigcup_k A_k \in \mathcal{F}_T$. □

Proposition 1.9. *Let S and T be two stopping times such that $S \leq T$. Then, $\mathcal{F}_S \subset \mathcal{F}_T$.*

Proof. Let $A \in \mathcal{F}_S$. Then, for all $n \geq 0$,

$$A\cap\{T = n\} = A\cap\{S \leq n\}\cap\{T = n\} = \bigcup_{k=0}^{n} A\cap\{S = k\}\cap\{T = n\} \in \mathcal{F}_n.$$ □

Definition 1.8. *Let $(X_n)_{n\geq 0}$ be an adapted stochastic process and T a stopping time. If $T < \infty$ a.s., we define the random variable X_T by*

$$X_T(\omega) = X_{T(\omega)}(\omega) = X_n(\omega) \quad \textit{if } T(\omega) = n.$$

Note that X_T is $\mathcal{F}_T$-measurable, since

$$\{X_T \in B\} \cap \{T = n\} = \{X_n \in B\} \cap \{T = n\} \in \mathcal{F}_n,$$

for any Borel set B.

4 Martingales transformations

Proposition 1.10. *Let (X_n) be an adapted process and (H_n) be a predictable process such that for all n, the r.v. $H_n(X_n - X_{n-1})$ is integrable. Let $(H.X)$ be the process defined by*

$$(H.X)_n = H_0X_0 + \sum_{k=1}^{n} H_k(X_k - X_{k-1}).$$

Then, if X is a martingale, $(H.X)$ is a martingale. If X is a upper- (resp. sub-) martingale, and if H is positive, then $(H.X)$ is a(n) upper- (resp. sub-) martingale.

Proof. Using the notation 1.1 , the process $(H.X)$ satisfies

$$(\Delta(H.X))_n = H_n(\Delta X)_n.$$

The proof is then left to the reader. □

In a casino for example, the process H corresponds to a player's strategy : according to all observations he has at time n, he bets at time $n+1$ an H_{n+1}, to earn a gain $H_{n+1}(X_{n+1} - X_n)$.

An important particular case of Proposition 1.10 is the following

Corollary 1.2. *Let (X_n) be a martingale (resp. a sub-, an upper-martingale), and let be a T stopping time. Then the process X^T defined by $X_n^T = X_{T\wedge n}$ is a martingale (resp. a sub-, an upper-martingale).*

Proof. It is enough to consider the predictable (right ?) process $H = \mathbf{1}_{[0,T]}$. In this case the process $(H.X)$ is nothing but X^T :

$$(H.X)_n = X_0 + \sum_{k=1}^{T\wedge n}(X_k - X_{k-1}) = X_0 + \sum_{k=1}^{n}(X_k - X_{k-1})\mathbf{1}_{k\leq T}.$$

□

Note that the process $T\wedge n$ is adapted to the filtration $\mathcal{G}_n = \mathcal{F}_{T\wedge n}$ smaller than $\mathcal{F}_n$.

Using the predictable process $H = \mathbf{1}_A \mathbf{1}_{[T,\infty[}$, for $A \in \mathcal{F}_T$, we obtain

Corollary 1.3. *If T is a stopping time, then $\mathbf{1}_A(X_{T\vee n} - X_{T)})$ is a martingale (resp. a sub-, an upper-martingale).*

5 Stopping theorem :bounded stopping time's case.

Theorem 1.2. *(**Stopping theorem**.)*

Let $(X_n, n \in \mathbb{N})$ be a martingale and S and T be two bounded stopping times (that means there exists an integer n such that $S \vee T \leq n$, almost surely). Then,

$$\mathbb{E}(X_T/\mathcal{F}_S) = X_{S \wedge T}. \tag{1.2}$$

If X is a sub (resp. an upper-)martingale,

$$X_{S \wedge T} \leq \ \textit{(resp} \geq \textit{)}\ \mathbb{E}(X_T/\mathcal{F}_S). \tag{1.3}$$

In particular if $(X_n, n \in \mathbb{N})$ is a sub-martingale and S and T are two bounded stopping time, then

$$\mathbb{E}(X_S \mathbf{1}_{S \leq T}) \leq \mathbb{E}(X_T \mathbf{1}_{S \leq T}). \tag{1.4}$$

We have the inverse inequality for an upper-martingale.

Proof. We give only the proof for the martingale case.

First, we study the case where

$T = n$ and $S \leq n$. The equality (1.2) to obtain can be written as

$$X_S = \mathbb{E}(X_n/\mathcal{F}_S).$$

By definition,

$$X_S = \sum_{k=0}^{n} X_k \mathbf{1}_{S=k}.$$

We know that X_S is $\mathcal{F}_S$ measurable; and also integrable as finite linear combination of integrable variables.

It is enough to prove that for all $A \in \mathcal{F}_S$,

$$\mathbb{E}(X_S \mathbf{1}_A) = \mathbb{E}(X_n \mathbf{1}_A).$$

This can be written as

$$\sum_{k=0}^{n} \mathbb{E}(X_k \mathbf{1}_{A \cap \{S=k\}}) = \sum_{k=0}^{n} \mathbb{E}(X_n \mathbf{1}_{A \cap \{S=k\}}).$$

But since $A \in \mathcal{F}_S$, then $A \cap \{S = k\} \in \mathcal{F}_k$, and using the martingale property we obtain, for all $k \leq N$,

$$\mathbb{E}(X_k \mathbf{1}_{A \cap \{S=k\}}) = \mathbb{E}(X_n \mathbf{1}_{A \cap \{S=k\}}).$$

We now study the general case. Let an integer n such that $S \vee T \leq n$.

Using the previous case for the stopped martingale X^T, and the stopping time S. We have $X_n^T = X_T$ since $T \leq n$, $X_S^T = X_{S \wedge T}$. We have

$$\mathbb{E}(X_T / \mathcal{F}_S) = X_{S \wedge T}.$$

Note that the variable $X_{S \wedge T}$ is $\mathcal{F}_{S \wedge T}$ measurable, and as a consequence

$$X_{S \wedge T} = \mathbb{E}(X_T / \mathcal{F}_{S \wedge T}).$$

To obtain inequality (1.4), it is enough to note that inequality (1.3) means that for all $A \in \mathcal{F}_S$

$$\mathbb{E}(X_{S \wedge T} \mathbf{1}_A) \leq \mathbb{E}(X_T \mathbf{1}_A).$$

We apply it to the set $A = \{S \leq T\}$.

□

Corollary 1.4. *Let (T_n) be an increasing sequence of bounded stopping time, and X be a martingale (resp. a sub-martingale, an upper-martingale) ; then $(X_{T_n}, n \in \mathbb{N})$ is a martingale (resp. a sub-martingale, an upper-martingale) for the filtration $(\mathcal{F}_{T_n}, n \in \mathbb{N})$.*

Proof. (on exercise) □

Corollary 1.5. *Let X be an integrable r.v., and let (X_n) be the martingale $\mathbb{E}(X/\mathcal{F}_n)$. If T is a bounded stopping time then*

$$\mathbb{E}(X/\mathcal{F}_T) = X_T.$$

If S et T are two bounded stopping time;

$$\mathbb{E}(X/\mathcal{F}_S/\mathcal{F}_T) = \mathbb{E}(X/\mathcal{F}_T/\mathcal{F}_S) = \mathbb{E}(X/\mathcal{F}_{S \wedge T}) = X_{S \wedge T}.$$

Proof. Let N such that $T \vee S \leq N$. Using successive conditioning for the martingale $X_n = \mathbb{E}(X/\mathcal{F}_n)$ $\mathbb{E}(X/\mathcal{F}_T) = \mathbb{E}(X_N/\mathcal{F}_T)$. The stopping theorem yields $\mathbb{E}(X_N/\mathcal{F}_T) = X_T$.

If S and T are two bounded stopping,

$$\mathbb{E}(X_T/\mathcal{F}_S) = X_{S\wedge T} = \mathbb{E}(X/\mathcal{F}_{S\wedge T}).$$

□

6 Finite stopping times

In this section we extend the stopping theorem to the case of finite stopping times. Its requires some additional integrability conditions on martingales (resp. sub-martingales, upper-martingales).

Proposition 1.11. *Let $(X_n, n \in \mathbb{N})$ be a martingale (resp. a sub-martingale) and T be S two almost surely finite stopping times.*

If the sequences $(X_{T\wedge n})$ are $(X_{S\wedge n})$ uniformly integrable, then

$$X_{S\wedge T} = E[X_T/\mathcal{F}_S]. \quad (\text{resp. } X_{S\wedge T} \leq E[X_T/\mathcal{F}_S]).$$

This is the case when there exists a r.v. $Y \in L^1$ such that for all n, $|X_{T\wedge n}| \leq Y$),or when (X_n) is uniformly integrable

In particular for the martingale, we have $\mathbb{E}(X_T) = \mathbb{E}(X_0)$ for all finite stopping time which satisfies this assumption.

Proof. We only study the martingale case.

First note that the r.v. X_T is integrable, as almost sure limit of uniformly integrable sequence $(X_{T\wedge n})$. (Note that T is a.s. finite.) It is the same for X_S. We have to prove that for all bounded variable Z $\mathcal{F}_S$-measurable , we have

$$\mathbb{E}(X_S Z) = \mathbb{E}(X_T Z).$$

We can use the monotone class theorem monotones and restrict ourself to the case where Z is $\mathcal{F}_{S\wedge n}$ measurable using the fact that $\mathcal{F}_S = \vee_n \mathcal{F}_{S\wedge n}$.

Let such an n. Using the Stopping Theorem 1.2 for the stopping time $S \wedge p$ and $T \wedge p$, and $p \geq n$ we obtain

$$\mathbb{E}(Z X_{S\wedge p}) = \mathbb{E}(Z X_{T\wedge p}).$$

Using uniform integrability we can let n going to infinity.

For the last point, if (X_n) is uniformly integrable, it is enough to note that $X_{T\wedge n} = \mathbb{E}(X_n/\mathcal{F}_T)$. The desired conclusion follows from the fact that a family of conditional expectation of uniformly integrable family is uniformly integrable □

7 Inequalities and convergence

7.1 Inequalities

Theorem 1.3. *(*DOOB*'s maximal inequality.) Let $(X_n, n \in \mathbb{N})$ be a positive sub martingale and $\lambda \geq 0$. let $X_n^* = \sup_{k=0}^n X_k$. Then*

$$\forall n \in \mathbb{N},\ \lambda \mathbb{P}\{X_n^* \geq \lambda\} \leq \mathbb{E}[X_n \mathbf{1}_{\{X_n^* \geq \lambda\}}] \leq E[X_n].$$

Proof. Let $T = \inf\{k \in \mathbb{N}, X_k \geq \lambda\}$ a stopping time. Then,

$$\{T \leq n\} = \{X_n^* \geq \lambda\}.$$

Take $S = T \wedge (n+1)$,which is a bounded stopping time . We have

$$A = \{S \leq n\} = \{T \leq n\} \in \mathcal{F}_S.$$

Using the Stopping Theorem for this sub-martingale, between n and $S \wedge n$.

$$\mathbb{E}(X_{S\wedge n} \mathbf{1}_A) \leq \mathbb{E}(X_n \mathbf{1}_A).$$

This can be written

$$\mathbb{E}(X_T \mathbf{1}_{T\leq n}) \leq \mathbb{E}(X_n \mathbf{1}_{T\leq n}). \tag{1.5}$$

On the set $\{T \leq n\}$, $X_T \geq \lambda$, hence $\lambda \mathbb{P}(T \leq n) \leq \mathbb{E}(X_T \mathbf{1}_{T\leq n})$. Then $\lambda \mathbb{P}(T \leq n) \leq\leq \mathbb{E}(X_n \mathbf{1}_{T\leq n})$ is the desired inequality. □

Corollary 1.6. *If $(X_n, n \in \mathbb{N})$ is a martingale, $(|X_n|, n \in \mathbb{N})$ is a positive sub-martingale and*

$$\forall n \in \mathbb{N},\ \lambda \mathbb{P}\{\max_{k\leq n} |X_k| \geq \lambda\} \leq E[|X_n| \mathbf{1}_{\max_{k\leq n} |X_k| \geq \lambda}] \leq E[|X_n|].$$

Theorem 1.4. *Let $(X_n, n \in \mathbb{N})$ be a positive sub-martingale and $p > 1$. Then, if $X_n \in L^p$,*

$$\|X_n^*\|_p \leq \frac{p}{p-1}\|X_n\|_p.$$

Proof. If $X_n \in L^p$ then variables $X_{k\in L^p}$ for $k \leq n$.

Let U be a positive r.v. in L^p,

$$E(U^p) = p\int_0^\infty t^{p-1}\mathbb{P}(U \geq t)dt.$$

Then

$$\begin{aligned} E[(X_n^*)^p] &= p\int_0^\infty t^{p-1}\mathbb{P}(X_n^* \geq t)dt \\ &\leq p\int_0^\infty t^{p-2}E[X_n\mathbf{1}_{\{X_n^*\geq t\}}]dt \\ &= pE[X_n\int_0^\infty t^{p-2}\mathbf{1}_{\{X_n^*\geq t\}}dt] \\ &= \frac{p}{p-1}E[X_n(X_n^*)^{p-1}]. \end{aligned}$$

Using Hölder inequality

$$E[X_n(X_n^*)^{p-1}] \leq \|X_n\|_p\|X_n^*\|_p^{p-1}.$$

Since X_n^* is bounded by $\sum_0^n X_k$, it belongs to L^p. The desired result is obtained by cancellation □

In particular

Corollary 1.7. *Let (X_n) be a positive sub-martingale bounded in L^1. Then the variable $X^* = \sup_n X_n$ is finite almost surely. If (X_n) is bounded in L^p $(p > 1)$, then X^* belongs to L^p. (This last result is false for $p = 1$.) The same conclusions hold for martingales (not necessary positive).*

Proof. The increasing sequence X_n^* converges towards X^*. It is enough to apply Doob's inequality and

$$\lambda\mathbb{P}(X_n^* > \lambda) \leq \sup_n \mathbb{E}(|X_n|) = K < \infty.$$

Letting n going to infinity

$$\lambda\mathbb{P}(X^* > \lambda) \leq K,$$

$\mathbb{P}(X^* > \lambda) \to 0$ $(\lambda \to \infty)$. The r.v. X^* is finite.

For the second part use the Theorem 1.4.

The case of martingales is obtained by applying the previous result to the positive sub-martingale $|X_n|$.

□

7.2 Convergences

Results

The following results are given without any proof.

Proposition 1.12. *Let X_n be a martingale, or a sub-martingale, or an upper-martingale, bounded in L^1. Then X_n converges almost surely towards a variable X_∞.*

Using Fatou's lemma, M_∞ the limit of a bounded in L^1 martingale M_n is integrable. In general $M_n \neq \mathbb{E}(M_\infty/\mathcal{F}_n)$.

It is the case for uniformly integrable martingales .

Proposition 1.13. *let M_n be a bounded martingale in L^1, and let M_∞ the limit of M_n when $n \to \infty$. The following statements are equivalent*

1. *M_n converges in L^1 towards M_∞.*
2. *$M_n$is uniformly integrable.*
3. *$M_n = \mathbb{E}[M_\infty/\mathcal{F}_n]$.*
4. *There exists an integrable r.v. M such that $M_n = \mathbb{E}[M/\mathcal{F}_n]$. Moreover in this case , $M_\infty = \mathbb{E}[M/\mathcal{F}_\infty]$.*

(Here $\mathcal{F}_\infty = \vee_n \mathcal{F}_n$.)

Proof. For a sequence of r.v. which converges almost surely, it is equivalent to converge in L^1 or to be uniformly integrable. For all integrable r.v. M, the set of the r.v. $\mathbb{E}(M/\mathcal{B})$, where $\mathcal{B}$ is running in all sub σ fields of $\mathcal{A}$ is an uniformly integrable family. It is enough to prove the following points:

1. If (M_n) is uniformly integrable, then $M_n = \mathbb{E}(M_\infty/\mathcal{F}_n)$;
2. If M is an integrable r.v., the martingale $M_n = \mathbb{E}(M/\mathcal{F}_n)$ converges towards $\mathbb{E}(M/\mathcal{F}_\infty)$.

For the first point note that for $p \geq n$ $M_n = \mathbb{E}(M_p/\mathcal{F}_n)$,letting p going to infinity using the fact that the expectation is continuous in L^1, and that M_p converges towards M_∞ in L^1 by assumption. We get the desired result.

For the second point, note that M_∞ is $\mathcal{F}_\infty$ measurable by construction. It is enough to show that, for a $A \in \mathcal{F}_\infty$, we have $\mathbb{E}(M_\infty \mathbf{1}_A) = \mathbb{E}(M\mathbf{1}_A)$. This is true when A belongs to sub σ fields of $\mathcal{F}_n$,since

$$\mathbb{E}(M\mathbf{1}_A) = \mathbb{E}(M_n\mathbf{1}_A) = \mathbb{E}(M_\infty\mathbf{1}_A).$$

The desired identity is then, true for all element of $\cup_n \mathcal{F}_n$, and for all element σ -field generated by $\cup_n \mathcal{F}_n$ using a monotone class theorem argument. The desired inequality is true for $\mathcal{F}_\infty$. □

Remarks

1. A similar statement as in Proposition 1.13 is true for sub-and upper-martingales; the proof is left to the reader.
2. A bounded martingale L^p for $p > 1$, is dominated by an L^p variable and converges in L^p.

We now are in position to enunciate the Stopping theorem for general stopping-times.

Theorem 1.5. *(***Stopping Theorem.***) Let M_n be a uniformly integrable martingale and let T be a stopping time (not necessarily finite). Then for $M_T = M_\infty$ on $\{T = \infty\}$,we have*

1. *$M_T = \mathbb{E}(M_\infty/\mathcal{F}_T)$.*
2. *The set (M_T), where T is a stopping time is uniformly integrable.*
3. *If S and T are two stopping time, we have*

$$\mathbb{E}(M_T/\mathcal{F}_S) = M_{S\wedge T}.$$

4. *Let M be a $\mathcal{A}$-measurable integrable r.v. and $M_n = \mathbb{E}(M/\mathcal{F}_n)$, then $M_T = \mathbb{E}(M/\mathcal{F}_T)$.*

Proof. For the first point, it is enough to write the proof of Stopping theorem in this case. If A belongs to $\mathcal{F}_T$, then

$$\begin{aligned}\mathbb{E}(M_T \mathbf{1}_A) &= \sum_{k\in\mathbb{N}\bigcup\infty} \mathbb{E}(M_k \mathbf{1}_{A\bigcap\{T=k\}}) \\ &= \sum_{k\in\mathbb{N}\bigcup\infty} \mathbb{E}(M_\infty \mathbf{1}_{A\bigcap\{T=k\}}) = \mathbb{E}(M_\infty \mathbf{1}_A).\end{aligned}$$

The family M_T is contained in the family $\mathbb{E}(M_\infty/\mathcal{B})$, where $\mathcal{B}$ is running in the subσ fields of $\mathcal{A}$. This last family is uniformly integrable.

The stopping martingale M^T is uniformly integrable. Using the Stopping theorem at time S, we obtain

$$\mathbb{E}(M_T/\mathcal{F}_S) = M_{S\wedge T}.$$

It is enough to write

$$\mathbb{E}(M/\mathcal{F}_T) = \mathbb{E}(\mathbb{E}(M/\mathcal{F}_\infty)/\mathcal{F}_T) = \mathbb{E}(M_\infty/\mathcal{F}_T) = M_T.$$

□

8 Exercises

1. Prove the claim 2 of examples 1 in section 1.
2. Recall a definition of uniform integrability (U.I.) that claims that X_i is U.I. if $\sup_i \mathbb{E}|X_i| < \infty$ and if a property sometimes called equiintegrability (to be recalled) is fulfilled.
3. Prove the claim 1 of examples 1 in section 1.
4. Show that M_n in the claim 3 of examples 1 in section 1 is square integrable. What is the Doob decomposition of M_n ?