Stochastic calculus lecture

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Chapter 1

Construction of Brownian Motion and first properties

1 Introduction

The aim of this lecture is to construct the stochastic integral. The primary motivation is to develop an integral and differential calculus capable of handling computations with random "noises." Historically, the concept of random "noises" originated from experimental sciences. In probability theory, the most classical example of "noise" is Brownian motion. Brownian motion was introduced by Robert Brown in 1828 to study the movement of pollen particles in water. Later, in 1905, Einstein used Brownian motion to model the trajectories of gas molecules. Additionally, Bachelier applied Brownian motion to model stock option prices.

Let us give a first definition.

Definition 1.1. *Brownian motion $(B_t)_{t\geq 0}$ is a $\mathbb{R}^d$ "process" $(d \geq 1)$ (i.e. a family of random variables in short r.v.) such that*

1. *$\forall n \in \mathbb{N}^*$ and $t_0 < t_1 < \ldots < t_n$ the r.v.'s $B_{t_0},\ B_{t_1} - B_{t_0}, \ldots, B_{t_{n+1}} - B_{t_n}$ are independent (BM is a process with independent increments PII in short.)*

2. *If $s < t$, $B_t - B_s$ is a centered Gaussian random vector with covariance matrix $(t-s)Id$.*

$(B_t)_{t\geq 0}$ starts from $0 \in \mathbb{R}^d$ if $B_0 = 0$, $\mathbb{P}a.s.$

Remark 1.1.

- *If* $d = 1$, $Var(B_t - B_s) = t - s$.
- *Random vectors from* $(\Omega, \mathcal{A}, P)$ *to* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, *Independence, Gaussian random vectors are supposed to be known.*
- *"Sample paths" : Heuristically we fix* $\omega \in \Omega$, *and we are interested in* $t \mapsto B_t(\omega)$ *Wiener (1923, 1924), Paley-Zygmund.* P *almost surely the sample paths are contnuous and nowhere differentiable.*
- *If* $B(t) = (B^1(t), \dots, B^d(t))$, *then* $\forall i = 1$ *to* d $B^i(t)$ *are real valued Brownian motions and if* $i \neq j$, B^i *is independent of* B^j.

The aim of stochastic calculus is to give a rigorous meaning to (stochastic) differential equations of the type

$$y_t = \int_0^t f(y_s)\dot{B}_s ds$$

which have many applications. We will show that the Brownian motion is a continuous martingale, and that integrals can be defined in this framework. Another goal is to have a chain rule associated to these integrals. It is called the Itô formula which claims that $\forall f \in C^2(\mathbb{R}^d, \mathbb{R})$,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

Actually one can show if $(X_t)_{t\geq 0}$ is real valued process with independent increments and if the distribution of $X_t - X_s$ does depend only of $t - s$, (stationary increments) with continuous sample paths then $X_t = X_0 + \sigma B_t$. More generally one can study PIIS process with stationary independent increments not with continuous sample paths. Another example of PIIS is the Poisson process.

Références :

- I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Springer Verlag, 1988.
- Damien Lamberton and Bernard Lapeyre. *Introduction to stochastic calculus applied to finance.* Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2008.

- Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

2 Definition and continuity of Brownian motion

2.1 Distribution of a process

Definition 1.2. *Let T be a set and $(E, \mathcal{E})$ a measurable set. A E valued stochastic process indexed by T is a family $(X_t,\ t \in T)$ of r.v.'s X_t : $(\Omega, \mathcal{A}, \mathbb{P}) \mapsto (E, \mathcal{E})$.*

Reminder : If $Y : (\Omega, \mathcal{A}, \mathbb{P}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is real valued random variable, the distribution $\mathbb{P}_Y$ of Y is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. It is the push forward of the probability $\mathbb{P}$ by the measurable function Y. It is defined by $\mathbb{P}_Y(A) = \mathbb{P}(Y \in A)$, $\forall A \in \mathscr{B}(\mathbb{R})$.

For a process a sample path is associated to every $\omega \in \Omega$, $t \mapsto X_t(\omega)$. Hence the distribution of a process is a probability on a set of functions from $T \mapsto E$ which is denoted by E^T, and endowed with the cylindrical sigma field. A Cylinder is indexed by $t_0 < t_1 < \ldots < t_n$, it is a subset of E^T,

$$C_{t_0,\, t_1,\, \ldots,\, t_n} = \{f \in E^T,\ (f(t_0),,\ \ldots,\ f(t_n)) \in A_0 \times \ldots \times A_n,\ \text{with } A_i \in \mathcal{E}\},$$

where $C_{t_0,\, t_1,\, \ldots,\, t_n}$ actually depends also on A_i's. Then the distribution of the process is a probability on E^T endowed with the smallest sigma field that contains all cylinders. This sigma field is called the cylindrical sigma field denoted by $\mathcal{E}^{\otimes T}$.

Definition 1.3. *The distribution of a E-valued process is a probability on E^T endowed with the smallest sigma field $\mathcal{E}^{\otimes T}$ that contains all cylinders. It is uniquely defined by*

$$\mathbb{P}(X(t_0) \in A_0,\ \ldots, X(t_n) \in A_n),$$

$\forall n \in \mathbb{N}^*$ and $t_0 < t_1 < \ldots < t_n$.

Example the Brownian motion that starts from 0 :

If $t_0 < t_1 < \ldots < t_n$, and f is bounded Borel function $(\mathbb{R}^d)^{n+1} \mapsto \mathbb{R}$, let

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(\frac{-\|x\|^2}{2t}\right), \tag{1.1}$$

where $\|.\|$ is the Euclidean norm on $\mathbb{R}^d$. Then $p_{t-s}(x)dx$ is the distribution of $B_t - B_s$, so

$$\mathbb{E}f(B_{t_0}, \ldots, B_{t_n}) = \int f(x_0, \ldots, x_n) d\mathbb{P}_{(B_{t_0}, \ldots, B_{t_n})}(x_0, \ldots, x_n)$$
$$= \int f(x_0, \ldots, x_n) d\mathbb{P}_{(B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}})}(x_0, x_1 - x_0, \ldots, x_n - x_{n-1})$$
$$= \int f(x_0, \ldots, x_n) p_{t_0}(x_0) p_{t_1 - t_0}(x_1 - x_0) \ldots p_{t_n - t_{n-1}}(x_n - x_{n-1}) dx_0 \ldots dx_n.$$

Hence

$$\mathbb{P}(B(t_0) \in A_0, \ \ldots, B(t_n) \in A_n)$$
$$= \int_{A_0 \times \ldots \times A_n} p_{t_0}(x_0) p_{t_1 - t_0}(x_1 - x_0) \ldots p_{t_n - t_{n-1}}(x_n - x_{n-1}) dx_0 \ldots dx_n \tag{1.2}$$

and

$$\mathbb{P}(B(t_0) \in A_0, \ \ldots, B(t_n) \in A_n) = \mathbb{P}(X_0 \in A_0, \ \ldots,, \ X_0 + \ldots + X_n \in A_n),$$

where $X_0 \overset{(d)}{=} \mathcal{N}(0, t_0)$, $X_i \overset{(d)}{=} \mathcal{N}(0, t_i - t_{i-1})$ for all $i = 1$ to n and X_i's are independent.

Remark 1.2. *This remark is also an exercise. If* $A \in \mathscr{B}(\mathbb{R}^d)^{\otimes \mathbb{R}_+}$, *then* $\exists (t_n)_{n \in \mathbb{N}}$ *and* $\exists B \in \mathscr{B}(\mathbb{R}^d)^{\otimes \mathbb{N}}$ *such that*

$$A = \{f \in (\mathbb{R}^d)^{\mathbb{R}_+}, \ (f(t_n))_{n \in \mathbb{N}} \in B\}.$$

Does exist on $(\mathbb{R}^d)^{\mathbb{R}_+}$ a probability such that

$$\mu(C_{t_0, t_1, \ldots, t_n}) = \mathbb{P}(B(t_0) \in A_0, \ \ldots, B(t_n) \in A_n)?$$

The answer"yes" is given by a Kolmogorov theorem. Let us first introduce definitions and a necessary condition.

Definition 1.4. *Let $\mathcal{T}$ be the set of finite increasing sequences $\underline{t} = (t_0, \dots, t_n)$ of numbers, where the length $n+1$ of these sequences ranges over the set of positive integers. Suppose that for each $\underline{t}$ of length $n+1$, we have a probability measure $Q_{\underline{t}}$ on $\left(\mathbb{R}^{n+1}, \mathscr{B}\left(\mathbb{R}^{n+1}\right)\right)$. Then the collection $\{Q_t\}_{t\in\mathcal{T}}$ is called a family of finite-dimensional distributions. This family is said to be consistent provided that the following condition is satisfied: If $\underline{t} = (t_0, t_1, \dots, t_n)$ with $n \geq 1, \underline{t}^i = (t_0, t_1, \dots, t_n)$, where t_i is missing then $\forall i \leq n$*

$$Q_{\underline{t}^i}(A_0 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) = Q_{\underline{t}}(A_0 \times \cdots \times \mathbb{R} \times \cdots \times A_n).$$

If we have a probability measure μ on $\left(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes[0,+\infty)}\right)$, then we can define a family of finite-dimensional distributions by

$$Q_{\underline{t}}(A) = \mu\left[\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \dots, \omega(t_n)) \in A\right], \tag{1.3}$$

where $A \in \mathscr{B}(\mathbb{R}^n)$ and $t = (t_1, \dots, t_n) \in T$. This family is easily seen to be consistent. We are interested in the converse of this fact, because it will enable us to construct a probability measure $\mathbb{P}$ from the finite-dimensional distributions of Brownian motion.

Theorem 1.1 ((Daniell (1918), Kolmogorov (1933))**.** *Let $\left\{Q_{\underline{t}}\right\}$ be a consistent family of finite-dimensional distributions. Then there is a probability measure $\mathbb{P}$ on $\left((\mathbb{R}^d)^{[0,\infty)}, \mathcal{B}(\mathbb{R}^d)^{\otimes[0,+\infty)}\right)$, such that (1.3) holds for every $t \in T$.*

Proof. A proof can be read in p 50 of Karatzas and Shreeve. One can use for instance Carathéodory theorem, that may also be used to construct Lebesgue measure. □

To verify that we can apply the Theorem to the construction of the Brownian motion, it is enough to show consistency in this case. Coming back to (1.2), we are left to check

$$\int_{\mathbb{R}^d} p_{t_i-t_{i-1}}(x_i - x_{i-1}) p_{t_{i+1}-t_i}(x_{i+1} - x_i) dx_i = p_{t_{i+1}-t_{i-1}}(x_{i+1} - x_{i-1}).$$

But it is the same as $p_{t_i-t_{i-1}} * p_{t_{i+1}-t_i} = p_{t_{i+1}-t_{i-1}}$, or $\mathcal{N}(0, t_i - t_{i-1}) + \mathcal{N}(0, t_{i+1} - t_i) = \mathcal{N}(0, t_{i+1} - t_{i-1})$ where independence is assumed on the left hand side.

2.2 Regularity of BM sample paths

In this section we want to convince ourselves that BM sample paths are almost surely continuous. We hope that $\mathcal{C}$ the set of continuous functions is of probability 1 under the distribution of BM.

Remark 1.3. *Unfortunately $\mathcal{C}$ is not measurable in $\mathscr{B}(\mathbb{R}^d)^{\otimes \mathbb{R}_+}$. Prove this fact as an exercise, that uses the previous Remark/exercise.*

To circumvent this problem we will show that there exists a process with almost sure continuous sample paths that has the same distribution as BM.

To clean a bit the situation some definitions are introduced.

Definition 1.5. *1. Processes X and Y have the same finite-dimensional distributions if, for any integer $n \geq 1$, real numbers $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, and $A \in \mathscr{B}\left(\mathbb{R}^{nd}\right)$, we have:*

$$\mathbb{P}\left[(X_{t_1}, \ldots, X_{t_n}) \in A\right] = \mathbb{P}\left[(Y_{t_1}, \ldots, Y_{t_n}) \in A\right].$$

2. Y is a modification of X if, for every $t \geq 0$, we have $\mathbb{P}\left[X_t = Y_t\right] = 1$.

3. X and Y are called indistinguishable if almost all their sample paths agree:

$$\mathbb{P}\left[X_t = Y_t; \forall 0 \leq t < \infty\right] = 1.$$

Exercise 1.1. *If X is a modification of X' then the distribution of X and X' are the same.*

If X and X' are indistinguishable then there are modifications of each other. The converse is false.

Example. *Let $\Omega = [0,1]$, $\mathcal{A} = \mathscr{B}([0,1])$, $\mathbb{P} = dx$ the Lebesgue measure. Let us take $X_t(\omega) = \mathbf{1}(\omega \neq t)$, and $Y_t = 1$. Then*

$$\mathbb{P}(X_t = Y_t) = \mathbb{P}(\omega \neq t) = 1.$$

Hence X is a modification of Y. But $\mathbb{P}(\forall t \in [0,1],\ X_t = Y_t) = 0$. They are not indistinguishable.

Exercise 1.2. *Let Y be a modification of X, and suppose that both processes have a.s. right-continuous sample paths. Then X and Y are indistinguishable.*

The next theorem will show that there exits a modification of the BM with almost sure continuous sample paths.

Theorem 1.2 (Kolmogorov, Čentsov (1956)).

Suppose that a process $\{X_t; 0 \le t \le T\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfies the condition

$$\mathbb{E}\,|X_t - X_s|^\alpha \le C|t-s|^{1+\beta}, \quad 0 \le s, t \le T,$$

for some positive constants α, β, and C. Then there exists a continuous modification $\tilde{X} = \left\{\tilde{X}_t; 0 \le t \le T\right\}$ of X, which is locally Hölder-continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$, i.e.,

$$\mathbb{P}\left[\omega; \sup_{\substack{0<t-s<h(\omega)\\ s,t\in[0,T]}} \frac{\left|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)\right|}{|t-s|^\gamma} \le \delta\right] = 1, \tag{1.4}$$

where $h(\omega)$ is an a.s. positive random variable and $\delta > 0$ is an appropriate constant.

Proof. For simplicity, we take $T = 1$. Much of what follows is a consequence of the Cebyšev inequality. First, for any $\varepsilon > 0$, we have

$$\mathbb{P}\left[|X_t - X_s| \ge \varepsilon\right] \le \frac{\mathbb{E}\,|X_t - X_s|^\alpha}{\varepsilon^\alpha} \le C\varepsilon^{-\alpha}|t-s|^{1+\beta},$$

and so $X_s \to X_t$ in probability as $s \to t$. Second, setting $t = k/2^n, s = (k-1)/2^n$, and $\varepsilon = 2^{-\gamma n}$ (where $0 < \gamma < \beta/\alpha$) in the preceding inequality, we obtain

$$\mathbb{P}\left[\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \ge 2^{-\gamma n}\right] \le C2^{-n(1+\beta-\alpha\gamma)},$$

and consequently,

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\le k\le 2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \ge 2^{-\gamma n}\right]\\
&\quad = \mathbb{P}\left[\bigcup_{k=1}^{2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \ge 2^{-\gamma n}\right]\\
&\quad \le C2^{-n(\beta-\alpha\gamma)}.
\end{aligned}$$

The last expression is the general term of a convergent series; by the Borel-Cantelli lemma, there is a set $\Omega^* \in \mathscr{F}$ with $\mathbb{P}(\Omega^*) = 1$ such that for each $\omega \in \Omega^*$,

$$\max_{1\le k\le 2^n} \left|X_{k/2^2}(\omega) - X_{(k-1)/2^n}(\omega)\right| < 2^{-\gamma n}, \quad \forall n \ge n^*(\omega), \tag{1.5}$$

where $n^*(\omega)$ is a positive, integer-valued random variable. For each integer $n \ge 1$, let us consider the partition $D_n = \{(k/2^n)\,; k = 0,\ 1, \ldots, 2^n\}$ of $[0,1]$, and let $D = \bigcup_{n=1}^{\infty} D_n$ be the set of dyadic rationals in $[0,1]$. We shall fix $\omega \in \Omega^*, n \ge n^*(\omega)$, and show that for every $m > n$, we have

$$|X_t(\omega) - X_s(\omega)| \le 2\sum_{j=n+1}^{m} 2^{-\gamma j}; \quad \forall t, s \in D_m, 0 < t-s < 2^{-n}. \tag{1.6}$$

For $m = n+1$, we can only have $t = (k/2^m)\,, s = ((k-1)/2^m)$, and (1.6) follows from (1.5). Suppose (1.6) is valid for $m = n+1, \ldots, M-1$. Take $s < t, s, t \in D_M$, consider the numbers $t^1 = \max\{u \in D_{M-1}; u \le t\}$ and $s^1 = \min\{u \in D_{M-1}; u \ge s\}$, and notice the relationships $s \le s^1 \le t^1 \le t, s^1 - s \le 2^{-M}, t - t^1 \le 2^{-M}$. From (1.5) we have $|X_{s^1}(\omega) - X_s(\omega)| \le 2^{-\gamma M}, |\ X_t(\omega) - X_{t^1}(\omega)\ | \le 2^{-\gamma M}$, and from (1.6) with $m = M-1$,

$$|X_{t^1}(\omega) - X_{s^1}(\omega)| \le 2\sum_{j=n+1}^{M-1} 2^{-\gamma j}.$$

We obtain (1.6) for $m = M$.

We can show now that $\{X_t(\omega); t \in D\}$ is uniformly continuous in t for every $\omega \in \Omega^*$. For any numbers $s, t \in D$ with $0 < t - s < h(\omega) \triangleq 2^{-n^*(\omega)}$, we select $n \ge n^*(\omega)$ such that $2^{-(n+1)} \le t - s < 2^{-n}$. We have from (1.6)

$$|X_t(\omega) - X_s(\omega)| \le 2\sum_{j=n+1}^{\infty} 2^{-\gamma j} \le \delta|t-s|^{\gamma}, \quad 0 < t-s < h(\omega), \tag{1.7}$$

where $\delta = 2/\left(1 - 2^{-\gamma}\right)$. This proves the desired uniform continuity. We define $\tilde{X}$ as follows. For $\omega \notin \Omega^*$, set $\tilde{X}_t(\omega) = 0, 0 \le t \le 1$. For $\omega \in \Omega^*$ and $t \in D$, set $\tilde{X}_t(\omega) = X_t(\omega)$. For $\omega \in \Omega^*$ and $t \in [0,1] \cap D^c$, choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$; uniform continuity and the Cauchy criterion imply that $\{X_{s_n}(\omega)\}_{n=1}^{\infty}$ has a limit which depends on t but not on the particular sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ chosen to converge to t, and we set

$\widetilde{X}_t(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$. The resulting process $\widetilde{X}$ is thereby continuous; indeed, $\widetilde{X}$ satisfies (1.7), so (1.4) is established.

To see that $\tilde{X}$ is a modification of X, observe that $\tilde{X}_t = X_t$ a.s. for $t \in D$; for $t \in [0,1] \cap D^c$ and $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, we have $X_{s_n} \to X_t$ in probability and $X_{s_n} \to \tilde{X}_t$ a.s., so $\widetilde{X}_t = X_t$ a.s. □