Stochastic calculus lecture

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September 12, 2025

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Chapter 1

Construction of Brownian Motion and first properties

1 Introduction

The aim of this lecture is to construct the stochastic integral. The primary motivation is to develop an integral and differential calculus capable of handling computations with random "noises." Historically, the concept of random "noises" originated from experimental sciences. In probability theory, the most classical example of "noise" is Brownian motion. Brownian motion was introduced by Robert Brown in 1828 to study the movement of pollen particles in water. Later, in 1905, Einstein used Brownian motion to model the trajectories of gas molecules. Additionally, Bachelier applied Brownian motion to model stock option prices.

Let us give a first definition.

Definition 1.1. *Brownian motion $(B_t)_{t\geq 0}$ is a $\mathbb{R}^d$ "process" $(d \geq 1)$ (i.e. a family of random variables in short r.v.) such that*

1. *$\forall n \in \mathbb{N}^*$ and $t_0 < t_1 < \ldots < t_n$ the r.v.'s $B_{t_0},\ B_{t_1} - B_{t_0}, \ldots, B_{t_{n+1}} - B_{t_n}$ are independent (BM is a process with independent increments PII in short.)*

2. *If $s < t$, $B_t - B_s$ is a centered Gaussian random vector with covariance matrix $(t-s)Id$.*

$(B_t)_{t\geq 0}$ starts from $0 \in \mathbb{R}^d$ if $B_0 = 0$, $\mathbb{P}a.s.$

Remark 1.1.

- *If* $d = 1$, $Var(B_t - B_s) = t - s$.
- *Random vectors from* $(\Omega, \mathcal{A}, P)$ *to* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$*, Independence, Gaussian random vectors are supposed to be known.*
- *"Sample paths" : Heuristically we fix* $\omega \in \Omega$*, and we are interested in* $t \mapsto B_t(\omega)$ *Wiener (1923, 1924), Paley-Zygmund. P almost surely the sample paths are contnuous and nowhere differentiable.*
- *If* $B(t) = (B^1(t), \ldots, B^d(t))$*, then* $\forall i = 1$ *to* d $B^i(t)$ *are real valued Brownian motions and if* $i \neq j$, B^i *is independent of* B^j.

The aim of stochastic calculus is to give a rigorous meaning to (stochastic) differential equations of the type

$$y_t = \int_0^t f(y_s)\dot{B}_s ds$$

which have many applications. We will show that the Brownian motion is a continuous martingale, and that integrals can be defined in this framework. Another goal is to have a chain rule associated to these integrals. It is called the Itô formula which claims that $\forall f \in C^2(\mathbb{R}^d, \mathbb{R})$,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

Actually one can show if $(X_t)_{t\geq 0}$ is real valued process with independent increments and if the distribution of $X_t - X_s$ does depend only of $t - s$, (stationary increments) with continuous sample paths then $X_t = X_0 + \sigma B_t$. More generally one can study PIIS process with stationary independent increments not with continuous sample paths. Another example of PIIS is the Poisson process.

Références :

- I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Springer Verlag, 1988.
- Damien Lamberton and Bernard Lapeyre. *Introduction to stochastic calculus applied to finance.* Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2008.

- Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

2 Definition and continuity of Brownian motion

2.1 Distribution of a process

Definition 1.2. *Let T be a set and $(E, \mathcal{E})$ a measurable set. A E valued stochastic process indexed by T is a family $(X_t,\ t \in T)$ of r.v.'s $X_t : (\Omega, \mathcal{A}, \mathbb{P}) \mapsto (E, \mathcal{E})$.*

Reminder : If $Y : (\Omega, \mathcal{A}, \mathbb{P}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is real valued random variable, the distribution $\mathbb{P}_Y$ of Y is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. It is the push forward of the probability $\mathbb{P}$ by the measurable function Y. It is defined by $\mathbb{P}_Y(A) = \mathbb{P}(Y \in A)$, $\forall A \in \mathscr{B}(\mathbb{R})$.

For a process a sample path is associated to every $\omega \in \Omega$, $t \mapsto X_t(\omega)$. Hence the distribution of a process is a probability on a set of functions from $T \mapsto E$ which is denoted by E^T, and endowed with the cylindrical sigma field. A Cylinder is indexed by $t_0 < t_1 < \ldots < t_n$, it is a subset of E^T,

$$C_{t_0,\, t_1,\, \ldots,\, t_n} = \{f \in E^T,\ (f(t_0),,\ \ldots,\ f(t_n)) \in A_0 \times \ldots \times A_n,\ \text{with } A_i \in \mathcal{E}\},$$

where $C_{t_0,\, t_1,\, \ldots,\, t_n}$ actually depends also on A_i's. Then the distribution of the process is a probability on E^T endowed with the smallest sigma field that contains all cylinders. This sigma field is called the cylindrical sigma field denoted by $\mathcal{E}^{\otimes T}$.

Definition 1.3. *The distribution of a E-valued process is a probability on E^T endowed with the smallest sigma field $\mathcal{E}^{\otimes T}$ that contains all cylinders. It is uniquely defined by*

$$\mathbb{P}(X(t_0) \in A_0,\ \ldots, X(t_n) \in A_n),$$

$\forall n \in \mathbb{N}^*$ and $t_0 < t_1 < \ldots < t_n$.

Example the Brownian motion that starts from 0 :

If $t_0 < t_1 < \ldots < t_n$, and f is bounded Borel function $(\mathbb{R}^d)^{n+1} \mapsto \mathbb{R}$, let

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(\frac{-\|x\|^2}{2t}\right), \tag{1.1}$$

where $\|.\|$ is the Euclidean norm on $\mathbb{R}^d$. Then $p_{t-s}(x)dx$ is the distribution of $B_t - B_s$, so

$$\mathbb{E}f(B_{t_0}, \ldots, B_{t_n}) = \int f(x_0, \ldots, x_n) d\mathbb{P}_{(B_{t_0}, \ldots, B_{t_n})}(x_0, \ldots, x_n)$$
$$= \int f(x_0, \ldots, x_n) d\mathbb{P}_{(B_{t_0}, B_{t_1}-B_{t_0}, \ldots, B_{t_n}-B_{t_{n-1}})}(x_0, x_1 - x_0, \ldots, x_n - x_{n-1})$$
$$= \int f(x_0, \ldots, x_n) p_{t_0}(x_0) p_{t_1-t_0}(x_1 - x_0) \ldots p_{t_n-t_{n-1}}(x_n - x_{n-1}) dx_0 \ldots dx_n.$$

Hence

$$\mathbb{P}(B(t_0) \in A_0, \ldots, B(t_n) \in A_n)$$
$$= \int_{A_0 \times \ldots \times A_n} p_{t_0}(x_0) p_{t_1-t_0}(x_1 - x_0) \ldots p_{t_n-t_{n-1}}(x_n - x_{n-1}) dx_0 \ldots dx_n \tag{1.2}$$

and

$$\mathbb{P}(B(t_0) \in A_0, \ldots, B(t_n) \in A_n) = \mathbb{P}(X_0 \in A_0, \ldots,, X_0 + \ldots + X_n \in A_n),$$

where $X_0 \stackrel{(d)}{=} \mathcal{N}(0, t_0)$, $X_i \stackrel{(d)}{=} \mathcal{N}(0, t_i - t_{i-1})$ for all $i = 1$ to n and X_i's are independent.

Remark 1.2. *This remark is also an exercise. If* $A \in \mathscr{B}(\mathbb{R}^d)^{\otimes \mathbb{R}_+}$*, then* $\exists (t_n)_{n\in\mathbb{N}}$ *and* $\exists B \in \mathscr{B}(\mathbb{R}^d)^{\otimes \mathbb{N}}$ *such that*

$$A = \{f \in (\mathbb{R}^d)^{\mathbb{R}_+}, \ (f(t_n))_{n\in\mathbb{N}} \in B\}.$$

Does exist on $(\mathbb{R}^d)^{\mathbb{R}_+}$ a probability such that

$$\mu(C_{t_0, t_1, \ldots, t_n}) = \mathbb{P}(B(t_0) \in A_0, \ldots, B(t_n) \in A_n)?$$

The answer"yes" is given by a Kolmogorov theorem. Let us first introduce definitions and a necessary condition.

Definition 1.4. *Let $\mathcal{T}$ be the set of finite increasing sequences $\underline{t} = (t_0, \ldots, t_n)$ of numbers, where the length $n+1$ of these sequences ranges over the set of positive integers. Suppose that for each $\underline{t}$ of length $n+1$, we have a probability measure $Q_{\underline{t}}$ on $\left(\mathbb{R}^{n+1}, \mathscr{B}\left(\mathbb{R}^{n+1}\right)\ \right)$. Then the collection $\{Q_t\}_{t\in\mathcal{T}}$ is called a family of finite-dimensional distributions. This family is said to be consistent provided that the following condition is satisfied: If $\underline{t} = (t_0, t_1, \ldots, t_n)$ with $n \geq 1, \underline{t}^i = (t_0, t_1, \ldots, t_n)$, where t_i is missing then $\forall i \leq n$*

$$Q_{\underline{t}^i}(A_0 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) = Q_{\underline{t}}(A_0 \times \cdots \times \mathbb{R} \times \cdots \times A_n).$$

If we have a probability measure μ on $\left(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes[0,+\infty)}\right)$, then we can define a family of finite-dimensional distributions by

$$Q_{\underline{t}}(A) = \mu\left[\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_n)) \in A\right], \tag{1.3}$$

where $A \in \mathscr{B}(\mathbb{R}^n)$ and $t = (t_1, \ldots, t_n) \in T$. This family is easily seen to be consistent. We are interested in the converse of this fact, because it will enable us to construct a probability measure $\mathbb{P}$ from the finite-dimensional distributions of Brownian motion.

Theorem 1.1 ((Daniell (1918), Kolmogorov (1933))**.** *Let $\left\{Q_{\underline{t}}\right\}$ be a consistent family of finite-dimensional distributions. Then there is a probability measure $\mathbb{P}$ on $\left((\mathbb{R}^d)^{[0,\infty)}, \mathcal{B}(\mathbb{R}^d)^{\otimes[0,+\infty)}\right)$, such that (1.3) holds for every $t \in T$.*

Proof. A proof can be read in p 50 of Karatzas and Shreeve. One can use for instance Carathéodory theorem, that may also be used to construct Lebesgue measure. □

To verify that we can apply the Theorem to the construction of the Brownian motion, it is enough to show consistency in this case. Coming back to (1.2), we are left to check

$$\int_{\mathbb{R}^d} p_{t_i - t_{i-1}}(x_i - x_{i-1}) p_{t_{i+1}-t_i}(x_{i+1} - x_i) dx_i = p_{t_{i+1}-t_{i-1}}(x_{i+1} - x_{i-1}).$$

But it is the same as $p_{t_i-t_{i-1}} * p_{t_{i+1}-t_i} = p_{t_{i+1}-t_{i-1}}$, or $\mathcal{N}(0, t_i - t_{i-1}) + \mathcal{N}(0, t_{i+1} - t_i) = \mathcal{N}(0, t_{i+1} - t_{i-1})$ where independence is assumed on the left hand side.

2.2 Regularity of BM sample paths

In this section we want to convince ourselves that BM sample paths are almost surely continuous. We hope that $\mathcal{C}$ the set of continuous functions is of probability 1 under the distribution of BM.

Remark 1.3. *Unfortunately $\mathcal{C}$ is not measurable in $\mathscr{B}(\mathbb{R}^d)^{\otimes \mathbb{R}_+}$. Prove this fact as an exercise, that uses the previous Remark/exercise.*

To circumvent this problem we will show that there exists a process with almost sure continuous sample paths that has the same distribution as BM.

To clean a bit the situation some definitions are introduced.

Definition 1.5. *1. Processes X and Y have the same finite-dimensional distributions if, for any integer $n \geq 1$, real numbers $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, and $A \in \mathscr{B}\left(\mathbb{R}^{nd}\right)$, we have:*

$$\mathbb{P}\left[(X_{t_1}, \ldots, X_{t_n}) \in A\right] = \mathbb{P}\left[(Y_{t_1}, \ldots, Y_{t_n}) \in A\right].$$

2. Y is a modification of X if, for every $t \geq 0$, we have $\mathbb{P}\left[X_t = Y_t\right] = 1$.

3. X and Y are called indistinguishable if almost all their sample paths agree:

$$\mathbb{P}\left[X_t = Y_t; \forall 0 \leq t < \infty\right] = 1.$$

Exercise 1.1. *If X is a modification of X' then the distribution of X and X' are the same.*

If X and X' are indistinguishable then there are modifications of each other. The converse is false.

Example. *Let $\Omega = [0,1]$, $\mathcal{A} = \mathscr{B}([0,1])$, $\mathbb{P} = dx$ the Lebesgue measure. Let us take $X_t(\omega) = \mathbf{1}(\omega \neq t)$, and $Y_t = 1$. Then*

$$\mathbb{P}(X_t = Y_t) = \mathbb{P}(\omega \neq t) = 1.$$

Hence X is a modification of Y. But $\mathbb{P}(\forall t \in [0,1],\ X_t = Y_t) = 0$. They are not indistinguishable.

Exercise 1.2. *Let Y be a modification of X, and suppose that both processes have a.s. right-continuous sample paths. Then X and Y are indistinguishable.*

The next theorem will show that there exits a modification of the BM with almost sure continuous sample paths.

Theorem 1.2 (Kolmogorov, Čentsov (1956)).

Suppose that a process $\{X_t; 0 \leq t \leq T\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfies the condition

$$\mathbb{E}\,|X_t - X_s|^{\alpha} \leq C|t-s|^{1+\beta}, \quad 0 \leq s, t \leq T,$$

for some positive constants α, β, and C. Then there exists a continuous modification $\tilde{X} = \left\{\tilde{X}_t; 0 \leq t \leq T\right\}$ of X, which is locally Hölder-continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$, i.e.,

$$\mathbb{P}\left[\omega; \sup_{\substack{0<t-s<h(\omega)\\ s,t\in[0,T]}} \frac{\left|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)\right|}{|t-s|^{\gamma}} \leq \delta\right] = 1, \tag{1.4}$$

where $h(\omega)$ is an a.s. positive random variable and $\delta > 0$ is an appropriate constant.

Proof. For simplicity, we take $T = 1$. Much of what follows is a consequence of the Cebyšev inequality. First, for any $\varepsilon > 0$, we have

$$\mathbb{P}\left[|X_t - X_s| \geq \varepsilon\right] \leq \frac{\mathbb{E}\,|X_t - X_s|^{\alpha}}{\varepsilon^{\alpha}} \leq C\varepsilon^{-\alpha}|t-s|^{1+\beta},$$

and so $X_s \to X_t$ in probability as $s \to t$. Second, setting $t = k/2^n, s = (k-1)/2^n$, and $\varepsilon = 2^{-\gamma n}$ (where $0 < \gamma < \beta/\alpha$) in the preceding inequality, we obtain

$$\mathbb{P}\left[\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}\right] \leq C2^{-n(1+\beta-\alpha\gamma)},$$

and consequently,

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\leq k\leq 2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}\right] \\
&\quad = \mathbb{P}\left[\bigcup_{k=1}^{2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}\right] \\
&\quad \leq C2^{-n(\beta-\alpha\gamma)}.
\end{aligned}$$

The last expression is the general term of a convergent series; by the Borel-Cantelli lemma, there is a set $\Omega^* \in \mathscr{F}$ with $\mathbb{P}(\Omega^*) = 1$ such that for each $\omega \in \Omega^*$,

$$\max_{1 \leq k \leq 2^n} \left| X_{k/2^2}(\omega) - X_{(k-1)/2^n}(\omega) \right| < 2^{-\gamma n}, \quad \forall n \geq n^*(\omega), \tag{1.5}$$

where $n^*(\omega)$ is a positive, integer-valued random variable. For each integer $n \geq 1$, let us consider the partition $D_n = \{(k/2^n)\,; k = 0,\ 1, \ldots, 2^n\}$ of $[0,1]$, and let $D = \bigcup_{n=1}^{\infty} D_n$ be the set of dyadic rationals in $[0,1]$. We shall fix $\omega \in \Omega^*, n \geq n^*(\omega)$, and show that for every $m > n$, we have

$$|X_t(\omega) - X_s(\omega)| \leq 2 \sum_{j=n+1}^{m} 2^{-\gamma j}; \quad \forall t, s \in D_m, 0 < t - s < 2^{-n}. \tag{1.6}$$

For $m = n+1$, we can only have $t = (k/2^m)\,, s = ((k-1)/2^m)$, and (1.6) follows from (1.5). Suppose (1.6) is valid for $m = n+1, \ldots, M-1$. Take $s < t, s, t \in D_M$, consider the numbers $t^1 = \max\{u \in D_{M-1}; u \leq t\}$ and $s^1 = \min\{u \in D_{M-1}; u \geq s\}$, and notice the relationships $s \leq s^1 \leq t^1 \leq t, s^1 - s \leq 2^{-M}, t - t^1 \leq 2^{-M}$. From (1.5) we have $|X_{s^1}(\omega) - X_s(\omega)| \leq 2^{-\gamma M}, |\ X_t(\omega) - X_{t^1}(\omega)\ | \leq 2^{-\gamma M}$, and from (1.6) with $m = M-1$,

$$|X_{t^1}(\omega) - X_{s^1}(\omega)| \leq 2 \sum_{j=n+1}^{M-1} 2^{-\gamma j}.$$

We obtain (1.6) for $m = M$.

We can show now that $\{X_t(\omega); t \in D\}$ is uniformly continuous in t for every $\omega \in \Omega^*$. For any numbers $s, t \in D$ with $0 < t - s < h(\omega) \triangleq 2^{-n^*(\omega)}$, we select $n \geq n^*(\omega)$ such that $2^{-(n+1)} \leq t - s < 2^{-n}$. We have from (1.6)

$$|X_t(\omega) - X_s(\omega)| \leq 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} \leq \delta |t - s|^{\gamma}, \quad 0 < t - s < h(\omega), \tag{1.7}$$

where $\delta = 2/\left(1 - 2^{-\gamma}\right)$. This proves the desired uniform continuity. We define $\tilde{X}$ as follows. For $\omega \notin \Omega^*$, set $\tilde{X}_t(\omega) = 0, 0 \leq t \leq 1$. For $\omega \in \Omega^*$ and $t \in D$, set $\tilde{X}_t(\omega) = X_t(\omega)$. For $\omega \in \Omega^*$ and $t \in [0,1] \cap D^c$, choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$; uniform continuity and the Cauchy criterion imply that $\{X_{s_n}(\omega)\}_{n=1}^{\infty}$ has a limit which depends on t but not on the particular sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ chosen to converge to t, and we set

$\widetilde{X}_t(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$. The resulting process $\widetilde{X}$ is thereby continuous; indeed, $\widetilde{X}$ satisfies (1.7), so (1.4) is established.

To see that $\tilde{X}$ is a modification of X, observe that $\tilde{X}_t = X_t$ a.s. for $t \in D$; for $t \in [0,1] \cap D^c$ and $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, we have $X_{s_n} \to X_t$ in probability and $X_{s_n} \to \tilde{X}_t$ a.s., so $\tilde{X}_t = X_t$ a.s. □

This long proof is an example of the "chaining" argument, which is used in many other proofs.

To show that BM has a continuous modification we still have to show that the bound on the expectations of the increments of BM that is an assumption of the Theorem is satisfied

Proposition 1.1. *If B is a real valued BM $d = 1$,*

$$\forall n \in \mathbb{N}, \forall t \ \in [0, +\infty),\ \mathbb{E}(B_t - B_s)^{2n} = C_n |t - s|^n \tag{1.8}$$

There exists a modification of the BM with locally Hölder continuous paths for every exponent $0 < \gamma < \frac{1}{2}$.

Proof.

$$B_t - B_s \stackrel{(d)}{=} \mathcal{N}(0, |t - s|) \stackrel{(d)}{=} \sqrt{|t - s|}\mathcal{N}(0, 1)$$

Hence $\mathbb{E}(B_t - B_s)^2 = |t - s|$ and $\mathbb{E}(B_t - B_s)^{2n} = C_n|t - s|^n$ where $C_n = \mathbb{E}X^{2n}$, $X \stackrel{(d)}{=} \mathcal{N}(0, 1)$. Applying (1.8) for n fixed we get Hölder continuity for $\gamma < \frac{n-1}{2n}$. □

Remark 1.4. *Further on we always take continuous modifications of BM.*

3 Quadratic variations of Brownian motions

We may wonder if the previous result is optimal. For instance could it be that sample paths of Brownian motions are locally Liptschitz continuous ? Actually elementary definitions of integrals of the type $\int H_s dB_s$ are possible if the sample paths have almost surely finite variations. Let us first recall some facts concerning functions with finite variations.

3.1 Reminder of functions with finite variations

If $f : (0,+\infty) \mapsto \mathbb{R}$ is a non decreasing function, right continuous, we can associate a measure μ on $(0,+\infty)$ with $\forall 0 < s < t$

$$\mu((s,t]) = f(t) - f(s)$$

and if g is a bounded Borel function one can define

$$\int_0^t g(s)df(s) \stackrel{def}{=} \int \mathbf{1}_{[0,t]}(s)g(s)d\mu(s).$$

If f is C^1 it is equal to $\int_0^t g(s)f'(s)ds$.

Definition 1.6. *For $t > 0$, Let $\mathcal{P}_t$ be the set of finite subdivisions Δ of $[0,t]$: $\Delta = (t_i)_{i=1,\dots,n} \in \mathcal{P}_t$, if $0 \le t_1 \le \dots t_n \le t$. The mesh of Δ is denoted by $|\Delta| = \sup_{i=1,\dots,n-1}(t_{i+1} - t_i)$. For $f : (0,+\infty) \mapsto \mathbb{R}$, the variation of f on $[0,t]$ is denoted by*

$$V_t(f) \stackrel{def}{=} \sup_{\Delta \in \mathcal{P}_t} \sum_{i=1}^n |f(t_{i+1}) - f(t_i)|$$

and is said to have finite variations if $\forall t > 0$, $V_t(f) < \infty$.

Example.
- *If f is monotone or a difference of non decreasing functions, f has finite variations.*
- *If f is locally Lipschitz, f has finite variations.*

We will admit two facts for Riemmann-Stieljes integral

- Every function f with finite variations is a difference of non decreasing functions f_1, f_2, one can write

$$\int_0^t g(s)df(s) \stackrel{def}{=} \int_0^t g(s)df_1(s) - \int_0^t g(s)df_2(s).$$

- If g is continuous and $(\Delta^n)_{n\in\mathbb{N}}$ a sequence of subdivisions with meshes $|\Delta^n| \to 0$,

$$\int_0^t g(s)df(s) = \lim_{n\to\infty} \sum_{t_i^n \in \Delta^n} g(t_i^n)(f(t_{i+1}^n) - f(t_i^n)).$$

3.2 Quadratic variations

Actually quadratic variations of BM sample paths are almost surely : positive finite and BM sample paths do not have finite variations.

Definition 1.7. *A real valued process X has finite quadratic variations denoted by $\langle X, X\rangle$ or $(\langle X\rangle)$ if $\forall(\Delta^n)_{n\in\mathbb{N}}$ sequence of subdivisions of $\mathcal{P}_t$ such that $|\Delta^n| \to 0$ and $\Delta^n = (t_i^n)_{1\le i\le N(n)}$ with $t_1^n = 0$ and $t_{N(n)}^n = t$ and*

$$T_{[0,t]}^{\Delta^n} \overset{def}{=} \sum_{t_i^n\in\Delta^n} (X_{t_{i+1}^n} - X_{t_i^n})^2 \overset{(P)}{\to} \langle X\rangle_t.$$

Proposition 1.2. *If B is a Brownian motion $\langle B\rangle_t = t$ a.s.*

Proof. We prove $T_{[0,t]}^{\Delta^n} \to t$ in $L^2(\Omega)$. If $\Delta^n = (t_i^n)_{1\le i\le N(n)}$

$B_{t_{i+1}^n} - B_{t_i^n} \overset{(d)}{=} \mathcal{N}(0, t_{i+1}^n - t_i^n) \overset{(d)}{=} \sqrt{t_{i+1}^n - t_i^n}\mathcal{N}(0,1)$. Then $\mathbb{E}((T_{[0,t]}^{\Delta^n}) = \sum_{t_i^n\in\Delta^n}\mathbb{E}(B_{t_{i+1}^n} - B_{t_i^n})^2 = \sum_{t_i^n\in\Delta^n} t_{i+1}^n - t_i^n = t$. Hence

$$\begin{aligned}
\mathbb{E}((T_{[0,t]}^{\Delta^n} - t)^2) &= Var(T_{[0,t]}^{\Delta^n} - t)\\
&= Var(\sum_{t_i^n\in\Delta^n} (B_{t_{i+1}^n} - B_{t_i^n})^2 - (t_{i+1}^n - t_i^n))\\
&= \sum_{t_i^n\in\Delta^n} Var((B_{t_{i+1}^n} - B_{t_i^n})^2 - (t_{i+1}^n - t_i^n))\\
&= \sum_{t_i^n\in\Delta^n} (t_{i+1}^n - t_i^n)^2 Var(N^2 - 1)\\
&\le C|\Delta^n| \sum_{t_i^n\in\Delta^n} (t_{i+1}^n - t_i^n) \to 0,
\end{aligned}$$

where $N = \mathcal{N}(0,1)$. This implies convergence in probability. □

To get almost sure convergence some additional assumptions are needed for Δ^n...

Proposition 1.3. *Almost surely the sample paths of Brownian motion have infinite variations on every intervals $[0,t]$ for $t > 0$.*

Proof. If ω is such that $V_{[0,t]}(B(\omega)) < +\infty$ then

$$\sum_{t_i^n\in\Delta^n} (B_{t_{i+1}^n} - B_{t_i^n})^2 \le \sup|(B_{t_{i+1}^n} - B_{t_i^n}| \sum_{t_i^n\in\Delta^n} |B_{t_{i+1}^n} - B_{t_i^n}|.$$

Since $s \mapsto B_s(\omega)$ is uniformly continuous on $[0,t]$ $\sup|B_{t^n_{i+1}} - B_{t^n_i}| \to 0$. Hence the quadratic variations of the sample paths should vanish, which is true only on a negligible set. □

4 Brownian motion as a Gaussian process

4.1 Elementary properties

Definition 1.8. *A real valued process $(X_t, t \in T)$ is a Gaussian process if $\forall n \in \mathbb{N},\ t_1, \ldots, t_n \in T,\ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n,\ \sum_{i=1}^n \alpha_i X_{t_i}$ is a Gaussian random variable. The process X is centered if $\forall t \in T,\ \mathbb{E}X_t = 0$ and $\Gamma(s,t) \stackrel{def}{=} Cov(X_s, X_t)$ is the covariance function.*

Remark 1.5.
- *If $\forall i = 1$ to n, $\alpha_i = 0$ $\sum_{i=1}^n \alpha_i X_{t_i} = 0$. It means that we consider $\mathcal{N}(0,0)$ as a generalized degenerated Gaussian random variable with variance 0. Gaussian processes are generalization of random Gaussian vectors (where T is a finite set). (See for instance N. Bouleau Processus stochastique et applications 1988.)*
- *If (X_1, X_2) is a Gaussian vector and $Cov(X_1, X_2) = 0$ then X_1 and X_2 are independent.*

Proposition 1.4. *The Brownian motion which starts from 0 is the unique centered Gaussian process with covariance $\Gamma(s,t) = \min(s,t)$.*

Proof. The proof relies on the fact that the covariance always characterizes the distribution of a Gaussian centered process. If X is a Gaussian centered process and $t_1, \ldots, t_n \in T$, then $(X_{t_1}, \ldots,\ X_{t_n})$ Gaussian vector implies the characteristic function

$$\mathbb{E}\exp(i\sum_{i=1}^n \alpha_i X_{t_i}) = \exp(-\frac{1}{2}\langle C\alpha, \alpha\rangle)$$

where $\alpha = (\alpha_1, \ldots,\ \alpha_n)$, $\langle .,. \rangle$ is The Euclidean scalar product in $\mathbb{R}^n$, and $C_{i,j} = \mathbb{E}(X_{t_i}X_{t_j}) = \Gamma(t_i, t_j)$. Hence the matrix $(\Gamma(t_i, t_j))$ characterizes the distribution of the finite dimensional margins $(X_{t_1}, \ldots,\ X_{t_n})$ $\forall t_1, \ldots, t_n \in T$, and henceforth the distribution of the process X.

Let us compute $\Gamma(s,t)$ for Brownian motion. Let $s \leq t$

$$\Gamma(s,t) = \mathbb{E}(B_sB_t) = \mathbb{E}(B_s(B_s + B_t - B_s)) = \mathbb{E}(B_s^2) + \mathbb{E}(B_s)\mathbb{E}(B_t - B_s) = s.$$

Hence $\Gamma(s,t) = \min(s,t)$.

□

Proposition 1.5. *Let $(B_t,\ t \geq 0)$ be a Brownian motion.*

1. $(B_{t+s} - B_s, \quad t \geq 0)$ *is a BM independent of* $\mathcal{F}_s^B = \sigma\left(B_u, \quad u \leq s\right)$.
2. $(-B_t, \quad t \geq 0)$ *is a Brownian motion,*
3. *(Self-similarity) for all* $\lambda > 0$, $\left(B_t^{(\lambda)}, \quad t \geq 0\right)$ *where* $B_t^{(\lambda)} := \frac{1}{\sqrt{\lambda}} B_{\lambda t}, \quad t \geq 0$ *is a Brownian motion.*

Proof. $W_t = B_{t+s} - B_s$ is a centered Gaussian process such that $W_0 = 0$ a.s. Its covariance

$\mathbb{E}(W_t W_{t'}) = \mathbb{E}((B_{t+s} - B_s)(B_{t'+s} - B_s)) = \min(t+s, t'+s) - s = \min(t,t')$.

$\forall u_1 \leq \ldots \leq u_n \leq s \leq t_1 \leq \ldots \leq t_n$ $(B_{u_1}, \ldots,\ B_{u_n},\ B_{t_1+s} - B_s, \ldots,\ B_{t_n+s} - B_s)$ is a Gaussian vector and $(B_{u_1}, \ldots,\ B_{u_n})$ is independent of $(B_{t_1+s} - B_s, \ldots,\ B_{t_n+s} - B_s)$ since $\forall i,\ j$

$$\mathbb{E}(B_{u_i}(B_{t_j+s} - B_s)) = 0.$$

Actually for Gaussian random vectors, a vanishing covariance yields independence. Then independence of sigma field is a consequence of independence of the random variable that generate them. To prove other parts of the Proposition, compute covariances. □

4.2 Brownian bridge

We may condition $(B_t,\ 0 \leq t \leq 1)$ to the event $B_1 = 0$. In this case we obtain a Brownian bridge.

Definition 1.9. *The process* $X_t^0 = B_t - tB_1$ *is called a Brownian bridge.*

Proposition 1.6. X^0 *is independent of* B_1.

Proof. $\forall 0 \leq t \leq 1,\ \mathbb{E}(X_t^0 - tB_1) = 0$. And it is a centered Gaussian process. Furthermore $\forall 0 \leq t \leq 1,\ \mathbb{E}(X_t^0 B_1) = \mathbb{E}((B_t - tB_1)B_1) = 0$. □

Proposition 1.7. *Let* $X_t^b = B_t - tB_1 + tb$. *The distribution of* X^b *is a regular version of the conditional distribution of* $(B_t,\ t \leq 1)$ *given* $B_1 = b$.

Proof. Reminder : $\mathbb{P}(dx, y)$ is a regular version of the conditional distribution of X given $Y = y$ if and only if (in short iff) $\forall \varphi$ bounded and measurable

$$\mathbb{E}(\varphi(X)|Y) = \int \varphi(x)\mathbb{P}(dx, Y)$$

$\mathbb{P}$ almost surely. It can be characterized by $\forall \varphi,\ g$ bounded and measurable

$$\mathbb{E}(\varphi(X)g(Y)) = \int (\int \varphi(x)\mathbb{P}(dx, y))g(y)d\mathbb{P}_Y(y).$$

In our case we have to show

$$\mathbb{E}(\varphi(B_s,\ s \leq 1)g(B_1)) = \int \mathbb{E}(\varphi(X_s^b,\ s \leq 1)g(b)\frac{e^{-\frac{b^2}{2}}db}{\sqrt{2\pi}}. \tag{1.9}$$

Let $\psi((f(s),\ s \leq 1), b) = \varphi((f(s) + sb, s \leq 1))$. Since $B_s = X_s^0 + sB_1$,

$$\begin{aligned}
\mathbb{E}(\varphi(B_s,\ s \leq 1)g(B_1)) &= \mathbb{E}(\psi((X_s^0,\ s \leq 1), B_1)g(B_1)) \\
&= \int \mathbb{E}(\psi((X_s^0,\ s \leq 1), b)g(b))\frac{e^{-\frac{b^2}{2}}db}{\sqrt{2\pi}} \\
&= \int \mathbb{E}(\varphi((X_s^b,\ s \leq 1))g(b))\frac{e^{-\frac{b^2}{2}}db}{\sqrt{2\pi}}.
\end{aligned}$$

□

4.3 Wiener integral

In this part the integral $\int f(s)dB_s$ is defined for a deterministic function using the Gaussianity of the Brownian motion. (Later the stochastic integral is defined for f a stochastic process.) If f is a simple function

$$f = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{(a_i, b_i]}$$

for $a_1 < b_1 \leq a_2 < b_2 \leq \ldots$ Then

$$\int f(s)dB_s \stackrel{def}{=} \sum_{i=1}^{n} \alpha_i(B(b_i) - B(a_i)), \tag{1.10}$$

this random variable is denoted by $I(f)$. It is a centered Gaussian random variable with variance

$$\mathbb{E}(I(f)^2) = \sum_{i=1}^{n} \alpha_i^2(b_i - a_i) = \|f\|^2_{L^2(0,\infty)}.$$

Theorem 1.3. *There exists a unique map I from $L^2(0,\infty)$ to $\mathcal{H}$ the set that is the closure in $L^2(\Omega,\mathcal{A},\mathbb{P})$ of the linear combinations $\sum_{i=1}^n \alpha_i(B(b_i)-B(a_i))$ such that*

1. $I(\mathbf{1}_{(a_i,b_i]}) = B(b_i) - B(a_i)$.
2. *I is a linear map*
3. *I is an isometry i.e.* $\forall f \in L^2(0,\infty)\ \|f\|_{L^2(0,\infty)} = \|I(f)\|_{L^2(\Omega,\mathcal{A},\mathbb{P})}$.

Proof. If $f \in L^2(0,\infty)$ $\exists (f_n)_{n\in\mathbb{N}}$ with f_n simple functions and $\lim_{n\to\infty} f_n = f$ in $L^2(0,\infty)$. Then let $I(f) \overset{def}{=} \lim_{n\to\infty} I(f_n)$ in $L^2(0,\infty)$. Please remark that $I(f)$ does not depend on the sequence $(f_n)_{n\in\mathbb{N}}$ since if $\lim_{n\to\infty} g_n = f$ then $\lim_{n\to\infty} I(g_n) - I(f_n) = 0$ because of the isometry property. Moreover we get the uniqueness of I by density of simple functions in L^2. □

Remark 1.6.

1. *Since I is an isometry $\forall f,\ g \in L^2(0,\infty)$,*

$$\langle f, g\rangle_{L^2(0,\infty)} = \langle I(f), I(g)\rangle_{L^2(\Omega,\mathcal{A},\mathbb{P})}. \tag{1.11}$$

   If we denote by $\int_0^\infty f(s)dB_s = I(f)$, the so-called Wiener integral this can be rewritten :

$$\int_0^\infty f(s)g(s)ds = \mathbb{E}(\int_0^\infty f(s)dB_s \int_0^\infty g(s)dB_s).$$

2. *Conversely if $J\ :\ L^2(0,\infty) \mapsto L^2(\Omega,\mathcal{A},\mathbb{P})$ is such that $J(f)$ is a centered Gaussian random variable and*

$$\int_0^\infty f(s)g(s)ds = \mathbb{E}(J(f)J(g))$$

   then J is a linear map. Moreover $J(\mathbf{1}_{(0,t]})$ is a real Brownian motion. (Since $(J(\mathbf{1}_{(0,t]}),\ t \le 0)$ is a centered Gaussian process and $\mathbb{E}(J(\mathbf{1}_{(0,s]})J(\mathbf{1}_{(0,t]}) = \min(s,t)$.) If we denote by $B_t = J(\mathbf{1}_{(0,t]})$, then J is the isometry I associated to the BM B.

The same construction can be generalized to all intervals I' and $L^2(I')$. For $I' = \mathbb{R}$, the process $X_t = I(\mathbf{1}_{(0,t]}), \forall t \in \mathbb{R}$ can be obtained from two independent real valued Brownian motion $X_t = B_t^1, \forall t \le 0$, and $X_t = B_t^2, \forall t \ge 0$. One can easily check that

$$\mathbb{E}(X_t - X_s)^2 = |t-s|,\ \forall t < 0 < s.$$

4.4 Second construction of Brownian motion

If $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis (ONB) of $L^2(0,1)$ then $(I(e_n))_{n\in\mathbb{N}}$ is a sequence of Gaussian independent random variables with distribution $\mathcal{N}(0,1)$. Actually $(1,\ \sqrt{2}\cos(2\pi ks),\ \sqrt{2}\sin(2\pi ks))_{k\in\mathbb{N}^*}$ is an ONB of $L^2(0,1)$, $\forall t \in (0,1)$,

$$\mathbf{1}_{(0,t]}(s) \overset{L^2(0,1)}{=} a_0(t) + \sum_{k=1}^{\infty} a_k(t)\sqrt{2}\cos(2\pi ks) + b_k(t)\sqrt{2}\sin(2\pi ks)$$

$$I(\mathbf{1}_{(0,t]}) \overset{L^2(\Omega,\mathcal{A},\mathbb{P})}{=} a_0(t)\xi_0 + \sum_{k=1}^{\infty} a_k(t)\xi_k + \sum_{k=1}^{\infty} b_k(t)\eta_k,$$

with $(\xi_0,\ \xi_k,\ \eta_k)$ i.i.d. Gaussian random variables with distribution $\mathcal{N}(0,1)$. Moreover $a_0(t) = \int_0^1 \mathbf{1}_{(0,t]}(s)ds = t$, $\forall k \geq 1$

$$a_k(t) = \sqrt{2}\int_0^t \cos(2\pi ks)ds = \frac{\sin(2\pi kt)}{\sqrt{2}\pi k}$$

$$b_k(t) = \sqrt{2}\int_0^t \sin(2\pi ks)ds = \frac{(1-\cos(2\pi kt))}{\sqrt{2}\pi k}.$$

Hence we get a series expansion of BM, a priori in L^2 sense...

$$I(\mathbf{1}_{(0,t]}) = t\xi_0 + \sum_{k=1}^{\infty} \xi_k \frac{\sin(2\pi ks)}{\sqrt{2}\pi k} + \sum_{k=1}^{\infty} \eta_k \frac{(1-\cos(2\pi ks))}{\sqrt{2}\pi k}. \tag{1.12}$$

Since $I(\mathbf{1}_{(0,1]}) = \xi_0$, hence (1.12) can be viewed as tB_1 plus the expansion of a Brownian bridge.

Theorem 1.4. *If $(\xi_0,\ \xi_k,\ \eta_k)$ are i.i.d. Gaussian random variables with distribution $\mathcal{N}(0,1)$*

$$t\xi_0 + \sum_{k=1}^{\infty} \xi_k \frac{\sin(2\pi kt)}{\sqrt{2}\pi k} + \sum_{k=1}^{\infty} \eta_k \frac{(1-\cos(2\pi kt))}{\sqrt{2}\pi k}$$

almost surely converges to a process $(B_t)_{t\in(0,1)}$ with the distribution of a BM.

Proof. We may refer to criteria for convergences of random Fourier series in Kahane Some random series of functions Theorem 2 p 236 second edition, we get almost surely the uniform (but not normal) convergence of the series. Since $I(\mathbf{1}_{(0,t]})$ is a BM we get the distribution of the limit of the series. □

Remark 1.7. *With this construction almost sure continuity of the sample paths is for free !*