Stochastic calculus lecture

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1 Introduction

The aim of this lecture is to construct the stochastic integral. The primary motivation is to develop an integral and differential calculus capable of handling computations with random "noises." Historically, the concept of random "noises" originated from experimental sciences. In probability theory, the most classical example of "noise" is Brownian motion. Brownian motion was introduced by Robert Brown in 1828 to study the movement of pollen particles in water. Later, in 1905, Einstein used Brownian motion to model the trajectories of gas molecules. Additionally, Bachelier applied Brownian motion to model stock option prices.

Let us give a first definition.

Definition 1.1. *Brownian motion $(B_t)_{t\geq 0}$ is a $\mathbb{R}^d$ "process" $(d \geq 1)$ (i.e. a family of random variables in short r.v.) such that*

1. *$\forall n \in \mathbb{N}^*$ and $t_0 < t_1 < \ldots < t_n$ the r.v.'s $B_{t_0},\ B_{t_1} - B_{t_0}, \ldots, B_{t_{n+1}} - B_{t_n}$ are independent (BM is a process with independent increments PII in short.)*
2. *If $s < t$, $B_t - B_s$ is a centered Gaussian random vector with covariance matrix $(t-s)Id$.*

$(B_t)_{t\geq 0}$ starts from $0 \in \mathbb{R}^d$ if $B_0 = 0$, $\mathbb{P}a.s.$

Remark 1.1.

- *If* $d = 1$, $Var(B_t - B_s) = t - s$.
- *Random vectors from* $(\Omega, \mathcal{A}, P)$ *to* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)$*, Independence, Gaussian random vectors are supposed to be known.*
- *"Sample paths" : Heuristically we fix* $\omega \in \Omega$*, and we are interested in* $t \mapsto B_t(\omega)$ *Wiener (1923, 1924), Paley-Zygmund.* P *almost surely the sample paths are contnuous and nowhere differentiable.*
- *If* $B(t) = (B^1(t), \ldots, B^d(t))$*, then* $\forall i = 1$ *to* d $B^i(t)$ *are real valued Brownian motions and if* $i \neq j$, B^i *is independent of* B^j.

The aim of stochastic calculus is to give a rigorous meaning to (stochastic) differential equations of the type

$$y_t = \int_0^t f(y_s)\dot{B}_s ds$$

which have many applications. We will show that the Brownian motion is a continuous martingale, and that integrals can be defined in this framework. Another goal is to have a chain rule associated to these integrals. It is called the Itô formula which claims that $\forall f \in C^2(\mathbb{R}^d, \mathbb{R})$,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

Actually one can show if $(X_t)_{t\geq 0}$ is real valued process with independent increments and if the distribution of $X_t - X_s$ does depend only of $t - s$, (stationary increments) with continuous sample paths then $X_t = X_0 + \sigma B_t$. More generally one can study PIIS process with stationary independent increments not with continuous sample paths. Another example of PIIS is the Poisson process.

Références :

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- Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

2 Definition and continuity of Brownian motion

2.1 Distribution of a process

Definition 1.2. *Let T be a set and $(E, \mathcal{E})$ a measurable set. A E valued stochastic process indexed by T is a family $(X_t,\ t \in T)$ of r.v.'s X_t : $(\Omega, \mathcal{A}, \mathbb{P}) \mapsto (E, \mathcal{E})$.*

Reminder : If $Y : (\Omega, \mathcal{A}, \mathbb{P}) \mapsto (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is real valued random variable, the distribution $\mathbb{P}_Y$ of Y is a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. It is the push forward of the probability $\mathbb{P}$ by the measurable function Y. It is defined by $\mathbb{P}_Y(A) = \mathbb{P}(Y \in A)$, $\forall A \in \mathscr{B}(\mathbb{R})$.

For a process a sample path is associated to every $\omega \in \Omega$, $t \mapsto X_t(\omega)$. Hence the distribution of a process is a probability on a set of functions from $T \mapsto E$ which is denoted by E^T, and endowed with the cylindrical sigma field. A Cylinder is indexed by $t_0 < t_1 < \ldots < t_n$, it is a subset of E^T,

$$C_{t_0,\,t_1,\,\ldots,\,t_n} = \{f \in E^T,\ (f(t_0),,\ \ldots,\ f(t_n)) \in A_0 \times \ldots \times A_n,\ \text{with } A_i \in \mathcal{E}\},$$

where $C_{t_0,\,t_1,\,\ldots,\,t_n}$ actually depends also on A_i's. Then the distribution of the process is a probability on E^T endowed with the smallest sigma field that contains all cylinders. This sigma field is called the cylindrical sigma field denoted by $\mathcal{E}^{\otimes T}$.

Definition 1.3. *The distribution of a E-valued process is a probability on E^T endowed with the smallest sigma field $\mathcal{E}^{\otimes T}$ that contains all cylinders. It is uniquely defined by*

$$\mathbb{P}(X(t_0) \in A_0,\ \ldots, X(t_n) \in A_n),$$

$\forall n \in \mathbb{N}^*$ and $t_0 < t_1 < \ldots < t_n$.

Example the Brownian motion that starts from 0 :

If $t_0 < t_1 < \ldots < t_n$, and f is bounded Borel function $(\mathbb{R}^d)^{n+1} \mapsto \mathbb{R}$, let

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(\frac{-\|x\|^2}{2t}\right), \tag{1.1}$$

where $\|.\|$ is the Euclidean norm on $\mathbb{R}^d$. Then $p_{t-s}(x)dx$ is the distribution of $B_t - B_s$, so

$$\mathbb{E}f(B_{t_0},\ldots,\ B_{t_n}) = \int f(x_0,\ldots,\ x_n)d\mathbb{P}_{(B_{t_0},\ldots,\ B_{t_n})}(x_0,\ldots,\ x_n)$$
$$= \int f(x_0,\ldots,\ x_n)d\mathbb{P}_{(B_{t_0},B_{t_1}-B_{t_0},\ldots,\ B_{t_n}-B_{t_{n-1}})}(x_0,\ x_1-x_0,\ldots,\ x_n-x_{n-1})$$
$$= \int f(x_0,\ldots,\ x_n)p_{t_0}(x_0)p_{t_1-t_0}(x_1-x_0)\ldots p_{t_n-t_{n-1}}(x_n-x_{n-1})dx_0\ldots dx_n.$$

Hence

$$\mathbb{P}(B(t_0)\in A_0,\ \ldots,B(t_n)\in A_n) = \int_{A_0\times\ldots\times A_n} p_{t_0}(x_0)p_{t_1-t_0}(x_1-x_0)\ldots p_{t_n-t_{n-1}}(x_n-x_{n-1})dx_0\ldots dx_n \tag{1.2}$$

and

$$\mathbb{P}(B(t_0)\in A_0,\ \ldots,B(t_n)\in A_n) = \mathbb{P}(X_0\in A_0,\ \ldots,,\ X_0+\ldots+X_n\in A_n),$$

where $X_0 \overset{(d)}{=} \mathcal{N}(0,t_0)$, $X_i \overset{(d)}{=} \mathcal{N}(0,t_i-t_{i-1})$ for all $i=1$ to n and X_i's are independent.

Remark 1.2. *This remark is also an exercise. If* $A\in\mathscr{B}(\mathbb{R}^d)^{\otimes\mathbb{R}_+}$, *then* $\exists(t_n)_{n\in\mathbb{N}}$ *and* $\exists B\in\mathscr{B}(\mathbb{R}^d)^{\otimes\mathbb{N}}$ *such that*

$$A = \{f\in(\mathbb{R}^d)^{\mathbb{R}_+},\ (f(t_n))_{n\in\mathbb{N}}\in B\}.$$

Does exist on $(\mathbb{R}^d)^{\mathbb{R}_+}$ a probability such that

$$\mu(C_{t_0,\,t_1,\,\ldots,\,t_n}) = \mathbb{P}(B(t_0)\in A_0,\ \ldots,B(t_n)\in A_n)?$$

The answer"yes" is given by a Kolmogorov theorem. Let us first introduce definitions and a necessary condition.

Definition 1.4. *Let $\mathcal{T}$ be the set of finite increasing sequences $\underline{t} = (t_0, \ldots, t_n)$ of numbers, where the length $n+1$ of these sequences ranges over the set of positive integers. Suppose that for each $\underline{t}$ of length $n+1$, we have a probability measure $Q_{\underline{t}}$ on $\left(\mathbb{R}^{n+1}, \mathscr{B}\left(\mathbb{R}^{n+1}\right)\ \right)$. Then the collection $\{Q_t\}_{t\in\mathcal{T}}$ is called a family of finite-dimensional distributions. This family is said to be consistent provided that the following condition is satisfied: If $\underline{t} = (t_0, t_1, \ldots, t_n)$ with $n \geq 1, \underline{t}^i = (t_0, t_1, \ldots, t_n)$, where t_i is missing then $\forall i \leq n$*

$$Q_{\underline{t}^i}(A_0 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) = Q_{\underline{t}}(A_0 \times \cdots \times \mathbb{R} \times \cdots \times A_n).$$

If we have a probability measure μ on $\left(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R})^{\otimes[0,+\infty)}\right)$, then we can define a family of finite-dimensional distributions by

$$Q_{\underline{t}}(A) = \mu\left[\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_n)) \in A\right], \tag{1.3}$$

where $A \in \mathscr{B}(\mathbb{R}^n)$ and $t = (t_1, \ldots, t_n) \in T$. This family is easily seen to be consistent. We are interested in the converse of this fact, because it will enable us to construct a probability measure $\mathbb{P}$ from the finite-dimensional distributions of Brownian motion.

Theorem 1.1 ((Daniell (1918), Kolmogorov (1933))**.** *Let $\left\{Q_{\underline{t}}\right\}$ be a consistent family of finite-dimensional distributions. Then there is a probability measure $\mathbb{P}$ on $\left((\mathbb{R}^d)^{[0,\infty)}, \mathcal{B}(\mathbb{R}^d)^{\otimes[0,+\infty)}\right)$, such that (1.3) holds for every $t \in T$.*

Proof. A proof can be read in p 50 of Karatzas and Shreeve. One can use for instance Carathéodory theorem, that may also be used to construct Lebesgue measure. □

To verify that we can apply the Theorem to the construction of the Brownian motion, it is enough to show consistency in this case. Coming back to (1.2), we are left to check

$$\int_{\mathbb{R}^d} p_{t_i - t_{i-1}}(x_i - x_{i-1}) p_{t_{i+1}-t_i}(x_{i+1} - x_i) dx_i = p_{t_{i+1}-t_{i-1}}(x_{i+1} - x_{i-1}).$$

But it is the same as $p_{t_i - t_{i-1}} * p_{t_{i+1}-t_i} = p_{t_{i+1}-t_{i-1}}$, or $\mathcal{N}(0, t_i - t_{i-1}) + \mathcal{N}(0, t_{i+1} - t_i) = \mathcal{N}(0, t_{i+1} - t_{i-1})$ where independence is assumed on the left hand side.

2.2 Regularity of BM sample paths

In this section we want to convince ourselves that BM sample paths are almost surely continuous. We hope that $\mathcal{C}$ the set of continuous functions is of probability 1 under the distribution of BM.

Remark 1.3. *Unfortunately $\mathcal{C}$ is not measurable in $\mathscr{B}(\mathbb{R}^d)^{\otimes \mathbb{R}_+}$. Prove this fact as an exercise, that uses the previous Remark/exercise.*

To circumvent this problem we will show that there exists a process with almost sure continuous sample paths that has the same distribution as BM.

To clean a bit the situation some definitions are introduced.

Definition 1.5. *1. Processes X and Y have the same finite-dimensional distributions if, for any integer $n \geq 1$, real numbers $0 \leq t_1 < t_2 < \cdots < t_n < \infty$, and $A \in \mathscr{B}\left(\mathbb{R}^{nd}\right)$, we have:*

$$\mathbb{P}\left[(X_{t_1}, \ldots, X_{t_n}) \in A\right] = \mathbb{P}\left[(Y_{t_1}, \ldots, Y_{t_n}) \in A\right].$$

2. Y is a modification of X if, for every $t \geq 0$, we have $\mathbb{P}\left[X_t = Y_t\right] = 1$.

3. X and Y are called indistinguishable if almost all their sample paths agree:

$$\mathbb{P}\left[X_t = Y_t; \forall 0 \leq t < \infty\right] = 1.$$

Exercise 1.1. *If X is a modification of X' then the distribution of X and X' are the same.*

If X and X' are indistinguishable then there are modifications of each other. The converse is false.

Example. *Let $\Omega = [0,1]$, $\mathcal{A} = \mathscr{B}([0,1])$, $\mathbb{P} = dx$ the Lebesgue measure. Let us take $X_t(\omega) = \mathbf{1}(\omega \neq t)$, and $Y_t = 1$. Then*

$$\mathbb{P}(X_t = Y_t) = \mathbb{P}(\omega \neq t) = 1.$$

Hence X is a modification of Y. But $\mathbb{P}(\forall t \in [0,1],\ X_t = Y_t) = 0$. They are not indistinguishable.

Exercise 1.2. *Let Y be a modification of X, and suppose that both processes have a.s. right-continuous sample paths. Then X and Y are indistinguishable.*

The next theorem will show that there exits a modification of the BM with almost sure continuous sample paths.

Theorem 1.2 (Kolmogorov, Čentsov (1956))**.**

Suppose that a process $\{X_t; 0 \le t \le T\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ satisfies the condition

$$\mathbb{E}\,|X_t - X_s|^\alpha \le C|t-s|^{1+\beta}, \quad 0 \le s, t \le T,$$

for some positive constants α, β, and C. Then there exists a continuous modification $\tilde{X} = \left\{\tilde{X}_t; 0 \le t \le T\right\}$ of X, which is locally Hölder-continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$, i.e.,

$$\mathbb{P}\left[\omega; \sup_{\substack{0<t-s<h(\omega) \\ s,t\in[0,T]}} \frac{\left|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)\right|}{|t-s|^\gamma} \le \delta\right] = 1, \tag{1.4}$$

where $h(\omega)$ is an a.s. positive random variable and $\delta > 0$ is an appropriate constant.

Proof. For simplicity, we take $T = 1$. Much of what follows is a consequence of the Cebyšev inequality. First, for any $\varepsilon > 0$, we have

$$\mathbb{P}\left[|X_t - X_s| \ge \varepsilon\right] \le \frac{\mathbb{E}\,|X_t - X_s|^\alpha}{\varepsilon^\alpha} \le C\varepsilon^{-\alpha}|t-s|^{1+\beta},$$

and so $X_s \to X_t$ in probability as $s \to t$. Second, setting $t = k/2^n, s = (k-1)/2^n$, and $\varepsilon = 2^{-\gamma n}$ (where $0 < \gamma < \beta/\alpha$) in the preceding inequality, we obtain

$$\mathbb{P}\left[\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \ge 2^{-\gamma n}\right] \le C2^{-n(1+\beta-\alpha\gamma)},$$

and consequently,

$$\begin{aligned}
&\mathbb{P}\left[\max_{1\le k\le 2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \ge 2^{-\gamma n}\right] \\
&\quad = \mathbb{P}\left[\bigcup_{k=1}^{2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \ge 2^{-\gamma n}\right] \\
&\quad \le C2^{-n(\beta-\alpha\gamma)}.
\end{aligned}$$

The last expression is the general term of a convergent series; by the Borel-Cantelli lemma, there is a set $\Omega^* \in \mathscr{F}$ with $\mathbb{P}(\Omega^*) = 1$ such that for each $\omega \in \Omega^*$,

$$\max_{1 \le k \le 2^n} \left| X_{k/2^2}(\omega) - X_{(k-1)/2^n}(\omega) \right| < 2^{-\gamma n}, \quad \forall n \ge n^*(\omega), \tag{1.5}$$

where $n^*(\omega)$ is a positive, integer-valued random variable. For each integer $n \ge 1$, let us consider the partition $D_n = \{(k/2^n)\,; k = 0,\ 1, \dots, 2^n\}$ of $[0,1]$, and let $D = \bigcup_{n=1}^{\infty} D_n$ be the set of dyadic rationals in $[0,1]$. We shall fix $\omega \in \Omega^*, n \ge n^*(\omega)$, and show that for every $m > n$, we have

$$|X_t(\omega) - X_s(\omega)| \le 2 \sum_{j=n+1}^{m} 2^{-\gamma j}; \quad \forall t, s \in D_m, 0 < t - s < 2^{-n}. \tag{1.6}$$

For $m = n+1$, we can only have $t = (k/2^m)\,, s = ((k-1)/2^m)$, and (1.6) follows from (1.5). Suppose (1.6) is valid for $m = n+1, \dots, M-1$. Take $s < t, s, t \in D_M$, consider the numbers $t^1 = \max\{u \in D_{M-1}; u \le t\}$ and $s^1 = \min\{u \in D_{M-1}; u \ge s\}$, and notice the relationships $s \le s^1 \le t^1 \le t, s^1 - s \le 2^{-M}, t - t^1 \le 2^{-M}$. From (1.5) we have $|X_{s^1}(\omega) - X_s(\omega)| \le 2^{-\gamma M}, |\ X_t(\omega) - X_{t^1}(\omega)\ | \le 2^{-\gamma M}$, and from (1.6) with $m = M-1$,

$$|X_{t^1}(\omega) - X_{s^1}(\omega)| \le 2 \sum_{j=n+1}^{M-1} 2^{-\gamma j}.$$

We obtain (1.6) for $m = M$.

We can show now that $\{X_t(\omega); t \in D\}$ is uniformly continuous in t for every $\omega \in \Omega^*$. For any numbers $s, t \in D$ with $0 < t - s < h(\omega) \triangleq 2^{-n^*(\omega)}$, we select $n \ge n^*(\omega)$ such that $2^{-(n+1)} \le t - s < 2^{-n}$. We have from (1.6)

$$|X_t(\omega) - X_s(\omega)| \le 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} \le \delta |t-s|^{\gamma}, \quad 0 < t - s < h(\omega), \tag{1.7}$$

where $\delta = 2/\left(1 - 2^{-\gamma}\right)$. This proves the desired uniform continuity. We define $\tilde{X}$ as follows. For $\omega \notin \Omega^*$, set $\tilde{X}_t(\omega) = 0, 0 \le t \le 1$. For $\omega \in \Omega^*$ and $t \in D$, set $\tilde{X}_t(\omega) = X_t(\omega)$. For $\omega \in \Omega^*$ and $t \in [0,1] \cap D^c$, choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$; uniform continuity and the Cauchy criterion imply that $\{X_{s_n}(\omega)\}_{n=1}^{\infty}$ has a limit which depends on t but not on the particular sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ chosen to converge to t, and we set

$\widetilde{X}_t(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$. The resulting process $\widetilde{X}$ is thereby continuous; indeed, $\widetilde{X}$ satisfies (1.7), so (1.4) is established.

To see that $\tilde{X}$ is a modification of X, observe that $\tilde{X}_t = X_t$ a.s. for $t \in D$; for $t \in [0,1] \cap D^c$ and $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, we have $X_{s_n} \to X_t$ in probability and $X_{s_n} \to \tilde{X}_t$ a.s., so $\widetilde{X}_t = X_t$ a.s. □

This long proof is an example of the "chaining" argument, which is used in many other proofs.

To show that BM has a continuous modification we still have to show that the bound on the expectations of the increments of BM that is an assumption of the Theorem is satisfied

Proposition 1.1. *If B is a real valued BM $d = 1$,*

$$\forall n \in \mathbb{N}, \forall t \ \in [0, +\infty), \ \mathbb{E}(B_t - B_s)^{2n} = C_n |t - s|^n \tag{1.8}$$

There exists a modification of the BM with locally Hölder continuous paths for every exponent $0 < \gamma < \frac{1}{2}$.

Proof.

$$B_t - B_s \overset{(d)}{=} \mathcal{N}(0, |t - s|) \overset{(d)}{=} \sqrt{|t - s|} \mathcal{N}(0, 1)$$

Hence $\mathbb{E}(B_t - B_s)^2 = |t - s|$ and $\mathbb{E}(B_t - B_s)^{2n} = C_n |t - s|^n$ where $C_n = \mathbb{E}X^{2n}$, $X \overset{(d)}{=} \mathcal{N}(0, 1)$. Applying (1.8) for n fixed we get Hölder continuity for $\gamma < \frac{n-1}{2n}$. □

Remark 1.4. *Further on we always take continuous modifications of BM.*

3 Quadratic variations of Brownian motions

We may wonder if the previous result is optimal. For instance could it be that sample paths of Brownian motions are locally Liptschitz continuous ? Actually elementary definitions of integrals of the type $\int H_s dB_s$ are possible if the sample paths have almost surely finite variations. Let us first recall some facts concerning functions with finite variations.

3.1 Reminder of functions with finite variations

If $f : (0, +\infty) \mapsto \mathbb{R}$ is a non decreasing function, right continuous, we can associate a measure μ on $(0, +\infty)$ with $\forall 0 < s < t$

$$\mu((s,t]) = f(t) - f(s)$$

and if g is a bounded Borel function one can define

$$\int_0^t g(s) df(s) \stackrel{def}{=} \int \mathbf{1}_{[0,t]}(s) g(s) d\mu(s).$$

If f is C^1 it is equal to $\int_0^t g(s) f'(s) ds$.

Definition 1.6. *For $t > 0$, Let $\mathcal{P}_t$ be the set of finite subdivisions Δ of $[0,t]$: $\Delta = (t_i)_{i=1,\dots,n} \in \mathcal{P}_t$, if $0 \leq t_1 \leq \dots t_n \leq t$. The mesh of Δ is denoted by $|\Delta| = \sup_{i=1,\dots,n-1}(t_{i+1} - t_i)$. For $f : (0, +\infty) \mapsto \mathbb{R}$, the variation of f on $[0,t]$ is denoted by*

$$V_t(f) \stackrel{def}{=} \sup_{\Delta \in \mathcal{P}_t} \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)|$$

and is said to have finite variations if $\forall t > 0$, $V_t(f) < \infty$.

Example.
- *If f is monotone or a difference of non decreasing functions, f has finite variations.*
- *If f is locally Lipschitz, f has finite variations.*

We will admit two facts for Riemmann-Stieljes integral

- Every function f with finite variations is a difference of non decreasing functions f_1, f_2, one can write

$$\int_0^t g(s) df(s) \stackrel{def}{=} \int_0^t g(s) df_1(s) - \int_0^t g(s) df_2(s).$$

- If g is continuous and $(\Delta^n)_{n \in \mathbb{N}}$ a sequence of subdivisions with meshes $|\Delta^n| \to 0$,

$$\int_0^t g(s) df(s) = \lim_{n \to \infty} \sum_{t_i^n \in \Delta^n} g(t_i^n)(f(t_{i+1}^n) - f(t_i^n)).$$

3.2 Quadratic variations

Actually quadratic variations of BM sample paths are almost surely : positive finite and BM sample paths do not have finite variations.

Definition 1.7. *A real valued process X has finite quadratic variations denoted by $\langle X, X\rangle$ or $(\langle X\rangle)$ if $\forall(\Delta^n)_{n\in\mathbb{N}}$ sequence of subdivisions of $\mathcal{P}_t$ such that $|\Delta^n|\to 0$ and $\Delta^n=(t_i^n)_{1\le i\le N(n)}$ with $t_1^n=0$ and $t_{N(n)}^n=t$ and*

$$T_{[0,t]}^{\Delta^n}\overset{def}{=}\sum_{t_i^n\in\Delta^n}(X_{t_{i+1}^n}-X_{t_i^n})^2\overset{(P)}{\to}\langle X\rangle_t.$$

Proposition 1.2. *If B is a Brownian motion $\langle B\rangle_t=t$ a.s.*

Proof. We prove $T_{[0,t]}^{\Delta^n}\to t$ in $L^2(\Omega)$. If $\Delta^n=(t_i^n)_{1\le i\le N(n)}$

$B_{t_{i+1}^n}-B_{t_i^n}\overset{(d)}{=}\mathcal{N}(0,t_{i+1}^n-t_i^n)\overset{(d)}{=}\sqrt{t_{i+1}^n-t_i^n}\mathcal{N}(0,1)$. Then $\mathbb{E}((T_{[0,t]}^{\Delta^n})=\sum_{t_i^n\in\Delta^n}\mathbb{E}(B_{t_{i+1}^n}-B_{t_i^n})^2=\sum_{t_i^n\in\Delta^n}t_{i+1}^n-t_i^n=t$. Hence

$$\begin{aligned}
\mathbb{E}((T_{[0,t]}^{\Delta^n}-t)^2)&=Var(T_{[0,t]}^{\Delta^n}-t)\\
&=Var(\sum_{t_i^n\in\Delta^n}(B_{t_{i+1}^n}-B_{t_i^n})^2-(t_{i+1}^n-t_i^n))\\
&=\sum_{t_i^n\in\Delta^n}Var((B_{t_{i+1}^n}-B_{t_i^n})^2-(t_{i+1}^n-t_i^n))\\
&=\sum_{t_i^n\in\Delta^n}(t_{i+1}^n-t_i^n)^2Var(N^2-1)\\
&\le C|\Delta^n|\sum_{t_i^n\in\Delta^n}(t_{i+1}^n-t_i^n)\to 0,
\end{aligned}$$

where $N=\mathcal{N}(0,1)$. This implies convergence in probability. □

To get almost sure convergence some additional assumptions are needed for Δ^n...

Proposition 1.3. *Almost surely the sample paths of Brownian motion have infinite variations on every intervals $[0,t]$ for $t>0$.*

Proof. If ω is such that $V_{[0,t]}(B(\omega))<+\infty$ then

$$\sum_{t_i^n\in\Delta^n}(B_{t_{i+1}^n}-B_{t_i^n})^2\le\sup|(B_{t_{i+1}^n}-B_{t_i^n}|\sum_{t_i^n\in\Delta^n}|B_{t_{i+1}^n}-B_{t_i^n}|.$$

Since $s \mapsto B_s(\omega)$ is uniformly continuous on $[0,t]$ $\sup |B_{t^n_{i+1}} - B_{t^n_i}| \to 0$. Hence the quadratic variations of the sample paths should vanish, which is true only on a negligible set. □

4 Brownian motion as a Gaussian process

4.1 Elementary properties

Definition 1.8. *A real valued process $(X_t, t \in T)$ is a Gaussian process if $\forall n \in \mathbb{N},\ t_1, \dots, t_n \in T,\ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n,\ \sum_{i=1}^n \alpha_i X_{t_i}$ is a Gaussian random variable. The process X is centered if $\forall t \in T,\ \mathbb{E}X_t = 0$ and $\Gamma(s,t) \stackrel{def}{=} Cov(X_s, X_t)$ is the covariance function.*

Remark 1.5.

- *If $\forall i = 1$ to n, $\alpha_i = 0$ $\sum_{i=1}^n \alpha_i X_{t_i} = 0$. It means that we consider $\mathcal{N}(0,0)$ as a generalized degenerated Gaussian random variable with variance 0. Gaussian processes are generalization of random Gaussian vectors (where T is a finite set). (See for instance N. Bouleau Processus stochastique et applications 1988.)*
- *If (X_1, X_2) is a Gaussian vector and $Cov(X_1, X_2) = 0$ then X_1 and X_2 are independent.*

Proposition 1.4. *The Brownian motion which starts from 0 is the unique centered Gaussian process with covariance $\Gamma(s,t) = \min(s,t)$.*

Proof. The proof relies on the fact that the covariance always characterizes the distribution of a Gaussian centered process. If X is a Gaussian centered process and $t_1, \dots, t_n \in T$, then $(X_{t_1}, \dots,\ X_{t_n})$ Gaussian vector implies the characteristic function

$$\mathbb{E}\exp(i \sum_{i=1}^n \alpha_i X_{t_i}) = \exp(-\frac{1}{2}\langle C\alpha, \alpha\rangle)$$

where $\alpha = (\alpha_1, \dots,\ \alpha_n)$, $\langle ., . \rangle$ is The Euclidean scalar product in $\mathbb{R}^n$, and $C_{i,j} = \mathbb{E}(X_{t_i} X_{t_j}) = \Gamma(t_i, t_j)$. Hence the matrix $(\Gamma(t_i, t_j))$ characterizes the distribution of the finite dimensional margins $(X_{t_1}, \dots,\ X_{t_n})$ $\forall t_1, \dots, t_n \in T$, and henceforth the distribution of the process X.

Let us compute $\Gamma(s,t)$ for Brownian motion. Let $s \le t$

$$\Gamma(s,t) = \mathbb{E}(B_s B_t) = \mathbb{E}(B_s(B_s + B_t - B_s)) = \mathbb{E}(B_s^2) + \mathbb{E}(B_s)\mathbb{E}(B_t - B_s) = s.$$

Hence $\Gamma(s,t) = \min(s,t)$.

□

Proposition 1.5. *Let $(B_t,\ t \geq 0)$ be a Brownian motion.*

1. *$(B_{t+s} - B_s, \quad t \geq 0)$ is a BM independent of $\mathcal{F}_s^B = \sigma(B_u, \quad u \leq s)$.*
2. *$(-B_t, \quad t \geq 0)$ is a Brownian motion,*
3. *(Self-similarity) for all $\lambda > 0$, $\left(B_t^{(\lambda)}, \quad t \geq 0\right)$ where $B_t^{(\lambda)} := \frac{1}{\sqrt{\lambda}} B_{\lambda t}, \quad t \geq 0$ is a Brownian motion.*

Proof. $W_t = B_{t+s} - B_s$ is a centered Gaussian process such that $W_0 = 0$ a.s. Its covariance

$$\mathbb{E}(W_t W_{t'}) = \mathbb{E}((B_{t+s} - B_s)(B_{t'+s} - B_s)) = \min(t+s, t'+s) - s = \min(t, t').$$

$\forall u_1 \leq \ldots \leq u_n \leq s \leq t_1 \leq \ldots \leq t_n$ $(B_{u_1}, \ldots,\ B_{u_n},\ B_{t_1+s} - B_s, \ldots,\ B_{t_n+s} - B_s)$ is a Gaussian vector and $(B_{u_1}, \ldots,\ B_{u_n})$ is independent of $(B_{t_1+s} - B_s, \ldots,\ B_{t_n+s} - B_s)$ since $\forall i,\ j$

$$\mathbb{E}(B_{u_i}(B_{t_j+s} - B_s)) = 0.$$

Actually for Gaussian random vectors, a vanishing covariance yields independence. Then independence of sigma field is a consequence of independence of the random variable that generate them. To prove other parts of the Proposition, compute covariances. □

4.2 Brownian bridge

We may condition $(B_t,\ 0 \leq t \leq 1)$ to the event $B_1 = 0$. In this case we obtain a Brownian bridge.

Definition 1.9. *The process $X_t^0 = B_t - tB_1$ is called a Brownian bridge.*

Proposition 1.6. *X^0 is independent of B_1.*

Proof. $\forall 0 \leq t \leq 1,\ \mathbb{E}(X_t^0 - tB_1) = 0$. And it is a centered Gaussian process. Furthermore $\forall 0 \leq t \leq 1,\ \mathbb{E}(X_t^0 B_1) = \mathbb{E}((B_t - tB_1)B_1) = 0$. □

Proposition 1.7. *Let $X_t^b = B_t - tB_1 + tb$. The distribution of X^b is a regular version of the conditional distribution of $(B_t,\ t \leq 1)$ given $B_1 = b$.*

Proof. Reminder : $\mathbb{P}(dx, y)$ is a regular version of the conditional distribution of X given $Y = y$ if and only if (in short iff) $\forall\varphi$ bounded and measurable

$$\mathbb{E}(\varphi(X)|Y) = \int \varphi(x)\mathbb{P}(dx, Y)$$

$\mathbb{P}$ almost surely. It can be characterized by $\forall\varphi,\ g$ bounded and measurable

$$\mathbb{E}(\varphi(X)g(Y)) = \int (\int \varphi(x)\mathbb{P}(dx, y))g(y)d\mathbb{P}_Y(y).$$

In our case we have to show

$$\mathbb{E}(\varphi(B_s,\ s \le 1)g(B_1)) = \int \mathbb{E}(\varphi(X_s^b,\ s \le 1)g(b)\frac{e^{-\frac{b^2}{2}}db}{\sqrt{2\pi}}. \tag{1.9}$$

Let $\psi((f(s),\ s \le 1), b) = \varphi((f(s) + sb, s \le 1))$. Since $B_s = X_s^0 + sB_1$,

$$\begin{aligned}\mathbb{E}(\varphi(B_s,\ s \le 1)g(B_1)) &= \mathbb{E}(\psi((X_s^0,\ s \le 1), B_1)g(B_1)) \\ &= \int \mathbb{E}(\psi((X_s^0,\ s \le 1), b)g(b))\frac{e^{-\frac{b^2}{2}}db}{\sqrt{2\pi}} \\ &= \int \mathbb{E}(\varphi((X_s^b,\ s \le 1))g(b))\frac{e^{-\frac{b^2}{2}}db}{\sqrt{2\pi}}.\end{aligned}$$

□

4.3 Wiener integral

In this part the integral $\int f(s)dB_s$ is defined for a deterministic function using the Gaussianity of the Brownian motion. (Later the stochastic integral is defined for f a stochastic process.) If f is a simple function

$$f = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{(a_i, b_i]}$$

for $a_1 < b_1 \le a_2 < b_2 \le \dots$ Then

$$\int f(s)dB_s \stackrel{def}{=} \sum_{i=1}^{n} \alpha_i(B(b_i) - B(a_i)), \tag{1.10}$$

this random variable is denoted by $I(f)$. It is a centered Gaussian random variable with variance

$$\mathbb{E}(I(f)^2) = \sum_{i=1}^{n} \alpha_i^2(b_i - a_i) = \|f\|^2_{L^2(0,\infty)}.$$

Theorem 1.3. *There exists a unique map I from $L^2(0,\infty)$ to $\mathcal{H}$ the set that is the closure in $L^2(\Omega,\mathcal{A},\mathbb{P})$ of the linear combinations $\sum_{i=1}^n \alpha_i(B(b_i)-B(a_i))$ such that*

1. $I(\mathbf{1}_{(a_i,b_i]}) = B(b_i) - B(a_i)$.
2. *I is a linear map*
3. *I is an isometry i.e.* $\forall f \in L^2(0,\infty)$ $\|f\|_{L^2(0,\infty)} = \|I(f)\|_{L^2(\Omega,\mathcal{A},\mathbb{P})}$.

Proof. If $f \in L^2(0,\infty)$ $\exists (f_n)_{n\in\mathbb{N}}$ with f_n simple functions and $\lim_{n\to\infty} f_n = f$ in $L^2(0,\infty)$. Then let $I(f) \overset{def}{=} \lim_{n\to\infty} I(f_n)$ in $L^2(0,\infty)$. Please remark that $I(f)$ does not depend on the sequence $(f_n)_{n\in\mathbb{N}}$ since if $\lim_{n\to\infty} g_n = f$ then $\lim_{n\to\infty} I(g_n) - I(f_n) = 0$ because of the isometry property. Moreover we get the uniqueness of I by density of simple functions in L^2. □

Remark 1.6.

1. *Since I is an isometry $\forall f,\ g \in L^2(0,\infty)$,*

$$\langle f, g\rangle_{L^2(0,\infty)} = \langle I(f), I(g)\rangle_{L^2(\Omega,\mathcal{A},\mathbb{P})}. \tag{1.11}$$

   If we denote by $\int_0^\infty f(s)dB_s = I(f)$, the so-called Wiener integral this can be rewritten :

$$\int_0^\infty f(s)g(s)ds = \mathbb{E}(\int_0^\infty f(s)dB_s \int_0^\infty g(s)dB_s).$$

2. *Conversely if $J : L^2(0,\infty) \mapsto L^2(\Omega,\mathcal{A},\mathbb{P})$ is such that $J(f)$ is a centered Gaussian random variable and*

$$\int_0^\infty f(s)g(s)ds = \mathbb{E}(J(f)J(g))$$

   then J is a linear map. Moreover $J(\mathbf{1}_{(0,t]})$ is a real Brownian motion. (Since $(J(\mathbf{1}_{(0,t]}),\ t \le 0)$ is a centered Gaussian process and $\mathbb{E}(J(\mathbf{1}_{(0,s]})J(\mathbf{1}_{(0,t]}) = \min(s,t)$.) If we denote by $B_t = J(\mathbf{1}_{(0,t]})$, then J is the isometry I associated to the BM B.

The same construction can be generalized to all intervals I' and $L^2(I')$. For $I' = \mathbb{R}$, the process $X_t = I(\mathbf{1}_{(0,t]}), \forall t \in \mathbb{R}$ can be obtained from two independent real valued Brownian motion $X_t = B_t^1, \forall t \le 0$, and $X_t = B_t^2, \forall t \ge 0$. One can easily check that

$$\mathbb{E}(X_t - X_s)^2 = |t-s|,\ \forall t < 0 < s.$$

4.4 Second construction of Brownian motion

If $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis (ONB) of $L^2(0,1)$ then $(I(e_n))_{n\in\mathbb{N}}$ is a sequence of Gaussian independent random variables with distribution $\mathcal{N}(0,1)$. Actually $(1,\ \sqrt{2}\cos(2\pi ks),\ \sqrt{2}\sin(2\pi ks))_{k\in\mathbb{N}^*}$ is an ONB of $L^2(0,1)$, $\forall t \in (0,1)$,

$$\mathbf{1}_{(0,t]}(s) \overset{L^2(0,1)}{=} a_0(t) + \sum_{k=1}^{\infty} a_k(t)\sqrt{2}\cos(2\pi ks) + b_k(t)\sqrt{2}\sin(2\pi ks)$$

$$I(\mathbf{1}_{(0,t]}) \overset{L^2(\Omega,\mathcal{A},\mathbb{P})}{=} a_0(t)\xi_0 + \sum_{k=1}^{\infty} a_k(t)\xi_k + \sum_{k=1}^{\infty} b_k(t)\eta_k,$$

with $(\xi_0,\ \xi_k,\ \eta_k)$ i.i.d. Gaussian random variables with distribution $\mathcal{N}(0,1)$. Moreover $a_0(t) = \int_0^1 \mathbf{1}_{(0,t]}(s)ds = t$, $\forall k \geq 1$

$$a_k(t) = \sqrt{2}\int_0^t \cos(2\pi ks)ds = \frac{\sin(2\pi kt)}{\sqrt{2}\pi k}$$

$$b_k(t) = \sqrt{2}\int_0^t \sin(2\pi ks)ds = \frac{(1-\cos(2\pi kt))}{\sqrt{2}\pi k}.$$

Hence we get a series expansion of BM, a priori in L^2 sense...

$$I(\mathbf{1}_{(0,t]}) = t\xi_0 + \sum_{k=1}^{\infty} \xi_k \frac{\sin(2\pi ks)}{\sqrt{2}\pi k} + \sum_{k=1}^{\infty} \eta_k \frac{(1-\cos(2\pi ks))}{\sqrt{2}\pi k}. \tag{1.12}$$

Since $I(\mathbf{1}_{(0,1]}) = \xi_0$, hence (1.12) can be viewed as tB_1 plus the expansion of a Brownian bridge.

Theorem 1.4. *If $(\xi_0,\ \xi_k,\ \eta_k)$ are i.i.d. Gaussian random variables with distribution $\mathcal{N}(0,1)$*

$$t\xi_0 + \sum_{k=1}^{\infty} \xi_k \frac{\sin(2\pi kt)}{\sqrt{2}\pi k} + \sum_{k=1}^{\infty} \eta_k \frac{(1-\cos(2\pi kt))}{\sqrt{2}\pi k}$$

almost surely converges to a process $(B_t)_{t\in(0,1)}$ with the distribution of a BM.

Proof. We may refer to criteria for convergences of random Fourier series in Kahane Some random series of functions Theorem 2 p 236 second edition, we get almost surely the uniform (but not normal) convergence of the series. Since $I(\mathbf{1}_{(0,t]})$ is a BM we get the distribution of the limit of the series. □

Remark 1.7. *With this construction almost sure continuity of the sample paths is for free !*

Chapter 2

Reminder for martingales indexed by $\mathbb{N}$

To integrate processes $H_s(\omega)$ against BM "$dB_s(\omega)$" we will assume that $H : [0, +\infty) \mapsto \mathbb{R}$, depends is "previsible". Roughly it means that $H(t, .)$ is measurable with respect to the sigma-field $\sigma(X_s,\ s < t)$ of the past. Then the time dependence of $t \mapsto \int_0^t H_s dB_s$, will be achieved so that $\int_0^t H_s dB_s$ is a martingale. First we recall results for martingales indexed by $\mathbb{N}$ especially convergence results. Then we will extend these results to martingales indexed by $[0, +\infty)$. The main issue in this case is that $[0, +\infty)$ is not denumerable.

1 Definitions and first examples

Definition 2.1. *A filtration on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a non-decreasing sequence of sub-*σ*-fields of* $\mathcal{F}$*:*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}.$$

One says that $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ *is a filtered probability space.*

Example 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0, 1[, \mathcal{B}([0, 1[), \lambda)$, where λ is Lebesgue measure. The filtration $(\mathcal{F}_n)_{n\geq 0}$ defined by

$$\mathcal{F}_n = \sigma\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right[, i = 0, \ldots, 2^n - 1\right), \quad n \geq 0$$

is called the dyadic filtration.

If the parameter n denotes time, then $\mathcal{F}_n$ is interpreted as available information up to time n.

Example 2.2. For a stochastic process $(X_n)_{n\geq 0}$, we define its natural filtration $\mathcal{F}^X = (\mathcal{F}_n^X)_{n\geq 0}$ by: for all $n \geq 0$,

$$\mathcal{F}_n^X = \sigma(X_0, X_1, \cdots, X_n),$$

which is the smallest σ-field such that $X_0, \ldots, X_n$ are measurable.

Definition 2.2. *We say that a stochastic process $X = (X_n)_{n\geq 0}$ is adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$, if for all $n \geq 0$, X_n is $\mathcal{F}_n$-measurable. We say that a stochastic process $(X_n)_{n\geq 0}$ is adapted if it is adapted to some filtration.*

A stochastic process is obviously adapted to its natural filtration.

Remark 2.1. If $(\mathcal{F}_n)_{n\geq 0}$ and $(\mathcal{G}_n)_{n\geq 0}$ are two filtrations such that $\mathcal{G}_n \subset \mathcal{F}_n$ for all $n \geq 0$, and if $(X_n)_{n\geq 0}$ is adapted to $(\mathcal{G}_n)_{n\geq 0}$, then $(X_n)_{n\geq 0}$ is adapted to $(\mathcal{F}_n)_{n\geq 0}$.

Definition 2.3. *Let $X = (X_n)_n$ be an adapted process on filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n, n \in \mathbb{N}), \mathbb{P})$ such that for all n, X_n is integrable.*

The process X is a **martingale** *if for all n,*

$$\mathbb{E}[X_{n+1}/\mathcal{F}_n] = X_n, \quad \textit{almost surely.}$$

The process $X$is a **sub-martingale** *if for all integer n n,*

$$\mathbb{E}[X_{n+1}/\mathcal{F}_n] \geq X_n, \quad \textit{almost surely.}$$

The process X is a **super martingale** *if for all integer n,*

$$\mathbb{E}[X_{n+1}/\mathcal{F}_n] \leq X_n, \quad \textit{almost surely.}$$

Examples

(See exercises at the end of the chapter for some proofs of the following properties are left to the reader.)

1. If $X \in L^1(\Omega, \mathcal{A})$, $X_n = \mathbb{E}[X/\mathcal{F}_n]$ is a martingale.This process is also uniformly integrable.

2. (**Fundamental example.**) Let $(Z_n, n \in \mathbb{N}^*)$ be a sequence of independent and integrable random variables and X_0 be an integrable random variable independent of the sequence (Z_n). (Most of the time, X_0 is constant.) Let $X_n := X_0 + \sum_{i=1}^{n} Z_i$. Then the filtrations $\mathcal{F}_n^X$ and $\mathcal{F}_n = \sigma(X_0, Z_1, \ldots, Z_n)$ are equal and for this filtration :
   (a) if for all integer n, $\mathbb{E}(Z_n) = 0$, X is a martingale;
   (b) if for all integer n, $\mathbb{E}(Z_n) \geq 0$, X is a sub martingale;
   (c) if for all integer n $\mathbb{E}(Z_n) \leq 0$, X is an super martingale;
   (d) if all r.v. Z_i have same expectation m, $X_n - nm$ is a martingale.
3. A special case of the example 2 comes from the game theory. In this case the distribution of the r.v. Z_n is the Bernoulli distribution with parameter p : $\mathbb{P}(Z_i = 1) = p$, $\mathbb{P}(Z_i = -1) = 1 - p$. with values $+1$ et -1. In this case X_n is the fortune of the player after n bets, when its initial fortune is X_0. The process $(M_n)_n$ where $M_n = X_n - n(2p - 1)$ is a martingale for its natural filtration $\mathcal{F}^X$.
4. In the example 2, if we assume that $\mathbb{E}[\exp(aZ_n)] := \exp(r_n)$ exists and is finite, let $R_n = r_1 + \cdots + r_n$. (Here $R_0 = 0$.)

   Then $M_n = \exp(aX_n - R_n)$ is a martingale for the natural filtration $\mathcal{F}^X$.

A process X can be a martingale (resp. super, resp sub) with respect to several filtrations.

Proposition 2.1. *If X is a martingale (resp. a super-martingale, a sub-martingale) with respect to a filtration $(\mathcal{F}_n)$ and the process X is adapted to an other filtration $(\mathcal{G}_n)$ smaller than$(\mathcal{F}_n)$ (that means for all n, $\mathcal{G}_n \subset \mathcal{F}_n$), Then X is a martingale (resp. a super-martingale, a sub-martingale)with respect to the filtration $\mathcal{G}_n$. A martingale (resp. a super-martingale, a sub–martingale) is a martingale (resp. a super-martingale, a sub–martingale)with respect to its natural filtration .*

Proof. Use successive conditioning. . □

We can also increase filtrations by adding to each σ fields $\mathcal{F}_n$ an independent σ field:

Proposition 2.2. *Let (X_n) be a martingale (resp. a sub-martingale, an super-martingale), with respect to a filtration $\mathcal{F}_n$. Let $\mathcal{B}$ be a σ field independent of $\mathcal{F}_\infty$, and let $\mathcal{G}_n = \mathcal{F}_n \vee \mathcal{B}$. Then (X_n) is a martingale (resp. a sub-martingale, an super-martingale) with respect to the filtration $\mathcal{G}_n$.*

Proof. Left to the reader. □

Notation 2.1. *In the sequel*

$$(\Delta X)_n := X_n - X_{n-1} \tag{2.1}$$

is the increments process of (X_n).

Proposition 2.3. *Let X be a $\mathcal{F}$-martingale. Then*

1. $\forall n \geq 0$, $\forall k \geq 0$, $\mathbb{E}[X_{n+k}/\mathcal{F}_n] = X_n$; $\mathbb{E}[X_n] = \mathbb{E}[X_0]$.
2. *If the martingale is square integrable the increments $(\Delta X)_n$ of X are orthogonal :*
$$n \neq m \implies \mathbb{E}[(\Delta X)_n(\Delta X)_m] = 0.$$
3. *If X is a super-martingale, $-X$ is a sub-martingale.*
4. *The set of martingales with respect to a given filtration is a linear space.*
5. *If X is a martingale and ϕ is a convex application such that $Y_n = \phi(X_n)$ is integrable then , Y_n is a sub-martingale.*
6. *If X is a sub-martingale, and if ϕ is increasing and convex, $\phi(X)$ is a sub-martingale if $\phi(X_n)$ is integrable.*

Proof. The proof is left to the reader.

The point 1 relies on successive conditioning and induction.

The point 2 is obtained by conditioning by $\mathcal{F}_{m-1}$ for $n < m$.

The points 3 et 4 are immediate.

The points 5 et 6 rely on Jensen conditionnal inequality.

□

For square integrable martingale, we have

Proposition 2.4. *If M_n is a square integrable martingale*

$$\forall n \leq p,\ \mathbb{E}[(M_p - M_n)^2] = \sum_{k=n+1}^{p} \mathbb{E}[(\Delta M)_k^2].$$

Proof. Apply the property of orthogonal increments

2 of Proposition 2.3. □

Corollary 2.1. *A martingale bounded in L^2 converges in L^2.*

Proof. By definition, since the martingale is bounded in L^2 there exists a constant C such that for all n,

$$\mathbb{E}(X_n^2) \leq C^2.$$

Then,

$$\mathbb{E}(X_n - X_0)^2 \leq 4C^2,$$

and Proposition 2.4 allows to prove that the series

$$\sum_k \mathbb{E}[(\Delta M)_k^2]$$

converges. As a consequence,

$$\lim_{n\to\infty} \sup_{p\geq q\geq n} \sum_q^p \mathbb{E}[(\Delta M)_k^2] = 0.$$

Using the previous proposition again

$$\lim_n \sup_{p,q\geq n} \mathbb{E}(M_p - M_q)^2 = 0,$$

and the sequence is Cauchy in L^2 and converges.

□

2 Doob's decomposition

Definition 2.4. *Let $(A_n)_{n\geq 0}$ be a process indexed by $\mathbb{N}$, A is predictable with respect to the sigma field $\mathcal{F}_n$ if $\forall n$ A_n is $\mathcal{F}_{n-1}$measurable.*

Theorem 2.1. D DOOB 'S DECOMPOSITION *: Let X be a sub-martingale ; there exists a martingale M and a predictable increasing process A, null at 0, unique, such that for all integer n, $X_n = M_n + A_n$.*

The process A is called "compensator" of X.

Proof. Let $A_0 = 0$ and $M_0 = X_0$. For $n \geq 1$, define A_n in the following way : let $\Delta_n = \mathbb{E}(X_n/\mathcal{F}_{n-1}) - X_{n-1}$, and

$$A_n = \Delta_1 + \cdots + \Delta_n.$$

Moreover$M_n = X_n - A_n$. By construction A_n is predictable, and since X_n is a sub-martingale, $\Delta_n \geq 0$, and A_n is increasing. Moreover,

$$\mathbb{E}(M_{n+1}/\mathcal{F}_n) = \mathbb{E}(X_{n+1}/\mathcal{F}_n) - A_{n+1} = X_n + \Delta_n - A_{n+1} = M_n.$$

and M_n is a martingale.

Uniqueness comes from the fact that if such a decomposition exists then

$$\mathbb{E}(X_{n+1} - X_n/\mathcal{F}_n) = A_{n+1} - A_n,$$

This characterize A_n if $A_0 = 0$. □

In the particular case of square integrable martingale we obtain the following.

Proposition 2.5. *Let M_n be a square integrable martingale. Recall (notation 2.1) and $(\Delta M)_n = M_n - M_{n-1}$ and let*

$$U_n = \mathbb{E}[(\Delta M)_n^2/\mathcal{F}_{n-1}).$$

Then $M_n^2 - \sum_{k=1}^n U_k$ is a martingale.

Proof. It is the Doob's decomposition applying to the sub-martingale M_n^2, since

$$\mathbb{E}[(\Delta M)_n^2/\mathcal{F}_{n-1}) = \mathbb{E}(M_n^2/\mathcal{F}_{n-1}] - M_{n-1}^2.$$

□

3 Stopping times

3.1 Definition

Definition 2.5. *A random variable* $T\colon \Omega \to \mathbb{N} \cup \{+\infty\}$ *is called a stopping time (with respect to the filtration* $(\mathcal{F}_n)_{n\geq 0}$*) if for all* $n \geq 0$*,*

$$\{T \leq n\} \in \mathcal{F}_n.$$

Remark 2.2. Since $\{T = n\} = \{T \leq n\} \setminus \{T \leq n-1\}$, T is a stopping time if and only if for all $n \geq 0$,

$$\{T = n\} \in \mathcal{F}_n.$$

Remark 2.3. A stopping time is thus a random time, which can be interpreted as a stopping rule for deciding whether to continue or stop a process on the basis of the present information and past events, for instance playing until you go broke or you break the bank, etc...

Example 2.3. 1. If $T = n$ a.s., then clearly T is a stopping time.

2. Let $(X_n)_{n\geq 0}$ be an adapted stochastic process, and consider the first time X_n reaches the borel set A:

$$T_A = \inf\{n \geq 0 \,|\, X_n \in A\},$$

with the convention that $\inf \varnothing = +\infty$. It is called the hitting time of A. Then T_A is a stopping time. Indeed,

$$\begin{aligned}\{T_A = n\} &= \{X_0 \notin A, X_1 \notin A, \ldots, X_{n-1} \notin A, X_n \in A\} \\ &= \bigcap_{k=0}^{n-1} \{X_k \notin A\} \cap \{X_n \in A\} \in \mathcal{F}_n.\end{aligned}$$

3. Show that $\tau_A = \sup\{n \geq 1 \,|\, X_n \in A\}$ the last passage time in A is not a stopping time in general.

Recall the notations: $x \wedge y = \inf(x, y)$ and $x \vee y = \max(x, y)$.

Proposition 2.6. *If* S *and* T *are two stopping times, then* $S \wedge T$*,* $S \vee T$ *and* $S + T$ *are also stopping times.*

Proof. Writing
$$\{S \wedge T \leq n\} = \{S \leq n\} \cup \{T \leq n\}$$
and
$$\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\}$$
gives the result for $S \wedge T$ and $S \vee T$. For $S + T$, we write:
$$\{S + T \leq n\} = \bigcup_{k \leq n} \{S = k\} \cap \{T \leq n - k\} \in \mathcal{F}_n,$$
since $\mathcal{F}_k \subset \mathcal{F}_n$ for all $k \leq n$. □

Remark 2.4. In particular, if T is a stopping time, then for all $n \geq 0$, $T \wedge n$ is a bounded stopping time.

Proposition 2.7. *If $(T_k)_k$ is a sequence of stopping times, then $\inf_k T_k$, $\sup_k T_k$, $\liminf_k T_k$ and $\limsup_k T_k$ are also stopping times.*

Proof. Exercise. □

Proposition 2.8. *Let T be a stopping time. Then,*
$$\mathcal{F}_T = \{A \in \mathcal{F} \,|\, \forall n \geq 0, A \cap \{T = n\} \in \mathcal{F}_n\}$$
is a σ-field, called the σ-field of T-past.

Remark 2.5. Obviously, T is $\mathcal{F}_T$-measurable.

Proof. It is obvious that $\Omega \in \mathcal{F}_T$. If $A \in \mathcal{F}_T$, then for all n,
$$A^c \cap \{T = n\} = \{T = n\} \setminus A = \{T = n\} \setminus (A \cap \{T = n\}) \in \mathcal{F}_n,$$
hence $A^c \in \mathcal{F}_T$. If $(A_k)_k$ is countable collection of $\mathcal{F}_T$-mesurable set, then
$$\left(\bigcup_k A_k\right) \cap \{T = n\} = \bigcup_k (A_k \cap \{T = n\}) \in \mathcal{F}_n,$$
hence $\bigcup_k A_k \in \mathcal{F}_T$. □

Proposition 2.9. *Let S and T be two stopping times such that $S \leq T$. Then, $\mathcal{F}_S \subset \mathcal{F}_T$.*

Proof. Let $A \in \mathcal{F}_S$. Then, for all $n \geq 0$,

$$A \cap \{T = n\} = A \cap \{S \leq n\} \cap \{T = n\} = \bigcup_{k=0}^{n} A \cap \{S = k\} \cap \{T = n\} \in \mathcal{F}_n. \quad \square$$

Definition 2.6. *Let $(X_n)_{n \geq 0}$ be an adapted stochastic process and T a stopping time. If $T < \infty$ a.s., we define the random variable X_T by*

$$X_T(\omega) = X_{T(\omega)}(\omega) = X_n(\omega) \quad \text{if } T(\omega) = n.$$

Note that X_T is $\mathcal{F}_T$-measurable, since

$$\{X_T \in B\} \cap \{T = n\} = \{X_n \in B\} \cap \{T = n\} \in \mathcal{F}_n,$$

for any Borel set B.

4 Martingales transformations

Proposition 2.10. *Let (X_n) be an adapted process and (H_n) be a predictable process such that for all n, the r.v. $H_n(X_n - X_{n-1})$ is integrable. Let $(H.X)$ be the process defined by*

$$(H.X)_n = H_0 X_0 + \sum_{k=1}^{n} H_k (X_k - X_{k-1}).$$

Then, if X is a martingale, $(H.X)$ is a martingale. If X is a super- (resp. sub-) martingale, and if H is positive, then $(H.X)$ is a(n) super- (resp. sub-) martingale.

Proof. Using the notation 2.1 , the process $(H.X)$ satisfies

$$(\Delta(H.X))_n = H_n (\Delta X)_n.$$

The proof is then left to the reader. $\square$

In a casino for example, the process H corresponds to a player's strategy : according to all observations he has at time n, he bets at time $n+1$ an H_{n+1}, to earn a gain $H_{n+1}(X_{n+1} - X_n)$.

An important particular case of Proposition 2.10 is the following

Corollary 2.2. *Let (X_n) be a martingale (resp. a sub-, an super-martingale), and let be a T stopping time. Then the process X^T defined by $X_n^T = X_{T\wedge n}$ is a martingale (resp. a sub-, an super-martingale).*

Proof. It is enough to consider the predictable (right ?) process $H = \mathbf{1}_{[0,T]}$. In this case the process $(H.X)$ is nothing but X^T :

$$(H.X)_n = X_0 + \sum_{k=1}^{T\wedge n}(X_k - X_{k-1}) = X_0 + \sum_{k=1}^{n}(X_k - X_{k-1})\mathbf{1}_{k\le T}.$$

□

Note that the process $T\wedge n$ is adapted to the filtration $\mathcal{G}_n = \mathcal{F}_{T\wedge n}$ smaller than $\mathcal{F}_n$.

Using the predictable process $H = \mathbf{1}_A\mathbf{1}_{[T,\infty[}$, for $A \in \mathcal{F}_T$, we obtain

Corollary 2.3. *If T is a stopping time, then $\mathbf{1}_A(X_{T\vee n} - X_{T)})$ is a martingale (resp. a sub-, an super-martingale).*

5 Stopping theorem :bounded stopping time's case.

Theorem 2.2. *(***Stopping theorem***.)*

Let $(X_n, n \in \mathbb{N})$ be a martingale and S and T be two bounded stopping times (that means there exists an integer n such that $S \vee T \le n$, almost surely). Then,

$$\mathbb{E}(X_T/\mathcal{F}_S) = X_{S\wedge T}. \tag{2.2}$$

If X is a sub (resp. an super-)martingale,

$$X_{S\wedge T} \le \ \ (resp \ge)\ \mathbb{E}(X_T/\mathcal{F}_S). \tag{2.3}$$

In particular if $(X_n, n \in \mathbb{N})$ is a sub-martingale and S and T are two bounded stopping time, then

$$\mathbb{E}(X_S\mathbf{1}_{S\le T}) \le \mathbb{E}(X_T\mathbf{1}_{S\le T}). \tag{2.4}$$

We have the inverse inequality for an super-martingale.

Proof. We give only the proof for the martingale case.

First, we study the case where

$T = n$ and $S \leq n$. The equality (2.2) to obtain can be written as

$$X_S = \mathbb{E}(X_n/\mathcal{F}_S).$$

By definition,

$$X_S = \sum_{k=0}^{n} X_k \mathbf{1}_{S=k}.$$

We know that X_S is $\mathcal{F}_S$ measurable; and also integrable as finite linear combination of integrable variables.

It is enough to prove that for all $A \in \mathcal{F}_S$,

$$\mathbb{E}(X_S \mathbf{1}_A) = \mathbb{E}(X_n \mathbf{1}_A).$$

This can be written as

$$\sum_{k=0}^{n} \mathbb{E}(X_k \mathbf{1}_{A \cap \{S=k\}}) = \sum_{k=0}^{n} \mathbb{E}(X_n \mathbf{1}_{A \cap \{S=k\}}).$$

But since $A \in \mathcal{F}_S$, then $A \cap \{S = k\} \in \mathcal{F}_k$, and using the martingale property we obtain, for all $k \leq N$,

$$\mathbb{E}(X_k \mathbf{1}_{A \cap \{S=k\}}) = \mathbb{E}(X_n \mathbf{1}_{A \cap \{S=k\}}).$$

We now study the general case. Let an integer n such that $S \vee T \leq n$.

Using the previous case for the stopped martingale X^T, and the stopping time S. We have $X_n^T = X_T$ since $T \leq n$, $X_S^T = X_{S \wedge T}$. We have

$$\mathbb{E}(X_T/\mathcal{F}_S) = X_{S \wedge T}.$$

Note that the variable $X_{S \wedge T}$ is $\mathcal{F}_{S \wedge T}$ measurable, and as a consequence

$$X_{S \wedge T} = \mathbb{E}(X_T/\mathcal{F}_{S \wedge T}).$$

To obtain inequality (2.4), it is enough to note that inequality (2.3) means that for all $A \in \mathcal{F}_S$

$$\mathbb{E}(X_{S \wedge T} \mathbf{1}_A) \leq \mathbb{E}(X_T \mathbf{1}_A).$$

We apply it to the set $A = \{S \leq T\}$.

□

Corollary 2.4. *Let (T_n) be an increasing sequence of bounded stopping time, and X be a martingale (resp. a sub-martingale, an super-martingale) ; then $(X_{T_n}, n \in \mathbb{N})$ is a martingale (resp. a sub-martingale, an super-martingale) for the filtration $(\mathcal{F}_{T_n}, n \in \mathbb{N})$.*

Proof. (on exercise) □

Corollary 2.5. *Let X be an integrable r.v., and let (X_n) be the martingale $\mathbb{E}(X/\mathcal{F}_n)$. If $T$is a bounded stopping time then*

$$\mathbb{E}(X/\mathcal{F}_T) = X_T.$$

If S et T are two bounded stopping time;

$$\mathbb{E}(X/\mathcal{F}_S/\mathcal{F}_T) = \mathbb{E}(X/\mathcal{F}_T/\mathcal{F}_S) = \mathbb{E}(X/\mathcal{F}_{S\wedge T}) = X_{S\wedge T}.$$

Proof. Let N such that $T \vee S \leq N$. Using successive conditioning for the martingale $X_n = \mathbb{E}(X/\mathcal{F}_n)$ $\mathbb{E}(X/\mathcal{F}_T) = \mathbb{E}(X_N/\mathcal{F}_T)$. The stopping theorem yields $\mathbb{E}(X_N/\mathcal{F}_T) = X_T$.

If S and T are two bounded stopping,

$$\mathbb{E}(X_T/\mathcal{F}_S) = X_{S\wedge T} = \mathbb{E}(X/\mathcal{F}_{S\wedge T}).$$

□

6 Finite stopping times

In this section we extend the stopping theorem to the case of finite stopping times. Its requires some additional integrability conditions on martingales (resp. sub-martingales, super-martingales).

Proposition 2.11. *Let $(X_n, n \in \mathbb{N})$ be a martingale (resp. a sub-martingale) and T be S two almost surely finite stopping times.*

If the sequences $(X_{T\wedge n})$ are $(X_{S\wedge n})$ uniformly integrable, then

$$X_{S\wedge T} = E[X_T/\mathcal{F}_S]. \quad \textit{(resp. } X_{S\wedge T} \leq E[X_T/\mathcal{F}_S]).$$

This is the case when there exists a r.v. $Y \in L^1$ such that for all n, $|X_{T\wedge n}| \leq Y$),or when (X_n) is uniformly integrable

In particular for the martingale, we have $\mathbb{E}(X_T) = \mathbb{E}(X_0)$ for all finite stopping time which satisfies this assumption.

Proof. We only study the martingale case.

First note that the r.v. X_T is integrable, as almost sure limit of uniformly integrable sequence $(X_{T\wedge n})$. (Note that T is a.s. finite.) It is the same for X_S. We have to prove that for all bounded variable Z $\mathcal{F}_S$-measurable , we have

$$\mathbb{E}(X_S Z) = \mathbb{E}(X_T Z).$$

We can use the monotone class theorem monotones and restrict ourself to the case where Z is $\mathcal{F}_{S\wedge n}$ measurable using the fact that $\mathcal{F}_S = \vee_n \mathcal{F}_{S\wedge n}$.

Let such an n. Using the Stopping Theorem 2.2 for the stopping time $S\wedge p$ and $T\wedge p$, and $p \geq n$ we obtain

$$\mathbb{E}(ZX_{S\wedge p}) = \mathbb{E}(ZX_{T\wedge p}).$$

Using uniform integrability we can let n going to infinity.

For the last point, if (X_n) is uniformly integrable, it is enough to note that $X_{T\wedge n} = \mathbb{E}(X_n/\mathcal{F}_T)$. The desired conclusion follows from the fact that a family of conditional expectation of uniformly integrable family is uniformly integrable □

7 Inequalities and convergence

7.1 Inequalities

Theorem 2.3. *(*Doob*'s maximal inequality.) Let* $(X_n, n \in \mathbb{N})$ *be a positive sub martingale and* $\lambda \geq 0$. *let* $X_n^* = \sup_{k=0}^n X_k$. *Then*

$$\forall n \in \mathbb{N},\ \lambda \mathbb{P}\{X_n^* \geq \lambda\} \leq \mathbb{E}[X_n \mathbf{1}_{\{X_n^* \geq \lambda\}}] \leq E[X_n].$$

Proof. Let $T = \inf\{k \in \mathbb{N}, X_k \geq \lambda\}$ a stopping time. Then,

$$\{T \leq n\} = \{X_n^* \geq \lambda\}.$$

Take $S = T \wedge (n+1)$,which is a bounded stopping time . We have

$$A = \{S \leq n\} = \{T \leq n\} \in \mathcal{F}_S.$$

Using the Stopping Theorem for this sub-martingale, between n and $S\wedge n$.

$$\mathbb{E}(X_{S\wedge n} \mathbf{1}_A) \leq \mathbb{E}(X_n \mathbf{1}_A).$$

This can be written

$$\mathbb{E}(X_T \mathbf{1}_{T\leq n}) \leq \mathbb{E}(X_n \mathbf{1}_{T\leq n}). \tag{2.5}$$

On the set $\{T \leq n\}$, $X_T \geq \lambda$, hence $\lambda\mathbb{P}(T \leq n) \leq \mathbb{E}(X_T \mathbf{1}_{T\leq n})$. Then $\lambda\mathbb{P}(T \leq n) \leq\leq \mathbb{E}(X_n \mathbf{1}_{T\leq n})$ is the desired inequality. □

Corollary 2.6. *If $(X_n, n \in \mathbb{N})$ is a martingale, $(|X_n|, n \in \mathbb{N})$ is a positive sub-martingale and*

$$\forall n \in \mathbb{N},\ \lambda\mathbb{P}\{\max_{k\leq n} |X_k| \geq \lambda\} \leq E[|X_n| \mathbf{1}_{\max_{k\leq n} |X_k|\geq\lambda}] \leq E[|X_n|].$$

Theorem 2.4. *Let $(X_n, n \in \mathbb{N})$ be a positive sub-martingale and $p > 1$. Then, if $X_n \in L^p$,*

$$\|X_n^*\|_p \leq \frac{p}{p-1} \|X_n\|_p.$$

Proof. If $X_n \in L^p$ then variables $X_{k\in L^p}$ for $k \leq n$.

Let U be a positive r.v. in L^p,

$$E(U^p) = p \int_0^\infty t^{p-1} \mathbb{P}(U \geq t)dt.$$

Then

$$\begin{aligned} E[(X_n^*)^p] &= p \int_0^\infty t^{p-1} \mathbb{P}(X_n^* \geq t)dt \\ &\leq p \int_0^\infty t^{p-2} E[X_n \mathbf{1}_{\{X_n^* \geq t\}}]dt \\ &= pE[X_n \int_0^\infty t^{p-2} \mathbf{1}_{\{X_n^* \geq t\}} dt] \\ &= \frac{p}{p-1} E[X_n (X_n^*)^{p-1}]. \end{aligned}$$

Using Hölder inequality

$$E[X_n (X_n^*)^{p-1}] \leq \|X_n\|_p \|X_n^*\|_p^{p-1}.$$

Since X_n^* is bounded by $\sum_0^n X_k$, it belongs to L^p. The desired result is obtained by cancellation □

In particular

Corollary 2.7. *Let (X_n) be a positive sub-martingale bounded in L^1. Then the variable $X^* = \sup_n X_n$ is finite almost surely. If (X_n) is bounded in L^p $(p > 1)$, then X^* belongs to L^p. (This last result is false for $p = 1$.) The same conclusions hold for martingales (not necessary positive).*

Proof. The increasing sequence X_n^* converges towards X^*. It is enough to apply Doob's inequality and

$$\lambda \mathbb{P}(X_n^* > \lambda) \leq \sup_n \mathbb{E}(|X_n|) = K < \infty.$$

Letting n going to infinity

$$\lambda \mathbb{P}(X^* > \lambda) \leq K,$$

$\mathbb{P}(X^* > \lambda) \to 0$ $(\lambda \to \infty)$. The r.v. X^* is finite.

For the second part use the Theorem 2.4.

The case of martingales is obtained by applying the previous result to the positive sub-martingale $|X_n|$.

□

7.2 Convergences

Results

The following results are given without any proof.

Proposition 2.12. *Let X_n be a martingale, or a sub-martingale, or an super-martingale, bounded in L^1. Then X_n converges almost surely towards a variable X_∞.*

Using Fatou's lemma, M_∞ the limit of a bounded in L^1 martingale M_n is integrable. In general $M_n \neq \mathbb{E}(M_\infty / \mathcal{F}_n)$.

It is the case for uniformly integrable martingales .

Proposition 2.13. *let M_n be a bounded martingale in L^1, and let M_∞ the limit of M_n when $n \to \infty$. The following statements are equivalent*

1. *M_n converges in L^1 towards M_∞.*

2. *M_n is uniformly integrable.*

3. $M_n = \mathbb{E}[M_\infty/\mathcal{F}_n]$.

4. *There exists an integrable r.v. M such that $M_n = \mathbb{E}[M/\mathcal{F}_n]$. Moreover in this case , $M_\infty = \mathbb{E}[M/\mathcal{F}_\infty]$.*

(Here $\mathcal{F}_\infty = \vee_n \mathcal{F}_n$.)

Proof. For a sequence of r.v. which converges almost surely, it is equivalent to converge in L^1 or to be uniformly integrable. For all integrable r.v. M, the set of the r.v. $\mathbb{E}(M/\mathcal{B})$, where $\mathcal{B}$ is running in all sub σ fields of $\mathcal{A}$ is an uniformly integrable family. It is enough to prove the following points:

1. If (M_n) is uniformly integrable, then $M_n = \mathbb{E}(M_\infty/\mathcal{F}_n)$;

2. If M is an integrable r.v., the martingale $M_n = \mathbb{E}(M/\mathcal{F}_n)$ converges towards $\mathbb{E}(M/\mathcal{F}_\infty)$.

For the first point note that for $p \geq n$ $M_n = \mathbb{E}(M_p/\mathcal{F}_n)$,letting p going to infinity using the fact that the expectation is continuous in L^1, and that M_p converges towards M_∞ in L^1 by assumption. We get the desired result.

For the second point, note that M_∞ is $\mathcal{F}_\infty$ measurable by construction. It is enough to show that, for a $A \in \mathcal{F}_\infty$, we have $\mathbb{E}(M_\infty \mathbf{1}_A) = \mathbb{E}(M \mathbf{1}_A)$. This is true when A belongs to sub σ fields of $\mathcal{F}_n$,since

$$\mathbb{E}(M\mathbf{1}_A) = \mathbb{E}(M_n\mathbf{1}_A) = \mathbb{E}(M_\infty\mathbf{1}_A).$$

The desired identity is then, true for all element of $\cup_n \mathcal{F}_n$, and for all element σ -field generated by $\cup_n \mathcal{F}_n$ using a monotone class theorem argument. The desired inequality is true for $\mathcal{F}_\infty$. □

Remarks

1. A similar statement as in Proposition 2.13 is true for sub-and super-martingales; the proof is left to the reader.

2. A bounded martingale L^p for $p > 1$, is dominated by an L^p variable and converges in L^p.

We now are in position to enunciate the Stopping theorem for general stopping-times.

Theorem 2.5. *(***Stopping Theorem.***) Let M_n be a uniformly integrable martingale and let T be a stopping time (not necessarily finite). Then for $M_T = M_\infty$ on $\{T = \infty\}$,we have*

1. *$M_T = \mathbb{E}(M_\infty/\mathcal{F}_T)$.*
2. *The set (M_T), where T is a stopping time is uniformly integrable.*
3. *If S and T are two stopping time, we have*

$$\mathbb{E}(M_T/\mathcal{F}_S) = M_{S\wedge T}.$$

4. *Let M be a $\mathcal{A}$-measurable integrable r.v. and $M_n = \mathbb{E}(M/\mathcal{F}_n)$, then $M_T = \mathbb{E}(M/\mathcal{F}_T)$.*

Proof. For the first point, it is enough to write the proof of Stopping theorem in this case. If A belongs to $\mathcal{F}_T$, then

$$\begin{aligned}\mathbb{E}(M_T\mathbf{1}_A) &= \sum_{k\in\mathbb{N}\bigcup\infty} \mathbb{E}(M_k\mathbf{1}_{A\bigcap\{T=k\}}) \\ &= \sum_{k\in\mathbb{N}\bigcup\infty} \mathbb{E}(M_\infty\mathbf{1}_{A\bigcap\{T=k\}}) = \mathbb{E}(M_\infty\mathbf{1}_A).\end{aligned}$$

The family M_T is contained in the family $\mathbb{E}(M_\infty/\mathcal{B})$, where $\mathcal{B}$ is running in the subσ fields of $\mathcal{A}$. This last family is uniformly integrable.

The stopping martingale M^T is uniformly integrable. Using the Stopping theorem at time S, we obtain

$$\mathbb{E}(M_T/\mathcal{F}_S) = M_{S\wedge T}.$$

It is enough to write

$$\mathbb{E}(M/\mathcal{F}_T) = \mathbb{E}(\mathbb{E}(M/\mathcal{F}_\infty)/\mathcal{F}_T) = \mathbb{E}(M_\infty/\mathcal{F}_T) = M_T.$$

□

8 Exercises

1. Prove the claim 2 of examples 1 in section 1.

2. Recall a definition of uniform integrability (U.I.) that claims that X_i is U.I. if $\sup_i \mathbb{E}|X_i| < \infty$ and if a property sometimes called equiintegrability (to be recalled) is fulfilled.

3. Prove the claim 1 of examples 1 in section 1.

4. Show that M_n in the claim 3 of examples 1 in section 1 is square integrable. What is the Doob decomposition of M_n^2 ?

Chapter 3

Martingales in continuous time

1 Filtrations in continuous time

The definition of filtrations in continuous time is given under the usual conditions. Those conditions are assumed to avoid nasty technical problems related to the fact that $(0, +\infty)$ is not denumerable and therefore there exist obstructions to measurabilty.

Definition 3.1. *A family of sigma fields $(\mathcal{F}_t)_{0\leq t\leq +\infty}$ all included in the sigma field $\mathcal{F}$ associated to a probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ is a filtration if $\forall s \leq t,\ \mathcal{F}_s \subset \mathcal{F}_t$. It satisfies the usual conditions if*

1. *$\mathcal{F}_0$ contains all negligible sets of $\mathcal{F}$ (it is complete)*
2. *$\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s, \forall t \geq 0,$ (it is right continuous.)*

In this lecture all filtrations satisfy the usual condition.

We introduce a measurability assumption for processes that states that the process depends only on the past of the filtration.

Definition 3.2. *The stochastic process X is adapted to the filtration $\{\mathscr{F}_t\}$ if, for each $t \geq 0, X_t$ is an $\mathscr{F}_t$-measurable random variable.*

Obviously, every process X is adapted to $\left\{\mathscr{F}_t^X\right\} \stackrel{def}{=} \sigma(X_s, 0 \leq s \leq t)$. Moreover, if X is adapted to $\{\mathscr{F}_t\}$ and Y is a modification of X, then Y is also adapted to $\{\mathscr{F}_t\}$ provided that $\mathscr{F}_0$ contains all the P-negligible sets in $\mathscr{F}$.

Definition 3.3. *The stochastic process X is called progressively measurable with respect to the filtration $\{\mathscr{F}_t\}$ if, for each $t \geq 0$ and $A \in \mathscr{B}\left(\mathbb{R}^d\right)$, the set $\{(s,\omega); 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$ belongs to the product σ-field $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$; in other words, if the mapping*

$$(s,\omega) \mapsto X_s(\omega) : ([0,t] \times \Omega, \mathscr{B}([0,t]) \otimes \mathscr{F}_t) \to \left(\mathbb{R}^d, \mathscr{B}\left(\mathbb{R}^d\right)\right)$$

is measurable, for each $t \geq 0$.

Proposition 3.1. *If the stochastic process X is adapted to the filtration $\{\mathscr{F}_t\}$ and every sample path is right-continuous or else every sample path is left continuous, then X is also progressively measurable with respect to $\{\mathscr{F}_t\}$.*

Proof. We treat the case of right-continuity. With $t > 0, n \geq 1, k = 0, 1, \ldots, 2^n - 1$, and $0 \leq s \leq t$, we define:

$$X_s^{(n)}(\omega) = X_{(k+1)t/2^n}(\omega) \text{ for } \quad \frac{kt}{2^n} < s \leq \frac{k+1}{2^n}t,$$

as well as $X_0^{(n)}(\omega) = X_0(\omega)$. The so-constructed map $(s,\omega) \mapsto X_s^{(n)}(\omega)$ from $[0,t] \times \Omega$ into $\mathbb{R}^d$ is demonstrably $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$-measurable. Besides, by right-continuity we have: $\lim_{n\to\infty} X_s^{(n)}(\omega) = X_s(\omega), \forall (s,\omega) \in [0,t] \times \Omega$. Therefore, the (limit) map $(s,\omega) \mapsto X_s(\omega)$ is also $\mathscr{B}([0,t]) \otimes \mathscr{F}_t$-measurable. □

2 Stopping times in continuous time

Definition 3.4. *Let us consider a measurable space $(\Omega, \mathscr{F})$ equipped with a filtration $\{\mathscr{F}_t\}$. A random variable $T : \Omega \mapsto [0, +\infty]$ is a stopping time of the filtration, if the event $\{T \leq t\}$ belongs to the σ-field $\mathscr{F}_t$, for every $t \geq 0$.*

Proposition 3.2. *Show that $\forall t \geq 0$, $\{T < t\}$ belongs to the σ-field $\mathscr{F}_t$ is equivalent to T is a stopping time.*

Proof. The proof is based on the observation $\{T < t\} = \bigcup_{n=1}^{\infty}\{T \leq t - (1/n)\} \in \mathscr{F}_t$, because if T is a stopping time, then $\{T \leq t - (1/n)\} \in \mathscr{F}_{t-(1/n)} \subseteq \mathscr{F}_t$ for $n \geq 1$. For the converse, suppose that $\forall t \geq 0$, $\{T <$

$t\} \in \mathscr{F}_t$ of the right-continuous filtration $\{\mathscr{F}_t\}$. Since for every positive integer m, we have $\{T \le t\} = \bigcap_{n=m}^{\infty}\{T < t + (1/n)\}$, we deduce that $\{T \le t\} \in \mathscr{F}_{t+(1/m)}$; whence $\{T \le t\} \in \cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$. □

Consider a subset $A \in \mathscr{B}\left(\mathbb{R}^d\right)$ of the state space of the process, and define the hitting time

$$D_A(\omega) = \inf\{t \ge 0; X_t(\omega) \in A\}$$

and

$$T_A(\omega) = \inf\{t > 0; X_t(\omega) \in A\}.$$

Remark 3.1. *By convention we set* $\inf(\emptyset) = +\infty$.

If $X_0 \notin A,\ D_A = T_A$.

Proposition 3.3. *1. If X is right continuous and adapted and A is open then T_A is a stopping time.*

2. If the process X is continuous, adapted and A is closed then D_A is a stopping time.

Proof. Let us prove the second claim.

$$\{D_A \le t\} = \cap_{n\in\mathbb{N}^*} \cup_{s\in\mathbb{Q},\, 0<s\le t} \{d(X_s, A) < \frac{1}{n}\}.$$

Let us first show that the set on the left is included in the set on the right. Since X is continuous and A closed $X_{D_A} \in A$. Moreover there exists a sequence $s_n \in \mathbb{Q}$ such that it is increasing to $D_A(\omega)$ and $d(X_{s_n}, X_{D_A}) < \frac{1}{n}$. Then $d(X_{s_n}, A) \le d(X_{s_n}, X_{D_A})$. The inclusion is proved, let us prove the inclusion the other way around.

$\forall n \in \mathbb{N}^*\ \exists s_n \le t$ and $d(X_{s_n}, A) < \frac{1}{n}$, We consider a susbsequence $s_{n_k} \to t' \le t$, then $d(X_{t'}, A) = 0$ by continuity of X and of $x \mapsto d(x, A)$. This implies $D_A \le t'$. Since $\{d(X_s, A) < \frac{1}{n}\} \in \mathcal{F}_s$ the proof is finished.

Let us prove the first claim. Because of the Proposition 3.2 it is enough to show that $\{T_A < t\} \in \mathcal{F}_t$.

$$\begin{aligned}\{T_A < t\} &= \{\omega,\ \exists s < t,\ X_s(\omega) \in A\}\\ &= \{\omega,\ \exists s < t, s \in \mathbb{Q}\ X_s(\omega) \in A\}\end{aligned}$$

since A is open and X right continuous. Then

$$\{T_A < t\} = \cup_{s<t, s\in\mathbb{Q}}\{\omega, X_s(\omega) \in A\}$$

and each of the set in the last union belongs to $\mathcal{F}_t$. □

Exercise 3.1. *If S and T are two stopping times, then $S \wedge T$, $S \vee T$, $T+S$ are also stopping times. (The last one is more difficult).*

Proposition 3.4. *Every stopping time is the limit of a nonincreasing sequence of stopping times that take only a finite number of values.*

Proof. Let us denote by T the stopping time and let us define for $n \in \mathbb{N}^*$ and $1 \leq k \leq 2^{2n}$,

$$T_n(\omega) = \frac{k}{2^n}$$

if $\frac{k-1}{2^n} < T \leq \frac{k}{2^n}$ and $T_n = +\infty$ if $T > 2^n$. On can check that T_n is a nonincreasing sequence converging to T. Moreover

$$\{T_n = \frac{k}{2^n}\} = \{T \leq \frac{k}{2^n}\} \setminus \{T \leq \frac{k-1}{2^n}\} \in \mathcal{F}_{\frac{k}{2^n}}.$$

If $t \geq 0$,

$$\{T_n \leq t\} = \cup_{k/2^n \leq t} \{T_n = \frac{k}{2^n}\}$$

then T_n is a stopping time. □

Definition 3.5. *Let T be a stopping time of the filtration $\{\mathscr{F}_t\}$. The σ-field $\mathscr{F}_T$ of events determined prior to the stopping time T consists of those events $A \in \mathscr{F}$ for which $A \cap \{T \leq t\} \in \mathscr{F}_t$ for every $t \geq 0$.*

Exercise 3.2. *Verify that $\mathscr{F}_T$ is actually a σ-field and T is $\mathscr{F}_T$-measurable. Show that if $T(\omega) = t$ for some constant $t \geq 0$ and every $\omega \in \Omega$, then $\mathscr{F}_T = \mathscr{F}_t$.*

Proposition 3.5. *If X is progressively measurable valued in $(E, \mathcal{E})$ and T is a stopping time then $X_T \mathbf{1}(T < \infty)$ is $\mathcal{F}_T$-measurable.*

We set $X_T \mathbf{1}(T < \infty) = 0$ if $T = \infty$.

Proof. Let us suppose $T < +\infty$. For a fixed $t \geq 0$, let us set

$$\tilde{\Omega} = \{\omega | T(\omega) \leq t\}$$

endowed with the sigma field $\mathcal{F}_t$ restricted to $\tilde{\Omega}$ that is $\mathcal{F}_t^{\tilde{\Omega}} \stackrel{def}{=} \mathcal{F}_t \cap \{T \leq t\}$. The map

$$\tilde{\Omega} \mapsto [0, t] \tag{3.1}$$

$$\omega \mapsto T(\omega) \tag{3.2}$$

is measurable from $\mathcal{F}_t^{\tilde{\Omega}}$ on $\mathcal{B}([0,t])$ because for $s \le t$ $\{T \le s\} \in \mathcal{F}_s$. Then Φ defined by

$$\tilde{\Omega} \mapsto [0,t] \times \Omega \tag{3.3}$$

$$\omega \mapsto (T(\omega), \omega) \tag{3.4}$$

is measurable if we endow $[0,t] \times \Omega$ with the sigma field $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$. Since X is progressively measurable $X_T = X \circ \Phi(\omega)$ is $\mathcal{F}_t^{\tilde{\Omega}}$ measurable i.e. $\forall A \in \mathcal{E}$, $\{\omega | X_{T(\omega)} \in A\} \cap \{\omega | T(\omega) \le t\} \in \mathcal{F}_t$, $\{\omega | T(\omega) < +\infty\}$ is $\mathcal{F}_T$ measurable hence $X_T \mathbf{1}(T < \infty)$ also.

Exercise 3.3. 1. *Show that $\mathcal{F}_T = \sigma(X_T,\ X \text{progressively measurable})$. The previous Proposition yields one inclusion out of two...*

2. *Show that if S, T stopping times and $S \le T$ a.s. $\mathcal{F}_S \subset \mathcal{F}_T$.*

□

3 Martingale in continuous time

In this section we shall consider exclusively real-valued processes $X = \{X_t; 0 \le t < \infty\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, adapted to a given filtration $\{\mathscr{F}_t\}$ and such that $\mathbb{E}\,|X_t| < \infty$ holds for every $t \ge 0$.

Definition 3.6. *The process $\{X_t, \mathscr{F}_t; 0 \le t < \infty\}$ is said to be a submartingale (respectively, a supermartingale) if, for every $0 \le s < t < \infty$, we have, a.s. $\mathbb{P}$; $\mathbb{E}\,(X_t \mid \mathscr{F}_s) \ge X_s$ (respectively, $\mathbb{E}\,(X_t \mid \mathscr{F}_s) \le X_s$).*

We shall say that $\{X_t, \mathscr{F}_t; 0 \le t < \infty\}$ is a martingale if it is both a submartingale and a supermartingale.

Example. 1. *If $(B_t)_{t \ge 0}$ is a Brownian Motion (BM) we consider the natural filtration $\mathcal{F}_t^B = \sigma(B_s,\ s \le t)$ and $\mathcal{F}_t = \sigma(\mathcal{F}_t^B, \mathcal{N})$, where $\mathcal{N}$ is the set of negligible sets. We take for granted that $\mathcal{F}_t$ is right continuous and hence satisfies the usual conditions. B_t* **is a $\mathcal{F}_t$ martingale** *since for $0 \le s \le t$*

$$\mathbb{E}(B_t | \mathcal{F}_s) = \mathbb{E}(B_t - B_s + B_s | \mathcal{F}_s) = \mathbb{E}(B_t - B_s) + B_s.$$

2. *$(B_t^2 - t)_{t \ge 0}$ is a $\mathcal{F}_t$ martingale. Actually $0 \le s \le t$ we have to show that $\mathbb{E}(B_t^2 - t | \mathcal{F}_s) = B_s^2 - s$, which is equivalent to $\mathbb{E}(B_t^2 - B_s^2 | \mathcal{F}_s) = t - s$. Observe that $\mathbb{E}((B_s + B_t - B_s)^2 - B_s^2 | \mathcal{F}_s) = \mathbb{E}(2B_s(B_t - B_s) + (B_t - B_s)^2 | \mathcal{F}_s) = t - s$.*

3. *$\forall \lambda \in \mathbb{C}$, $M_\lambda(t) = e^{\lambda B_t - \frac{\lambda^2}{2}t}$ is a $\mathcal{F}_t$ martingale. (It is important to allow λ to be complex valued since we will take $\lambda = iu$ where $u \in \mathbb{R}$ which is related to the characteristic function $\mathbb{E}e^{iuB_t}$; the definition of complex valued martingales just amounts to say that the real part and the imaginary part of the process are martingales.) To ensure integrability of $M_\lambda(t)$ we recall that the Laplace transform of Gaussian random variables is always finite. Let us show that $\mathbb{E}(\frac{M_\lambda(t)}{M_\lambda(s)}|\mathcal{F}_s) = 1$.*

$$\begin{aligned}\mathbb{E}(e^{\lambda(B_t - B_s) - \frac{\lambda^2}{2}(t-s)}|\mathcal{F}_s) &= \mathbb{E}(e^{\lambda(B_t-B_s)})e^{-\frac{\lambda^2}{2}(t-s)} \\ &= 1.\end{aligned}$$

Let $X = \{X_t; 0 \leq t < \infty\}$ be a real-valued stochastic process. Consider two numbers $\alpha < \beta$ and a finite subset F of $[0, \infty)$. We define the number of upcrossings $U_F(\alpha, \beta; X(\omega))$ of the interval $[\alpha, \beta]$ by the restricted sample path $\{X_i; t \in F\}$ as follows. Set

$$\tau_1(\omega) = \min\{t \in F; X_t(\omega) \leq \alpha\},$$

and define recursively for $j = 1, 2, \ldots$

$$\begin{aligned}\sigma_j(\omega) &= \min\{t \in F; t \geq \tau_j(\omega), X_t(\omega) > \beta\} \\ \tau_{j+1}(\omega) &= \min\{t \in F; t \geq \sigma_j(\omega), X_t(\omega) < \alpha\}\end{aligned}$$

The convention here is that the minimum of empty set is $+\infty$, and we denote by $U_F(\alpha, \beta; X(\omega))$ the largest integer j for which $\sigma_j(\omega) < \infty$. If $I \subset [0, \infty)$ is not necessarily finite, we define

$$U_I(\alpha, \beta; X(\omega)) = \sup\{U_F(\alpha, \beta; X(\omega)); F \subseteq I, F \text{ is finite }\}$$

The number of downcrossings $D_I(\alpha, \beta; X(\omega))$ is defined similarly.

The following theorem extends to the continuous-time case results of discrete martingales.

Theorem 3.1. *Let $\{X_t, \mathscr{F}_t; 0 \leq t < \infty\}$ be a submartingale whose every path is right-continuous, let $[\sigma, \tau]$ be a subinterval of $[0, \infty)$, and let $\alpha < \beta, \lambda > 0$ be real numbers. We have the following results:*

1. *Submartingale inequality:*

$$\lambda \cdot \mathbb{P}\left[\sup_{\sigma \le t \le \tau} X_t \ge \lambda\right] \le \mathbb{E}\left(X_\tau^+\right).$$

2. *Upcrossings and downcrossings inequalities:*

$$\mathbb{E}U_{[\sigma,\tau]}(\alpha,\beta;X(\omega)) \le \frac{\mathbb{E}\left(X_\tau^+\right) + |\alpha|}{\beta - \alpha}, \quad \mathbb{E}D_{[\sigma,\tau]}(\alpha,\beta;X(\omega)) \le \frac{\mathbb{E}\left(X_\tau - \alpha\right)^+}{\beta - \alpha}.$$

3. *Doob's maximal inequality:*

$$\mathbb{E}\left(\sup_{\sigma \le t \le \tau} X_t\right)^p \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left(X_\tau^p\right), \quad p > 1$$

provided $X_t \ge 0$ *a.s.* $\mathbb{P}$ *for every* $t \ge 0$, *and* $\mathbb{E}\left(X_\tau^p\right) < \infty$.

There exist modifications of martingales, which are right continuous.

Theorem 3.2. *If* X_t *is a* $\mathcal{F}_t$ *(with the usual conditions) sub-martingale there exists a modification of* X_t *with right continuous paths iff* $t \mapsto \mathbb{E}X_t$ *is right continuous. Moreover this modification has paths with left limits and it is a sub-martingale.*

Corollary 3.1. *If* X_t *is a* $\mathcal{F}_t$ *(with the usual conditions) martingale there exists a modification of* X_t *with r.c.l.l. right continuous left limits paths and it is a martingale*

The proof of the Theorem 3.2 can be found in Karatzas and Shreeve. It uses the upcrossings inequality and the backward martingales that I did not recalled.

We have convergence results for martingales in continuous time similar to those in discrete time. We always use the right continuous modification.

Theorem 3.3. 1. *If* X_t *is a* $\mathcal{F}_t$ *sub-martingale and* $\sup \mathbb{E}(X_t^+) < \infty$ *then* $\lim_{t \to +\infty} X_t = X_\infty$ *exists a.s. and* $\mathbb{E}|X_\infty| < \infty$.

2. *If* X_t *is a* $\mathcal{F}_t$ *martingale the following properties are equivalent.*

   (a) X_t *converges in* L^1 *towards* X_∞.

   (b) X_t *is uniformly integrable.*

   (c) $\exists X_\infty \in L^1$ *and* $X_t = \mathbb{E}[X_\infty / \mathcal{F}_t]$.

Under any of this hypothesis we have also a.s. convergence of X_t when $t \to \infty$.

Please remark the slightly different assumption for the first part of the theorem, when it is compared to the result for martingales indexed by $\mathbb{N}$. This assumption works also for martingales indexed by $\mathbb{N}$.

4 Stopping time theorems

Let us generalize the stopping time theorems already obtained for martingales indexed by $\mathbb{N}$ to martingales in continuous time. Here we consider the case when X_t is uniformly integrable.

Definition 3.7. *If X_t is a martingale uniformly integrable and T a stopping time, let us define $X_T(\omega) = X_\infty(\omega)$ if $T(\omega) = \infty$ and $X_T(\omega) = X_{T(\omega)}(\omega)$ if $T(\omega) < \infty$.*

Theorem 3.4. *If X_t is a martingale uniformly integrable then the family of (X_S) where S is any stopping time is also uniformly integrable. If $S \leq T$ are two stopping times $X_S = \mathbb{E}(X_T|\mathcal{F}_S) = \mathbb{E}(X_\infty|\mathcal{F}_S)$.*

Proof. Let us first show

$$X_S = \mathbb{E}(X_\infty|\mathcal{F}_S). \tag{3.5}$$

For the first step I assume that S only takes a finite number of values $\{s_1 < \ldots < s_n\}$, and add by convention $s_0 = -\infty$. Let $Y_k = X_{s_k}$ for $k \leq n$ and $Y_{n+1} = X_\infty$. Then Y is a martingale for $\mathcal{F}_{s_k}$ and $Y_k = \mathbb{E}(X_\infty|\mathcal{F}_{s_k})$. Let $A \in \mathcal{F}_S$, i.e. $A \cap \{S \leq t\} \in \mathcal{F}_t$

$$\begin{aligned} X_S\mathbf{1}_A &= \sum_{k=1}^{n+1} Y_k \mathbf{1}_{\{S=s_k\}\cap A} \\ &= \sum_{k=1}^{n} Y_k(\mathbf{1}_{\{S\leq s_k\}\cap A} - \mathbf{1}_{\{S\leq s_{k-1}\}\cap A}) + Y_{n+1}\mathbf{1}_{\{S>s_n\}\cap A}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}(X_S\mathbf{1}_A) &= \sum_{k=1}^{n+1} \mathbb{E}(X_\infty(\mathbf{1}_{\{S\leq s_k\}\cap A} - \mathbf{1}_{\{S\leq s_{k-1}\}\cap A})) + \mathbb{E}(X_\infty \mathbf{1}_{\{S>s_n\}\cap A}) \\ &= \mathbb{E}(X_\infty \mathbf{1}_A). \end{aligned}$$

In the general case S is the limit of a non increasing sequence of stopping times S_n that take a finite number of values and $\forall n \in \mathbb{N},\ X_{S_n} = \mathbb{E}(X_\infty|\mathcal{F}_{S_n})$. Since X is right continuous, $X_S = \lim_{n\to\infty} X_{S_n}$. Moreover $\mathcal{F}_{S_{n+1}} \subset \mathcal{F}_{S_n}$ hence X_{S_n} is a backward martingale and we admit that every backward martingale converges in L^1. So

$$X_S = \lim_{N\to\infty} \mathbb{E}(X_\infty| \cap_{n=1}^N \mathcal{F}_{S_n})$$

in L^1. Since $\mathcal{F}_S \subset \cap_{n=1}^N \mathcal{F}_{S_n}$,

$$\begin{aligned} X_S &= \mathbb{E}(\lim_{N\to\infty} \mathbb{E}(X_\infty| \cap_{n=1}^N \mathcal{F}_{S_n})|\mathcal{F}_S) \\ &= \lim_{N\to\infty} \mathbb{E}(\mathbb{E}(X_\infty| \cap_{n=1}^N \mathcal{F}_{S_n})|\mathcal{F}_S) \quad = \lim_{N\to\infty} \mathbb{E}(X_\infty|\mathcal{F}_S) = \mathbb{E}(X_\infty|\mathcal{F}_S). \end{aligned}$$

Since $\mathbb{E}(X_\infty|\mathcal{F}_S)$ is uniformly integrable so the family of (X_S) where S is any stopping time is also uniformly integrable. Moreover, if $S \leq T$,

$$\mathbb{E}(X_T|\mathcal{F}_S) = \mathbb{E}(\mathbb{E}(X_\infty|\mathcal{F}_T)|\mathcal{F}_S) = \mathbb{E}(X_\infty|\mathcal{F}_S) = X_S$$

because $\mathcal{F}_S \subset \mathcal{F}_T$. □

Corollary 3.2. *If X is a martingale and $S \leq T$ bounded stopping times then $\mathbb{E}(X_T|\mathcal{F}_S) = X_S$.*

Proof. Let $M > 0$ such that $0 \leq S(\omega) \leq T(\omega) \leq M$ a.s. Let us define $Y_t = X_{t\wedge M}, \forall t \geq 0$. Then $Y_t = \mathbb{E}(Y_M|\mathcal{F}_t)$, it is uniformly integrable and we apply the previous theorem. Then $\mathbb{E}(Y_T|\mathcal{F}_S) = Y_S$ which is also $\mathbb{E}(X_{T\wedge M}|\mathcal{F}_S) = X_{S\wedge M}$. Hence $\mathbb{E}(X_T|\mathcal{F}_S) = X_S$ since $0 \leq S(\omega) \leq T(\omega) \leq M$. □

Remark 3.2. *But, if X is not uniformly integrable, one cannot only suppose that $S \leq T$ finite a.s.*

Exercise 3.4. *Let $B_t,\ t \geq 0$ be a BM starting from 0 and for $a > 0$ $T_a = \inf\{t > 0,\ B_T = a\}$. Here we assume that $T_a < \infty$ a.s. Compute $\mathbb{E}(e^{-\mu T_a})$ for $\mu \geq 0$. Deduce that $\mathbb{E}T_a = +\infty$.*

Solution :

Let $M^\lambda(t) = e^{\lambda B_t - \frac{\lambda^2}{2}t}$ be the exponential martingale for $\lambda \geq 0$. The stopping time $t \wedge T_a$ is bounded. Since $s \wedge T_a \leq t \wedge T_a$, then

$$\mathbb{E}(M^\lambda_{t\wedge T_a}|\mathcal{F}_{s\wedge T_a}) = M^\lambda_{s\wedge T_a}.$$

Hence $M^{\lambda}_{t\wedge T_a}$ is a martingale. So $\mathbb{E}M^{\lambda}_{t\wedge T_a} = \mathbb{E}M^{\lambda}_0 = 1$. In

$$\mathbb{E}(e^{\lambda B_{t\wedge T_a} - \frac{\lambda^2}{2}(t\wedge T_a)}) = 1,$$

we let $t \to +\infty$, use $T_a < \infty$. a.s. and a dominated convergence argument, so that $\mathbb{E}(e^{\lambda B_{T_a} - \frac{\lambda^2}{2}T_a}) = 1$. Then we remark that $B_{T_a} = a$ and we get

$$\mathbb{E}(e^{-\frac{\lambda^2}{2}T_a}) = e^{-\lambda a}.$$

Let us take $\mu = \frac{\lambda^2}{2}$, then $\mathbb{E}(e^{-\mu T_a}) = e^{-\sqrt{2\mu}a}$. We have the Laplace transform of the positive random variable T_a. Classically $\mathbb{E}T_a$ is obtained as the derivative of this Laplace transform for $\mu = 0$ which is here $+\infty$. Hence $\mathbb{E}T_a = +\infty$.

Theorem 3.5. *If X_t is a non-negative supermartingale and $S \leq T$ are two stopping times $X_S \geq \mathbb{E}(X_T|\mathcal{F}_S)$.*

Remark 3.3. *We know that the* $\lim_{t\to+\infty} X_t = X_\infty$ *exists a.s. so we don't need to have the stopping time a.s. finite in this case.*

We admit the proof.

Chapter 4

Stochastic integral

The results for martingales are used to build $X_t = \int_0^t H_s dMs$ where H is progressively measurable and M is a martingale. Because of the martingale transform in the discrete case we expect X_t to be a martingale. But the real life is more complicated for integrability reasons... Hence we are forced to define local martingale associated to a sequence of stopping times i.e. we assume that there is a non decreasing sequence of stopping times T_n such that $X_{t\wedge T_n}$ is a martingale uniformly integrable. Our basic tool will be quadratic variations that we will generalize from Brownian motion to continuous martingales. In this part we assume the processes are a.s. continuous.

1 Quadratic variations

Using Riemmann Stieljes integral we know how to integrate with respect to processes A with finite variations. We will show that we can define an integral with respect to local martingales M. Once we have done that we will be able to define integrals with respect to processes of the form $M + A$... That is the goal of the chapter. Let us come back to a technical question "Do the martingales have finite variations ?". First we show that a process cannot be a continuous martingale and with finite variations unless it is trivial.

Proposition 4.1. *Every continuous martingale with finite variations is constant.*

Proof. Let Δ be a subdivision of $[0,t]$ and

$$V_t(M) = \sup_{\Delta \in \mathcal{P}_t} \sum_{i=1}^{n} |M_{t_{i+1}} - M_{t_i}|$$

where M is a continuous martingale with finite variations. Let $T_n = \inf\{t \geq 0,\ V_t \geq n \text{ or } |M_t| \geq n\}$. As a hitting time T_n is a stopping time and we can show by contradiction that T_n almost surely converges to $+\infty$. We denote by $M_t^{T_n} = M_{t\wedge T_n}$ the stopped martingale : it is a bounded by n continuous process. It is also a martingale because of the stopping time theorem for bounded stopping time. If $s \leq t,\ \ s \wedge T_n \leq t \wedge T_n$

$$\mathbb{E}(M_{t\wedge T_n} | \mathcal{F}_{s\wedge T_n}) = M_{s\wedge T_n}.$$

Hence we can assume without a loss of generality that M is a bounded continuous martingale with bounded variations. If $\Delta \in \mathcal{P}_t$, by orthogonality of the increments of L^2 martingales

$$\begin{aligned}\mathbb{E}(M_t - M_s)^2 &= \sum_{t_i \in \Delta} \mathbb{E}(M_{t_{i+1}} - M_{t_i})^2 \\ &\leq \mathbb{E}(V_t \sup_{t_i \in \Delta} |M_{t_{i+1}} - M_{t_i}|).\end{aligned}$$

Since M is uniformly continuous on $[0,t]$ $M_t = M_0$ a.s. □

We have checked that $B_t^2 - t$ is martingale, we will show that $M^2 - \langle M, M\rangle$ is a martingale. In this chapter we consider Δ with an infinite number of points t_i such that $\lim_{i\to\infty} t_i = +\infty$ and rewrite $T^{\Delta}_{[0,t]} \stackrel{def}{=} \sum_{t_i\in\Delta}(X_{t_{i+1}\wedge t} - X_{t_i \wedge t})^2$.

Theorem 4.1. *A continuous and bounded martingale M is of finite quadratic variation and $\langle M, M\rangle$ is the unique continuous non decreasing adapted process vanishing at zero such that $M^2 - \langle M, M\rangle$ is a martingale.*

Proof. Uniqueness is an easy consequence of Proposition 4.1, since if there were two such processes A and B, then $A - B$ would be a continuous martingale vanishing at zero with finite variations.

To prove the existence of $\langle M, M\rangle$, we first observe that since for $t_i < s < t_{i+1}$,

$$E\left[\left(M_{t_{i+1}} - M_{t_i}\right)^2 \mid \mathcal{F}_s\right] = E\left[\left(M_{t_{i+1}} - M_s\right)^2 \mid \mathscr{F}_s\right] + (M_s - M_{t_i})^2$$

it is easily proved that

$$E\left[T_t^{\Delta}(M) - T_s^{\Delta}(M) \mid \mathscr{F}_s\right] = E\left[(M_t - M_s)^2 \mid \mathscr{F}_s\right]$$
$$= E\left[M_t^2 - M_s^2 \mid \mathscr{F}_s\right] \tag{4.1}$$

As a result, $M_t^2 - T_t^{\Delta}(M)$ is a continuous martingale. In the sequel, we write T_t^{Δ} instead of $T_t^{\Delta}(M)$.

We now fix $a > 0$ and we are going to prove that if $\{\Delta_n\}$ is a sequence of subdivisions of $[0, a]$ such that $|\Delta_n|$ goes to zero, then $\left\{T_a^{\Delta_n}\right\}$ converges in L^2.

If Δ and Δ' are two subdivisions we call $\Delta\Delta'$ the subdivision obtained by taking all the points of Δ and Δ'. By (4.1) the process $X = T^{\Delta} - T^{\Delta'}$ is a martingale and, by (4.1) again, applied to X instead of M, we have

$$E\left[X_a^2\right] = E\left[\left(T_a^{\Delta} - T_a^{\Delta'}\right)^2\right] = E\left[T_a^{\Delta\Delta'}(X)\right]$$

Because $(x+y)^2 \leq 2\left(x^2 + y^2\right)$ for any pair (x, y) of real numbers,

$$T_a^{\Delta\Delta'}(X) \leq 2\left\{T_a^{\Delta\Delta'}\left(T^{\Delta}\right) + T_a^{\Delta\Delta'}\left(T^{\Delta'}\right)\right\}$$

and to prove our claim, it is enough to show that $E\left[T_a^{\Delta\Delta'}\left(T^{\Delta}\right)\right]$ converges to 0 as $|\Delta| + |\Delta'|$ goes to zero.

Let then s_k be in $\Delta\Delta'$ and t_l be the rightmost point of Δ such that $t_l \leq s_k < s_{k+1} \leq t_{l+1}$; we have

$$T_{s_{k+1}}^{\Delta} - T_{s_k}^{\Delta} = \left(M_{s_{k+1}} - M_{t_l}\right)^2 - \left(M_{s_k} - M_{t_l}\right)^2$$
$$= \left(M_{s_{k+1}} - M_{s_k}\right)\left(M_{s_{k+1}} + M_{s_k} - 2M_{t_l}\right)$$

and consequently,

$$T_a^{\Delta\Delta'}\left(T^{\Delta}\right) \leq \left(\sup_k \left|M_{s_{k+1}} + M_{s_k} - 2M_{t_t}\right|^2\right) T_a^{\Delta\Delta'}$$

By Schwarz's inequality,

$$E\left[T_a^{\Delta\Delta'}\left(T^{\Delta}\right)\right] \le E\left[\sup_k \left|M_{s_{k+1}} + M_{s_k} - 2M_{t_l}\right|^4\right]^{1/2} E\left[\left(T_a^{\Delta\Delta'}\right)^2\right]^{1/2}$$

Whenever $|\Delta| + |\Delta'|$ tends to zero, the first factor goes to zero because M is continuous; it is therefore enough to prove that the second factor is bounded by a constant independent of Δ and Δ'. To this end, we write with $a = t_n$,

$$\begin{aligned}\left(T_a^{\Delta}\right)^2 &= \left(\sum_{k=1}^{n}\left(M_{t_k} - M_{t_{k-1}}\right)^2\right)^2 \\ &= 2\sum_{k=1}^{n}\left(T_a^{\Delta} - T_{t_k}^{\Delta}\right)\left(T_{t_k}^{\Delta} - T_{t_{k-1}}^{\Delta}\right) + \sum_{k=1}^{n}\left(M_{t_k} - M_{t_{k-1}}\right)^4\end{aligned}$$

Because of (4.1), we have $E\left[T_a^{\Delta} - T_{t_k}^{\Delta} \mid \mathscr{F}_{t_k}\right] = E\left[\left(M_a - M_{t_k}\right)^2 \mid \cdot \mathscr{F}_{t_k}\right]$ and consequently

$$\begin{aligned}E\left[\left(T_a^{\Delta}\right)^2\right] =& 2\sum_{k=1}^{n} E\left[\left(M_a - M_{t_k}\right)^2\left(T_{t_k}^{\Delta} - T_{t_{k-1}}^{\Delta}\right)\right] \\ &+ \sum_{k=1}^{n} E\left[\left(M_{t_k} - M_{t_{k-1}}\right)^4\right] \\ \le& E\left[\left(2\sup_k \left|M_a - M_{t_k}\right|^2 + \sup_k \left|M_{t_k} - M_{t_{k-1}}\right|^2\right) T_a^{\Delta}\right]\end{aligned}$$

Let C be a constant such that $|M| \le C$; by (4.1), it is easily seen that $E\left[T_a^{\Delta}\right] \le 4C^2$ and therefore

$$E\left[\left(T_a^{\Delta}\right)^2\right] \le 12C^2 E\left[T_a^{\Delta}\right] \le 48C^4.$$

We have thus proved that for any sequence $\{\Delta_n\}$ such that $|\Delta_n| \to 0$, the sequence $\left\{T_a^{\Delta_n}\right\}$ has a limit $\langle M, M\rangle_a$ in L^2 hence in probability. It remains to prove that $\langle M, M\rangle_a$ may be chosen within its equivalence class in such a way that the resulting process $\langle M, M\rangle$ has the required properties.

Let $\{\Delta_n\}$ be as above; by Doob's inequality applied to the martingale $T^{\Delta_n} - T^{\Delta_m}$,

$$E\left[\sup_{t \leq a} \left| T_t^{\Delta_n} - T_t^{\Delta_m} \right|^2 \right] \leq 4E\left[\left(T_a^{\Delta_n} - T_a^{\Delta_m} \right)^2 \right].$$

Since, from a sequence converging in L^2, one can extract a subsequence converging a.s., there is a subsequence $\{\Delta_{n_k}\}$ such that $T_t^{\Delta_{e_k}}$ converges a.s. uniformly on $[0, a]$ to a limit $\langle M, M \rangle_t$ which perforce is a.s. continuous. Moreover, the original sequence might have been chosen such that Δ_{n+1} be a refinement of Δ_n and $\bigcup_n \Delta_n$ be dense in $[0, a]$. For any pair (s, t) in $\bigcup_n \Delta_n$ such that $s < t$, there is an n_0 such that s and t belong to Δ_n for any $n \geq n_0$. We then have $T_s^{\Delta_n} \leq T_t^{\Delta_n}$ and as a result $\langle M, M \rangle$ is non decreasing on $\bigcup_n \Delta_n$; as it is continuous, it is increasing everywhere (although the T^{Δ_n} are not necessarily non decreasing).

Finally, that $M^2 - \langle M, M \rangle$ is a martingale follows upon passing to the limit in (4.1). The proof is thus complete. □

To enlarge the scope of the above result we will need the

Proposition 4.2. *Under the assumptions of the previous theorem, for every stopping time T,*

$$\left\langle M^T, M^T \right\rangle = \langle M, M \rangle^T$$

Much as it is interesting, Theorem 4.1 is not sufficient for our purposes; it does not cover, for instance, the case of the Brownian motion B which is not a bounded martingale. Nonetheless, we have seen that B has a "quadratic variation", namely t, and that $B_t^2 - t$ is a martingale exactly as in Theorem 4.1. We now show how to subsume the case of BM and the case of bounded martingales in a single result by using the fecund idea of localization.

Definition 4.1. *An adapted, right-continuous process X is an $(\mathscr{F}_t, \mathbb{P})$-local martingale if there exist stopping times $T_n, n \geq 1$, such that*

1. *the sequence $\{T_n\}$ is increasing and $\lim_n T_n = +\infty$ a.s.*
2. *for every n, the process $X^{T_n} 1_{[T_n > 0]}$ is a uniformly integrable $(\mathscr{F}_t, \mathbb{P})$-martingale.*

We will drop $(\mathscr{F}_t, \mathbb{P})$ when there is no risk of ambiguity.

In condition 2. we can drop the uniform integrability and ask only that $X^{T_n} 1_{[T_n>0]}$ be a martingale; indeed, one can always replace T_n by $T_n \wedge n$ to obtain a u.i. martingale.

Likewise, if X is continuous as will nearly always be in this book, by setting $S_n = \inf\{t : |X_t| = n\}$ and replacing T_n by $T_n \wedge S_n$, we may assume the martingales in 2. to be bounded. This will be used extensively in the sequel.

We further say that the stopping time T reduces X if $X^T 1_{[T>0]}$ is a u.i. martingale.

This property can be decomposed in two parts if one introduces the process $Y_t = X_t - X_0 : T$ reduces X if and only if

i) X_0 is integrable on $\{T > 0\}$;

ii) Y^T is a u.i. martingale.

A common situation however is that in which X_0 is constant this explains why in the sequel we will often drop the qualifying $1_{[T>0]}$. As an exercise, the reader will show the following simple properties :

Exercise 4.1. *i) if T reduces X and $S \leq T$, then S reduces X;*

ii) the sum of two local martingales is a local martingale;

iii) if Z is a $\mathscr{F}_0$-measurable r.v. and X is a local martingale then, so is ZX; in particular, the set of local martingales is a vector space;

iv) a stopped local martingale is a local martingale;

v) a positive local martingale is a supermartingale.

We can now extend the quadratic variations to local martingales.

Theorem 4.2. *If M is a continuous local martingale, there exists a unique increasing continuous process $\langle M, M\rangle$, vanishing at zero, such that $M^2 - \langle M, M\rangle$ is a continuous local martingale. Moreover, for every t and for any sequence $\{\Delta_n\}$ of subdivisions of $[0, t]$ such that $|\Delta_n| \to 0$, the r.v.'s*

$$\sup_{s \leq t} \left| T_s^{\Delta_n}(M) - \langle M, M\rangle_s \right|$$

converge to zero in probability.

Proof. Let $\{T_n\}$ be a sequence of stopping times increasing to $+\infty$ and such that $X_n = M^{T_n} 1_{[T_n>0]}$ is a bounded martingale. By Theorem 4.1, there is, for each n, a continuous process A_n with finite variations vanishing at zero and such that $X_n^2 - A_n$ is a martingale. Now, $\left(X_{n+1}^2 - A_{n+1}\right)^{T_n} 1_{[T_n>0]}$ is a martingale and is equal to $X_n^2 - A_{n+1}^{T_n} 1_{[T_n>0]}$. By the uniqueness property in Theorem 4.1, we have $A_{n+1}^{T_n} = A_n$ on $[T_n > 0]$ and we may therefore define unambiguously a process $\langle M, M\rangle$ by setting it equal to A_n on $[T_n > 0]$. Obviously, $\left(M^{T_n}\right)^2 1_{[T_n>0]} - \langle M, M\rangle^{T_n}$ is a martingale and therefore $\langle M, M\rangle$ is the sought-after process. The uniqueness follows from the uniqueness on each interval $[0, T_n]$.

To prove the second statement, let $\delta, \varepsilon > 0$ and t be fixed. One can find a stopping time S such that $M^S 1_{[S>0]}$ is bounded and $P[S \le t] \le \delta$. Since $T^\Delta(M)$ and $\langle M, M\rangle$ coincide with $T^\Delta\left(M^S\right)$ and $\left\langle M^S, M^S\right\rangle$ on $[0, S]$, we have

$$P\left[\sup_{s\le t}\left|T_s^\Delta(M) - \langle M, M\rangle_s\right| > \varepsilon\right] \le \delta + P\left[\sup_{s\le t}\left|T_s^\Delta\left(M^S\right) - \left\langle M^S, M^S\right\rangle_s\right| > \varepsilon\right]$$

and the last term goes to zero as $|\Delta|$ tends to zero. □