M2R Examination A8 Novembre the 28 th 2024,

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3 hours examination.

You can write in French or in English.

Exercice 1 (8 points)

Let $\left(X_t^H\right)_{t\in\mathbb{R}}$ be a Gaussian process such that

$$\mathbb{E}\left(\mathrm{X}_s^H \mathrm{X}_t^H\right) = 1/2\left\{|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right\}$$

for $0 < H < 1$ and $\forall t \in \mathbb{R}, \quad \mathbb{E}X_t^H = 0$.

1. Show that $\forall a > 0$, $\left(\mathrm{X}_{at}^H\right)_{t\in\mathbb{R}} \overset{(d)}{=} a^H \left(\mathrm{X}_t^H\right)_{t\in\mathbb{R}}$ where $\overset{(d)}{=}$ is the equality of the distribution of the processes.

2. Show that $\mathbf{E}\left(\mathrm{X}_s^H - X_t^H\right)^2 = |t-s|^{2H}$.

3. Show that if $H > \frac{1}{2}$ $\quad \mathbb{P}$ a.s. there exists a modification of $\left(\mathrm{X}_t^H\right)_{0\le t\le T}$ which is locally Hölder with exponent $\gamma < H - 1/2$.

4. Show that

$$\mathrm{E}\left(\mathrm{X}_s^H - X_t^H\right)^{2n} = C_n|t-s|^{2Hn}$$

where $C_n = \mathbf{E}N^{2n}$ and N is a standard centered Gaussian random variable. Deduce from this fact that we can improve the result from 3. so that $\gamma < H$.

Exercise 2 (8 points)

Let $B = (B_t, t \ge 0)$ be a Brownian motion for a filtration $\mathcal{F} = (\mathcal{F}_t, t \ge 0)$ which satisfies the usual conditions. Let

$$M_t = \frac{e^{-B_t^2/2(1-t)}}{\sqrt{1-t}} \quad t \in [0,1[.$$

1. Show that almost surely $\lim_{t\to 1^-} M_t = 0$.

2. Show that $(M_t, 0 \le t < 1)$ is a local martingale.

3. Compute $\langle M, M\rangle_t$ pour $t < 1$. Deduce that $(M_t, 0 \le t < 1)$ is a martingale.

4. Deduce that $\mathbb{E}\left[\sup_{0\le t<1} M_t\right] = +\infty$. (You can argue by contradiction)

Exercise 3 (8 points)

Let $(X_t)_{t\ge 0}$ be a standard Brownian motion starting from 0. Let

$$A_n(K,N) = \left\{\omega, \exists s \in \left[0, N - N/2^{n-1}\right[\text{ such that } |t-s| \le N/2^{n-1} \Rightarrow |X_t(\omega) - X_s(\omega)| \le K|t-s|\right\}, \quad (1)$$

for K, N positive integers and n an integer such that $n \ge 2$.

1. Show that$\forall N \in \mathbb{N}^*$,

$$\{\omega, \exists t_0 \in [0, N[\text{ such that } X_t(\omega) \text{ is differentiable at } t_0\} \subset \cup_{K=1}^{+\infty} \limsup_{n\to+\infty} A_n(K,N). \quad (2)$$

Let us consider that differentiable at 0 means right differentiable at 0. Let us fix K et N and shorten $A_n(K,N)$ in A_n in what follows.

2. Let $Y_{n,k} = \max_{l=0,1,2}\left|X_{(k+l)N/2^n} - X_{(k+l-1)N/2^n}\right|$ for $k = 1, \dots, 2^n - 2$, and for $k = 0$ let $Y_{n,0} = \max_{l=1,2}\left|X_{lN/2^n} - X_{(l-1)N/2^n}\right|$.

Let us define $B_n = \cup_{k=0}^{2^n-2}\{Y_{n,k} \le 4KN/2^n\}$ for $n \ge 2$. Show that $A_n \subset B_n$.

3. Let $a_n = \mathbf{P}\left(\left|X_{N/2^n}\right| \le 4KN/2^n\right)$. Show that $\mathbf{P}(B_n) \le a_n^2 + (2^n - 2)a_n^3$ for $n \ge 2$.

4. Show that $a_n \le 8K(N/2^n)^{1/2}(2\pi)^{-1/2}$.

5. Show that $\sum_{n=2}^{\infty} \mathbf{P}(B_n) < +\infty$.

6. Deduce that $\mathbf{P}\left(\limsup_{n\to+\infty} B_n\right) = 0$.

7. Deduce that the sample paths of a standard Brownian motion starting from zero are almost surely nowhere differentiable.