Correction Feuille 1 : Systèmes linéaires, Pivot de Gauss

19 Janvier 2016

Correction 1. 0.

$$\left\{\begin{array}{rrrcr} & 2y & +3z & = & 2 \\ -2x & -4y & +3z & = & 4 \\ x & +y & -3z & = & -3 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{rrrcr} x & +y & -3z & = & -3 \\ & 2y & +3z & = & 2 \\ -2x & -4y & +3z & = & 4 \end{array}\right. \quad (L1) \longleftrightarrow (L3)$$

$$\Longleftrightarrow \left\{\begin{array}{rrrcr} x & +y & -3z & = & -3 \\ & 2y & +3z & = & 2 \\ & -2y & -3z & = & -2 \end{array}\right. \quad (L3) \longleftarrow (L3) + 2(L1)$$

$$\Longleftrightarrow \left\{\begin{array}{rrrcr} x & +y & -3z & = & -3 \\ & 2y & +3z & = & 2 \\ & & 0 & = & 0 \end{array}\right. \quad (L3) \longleftarrow (L3) + (L2)$$

$$\Longleftrightarrow \left\{\begin{array}{rrrcr} x & +y & -3z & = & -3 \\ & 2y & +3z & = & 2 \end{array}\right. \quad \begin{array}{c}\text{(pivot nul pour la troisième variable} \\ = \text{variable indépendante)}\end{array}$$

$$\Longleftrightarrow \left\{\begin{array}{rcl} x & = & \frac{9}{2} - 4\lambda \\ y & = & 1 - \frac{3}{2}\lambda \\ z & = & \lambda,\ \lambda \in \mathbb{R} \end{array}\right.$$

L'ensemble des solutions est donc

$$\Sigma_0 = \left\{(4,1,0) + \lambda\left(\frac{9}{2}, -\frac{3}{2}, 1\right), \lambda \in \mathbb{R}\right\}.$$

1.

$$\left\{\begin{array}{rrrcr} x & & +z & = & 1 \\ & y & +z & = & 0 \\ x & +y & & = & 1 \\ 2x & +3y & & = & 0 \end{array}\right. \Longleftrightarrow \left\{\begin{array}{rrrcr} x & & +z & = & 1 \\ & y & +z & = & 0 \\ & y & -z & = & 0 \\ & 3y & -2z & = & -2 \end{array}\right. \quad \begin{array}{l} \\ \\ (L3) \longleftarrow (L3) - (L1) \\ (L4) \longleftarrow (L4) - 2(L1) \end{array}$$

$$\Longleftrightarrow \left\{\begin{array}{rrrcr} x & & +z & = & 1 \\ y & = & z & = & 0 \\ & 3y & -2z & = & -2 \end{array}\right.$$

On en déduit que le système (S_1) n'admet pas de solution.

2.

$$\left\{\begin{array}{rrrrrr} x & +y & +z & +t & = & 1 \\ x & -y & -z & +t & = & 1 \\ -x & -y & +z & +t & = & -3 \\ -3x & +y & -3z & -6t & = & 2 \end{array}\right. \iff \left\{\begin{array}{rrrrrrl} x & +y & +z & +t & = & 1 & \\ & -2y & -2z & & = & 0 & (L2) \longleftarrow (L2)-(L1) \\ & & 2z & +2t & = & -2 & (L3) \longleftarrow (L3)+(L1) \\ & 4y & & -3t & = & 5 & (L4) \longleftarrow (L4)+3(L1) \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrrrrl} x & +y & +z & +t & = & 1 & \\ & -2y & -2z & & = & 0 & \\ & & 2z & +2t & = & -2 & \\ & & -4z & -3t & = & 5 & (L4) \longleftarrow (L4)+2(L1) \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrrrrl} x & +y & +z & +t & = & 1 & \\ & -2y & -2z & & = & 0 & \\ & & 2z & +2t & = & -2 & \\ & & & t & = & 1 & (L4) \longleftarrow (L4)+2(L2) \end{array}\right.$$

$$\iff \left\{\begin{array}{rrr} x & = & 0 \\ y & = & 2 \\ z & = & -2 \\ t & = & 1 \end{array}\right.$$

L'ensemble des solutions est donc

$$\Sigma_2 = \{(0,2,-2,1)\}.$$

3.

$$\left\{\begin{array}{rrrrr} -8x & +11y & +5z & = & 1 \\ -2x & +3y & +3z & = & 0 \\ x & -2y & -5z & = & -1 \end{array}\right. \iff \left\{\begin{array}{rrrrrl} x & -2y & -5z & = & -1 & (L1) \longleftrightarrow (L3) \\ -8x & +11y & +5z & = & 1 & \\ -2x & +3y & +3z & = & 0 & \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrrrl} x & -2y & -5z & = & -1 & \\ & -5y & -35z & = & -7 & (L2) \longleftarrow (L2)+8(L1) \\ & -y & -7z & = & -2 & (L3) \longleftarrow (L3)+2(L1) \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrrrl} x & -2y & -5z & = & -1 & \\ & -y & -7z & = & -2 & (L2) \longleftrightarrow (L3) \\ & -5y & -35z & = & -7 & \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrrrl} x & -2y & -5z & = & -1 & \\ & -y & -7z & = & -2 & \\ & & 0 & = & 3 & (L3) \longleftarrow (L3)-5(L5) \end{array}\right.$$

4.

$$\left\{\begin{array}{rrrcr} 2x & +5y & -8z & = & 8 \\ 4x & +3y & -9z & = & 9 \\ 2x & +3y & -5z & = & 7 \\ x & +8y & -7z & = & 12 \end{array}\right. \iff \left\{\begin{array}{rrrcrl} x & +8y & -7z & = & 12 & (L1) \longleftarrow (L2) \\ 2x & +5y & -8z & = & 8 & \\ 4x & +3y & -9z & = & 9 & \\ 2x & +3y & -5z & = & 7 & \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrcrl} x & +8y & -7z & = & 12 & \\ & -11y & +6z & = & -16 & (L2) \longleftarrow (L2) - 2(L1) \\ & -29y & +19z & = & -39 & (L3) \longleftarrow (L3) - 4(L1) \\ & -13y & +9z & = & -17 & (L4) \longleftarrow (L4) - 2(L1) \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrcrl} x & +8y & -7z & = & 12 & \\ & 2y & -3z & = & 1 & (L2) \longleftarrow (L2) - (L3) \\ & -3y & +z & = & -5 & (L3) \longleftarrow (L3) - 2(L2) \\ & -13y & +9z & = & -17 & \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrcrl} x & +8y & -7z & = & 12 & \\ & 2y & -3z & = & 1 & \\ & -3y & +z & = & -5 & \\ & -y & +5z & = & 3 & (L4) \longleftarrow (L4) - 4(L3) \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrcrl} x & +8y & -7z & = & 12 & \\ & y & -5z & = & -3 & (L2) \longleftrightarrow (L3) \\ & 2y & -3z & = & 1 & \\ & -3y & +z & = & -5 & \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrcrl} x & +8y & -7z & = & 12 & \\ & y & -5z & = & -3 & \\ & & 7z & = & 7 & (L3) \longleftarrow (L3) - 2(L2) \\ & & -14z & = & -14 & (L4) \longleftarrow (L4) + 3(L2) \end{array}\right.$$

$$\iff \left\{\begin{array}{rcr} x & = & 3 \\ y & = & 2 \\ z & = & 1 \end{array}\right.$$

Correction 2. Soient x et y les quantités d'alliages 1 et 2. Pour que l'alliage obtenu en fondant x.g de l'alliage 1 et y.g de l'alliage 2 soit de 100.g et composé à moitié d'argent, (x, y) doit être solution du système

$$\left\{\begin{array}{rrcr} x & +y & = & 100 \\ 35x & +60y & = & 5000 \end{array}\right. \iff \left\{\begin{array}{rrcr} x & +y & = & 100 \\ & 25y & = & 1500 \end{array}\right. \iff \left\{\begin{array}{rcr} x & = & 40 \\ y & = & 60 \end{array}\right.$$

Correction 3.

$$(x,y,z,t)\in\Sigma_m \iff \left\{\begin{array}{rrrrcl} x & +2y & -z & +2t & = & 1\\ 3x & +y & +z & +2t & = & 3\\ x & -3y & +3z & -t & = & 1\\ 5x & +5y & -z & +7t & = & m \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrrclr} x & +2y & -z & +2t & = & 1 & \\ & -5y & +4z & -4t & = & 0 & (L2)\longleftarrow(L2)-3(L1)\\ & -5y & +4z & -3t & = & 0 & (L3)\longleftarrow(L3)-(L1)\\ & -5y & +4z & -3t & = & m-5 & (L4)\longleftarrow(L4)-5(L1) \end{array}\right.$$

On en déduit que si $m\neq 5$, alors $\Sigma_m=\emptyset$. Supposons maintenant $m=5$.

$$(x,y,z,t)\in\Sigma_5 \iff \left\{\begin{array}{rrrrcl} x & +2y & -z & +2t & = & 1\\ & -5y & +4z & -4t & = & 0\\ & & & t & = & 0 \end{array}\right.$$

$$\iff \left\{\begin{array}{rcll} x & = & 1-\frac{3}{5}\lambda & \\ y & = & \frac{4}{5}\lambda & \\ z & = & \lambda, & \lambda\in\mathbb{R},\\ t & = & 0 & \end{array}\right.$$

On conclut que le système (S) admet au moins une solution si et seulement si $m=5$.

Correction 4. Soit Σ_p l'ensemble des solutions du système considéré.

$$(x,y,z)\in\Sigma_p \iff \left\{\begin{array}{rrrcl} x & +py & +(2+p)z & = & 1\\ & y & +(3p-1)z & = & 1\\ x & +2y & +5pz & = & 3 \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrclr} x & +py & +(2+p)z & = & 1 & \\ & y & +(3p-1)z & = & 1 & \\ & (2-p)y & +2(2p-1)z & = & 2 & (L3)\longleftarrow(L3)-(L1) \end{array}\right.$$

$$\iff \left\{\begin{array}{rrrclr} x & +py & +(2+p)z & = & 1 & \\ & y & +(3p-1)z & = & 1 & \\ & & 3p\,(p-1)\,z & = & p & (L3)\longleftarrow(L3)-(2-p)(L2) \end{array}\right.$$

Si $p=1$, alors le système n'a pas de solution. Si $p=0$, on a

$$(x,y,z)\in\Sigma_0 \iff \left\{\begin{array}{rrrcl} x & & +2z & = & 1\\ & y & -z & = & 1 \end{array}\right.$$

$$\iff \left\{\begin{array}{rcl} x & = & 1-2\lambda\\ y & = & 1+\lambda\\ z & = & \lambda,\lambda\in\mathbb{R} \end{array}\right.$$

Supposons $p \notin \{0,1\}$. On a alors,

$$(x,y,z) \in \Sigma_p \iff \left\{\begin{array}{ccccc} x & +py & +(2+p)z & = & 1 \\ & y & +(3p-1)z & = & 1 \\ z & = & \frac{1}{3(p-1)} & & \end{array}\right.$$

On en déduit que Σ_p admet un unique élément. En conclusion : si $p = 1$ alors le système n'a pas de solution. Si $p = 0$, il en admet une infinité. Si $p \notin \{0,1\}$, alors il admet une unique solution.

Correction 5. Soit $\mathcal{E}_a$ l'ensemble des solutions du système (S_a).

$$(x,y,z) \in \mathcal{E}_a \iff \left\{\begin{array}{ccccc} ax & +y & +z & = & 1 \\ x & +ay & +z & = & a \\ x & +y & +az & = & a^2 \\ x & +ay & +az & = & a \\ ax & +y & +az & = & 1 \end{array}\right.$$

$$\iff \left\{\begin{array}{ccccc} x & +ay & +z & = & a \\ ax & +y & +z & = & 1 \\ x & +y & +az & = & a^2 \\ x & +ay & +az & = & a \\ ax & +y & +az & = & 1 \end{array}\right. \qquad (L1) \longleftrightarrow (L2)$$

$$\iff \left\{\begin{array}{ccccc} x & +ay & +z & = & a \\ & (1-a^2)y & +(1-a)z & = & 1-a^2 \\ & (1-a)y & -(1-a)z & = & a(a-1) \\ & & -(1-a)z & = & 0 \\ & (1-a^2)y & & = & 1-a^2 \end{array}\right. \quad \begin{array}{l} \\ (L2) \longleftarrow (L2) - a(L1) \\ (L3) \longleftarrow (L3) - (L1) \\ (L4) \longleftarrow (L4) - (L1) \\ (L5) \longleftarrow (L5) - a(L1) \end{array}$$

Supposons $a = 1$. On a alors

$$(x,y,z) \in \mathcal{E}_1 \iff x+y+z = 1$$

$$\iff \left\{\begin{array}{ccc} x & = & 1-\lambda-\mu \\ y & = & \lambda \\ z & = & \mu, \ \lambda,\mu \in \mathbb{R} \end{array}\right.$$

On en déduit que

$$\mathcal{E}_1 = \{(1-\lambda-\mu,\lambda,\mu) : \lambda,\mu \in \mathbb{R}\}.$$

Supposons $a = -1$. On a alors,

$$(x,y,z) \in \mathcal{E}_{-1} \iff \left\{\begin{array}{ccccc} x & -y & +z & = & -1 \\ & & 2z & = & 0 \\ & 2y & -2z & = & 2 \end{array}\right.$$

$$\iff \left\{\begin{array}{ccc} x & = & 0 \\ y & = & 1 \\ z & = & 0 \end{array}\right.$$

On en conclut que
$$\mathcal{E}_{-1} = \{(0,1,0)\}\,.$$
Supposons maintenant $a \notin \{1,-1\}$.

$$(x,y,z)\in\mathcal{E}_a \iff \left\{\begin{array}{ccccl} x & +ay & +z & = & a \\ & (1+a)y & +z & = & 1+a \qquad (L_2)\longleftarrow \frac{1}{1-a}(L_2) \\ & y & -z & = & -a \qquad (L_3)\longleftarrow \frac{1}{1-a}(L_3) \\ & & z & = & 0 \qquad (L_4)\longleftarrow \frac{1}{a-1}(L_4) \\ & & y & = & 1 \end{array}\right.$$
$$\iff \left\{\begin{array}{ccc} x+ay+z & = & a \\ (1+a)y+z & = & (1+a) \\ a & = & -1 \\ z & = & 0 \\ y & = & 1 \end{array}\right.$$

On conclut que le système (S_a) n'admet pas de solution quand $a \notin \{1,-1\}$.

Correction 6. 1. (d). Si $(x_1,...,x_n)$ est une solution non nulle d'un système homogène, alors $(\lambda x_1,...,\lambda x_n)$ est une solution pour tout $\lambda\in\mathbb{R}$.
2. (b). (pas encore les outils pour justifier)
3. (b), (c) avec $m \neq 0$.
4. (c).

Correction 7. Soient x,y,z les coefficients de mathématiques, Informatique, et Physique respectivement. Ces coefficients sont solutions du système suivant

$$\left\{\begin{array}{ccccc} 10x & +9y & +12z & = & 10 \\ 5x & +10y & +13z & = & 8 \\ 12x & +8y & +20z & = & 12 \end{array}\right. \iff \left\{\begin{array}{ccccc} 5x & +10y & +13z & = & 8 \\ 10x & +9y & +12z & = & 10 \\ 12x & +8y & +20z & = & 12 \end{array}\right. \quad (L1)\longleftrightarrow(L2)$$
$$\iff \left\{\begin{array}{ccccl} 5x & +10y & +13z & = & 8 \\ & -11y & -14z & = & -6 \qquad (L2)\longleftarrow(L2)-2(L1) \\ & -16y & -\frac{56}{5}z & = & -\frac{36}{5} \qquad (L3)-\longleftarrow(L3)-\frac{12}{5}(L1) \end{array}\right.$$
$$\iff \left\{\begin{array}{ccccl} 5x & +10y & +13z & = & 8 \\ & -11y & -14z & = & -6 \\ & & \frac{56.9}{55}z & = & \frac{14.6}{55} \qquad (L3)\longleftarrow(L3)+\frac{16}{11}(L2) \end{array}\right.$$
$$\iff \left\{\begin{array}{ccc} x & = & \frac{1}{2} \\ y & = & \frac{1}{3} \\ z & = & \frac{1}{6} \end{array}\right.$$

Une autre interprétation (plus juste) du problème est que les coefficients x,y,z

satisfont

$$\left\{\begin{array}{rrrcl} 10x & +9y & +12z & = & 10(x+y+z)\\ 5x & +10y & +13z & = & 8(x+y+z)\\ 12x & +8y & +20z & = & 12(x+y+z)\end{array}\right. \iff \left\{\begin{array}{rrrcl} & -y & +2z & = & 0\\ -3x & +2y & +5z & = & 0\\ & -4y & +8z & = & 0\end{array}\right.$$

$$\iff \left\{\begin{array}{rrrcl} -3x & +2y & +5z & = & 0\\ & -y & +2z & = & 0\end{array}\right. \qquad (L1) \longleftrightarrow (L2)$$

On a supprimé la dernière équation car elle est identique à la première. On en déduit que l'on peut paramétrer l'ensemble des solutions à l'aide d'un paramètre (ici la variable z) :

$$\mathcal{E} = \{(3\lambda, 2\lambda, \lambda) : \lambda \in \mathbb{R}\}.$$

On en déduit que les coefficients sont $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. (On choisit des coefficients dont la somme est égale à 1).

Correction 8. Le système se traduit par la donnée de

$$\begin{pmatrix} 1 & m & m^2 & m^3\\ m & m^2 & m^3 & 1\\ m^2 & m^3 & 1 & m\\ m^3 & 1 & m & m^2\end{pmatrix} \quad \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Le premier pivot est 1. On effectue les opérations
$\bullet (L_2) \longleftarrow (L_2) - m(L_1)$
$\bullet (L_3) \longleftarrow (L_3) - m^2(L_1)$
$\bullet (L_4) \longleftarrow (L_4) - m^3(L_1)$
pour obtenir :

$$\begin{pmatrix} 1 & m & m^2 & m^3\\ 0 & 0 & 0 & 1-m^4\\ 0 & 0 & 1-m^4 & m(1-m^4)\\ 0 & 1-m^4 & m(1-m^4) & m^2(1-m^4)\end{pmatrix} \quad \begin{pmatrix}1\\1-m\\1-m^2\\1-m^3\end{pmatrix}$$

On applique $(L_2) \longleftrightarrow (L_4)$,

$$\begin{pmatrix} 1 & m & m^2 & m^3\\ 0 & 1-m^4 & m(1-m^4) & m^2(1-m^4)\\ 0 & 0 & 1-m^4 & m(1-m^4)\\ 0 & 0 & 0 & 1-m^4\end{pmatrix} \quad \begin{pmatrix}1\\1-m^3\\1-m^2\\1-m\end{pmatrix}$$

On effectue l'opération $(L_2) \longleftarrow (L_2) - m(L_3)$, pour obtenir,

$$\begin{pmatrix} 1 & m & m^2 & m^3\\ 0 & 1-m^4 & 0 & 0\\ 0 & 0 & 1-m^4 & m(1-m^4)\\ 0 & 0 & 0 & 1-m^4\end{pmatrix} \quad \begin{pmatrix}1\\1-m\\1-m^2\\1-m\end{pmatrix}$$

On applique $(L_3) \longleftarrow (L_3) - m(L_4)$,

$$\begin{pmatrix} 1 & m & m^2 & m^3 \\ 0 & 1-m^4 & 0 & 0 \\ 0 & 0 & 1-m^4 & 0 \\ 0 & 0 & 0 & 1-m^4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1-m \\ 1-m \\ 1-m \end{pmatrix}$$

Ceci correspond au système

$$\left\{ \begin{array}{ccc} x + my + m^2z + m^3t & = & 1 \\ (1-m^4)y & = & 1-m \\ (1-m^4)z & = & 1-m \\ (1-m^4)t & = & 1-m \end{array} \right.$$

qui est équivalent à

$$\left\{ \begin{array}{ccccc} x & = & 1-(m+m^2+m^3)\frac{1-m}{1-m^4} & = \frac{1-m}{1-m^4} \\ y & = & \frac{1-m}{1-m^4} \\ z & = & \frac{1-m}{1-m^4} \\ t & = & \frac{1-m}{1-m^4} \end{array} \right.$$

L'ensemble des solutions est donc

$$\mathcal{E} = \left\{ \left(\frac{1-m}{1-m^4}, \frac{1-m}{1-m^4}, \frac{1-m}{1-m^4}, \frac{1-m}{1-m^4} \right) \right\}.$$