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AN ASYMPTOTIC MODEL FOR THE TRANSPORT OF AN ELECTRON GAS IN A SLAB

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We study the limiting behavior of a Schrödinger–Poisson system describing a three-dimensional quantum gas that is confined along the vertical z-direction in a fine slab. The starting point is the three-dimensional Schrödinger–Poisson system with Dirichlet conditions on two horizontal planes $z = 0$ and $z = \varepsilon$, where the small parameter ε is the scale width of the slab. The limit $\varepsilon \to 0$ appears to be an infinite system of two-dimensional nonlinear Schrödinger equations. Our strategy combines a refined analysis of the Poisson kernel acting on strongly confined densities and a time-averaging process that allows us to deal with the fast time oscillations.

Keywords: Schrödinger–Poisson system; asymptotic analysis; singular perturbation; time averaging; quantum transport; slab; nanoelectronics.

AMS Subject Classification: 35Q40, 35Q55, 35B40, 35B25, 82D37

1. Introduction

1.1. *The singular perturbation problem*

In this paper, we study the behavior of a quantum gas that evolves in a fine slab. These confined electron gas are of great interest for the nanoelectronic industry as the functioning of many nanoelectronic devices relies on the confined transport of electron gas. Here, we are interested in monodimensional confinement, meaning that the transport of charged particles remains typically bidimensional. This work is more precisely devoted to the rigorous derivation of a dynamic two-dimensional quantum model with space-charge effects describing the transport of electrons confined in a fine slab, say of thickness ε.

Even if the transport of the electron gas seems typically bidimensional in our case, the space-charge effects remain three-dimensional. Our starting model is thus the three-dimensional Schrödinger–Poisson system. Let $\varepsilon > 0$ be a small parameter measuring the typical extension of the two-dimensional electron gas in the z direction, the three-dimensional space variables are denoted by $(x, z) \in \mathbb{R}^2 \times \mathbb{R}$. The electron gas is confined along the horizontal x directions and *strongly* confined along the vertical z direction between the planes $z = 0$ and $z = \varepsilon$. It is therefore subject to three different effects: both confinements along the respective vertical z direction and the horizontal x directions, and the self-consistent Poisson potential.

In order to model both confinements, the first idea is to introduce, in the three-dimensional Schrödinger–Poisson system, two smooth confining exterior potentials at two different scales.

- The horizontal confinement (in the x directions) is modeled by the potential $V_1(x)$, where V_1 is meant to be a real positive function that goes to infinity with $|x|$ (the assumptions on V_1 will be made precise later on).
- The vertical confinement (in the z direction) is modeled by a smooth potential $V_2^\varepsilon(z)$ of the form $V_2^\varepsilon(z) = (1/\varepsilon^2)\, V_2(z/\varepsilon)$, where V_2 is meant to be a smooth positive function.

Precise assumptions on these confinement potentials will be made clear later on. We therefore start with the following dimensionless Schrödinger–Poisson system (where a rescaling in the z direction has been performed):

$$
\begin{aligned}
i\partial_t \Psi^\varepsilon = {} & (-\Delta_x + V_1(x))\Psi^\varepsilon + \frac{1}{\varepsilon^2}\left(-\partial_z^2 + V_2(z)\right)\Psi^\varepsilon \\
& + V^\varepsilon \Psi^\varepsilon, \quad t > 0, \ \ (x, z) \in \Omega, \qquad (1.1)
\end{aligned}
$$

$$
\Psi^\varepsilon(t, x, 0) = \Psi^\varepsilon(t, x, 1) = 0, \quad t > 0, \ \ x \in \mathbb{R}^2, \qquad (1.2)
$$

$$
\Psi^\varepsilon(0, x, z) = \Psi_0^\varepsilon(x, z), \quad (x, z) \in \Omega, \qquad (1.3)
$$

$$
-\partial_z^2 V^\varepsilon - \varepsilon^2 \Delta_x V^\varepsilon(t) = |\Psi^\varepsilon(t)|^2, \quad t > 0, \ \ (x, z) \in \Omega, \qquad (1.4)
$$

$$
V^\varepsilon(t, x, 0) = V^\varepsilon(t, x, 1) = 0, \quad t > 0, \ \ x \in \mathbb{R}^2. \qquad (1.5)
$$

The unknown is the pair $(\Psi^\varepsilon, V^\varepsilon)$ made of the electronic wave function Ψ^ε and the self-consistent potential V^ε that models the space-charge effects. The set Ω is the dimensionless set that corresponds to the physical extension $\mathbb{R}^2 \times (0, \varepsilon)$ of the gas

$$
\Omega := \mathbb{R}^2 \times (0, 1).
$$

The main modeling assumptions, in this context, are therefore the choice of a scale between both confinement terms $V_1(x)$ and $(1/\varepsilon^2)\, V_2(z/\varepsilon)$ as well as precise assumptions on both functions V_1 and V_2.

Let us first give a few words about the rescaling that has been performed to obtain (1.1)–(1.5). We refer the reader to Refs. 3 and 21 for a model where no Dirichlet boundary condition is imposed and to Ref. 13 where a magnetic potential is added. The main idea is to introduce two characteristic energies E_{transp} and E_{conf}. Thus, E_{transp} is chosen as the typical energy of the longitudinal transport (in the x directions), the confinement potential V_1, the self-consistant effects and the time scale, whereas the kinetic energy along z and the transversal confinement potential V_2 are set to the scale $E_{\rm conf}$. Finally, consider the Schrödinger–Poisson system in physical variables

$$i\hbar\partial_{\mathbf{t}}\mathbf{\Psi} = -\frac{\hbar^2}{2m}\Delta\mathbf{\Psi} + e\mathbf{V_1}(\mathbf{x})\mathbf{\Psi} + \mathbf{V_2}(\mathbf{z})\mathbf{\Psi} + \mathbf{V}(\mathbf{x},\mathbf{z})\mathbf{\Psi}, \tag{1.6}$$

$$-\Delta\mathbf{V} = \frac{e}{\epsilon_0}\,|\mathbf{\Psi}|^2, \tag{1.7}$$

where m is the effective mass, e the elementary charge of the electron and ϵ_0 denotes the electric permittivity of the material. If ε^2 measures the rate $E_{\rm trans}/E_{\rm conf}$, then (1.6)–(1.7), respectively, becomes

$$i\partial_t\Psi^\varepsilon = (-\Delta_x + V_1(x))\Psi^\varepsilon + \frac{1}{\varepsilon^2}(-\partial_z^2\Psi^\varepsilon + V_2(z))\Psi^\varepsilon + V^\varepsilon\Psi^\varepsilon,$$

$$-\varepsilon^2\Delta_x V^\varepsilon - \partial_z^2 V^\varepsilon = \frac{e\overline{x}^2}{\epsilon_0 E_{\rm transp}} N\varepsilon^2|\Psi^\varepsilon|^2.$$

In order to avoid a trivial formal limit of the Poisson equation, we choose to work with *high densities* and set

$$N = \frac{\epsilon_0 E_{\rm transp}}{e^2\overline{x}^2}\frac{1}{\varepsilon^2},$$

which finally leads to (1.1)–(1.5).

Let us now introduce the following assumptions on both confinement potentials V_1 and V_2.

Assumption 1.1. Both potentials V_1 and V_2 are $\mathcal{C}^\infty$ non-negative functions. Moreover, the longitudinal V_1 potential satisfies

$$V_1(x) \underset{|x|\to\infty}{\longrightarrow} \infty. \tag{1.8}$$

For later functional analysis purposes, we shall assume a reinforced version of the longitudinal confinement (in the $x \in \mathbb{R}^2$ directions). We will make them clear later on (see Assumption 1.2). Note that a smooth potential of the form $V_1(x) = C|x|^s$ for $|x| \geq |x_0|$, with $C > 0$ and $s > 0$, satisfies these assumptions. In particular, we keep in mind, throughout the paper, the example of the harmonic potential $V_1 = a^2|x|^2$ that fits these conditions. What could be surprising here is, the lack of growth-at-infinity assumptions for the confinement potential in the z direction. The point is, the confinement in the z direction is due to the boundary Dirichlet conditions. In the present case, the variable z indeed lies in [0,1].

This paper aims at exhibiting an asymptotic bidimensional system for Eqs. (1.1)–(1.5) as $\varepsilon \to 0$. Let us now give short bibliographical notes. First, confined quantum electron gas has been studied in a linear setting for a long time and by several authors (see Refs. 12, 14, 16, 25 and references therein). Nonlinear problems linked to the confinement of an electron gas have been studied more recently. Indeed, the approximation of the Schrödinger–Poisson system describing an electron gas constraint in a plane was studied in Refs. 3 and 21, and when the gas is confined along a line in Ref. 2.

The problem of finding a hierarchy of asymptotic models for the transport of an electron gas confined along a plane has been treated in Ref. 3. In that paper, the authors model the confinement on a plane with a three-dimensional Schrödinger–Poisson system on the whole space $\mathbb{R}^3$ singularly perturbed with a confinement potential of form $(1/\varepsilon^2)\,V_c(z/\varepsilon)$. The key tools are a refined analysis of the Poisson nonlinearity and techniques based on the projection upon the eigenmodes of the transverse Hamiltonian. The main difference with our current problem here comes from the fact that the solution to the Poisson equation is not of the same order whether the equation holds on the whole space $\mathbb{R}^3$ or on $\mathbb{R}^2 \times (0,1)$. It indeed leads to two different range of densities: *high densities* or *densities of order* 1 (see the further discussion).

In Ref. 13, the authors give an asymptotic model for the Schrödinger–Poisson system describing a three-dimensional electron gas confined on a plane and subject to a strong uniform magnetic field lying in the transport plane. In order to deal with the fast oscillations due to the magnetic potential, they use second-order long-time averaging techniques and a Sobolev scale adapted to the confinement operator.

When the nonlinearity depends locally on the density (it is not the case of the Poisson nonlinearity here), an asymptotic model for confined Bose–Einstein condensates is studied in Refs. 4 and 1. In Ref. 7, the authors present a model describing Bose–Einstein condensation of trapped dipolar quantum gases. This model takes the form of a time-dependent Schrödinger equation including a cubic nonlinearity and a nonlocal nonlinearity under the form of a convolution of the density with a dipole-interaction kernel.

1.2. *Heuristic approach of the asymptotic model*

In this section, we aim at heuristically exhibiting an asymptotic model for the Schrödinger–Poisson system with Dirichlet conditions (1.1)–(1.5).

First of all, Eqs. (1.4) and (1.5) allow us to expect the formal limit of the three-dimensional Poisson potential V^ε to be the solution $W(|\psi^\varepsilon|^2)$ of

$$\begin{aligned} -\partial_z^2 W(t,\cdot) &= |\psi^\varepsilon(t,\cdot)|^2, \quad t \geq 0,\ (x,z) \in \Omega, \\ W(t,x,0) &= W(t,x,1) = 0, \quad t > 0,\ x \in \mathbb{R}^2. \end{aligned}$$

Consider the following model in which the Poisson equations (1.4) and (1.5) is replaced by its formal asymptotic. It will be referred to as the *intermediate model* in what follows:

$$i\partial_t \psi^\varepsilon = H_x\psi^\varepsilon + \frac{1}{\varepsilon^2} H_z\psi^\varepsilon + W(|\psi^\varepsilon|^2)\psi^\varepsilon, \quad t > 0,\ (x,z) \in \Omega, \tag{1.9}$$

$$\psi^\varepsilon(t,x,0) = \psi^\varepsilon(t,x,1) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \tag{1.10}$$

$$-\partial_z^2 W(t,\cdot) = |\psi^\varepsilon(t,\cdot)|^2, \quad t \geq 0,\ (x,z) \in \Omega, \tag{1.11}$$

$$W(t,x,0) = W(t,x,1) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \tag{1.12}$$

where H_x and H_z, respectively, denote the longitudinal Hamiltonian defined by

$$H_x := -\Delta_x + V_1(x) \quad \text{with domain } \mathcal{D}(H_x) := \{u \in H^2(\mathbb{R}^2),\, V_1 u \in L^2(\mathbb{R}^2)\} \tag{1.13}$$

and the transversal Hamiltonian defined by

$$H_z := -\partial_z^2 + V_2(z) \quad \text{with homogeneous Dirichlet boundary conditions} \tag{1.14}$$

and with domain $\mathcal{D}(H_z) := H^2 \cap H_0^1(0,1)$.

In that case, $W(|\psi^\varepsilon|^2)$ is explicit and reads, for $t > 0$ and $(x,z) \in \Omega$,

$$W(|\psi^\varepsilon|^2)(t,x,z) = \int_0^1 K(z,z')|\psi^\varepsilon(t,x,z')|^2\,dz', \tag{1.15}$$

where the kernel K denotes

$$\forall\, z,z' \in (0,1), \quad K(z,z') := z(1-z') - (z-z')\mathbb{1}_{z'\le z}.$$

Remark 1.1. The Dirichlet boundary conditions have here lead us to make high density assumptions. In Refs. 3, 21 and 13 the authors study the transport of an electron gas that is strongly confined in the z direction, where no Dirichlet boundary condition is imposed. They work with *low densities* and therefore introduce the following system that will be referred to as the "soft wall potential model" in what follows:

$$\begin{aligned} i\partial_t \Psi^\varepsilon &= (-\Delta_x + V_1(x))\Psi^\varepsilon + \frac{1}{\varepsilon^2}(-\partial_z^2\Psi^\varepsilon + V_2(z))\Psi^\varepsilon + V^\varepsilon\Psi^\varepsilon, \\ -\varepsilon^2\Delta_x V^\varepsilon &- \partial_z^2 V^\varepsilon = \varepsilon|\Psi^\varepsilon|^2. \end{aligned}$$

The Poisson equation can, in that case, be rewritten with a convolution as

$$V^\varepsilon = \frac{1}{4\pi\sqrt{|x|^2+\varepsilon^2 z^2}} * |\Psi^\varepsilon|^2,$$

whose asymptotic behavior is given by

$$V^\varepsilon(t,x,z) \sim \frac{1}{4\pi|x|} * |\psi^\varepsilon(t,x,z)|^2 = \frac{1}{4\pi|x|} *_x \left(\int_{\mathbb{R}} |\psi^\varepsilon(t,\cdot,z')\,dz'|^2\right).$$

Therefore, the asymptotic of the Poisson potential in the soft-wall potential case does not depend on z. Note that, on the contrary, in our "hard-wall potential case" the nonlinearity $W(|\psi^\varepsilon|^2)$ defined by (1.15) obviously depends on z. This dependence in z is crucial as it radically changes the nature of the analysis at hand in our work. It indeed induces fast oscillating in time terms that will not be dealt with as easily as previously.

The first step of our work is to prove the well-posedness of both systems (1.1)–(1.5) and (1.9)–(1.12) and then estimate the difference between their respective solutions in an adapted functional framework. This will *a posteriori* justify the approximation of (1.1)–(1.5) by (1.9)–(1.12). In a second part, we focus on the asymptotic behavior of the intermediate system (1.9)–(1.12).

By Assumption 1.1, $V_2(z)$ is a smooth non-negative function, and thus, the operator H_z has a discrete spectrum. In what follows, the collection of its eigenvalues is denoted by $E_p \geq 0$ and their associated eigenfunctions, chosen so as to form a Hilbertian basis of $\mathcal{D}(H_z)$, are denoted by $\chi_p(z)$, as p runs in $\mathbb{N}$. They satisfy, for any index p,

$$H_z\chi_p = (-\partial_z^2 + V_2(z))\chi_p = E_p\chi_p, \quad \chi_p(0) = \chi_p(1) = 0.$$

The second step consists in studying the asymptotics of the intermediate model (1.9)–(1.12) as ε tends to zero. The probably most natural approach is to first project the Schrödinger equation (1.9) over the orthonormal basis $(\chi_p)_{p\geq 0}$. Its decomposition over $(\chi_p)_{p\geq 0}$ reads

$$\psi^\varepsilon(t,x,z) = \sum_{p\geq 0} \psi_p^\varepsilon(t,x)\chi_p(z) \qquad \text{with } \psi_p^\varepsilon(t,x) = \langle \psi^\varepsilon(t,x,\cdot)\chi_p\rangle,$$

where we used the notation

$$\langle f\rangle := \int_0^1 f(z)\,dz.$$

Now, inserting this decomposition in the Schrödinger equation (1.9) and formally projecting over the $(\chi_p)_{p\geq 0}$ basis leads to the following infinite system of coupled nonlinear Schrödinger equations:

$$i\partial_t\psi_p^\varepsilon = H_x\psi_p^\varepsilon + \frac{E_p}{\varepsilon^2}\psi_p^\varepsilon + \sum_{s\geq 0}\psi_s^\varepsilon\left\langle W\left(\left|\sum_{q\geq 0}\psi_q^\varepsilon\chi_q\right|^2\right)\chi_s\chi_p\right\rangle. \tag{1.16}$$

In view of (1.16), $\partial_t\psi_p^\varepsilon$ has size $\mathcal{O}(1/\varepsilon^2)$. For this reason, it seems natural to filter out the time oscillations induced by the $(E_p/\varepsilon^2)\psi_p^\varepsilon$ term. Therefore, let ϕ_p^ε be defined as the filtered ψ_p^ε as follows:

$$\phi_p^\varepsilon(t,x) = \exp(itE_p/\varepsilon^2)\psi_p^\varepsilon(t,x).$$

The ϕ_p^ε's then satisfy the filtered system

$$i\partial_t\phi_p^\varepsilon = H_x\phi_p^\varepsilon + \sum_{s\geq 0} e^{-it(E_s-E_p)/\varepsilon^2}\left\langle W\left(\left|\sum_{q\geq 0} e^{-i(t/\varepsilon^2)E_q}\phi_q^\varepsilon\chi_q\right|^2\right)\chi_s\chi_p\right\rangle\phi_s^\varepsilon. \tag{1.17}$$

However, according to definition (1.15), we have

$$\begin{aligned} W\left(\left|\sum_{q\geq 0} e^{-i(t/\varepsilon^2)E_q}\phi_q^\varepsilon\chi_q\right|^2\right) &= W\left(\sum_{q\geq 0}\sum_{r\geq 0} e^{-it(E_r-E_q)/\varepsilon^2}\phi_r^\varepsilon\overline{\phi_q^\varepsilon}\chi_r\chi_q\right) \\ &= \sum_{q\geq 0}\sum_{r\geq 0}\langle K(z,\cdot)\chi_r\chi_q\rangle\, e^{-it(E_r-E_q)/\varepsilon^2}\phi_r^\varepsilon\overline{\phi_q^\varepsilon}. \end{aligned} \tag{1.18}$$

Finally, combining (1.17) with (1.18) allows us to conclude under nice regularity assumptions, and provided the series at hand in (1.18) converge, that the ϕ_p^ε satisfy the following infinite system:

$$\forall\, p \geq 0, \quad i\partial_t \phi_p^\varepsilon = H_x \phi_p^\varepsilon + \sum_{s\geq 0}\sum_{q\geq 0}\sum_{r\geq 0} \alpha_{p,q,r,s}\, e^{it(E_p+E_q-E_r-E_s)/\varepsilon^2} \phi_r^\varepsilon \overline{\phi_q^\varepsilon} \phi_s^\varepsilon, \tag{1.19}$$

$$\phi_p^\varepsilon(0,x) = \psi_{0,p} := \langle \psi_0(x,\cdot)\chi_p\rangle, \tag{1.20}$$

where

$$\forall\, p,q,r,s \geq 0, \quad \alpha_{p,q,r,s} := \int_0^1\int_0^1 K(z,z')\chi_r(z')\chi_q(z')\chi_s(z)\chi_p(z)\,dz dz'. \tag{1.21}$$

Now that each $\partial_t \phi_p^\varepsilon$ is of order $\mathcal{O}(1)$, note that the infinite system of coupled nonlinear Schrödinger equation satisfied by the ϕ_p^ε's ($p \in \mathbb{N}$) is of the form

$$\partial_t u^\varepsilon = Au^\varepsilon + B(t/\varepsilon^2, u^\varepsilon), \tag{1.22}$$

where the nonlinearity B happens to have some kind of periodicity in time due to the oscillatory $e^{it(E_p+E_q-E_r-E_s)/\varepsilon^2}$ factor. More precisely, as we will see in the following parts, the nonlinearity is *almost periodic* in time.

It now becomes quite tempting to average in time Eq. (1.17) or, equivalently, the toy model (1.22). Here, we use a key tool developed in Ref. 1, adapted from the well-detailed work on the ODEs in Ref. 23 and from Schochet's work Ref. 24. Assume that the function $B(\tau, u)$ entering in (1.22) possesses some ergodicity in time, i.e. that one can define, in a functional framework (precised later on), the limit

$$B_{\mathrm{av}}(u) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T B(\tau,u)\,d\tau.$$

Then, the toy model (1.22) converges, as ε tends to zero, toward the following limit system:

$$\partial_t u = Au + B_{\mathrm{av}}(u). \tag{1.23}$$

For these reasons, and despite the differential system satisfied by the ϕ_p^ε's is infinite, we can expect the ϕ_p^ε's solving (1.17) to converge at least formally towards the solution of the following infinite averaged system:

$$i\partial_t \phi_p = H_x\phi_p + \sum\sum\sum_{(q,r,s)\in\Lambda_p} \alpha_{p,q,r,s}\phi_r \overline{\phi_q}\phi_s, \quad t>0,\ (x,z)\in\Omega, \tag{1.24}$$

$$\phi_p(0,x) = \langle \psi_0(x,\cdot)\chi_p\rangle, \quad x \in \mathbb{R}^2, \tag{1.25}$$

where

$$\forall\, p\geq 0, \quad \Lambda_p := \{(q,r,s)\in\mathbb{N}^3,\ E_p+E_q = E_r+E_s\}. \tag{1.26}$$

This paper therefore aims at rigorously proving the convergence towards (1.24) in an appropriate framework.

1.3. *Statement of the main results and sketch of the proof*

Let us add the following technical assumptions on the longitudinal confinement potential $V_1(x)$.

Assumption 1.2.

$$\forall\, \alpha \in \mathbb{N}^2, \quad \frac{\partial^\alpha V_1}{\partial x^\alpha}(x) = \mathcal{O}(V_1(x)) \ \text{ as } |x| \to \infty,$$
$$\exists\, M_x > 0, \quad V_1(x) = \mathcal{O}\big(|x|^{M_x}\big) \ \text{ as } |x| \to \infty,$$
$$\exists\, M_x' > 0, \quad \frac{|\nabla_x V_1(x)|}{V_1(x)} = \mathcal{O}\big(|x|^{-M_x'}\big) \ \text{ as } |x| \to \infty.$$

Note that these assumptions are purely technical helps in order to carry out a functional analysis, inspired by Ref. 2, which will help us identify the Sobolev spaces well-adapted to our operators H_x and H_z.

Theorem 1.1. *Convergence towards the asymptotic model. Under Assumptions* 1.1 *and* 1.2, *fix* $\varepsilon > 0$ *and consider a function* $\psi_0(x, z)$ *in*

$$\mathcal{X} := \{u \in H^2(\Omega) \cap H_0^1(\Omega),\, V_1 u \in L^2(\Omega)\} \tag{1.27}$$

equipped with the norm

$$\|u\|_{\mathcal{X}}^2 := \|u\|_{H^2(\Omega)}^2 + \|V_1 u\|_{L^2(\Omega)}^2. \tag{1.28}$$

Then, there exists $T > 0$ *depending only on* $\|\psi_0\|_{\mathcal{X}}$ *such that the following holds.*

- *The initial Schrödinger–Poisson system* (1.1)–(1.5) *with initial datum* ψ_0 *possess a unique solution denoted by* $(\psi^\varepsilon, V^\varepsilon)$, *that is bounded in* $\mathcal{C}([0, T], \mathcal{X})$ *uniformly in* ε.
- *The asymptotic system* (1.24)–(1.25) *with initial datum* ψ_0 *admits a unique solution denoted by the set of functions* $(\phi_p)_{p\geq 0} \in \mathcal{C}([0, T], \mathcal{D}(H_x))$ *where* $\mathcal{D}(H_x)$ *is defined by* (1.13).
- *If* ψ^ε *and* $(\phi_p)_{p\geq 0}$ *denote the respective solutions to* (1.1)–(1.5) *and* (1.24)–(1.25), *then the following convergence holds*:

$$\left\| \psi^\varepsilon(t, x, z) - \sum_{p\geq 0} \phi_p(t, x) e^{-itE_p/\varepsilon^2} \chi_p(z) \right\|_{\mathcal{C}([0,T],\mathcal{X})} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{1.29}$$

Sketch of the proof. The present paper is devoted to rigorously proving the convergence of (1.1)−(1.5) towards (1.24)−(1.25) with given initial data, in three steps. As a first step, we follow the ideas already used in Ref. 2 in the case of a Schrödinger−Poisson equation with a confinement potential that models the two-dimensional confinement in a nanowire, and in Ref. 13 in the case of a strongly confined bidimensional electron gas under a strong magnetic field. The first key tool is a refined analysis of the rescaled Poisson potential, defined in (1.4)−(1.5) and its asymptotics. First, we prove tame estimates for both nonlinearities V^ε and W, respectively, defined by (1.4)−(1.5) and (1.11)−(1.12). These estimates allow us to study the well-posedness and regularity of solutions to both initial (1.1)−(1.5) and intermediate (1.9)−(1.12) models (see Lemma 2.4). Then, we rewrite (1.1) as a perturbation of (1.9) by rewriting $V^\varepsilon(t,x,z)$ as a perturbation of W and estimating the remainder $V^\varepsilon - W$ as ε tends to zero.

As a second step, we prove the convergence of the solution to the intermediate system towards the function $\sum_p \phi_p e^{-itE_p/\varepsilon^2}\chi_p$, where the $(\phi_p)_{p\geq 0}$ solve (1.24)−(1.26). The difficulty of making the heuristic arguments rigorous is twofold. First, it requires to decompose ψ^ε over the χ_p's and therefore to write down series expansion of form $\sum_p \cdots$ as in (1.17), (1.18) or (1.19). However, it happens to be very difficult to control the convergence of these series expansions, even when nice estimates on the ψ_p^ε's are at hand. This is due to the lack of information on the behavior of the term $\langle W(|\psi^\varepsilon|^2)\chi_r, \chi_p\rangle$ for large values of p and r. Secondly, independent of the Schrödinger equation, when proving the convergence of systems of form (1.22) towards (1.23), one usually needs *small denominator estimates* which turn out to be very difficult to handle with in the present context. Here, we follow the same lines as is done in Ref. 1 in the case of a general nonlinear Schrödinger equation with a nonlinearity of the form $F(u)$ where F is a $\mathcal{C}^\infty$-function. To sum up, the first idea is to filter out the time-oscillations in (1.9) by defining

$$\phi^\varepsilon(t,x,z) := e^{itH_z/\varepsilon^2}\psi^\varepsilon(t,x,z). \tag{1.30}$$

It now satisfies

$$i\partial_t\phi^\varepsilon = H_x\phi^\varepsilon + e^{itH_z/\varepsilon^2}\, V\big(|e^{-itH_z/\varepsilon^2}\phi^\varepsilon|^2\big)\, e^{-itH_z/\varepsilon^2}\phi^\varepsilon. \tag{1.31}$$

Then, introducing the nonlinearity

$$\tau \geq 0 \mapsto G(\tau,u) := e^{i\tau H_z}\, W(|e^{-i\tau H_z}u|^2)\, e^{-i\tau H_z}u, \tag{1.32}$$

Eq. (1.31) can therefore be approached, by the first step by

$$i\partial_t\phi^\varepsilon = H_x\phi^\varepsilon + G\left(\frac{t}{\varepsilon^2}, \phi^\varepsilon\right), \tag{1.33}$$

$$\phi^\varepsilon(t,x,0) = \phi^\varepsilon(t,x,1) = 0, \quad t>0,\ x\in\mathbb{R}^2, \tag{1.34}$$

$$\phi^\varepsilon(0,x,z) = \psi_0(x,z), \quad (x,z)\in\Omega, \tag{1.35}$$

which is of form (1.22). The key point is therefore to define a functional framework, say a functional space Z such that if $u\in Z$, then the to-be-averaged function $G(\tau,u)$ is *almost-periodic in time with values in* Z. This roughly means that $G(\tau,u)$ has *comptably many frequencies in* τ, which in fact translates the fact that H_z has a discrete spectrum. Indeed, the only oscillation terms that appear in the definition of $G(\tau,u)$ are due to the propagator $e^{\pm i\tau H_z}$. The important fact about almost-periodic functions is that they possess a *well-defined long time average*, and the formula

$$G_{\mathrm{av}}(u) = \lim_{T\to\infty}\frac{1}{T}\int_0^T G(\tau,u)\,d\tau$$

makes sense in appropriate functional spaces. Section 3 is therefore devoted to the time averaging of Eq. (1.33). In order to carry out the time-averaging procedure, we prove that we are dealing with almost-periodic nonlinearities (hence the possibility of defining long-time averages), with values in good Sobolev spaces (spaces that are adapted to our operators and allow us to carry out the nonlinear analysis). These spaces are defined in Sec. 2.1, and the properties of almost periodicity of our nonlinearities are proved in Proposition 3.3. The time-averaging process leads to the convergence result stated in Proposition 3.4 for an initial datum that lies in a regularized space denoted by $\mathcal{Y}$.

As a third step, Sec. 4 is devoted to gathering the results of both first and second step. In this section, we therefore prove Proposition 4.1 that states a convergence result in $\mathcal{X}$ of the initial system towards the limit one with an initial datum in the regularized space $\mathcal{Y}$. The second part of Sec. 4 ends the proof of the main theorem with a regularizing procedure.

2. Approximation by the Intermediate System

In this section, we focus our study on the approximation of the initial system (1.1)–(1.5) by the intermediate system (1.9)–(1.12). Lemma 2.1 first introduces a functional framework that is well-adapted to the operators H_x and H_z in order to deal with both nonlinearities $V^\varepsilon(|\psi^\varepsilon|^2)$ and $W(|\psi^\varepsilon|^2)$ (defined by (1.4)–(1.5) and (1.11)–(1.12), respectively). Then, Lemma 2.4 states regularity results for the Poisson equation that allow us to prove tame estimates on both nonlinearities in Corollary 2.1. Finally, Proposition 2.1 proves the well-posedness of both systems and estimates, in this framework, the difference between the solutions of the initial system (1.1)–(1.5) and the intermediate system (1.9)–(1.10).

2.1. *Preliminaries: The functional framework*

In this section, we aim at defining a Sobolev scale adapted to both operators H_x and H_z. Indeed, the only uniform-in-ε bound at hand on ψ^ε, solution to (1.1)–(1.5) reads

$$\|\psi^\varepsilon\|^2_{L^2(\Omega)} + \|H_x\psi^\varepsilon\|^2_{L^2(\Omega)} + \|H_z\psi^\varepsilon\|^2_{L^2(\Omega)} = \mathcal{O}(1)$$

on some nontrivial time interval $[0, t]$ whenever $\|\psi_0\|^2_{L^2(\Omega)} + \|H_x\psi_0\|^2_{L^2(\Omega)} + \|H_z\psi_0\|^2_{L^2(\Omega)}$ is bounded, ψ_0 denoting the initial datum. All other energy estimates (obtained by simply applying the operators $\partial_z, \nabla_x, V_1(x)$ or $V_2(z)$ to Eqs. (1.1)–(1.5) and integrating by part) give rise to commutators, hence diverging factors of order $\mathcal{O}(1/\varepsilon^2)$ due to the term $(1/\varepsilon^2)H_z$. Therefore, as they only give access to bounds of order $\mathcal{O}(1/\varepsilon)$, they are barely useless here. It therefore seems natural to consider the energy space

$$\{u \in L^2(\Omega),\ H_x u \in L^2(\Omega),\ H_z u \in L^2(\Omega)\}$$

equipped with the norm

$$\|u\|^2 := \|u\|^2_{L^2(\Omega)} + \|H_x u\|^2_{L^2(\Omega)} + \|H_z u\|^2_{L^2(\Omega)}. \tag{2.1}$$

The first task here lies in the identification of this space and $(\mathcal{X}, \| \ \|_{\mathcal{X}})$ defined by Eqs. (1.27) and (1.28). We indeed show that these spaces can be identified and that both norms are equivalent. Moreover, we also need a regularization space that we denote by $\mathcal{Y}$, i.e.

$$\mathcal{Y} := \{u \in L^2(\Omega), H_x u \in \mathcal{X}, H_z^{1+\alpha/2} u \in L^2(\Omega)\} \tag{2.2}$$

equipped with the norm

$$\|u\|_{\mathcal{Y}}^2 := \|u\|_{L^2(\Omega)}^2 + \|H_x u\|_{\mathcal{X}}^2 + \|H_z^{1+\alpha/2} u\|_{L^2(\Omega)}^2, \tag{2.3}$$

where $\alpha \in \mathbb{R}$, such that $0 < \alpha < 1/2$, that also needs to be identified as a Sobolev space with additional growth at infinity assumptions (of kind $V_1^{1+\alpha/2} u \in L^2(\Omega)$). The equivalence of both norms $\|u\|_{\mathcal{X}}$ and $\|u\|$ and the identification of $\|u\|_{\mathcal{Y}}$ may be technically delicate, yet, it is absolutely crucial here. In that prospect, we refer the reader to Ref. 1. In this paper, the authors identify the Sobolev scale adapted to their own operators that are $-\Delta_x + V_1(x)$ with domain $\{u \in L^2(\mathbb{R}^2), -\Delta_x u \in L^2(\mathbb{R}^3), V_1 u \in L^2(\mathbb{R}^2)\}$, and $-\partial_z^2 + V_2$ with domain $\{u \in L^2(\mathbb{R}), \partial_z^2 u \in L^2(\mathbb{R}), V_2 u \in L^2(\mathbb{R})\}$. The only difference with our situation is therefore the fact that the transverse operator acts on $L^2(\mathbb{R})$, instead of $L^2(0,1)$ with boundary Dirichlet conditions. The key tool they use is the Weyl–Hörmander calculus, and, following the same arguments, we can state the following lemma.

Lemma 2.1. (Equivalence of norms) *For all $u \in \mathcal{X}$, both norms*

$$\|u\|^2 := \|u\|_{L^2(\Omega)}^2 + \|H_x u\|_{L^2(\Omega)}^2 + \|H_z u\|_{L^2(\Omega)}^2$$

and

$$\|u\|_{\mathcal{X}}^2 := \|u\|_{H^2(\Omega)}^2 + \|V_1 u\|_{L^2(\Omega)}^2$$

are equivalent. Moreover, if $u \in \mathcal{Y}$ defined by (2.2), then the following equivalence holds:

$$\|u\|_{\mathcal{Y}}^2 \sim \|u\|_{H^{2+\alpha}(\Omega)}^2 + \|H_x u\|_{\mathcal{X}}^2. \tag{2.4}$$

Remark 2.1. Let $0 < \alpha < 1/2$ and $u = \sum_{p\geq 0} u_p \chi_p$ be in $H_0^1(\Omega)$, with $u_p(x) = \int_0^1 u(x,z)\chi_p(z)\,dz$. Note that, according to Ref. 17 or Ref. 18,

$$\|H_z^{1+\alpha/2} u\|_{L^2(\Omega)}^2 = \sum_{p\geq 0} E_p^{2+\alpha} \|u_p\|_{L^2(\mathbb{R}^2)}^2.$$

Therefore, definitions (2.1) and (2.3) become

$$\|u\|^2 = \sum_{p\geq 0} (1 + E_p^2) \|u_p\|_{L^2(\mathbb{R}^2)}^2 + \|H_x u\|_{L^2(\Omega)}^2$$

and

$$\|u\|_{\mathcal{Y}}^2 = \sum_{p\geq 0} (1 + E_p^{2+\alpha}) \|u_p\|_{L^2(\mathbb{R}^2)}^2 + \|H_x u\|_{\mathcal{X}}^2. \tag{2.5}$$

Combined with the equivalences stated in Lemma 2.1, the following equivalences hold:

$$\|u\|_{\mathcal{X}}^2 \sim \sum_{p\geq 0}(1+E_p^2)\|u_p\|_{L^2(\mathbb{R}^2)}^2 + \|H_x u\|_{L^2(\Omega)}^2 \tag{2.6}$$

and

$$\|u\|_{\mathcal{Y}}^2 \sim \|u\|_{H^2(\Omega)}^2 + \|\Delta_x u\|_{H^2(\Omega)}^2 + \|V_1^2 u\|_{L^2(\Omega)}^2 + \sum_{p\geq 0} E_p^{2+\alpha}\|u_p\|_{L^2(\mathbb{R}^2)}^2. \tag{2.7}$$

Proof of Lemma 2.1. This lemma can be proved by combining Proposition 2.5 in Ref. 1 in order to prove the following equivalence:

$$H_x u \in L^2(\Omega) \Leftrightarrow \|u\|_{H^2(\Omega)}^2 + \|V_1 u\|_{L^2(\Omega)}^2 < \infty. \qquad \square$$

Lemma 2.2. (Properties of the Sobolev spaces $\mathcal{X}$ and $\mathcal{Y}$) *For any fixed* $0<\alpha<1/2$, $\mathcal{X}$ *and* $\mathcal{Y}$ *are continuously embedded in* $L^\infty(\Omega)$. *Moreover*, $\mathcal{X}$ *and* $\mathcal{Y}$ *are algebras, and the embedding* $\mathcal{Y}\subset\mathcal{X}$ *is compact.*

Proof. The fact that $\mathcal{X}$ and $\mathcal{Y}$ are continuously injected in $L^\infty(\Omega)$ readily comes from the fact that they are continuously embedded in $H^2(\Omega)$ that is continuously embedded in $L^\infty(\Omega)$.

Secondly, it is clear that $H^2(\Omega)\cap H_0^1(\Omega)$ is an algebra. Therefore, $\mathcal{X}$ clearly is an algebra too according to definition (1.27). As far as $\mathcal{Y}$ is concerned, if $u, v \in \mathcal{Y}$, then, $uv \in H^2(\Omega)\cap H_0^1(\Omega)$ and $V_1^2 uv \in L^2(\Omega)$ since $V_1^2 u \in L^2(\Omega)$ and $\mathcal{Y}\subset L^\infty(\Omega)$ with continuous embedding. The same arguments allows us to prove that, if

$$uv = \sum_{p\geq 0}(uv)_p\chi_p \quad \text{with } (uv)_p := \int_0^1 u(x,z)v(x,z)\chi_p(z)\,dz$$

then

$$\sum_{p\geq 0} E_p^{2+\alpha}\|(uv)_p\|_{L^2(\mathbb{R}^2)}^2 \leq \|u\|_{L^\infty}^2 \sum_{p\geq 0} E_p^{2+\alpha}\|v_p\|_{L^2(\mathbb{R}^2)}^2 \leq C\|u\|_{\mathcal{Y}}^2\|v\|_{\mathcal{Y}}^2.$$

In order to prove that $uv\in\mathcal{Y}$, we also need to prove that $\Delta_x(uv)\in\mathcal{X}$. In that view, let us write

$$\Delta_x(uv) = (\Delta_x u)v + u(\Delta_x v) + 2\nabla_x u\cdot\nabla_x v.$$

However, $\Delta_x u \in H^2(\Omega)$ and $v\in H^2(\Omega)$, therefore,

$$(\Delta_x u)v \in H^2(\Omega) \quad \text{and} \quad \|(\Delta_x u)v\|_{H^2(\Omega)} \leq \|u\|_{\mathcal{Y}}^2\|v\|_{\mathcal{Y}}^2.$$

Similarly, $u(\Delta_x v)\in H^2(\Omega)$. Finally, if $uv\in\mathcal{Y}$, then

$$\nabla_x u, \nabla_x v \in H^2(\Omega) \quad \text{and} \quad \|\nabla_x u\cdot\nabla_x v\|_{H^2(\Omega)}^2 \leq 2\|u\|_{\mathcal{Y}}^2\|v\|_{\mathcal{Y}}^2.$$

Consequently,

$$\Delta_x(uv)\in H^2(\Omega), \quad \|\Delta_x(uv)\|_{H^2(\Omega)} \leq C\|u\|_{\mathcal{Y}}\|v\|_{\mathcal{Y}},$$

where $C > 0$ does not depend on u or v. To conclude, spaces $\mathcal{X}$ and $\mathcal{Y}$ are both algebras and we have

$$\forall\, u, v \in \mathcal{X}, \quad \|uv\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}},$$
$$\forall\, u, v \in \mathcal{Y}, \quad \|uv\|_{\mathcal{Y}} \le C\|u\|_{\mathcal{Y}}\|v\|_{\mathcal{Y}},$$

where $C > 0$ does not depend on u and v.

Finally, we clearly have the embedding $\mathcal{Y} \subset \mathcal{X}$. Its compactness is due to the fact that the embedding $H^{2+\alpha}(\Omega) \cap H_0^1(\Omega) \subset H^2(\Omega) \cap H_0^1(\Omega)$ is locally compact since $2 < 2 + \alpha < 5/2$ together with the fact that $V_1(x)$ tends to infinity at infinity. □

We end this section by the following lemma.

Lemma 2.3.

$$\forall\, u \in \mathcal{Y}, \quad \|\nabla_x u\|_{\mathcal{X}}^2 \le C\|u\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}.$$

Proof. In order to prove this estimate, let us consider $u \in \mathcal{Y}$. Then,

$$\|\nabla_x u\|_{\mathcal{X}}^2 \le \|H_x^{1/2}u\|_{\mathcal{X}}^2 = \|H_x^{1/2}u\|_{L^2(\Omega)}^2 + \|H_x^{3/2}u\|_{L^2(\Omega)}^2 + \|H_z H_x^{1/2}u\|_{L^2(\Omega)}^2. \tag{2.8}$$

However,

$$\begin{aligned}
\|H_x^{1/2}u\|_{L^2(\Omega)}^2 &:= \langle H_x^{1/2}u, H_x^{1/2}u\rangle_{L^2(\Omega)} = |\langle H_x u, u\rangle_{L^2(\Omega)}| \\
&\le \|H_x u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \le C\|u\|_{\mathcal{X}}^2, \\
\|H_x^{3/2}u\|_{L^2(\Omega)}^2 &:= \langle H_x^{3/2}u, H_x^{3/2}u\rangle_{L^2(\Omega)} = |\langle H_x^2 u, H_x u\rangle_{L^2(\Omega)}| \\
&\le \|H_x^2 u\|_{L^2(\Omega)}\|H_x u\|_{L^2(\Omega)} \le \|u\|_{\mathcal{Y}}\|u\|_{\mathcal{X}}
\end{aligned}$$

and, finally, using the fact that both operators H_x and H_z commute, we obtain

$$\begin{aligned}
\|H_z H_x^{1/2}u\|_{L^2(\Omega)}^2 &:= \langle H_z H_x^{1/2}u, H_z H_x^{1/2}u\rangle_{L^2(\Omega)} = |\langle H_x H_z u, H_z u\rangle_{L^2(\Omega)}| \\
&\le \|H_z H_x u\|_{L^2(\Omega)}\|H_z u\|_{L^2(\Omega)} \le \|u\|_{\mathcal{Y}}\|u\|_{\mathcal{X}},
\end{aligned}$$

which combined with (2.8) allows us to conclude. □

2.2. A priori *estimates*

In this subsection, we state *a priori* estimates on both nonlinearities V^ε and W defined in (1.4)–(1.5) and (1.11)–(1.12), respectively. In that view, we state the following regularity result.

Lemma 2.4. *Consider any real number $\varepsilon \in [0, 1]$ and any function $f \in \mathcal{X}$. Then, the equations*

$$-\partial_z^2 u^\varepsilon - \varepsilon^2 \Delta_x u^\varepsilon = f, \quad (x, z) \in \Omega, \tag{2.9}$$

$$u^\varepsilon(x, 0) = u^\varepsilon(x, 1) = 0, \quad x \in \mathbb{R}^2, \tag{2.10}$$

admit a unique solution u^ε *and, for* ε *small enough, there exists* $C > 0$ *independent of* ε *such that the following holds*:

$$\|u^\varepsilon\|_{\mathcal{X}} \le C\|f\|_{\mathcal{X}}, \tag{2.11}$$

$$\|\partial_z^2 u^\varepsilon\|_{\mathcal{X}} \le C\|f\|_{\mathcal{X}}. \tag{2.12}$$

To be more readable, the proof of this lemma is postponed to the Appendix.

Corollary 2.1. *Let* $\varepsilon \in [0,1]$, *and define the nonlinearity* $F^\varepsilon(u)$ *as either* $V^\varepsilon(|u|^2)$ *or* $W(|u|^2)$. *Then, the following holds*:

$$\forall\, u \in \mathcal{X}, \quad \|F^\varepsilon(u)\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}^2, \tag{2.13}$$

$$\forall\, u \in \mathcal{X}, \quad \|F^\varepsilon(u)u\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}^3, \tag{2.14}$$

$$\forall\, u,v \in \mathcal{X}, \quad \|F^\varepsilon(u)u - F^\varepsilon(v)v\|_{\mathcal{X}} \le C(\|u\|_{\mathcal{X}}^2 + \|v\|_{\mathcal{X}}^2)\|u-v\|_{\mathcal{X}}. \tag{2.15}$$

Moreover,

$$\forall\, u \in \mathcal{Y}, \quad \|W(|u|^2)\|_{\mathcal{Y}} \le C\|u\|_{\mathcal{Y}}\|u\|_{\mathcal{X}}, \tag{2.16}$$

$$\forall\, u,v \in \mathcal{Y}, \quad \|W(uv)\|_{\mathcal{Y}} \le C\|u\|_{\mathcal{Y}}\|v\|_{\mathcal{Y}}, \tag{2.17}$$

$$\forall\, u \in \mathcal{Y}, \quad \|W(|u|^2)u\|_{\mathcal{Y}} \le C\|u\|_{\mathcal{X}}^2\|u\|_{\mathcal{Y}}, \tag{2.18}$$

$$\forall\, u,v \in \mathcal{Y}, \quad \|W(|u|^2)u - W(|v|^2)v\|_{\mathcal{Y}} \le C(\|u\|_{\mathcal{Y}}^2 + \|v\|_{\mathcal{Y}}^2)\|u-v\|_{\mathcal{Y}}. \tag{2.19}$$

Proof. In order to prove the first part of Corollary 2.1, we fix $\varepsilon \ge 0$ and $u \in \mathcal{X}$, and we apply Lemma 2.4 to the nonlinearity $F^\varepsilon(u)$. Indeed, $V^\varepsilon(|u|^2)$ and $W(|u|^2)$ solve the system (2.9)–(2.10) for $f = u$, ε and $\varepsilon = 0$, respectively. We therefore get

$$\|F^\varepsilon(u)\|_{\mathcal{X}} \le C\||u|^2\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}^2 \quad \text{and} \quad \|F^\varepsilon(u)u\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}^3,$$

where we used the fact that $\mathcal{X}$ is an algebra. Estimates (2.13) and (2.14) are proved. As far as the estimate (2.15) is concerned, first note that, if $u,v \in \mathcal{X}$, then

$$F^\varepsilon(u)u - F^\varepsilon(v)v = (F^\varepsilon(u) - F^\varepsilon(v))u + F^\varepsilon(v)(v-u). \tag{2.20}$$

Moreover, $F^\varepsilon(u) - F^\varepsilon(v)$ readily satisfies the following system:

$$-\partial_z^2(F^\varepsilon(u) - F^\varepsilon(v)) - \varepsilon^2\Delta_x(F^\varepsilon(u) - F^\varepsilon(v)) = |u|^2 - |v|^2,$$
$$(F^\varepsilon(u) - F^\varepsilon(v))(x,0) = (F^\varepsilon(u) - F^\varepsilon(v))(x,1) = 0.$$

Therefore, applying Lemma 2.4 yields

$$\begin{aligned}\|F^\varepsilon(u) - F^\varepsilon(v)\|_{\mathcal{X}} &\le C\||u|^2 - |v|^2\|_{\mathcal{X}} \le C\|(|u|+|v|)(|u|-|v|)\|_{\mathcal{X}} \\ &\le C(\|u\|_{\mathcal{X}} + \|v\|_{\mathcal{X}})\|u-v\|_{\mathcal{X}}.\end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21) with (2.13) finally provides us with estimate (2.15).

In order to prove the second part of the corollary, let us consider $u \in \mathcal{Y}$. We have already proved that

$$\|W(|u|^2)u\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}^3. \tag{2.22}$$

Applying operator H_x to Eqs. (1.11)–(1.12) gives

$$-\partial_z^2 H_x(W(|u|^2)) = H_x(|u|^2), \quad (x,z) \in \Omega,$$
$$H_x(W(|u|^2))(x,0) = H_x(W(|u|^2))(x,1) = 0.$$

Thus, combining Lemmas 2.4 and 2.3 gives

$$\|H_x(W(|u|^2))\|_{\mathcal{X}} \leq C\|H_x(|u|^2)\|_{\mathcal{X}} \leq C\|u\|_{\mathcal{Y}}\|u\|_{\mathcal{X}}. \tag{2.23}$$

Now, note that

$$H_z(W(|u|^2)) = -\partial_z^2 W(|u|^2) + V_2 W(|u|^2) = |u|^2 + V_2 W(|u|^2) \in \mathcal{X}.$$

Moreover, we have

$$\|H_z^2(W(|u|^2))\|_{L^2(\Omega)} \leq \|H_z(W(|u|^2))\|_{\mathcal{X}},$$

which applying the estimate (2.12) of Lemma 2.4 leads to

$$\|H_z^2(W(|u|^2))\|_{L^2(\Omega)} \leq C\|W(|u|^2)\|_{\mathcal{X}} + C\|\partial_z^2(W(|u|^2))\|_{\mathcal{X}} \leq C\|u\|_{\mathcal{X}}^2. \tag{2.24}$$

Now, define

$$W(|u|^2)_p(x) = \langle W(|u|^2)(x,\cdot)\chi_p\rangle,$$

Definition (2.5) gives

$$\begin{aligned}
\|H_z^{1+\alpha/2}(W(|u|^2))\|_{L^2(\Omega)}^2 &= \sum_{p\geq 0} E_p^{2+\alpha}\|W(|u|^2)_p\|_{L^2(\Omega)}^2 \\
&\leq \left(\sum_{p\geq 0} E_p^4\|W(|u|^2)_p\|_{L^2(\Omega)}^2\right)^{1/2} \\
&\quad \cdot \left(\sum_{p\geq 0} E_p^{2\alpha}\|W(|u|^2)_p\|_{L^2(\Omega)}^2\right)^{1/2} \\
&\leq \|H_z^2(W(|u|^2))\|_{L^2(\Omega)}\left(\sum_{p\geq 0} E_p^2\|W(|u|^2)_p\|_{L^2(\Omega)}^2\right)^{1/2} \\
&\leq C\|u\|_{\mathcal{X}}^2\|H_z(W(|u|^2))\|_{L^2(\Omega)} \leq C\|u\|_{\mathcal{X}}^4,
\end{aligned}$$

where we used (2.24), the fact that $\alpha < 1/2$ and that E_p is increasing and tends to infinity with p. Finally,

$$\|H_z^{1+\alpha/2}(W(|u|^2))\|_{L^2(\Omega)} \leq C\|u\|_{\mathcal{X}}^2. \tag{2.25}$$

Combining (2.22) with (2.23) and (2.25) finally provides us with the following tame estimate, according to (2.3):

$$\forall\, u \in \mathcal{Y}, \quad \|W(|u|^2)\|_{\mathcal{Y}} \leq C\|u\|_{\mathcal{Y}}\|u\|_{\mathcal{X}},$$

where $C > 0$ does not depend on u. Following the exact same lines, (2.17) can also easily be proved. Now, let us consider any function $u \in \mathcal{Y}$ and prove estimate (2.18) by the equivalence (2.4). First, we know that

$$\begin{aligned}\|W(|u|^2)u\|_{H^{2+\alpha}(\Omega)} &\le \|W(|u|^2)\|_{H^{2+\alpha}(\Omega)}\|u\|_{L^\infty(\Omega)} + \|u\|_{H^{2+\alpha}(\Omega)}\|W(|u|^2)\|_{L^\infty(\Omega)}\\ &\le \|W(|u|^2)\|_{\mathcal{Y}}\|u\|_{\mathcal{X}} + \|u\|_{\mathcal{Y}}\|W(|u|^2)\|_{\mathcal{X}},\end{aligned}$$

where we used the continuous embeddings $\mathcal{X} \subset L^\infty(\Omega)$ and $\mathcal{Y} \subset H^{2+\alpha}(\Omega)$. Applying (2.16) and (2.13) gives

$$\|W(|u|^2)u\|_{H^{2+\alpha}(\Omega)} \le C\|u\|_{\mathcal{X}}^2\|u\|_{\mathcal{Y}}. \tag{2.26}$$

As far as the norm $H_x(W(|u|^2)u)$ is concerned, we have

$$H_x(W(|u|^2)u) = H_x(W(|u|^2))u - W(|u|^2)\Delta_x u - 2\nabla_x(W(|u|^2)) \cdot \nabla_x u. \tag{2.27}$$

However, since $\mathcal{X}$ is an algebra,

$$\begin{aligned}\|H_x\big(W(|u|^2)\big)u\|_{\mathcal{X}} &\le C\|H_x\big(W(|u|^2)\big)\|_{\mathcal{X}}\|u\|_{\mathcal{X}}\\ &\le C\|u\|_{\mathcal{Y}}\|u\|_{\mathcal{X}}^2,\end{aligned} \tag{2.28}$$

where we used (2.18) and the equivalence (2.4).

Moreover,

$$\begin{aligned}\|W(|u|^2)\Delta_x u\|_{\mathcal{X}} &\le C\|W(|u|^2)\|_{\mathcal{X}}\|\Delta_x u\|_{\mathcal{X}}\\ &\le C\|u\|_{\mathcal{Y}}\|u\|_{\mathcal{X}}^2,\end{aligned} \tag{2.29}$$

where we used (2.13) and the equivalence (2.4). Finally,

$$\nabla_x(W(|u|^2)) \cdot \nabla_x u = \sum_{i=1}^{2} \partial_i(W(|u|^2))\partial_i u + W(|u|^2)\partial_i u. \tag{2.30}$$

Fix $i \in \{1,2\}$, we have

$$\partial_i(W(|u|^2)) = W(\partial_i(|u|^2))$$

and applying Lemma 2.4 with $f = \partial_1(|u|^2)$ and $\varepsilon = 0$ yields

$$\|W(\partial_i(|u|^2))\|_{\mathcal{X}} \le C\|\partial_i(|u|^2)\|_{\mathcal{X}}.$$

Therefore, combined with (2.30), this leads to

$$\|\nabla_x(W(|u|^2)) \cdot \nabla_x u\|_{\mathcal{X}} \le C\sum_{i=1}^{2} \|\partial_i(|u|^2)\|_{\mathcal{X}}\|\partial_i u\|_{\mathcal{X}} \le C\sum_{i=1}^{2} \|\partial_i u\|_{\mathcal{X}}^2\|u\|_{\mathcal{X}}. \tag{2.31}$$

Applying Lemma 2.3, combined with (2.31) allows us to conclude that

$$\|\nabla_x(W(|u|^2)) \cdot \nabla_x u\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}^2\|u\|_{\mathcal{Y}}. \tag{2.32}$$

Finally, combining (2.32) with (2.27), (2.28) and (2.29) yields

$$\|H_x(W(|u|^2)u)\|_{\mathcal{X}} \le C\|u\|_{\mathcal{X}}^2\|u\|_{\mathcal{Y}},$$

which combined with (2.26) and the equivalence (2.4) concludes the proof of (2.18). Estimates (2.16) and (2.18) are now proved. As far as (2.19) is concerned, we write

$$W(|u|^2)u - W(|v|^2)v = (W(|u|^2) - W(|v|^2))u + W(|v|^2)(u - v).$$

Then, as $\mathcal{Y}$ is an algebra,

$$\begin{aligned}\|W(|u|^2)u - W(|v|^2)v\|_{\mathcal{Y}} &\leq \|W(|u|^2) - W(|v|^2)\|_{\mathcal{Y}}\|u\|_{\mathcal{Y}} + \|W(|v|^2)\|_{\mathcal{Y}}\|u - v\|_{\mathcal{Y}} \\ &\leq \|W((|u| + |v|)|u - v|)\|_{\mathcal{Y}}\|u\|_{\mathcal{Y}} + \|W(|v|^2)\|_{\mathcal{Y}}\|u - v\|_{\mathcal{Y}} \\ &\leq C\||u| + |v|\|_{\mathcal{Y}}\|u - v\|_{\mathcal{Y}}\|u\|_{\mathcal{Y}} + C\|v\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}\|u - v\|_{\mathcal{Y}},\end{aligned}$$

where we used (2.16) and (2.17). Therefore, we get

$$\|W(|u|^2)u - W(|v|^2)v\|_{\mathcal{Y}} \leq C(\|u\|_{\mathcal{Y}}^2 + \|v\|_{\mathcal{Y}}^2)\|u - v\|_{\mathcal{Y}},$$

which ends the proof of (2.19). □

Now that we have obtained *a priori* estimates on both V^ε and W nonlinearities, we focus on the existence and uniqueness results for both initial (1.1)–(1.5) and intermediate system (1.9)–(1.12).

2.3. *Approximation result*

Proposition 2.1. *Let $\varepsilon > 0$ be fixed and consider two initial data $\psi_0 \in \mathcal{X}$ and $\tilde{\psi}_0 \in \mathcal{Z}$, where $\mathcal{Z}$ denotes either $\mathcal{X}$ or $\mathcal{Y}$.*

Then, there exists a common $T_0 > 0$ depending only on $\|\psi_0\|_{\mathcal{X}}$ and $\|\tilde{\psi}_0\|_{\mathcal{X}}$ (in particular T_0 does not depend on ε) such that both initial Schrödinger–Poisson system (1.1)–(1.5) and intermediate system (1.9)–(1.12) with initial data ψ_0 and $\tilde{\psi}_0$, respectively, possess unique solutions denoted by $\psi^\varepsilon \in \mathcal{C}([0, T_0], \mathcal{X})$ and $\tilde{\psi}^\varepsilon \in \mathcal{C}([0, T_0], \mathcal{Z})$.

Moreover, there exists a common bound $M > 0$ depending only on $\|\psi_0\|_{\mathcal{X}}$ and $\|\tilde{\psi}_0\|_{\mathcal{Z}}$ such that

$$\sup_{0<\varepsilon<1} (\|\psi^\varepsilon\|_{\mathcal{C}((0,T_0),\mathcal{X})}, \|\tilde{\psi}^\varepsilon\|_{\mathcal{C}((0,T_0),\mathcal{Z})}) \leq M. \tag{2.33}$$

Besides, if $\tilde{\psi}_0 \in \mathcal{Y}$, then the following holds:

$$\forall\, t \in [0, T_0], \quad \|\psi^\varepsilon(t, \cdot) - \tilde{\psi}^\varepsilon(t, \cdot)\|_{\mathcal{X}} \leq C(\|\psi_0 - \tilde{\psi}_0\|_{\mathcal{X}} + \varepsilon^2), \tag{2.34}$$

where $C > 0$ does not depend on ε.

Proof. As a first step, let us prove the existence and uniqueness of solutions ψ^ε (respectively $\tilde{\psi}^\varepsilon$) to (1.1)–(1.5) (respectively (1.9)–(1.12)) on time intervals $[0, T_\varepsilon)$ (respectively $[0, \tilde{T}_\varepsilon)$). In that view, we define the following problem, in which $F^\varepsilon(u^\varepsilon)$ denotes either $V^\varepsilon(|u^\varepsilon|^2)$ or $W(|u^\varepsilon|^2)$:

$$i\partial_t u^\varepsilon = H_x u^\varepsilon + \frac{1}{\varepsilon^2} H_z u^\varepsilon + F^\varepsilon(u^\varepsilon)u^\varepsilon, \quad t > 0,\ (x, z) \in \Omega, \tag{2.35}$$

$$u^\varepsilon(t, x, 0) = u^\varepsilon(t, x, 1) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \tag{2.36}$$

$$u^\varepsilon(0, x, z) = u_0, \quad (x, z) \in \Omega. \tag{2.37}$$

Of course, we look for a solution u^ε of this Cauchy problem as a solution to the following fixed-point equation given by the Duhamel formula:

$$u^\varepsilon(t,x,z) = e^{-itH^\varepsilon} u_0(x,z) - i\int_0^t e^{-i(t-s)H^\varepsilon} F^\varepsilon(u^\varepsilon)u^\varepsilon(s,x,z)\,ds, \tag{2.38}$$

where $H^\varepsilon := H_x + (1/\varepsilon^2)H_z$. In order to state the existence of a unique solution in $\mathcal{X}$ or $\mathcal{Y}$ on a time interval $[0, T_\varepsilon]$ to this fixed-point equation, we use the tame estimates (2.15) and (2.19) stated in Corollary 2.1.

Even more, if $t > 0$ is fixed and if $u, v \in \mathcal{C}([0,t), \mathcal{X})$, then by the fact that e^{itH_ε} is unitary in $\mathcal{X}$ (as H_z and H_x commute with H^ε), the following Lipschitz estimate holds true:

$$\begin{aligned}
&\left\|\int_0^t e^{-i(t-s)H^\varepsilon}[F^\varepsilon(u)u(s) - F^\varepsilon(v)v(s)]\,ds\right\|_{\mathcal{X}}\\
&\quad\le t\times \sup_{s\in[0,t]}\|F^\varepsilon(u)u(s) - F^\varepsilon(v)v(s)\|_{\mathcal{X}}\\
&\quad\le t\times C(\|u\|^2_{\mathcal{C}([0,t],\mathcal{X})} + \|v\|^2_{\mathcal{C}([0,t],\mathcal{X})})\|u-v\|_{\mathcal{C}([0,t],\mathcal{X})},
\end{aligned}$$

where we used the tame estimate (2.15) at hand on $F^\varepsilon(u^\varepsilon)$. Similarly, we prove by the tame estimate (2.19) at hand for $W(|u^\varepsilon|^2)$ that, for all $u, v \in \mathcal{C}([0,t), \mathcal{Y})$,

$$\begin{aligned}
&\left\|\int_0^t e^{-i(t-s)H^\varepsilon}[W(|u|^2)u(s) - W(|v|^2)v(s)]\,ds\right\|_{\mathcal{Y}}\\
&\quad\le t\times C(\|u\|^2_{\mathcal{C}([0,t],\mathcal{Y})} + \|v\|^2_{\mathcal{C}([0,t],\mathcal{Y})})\|u-v\|_{\mathcal{C}([0,t],\mathcal{Y})}.
\end{aligned}$$

It is now easy (see, for example, Ref. 11) to conclude that, for any $\varepsilon > 0$, there exists a possibly small time $T_\varepsilon > 0$, and a unique solution $u^\varepsilon \in \mathcal{C}([0, T_\varepsilon], \mathcal{X})$ to the integral equation (2.38), and this solution incidently provides the unique solution to the nonlinear Schrödinger system (2.35)–(2.37). We similarly prove the existence and uniqueness on a possibly small time interval $[0, T_\varepsilon]$ of a solution $u^\varepsilon \in \mathcal{C}([0, T_\varepsilon], \mathcal{Y})$ to the system (2.35)–(2.37), where $u_0 \in \mathcal{Y}$ and $F^\varepsilon(u^\varepsilon) = W(|u^\varepsilon|^2)$. This task is left to the reader.

Let us now prove that there is a common lower bound T_0 for all these T_ε as ε fluctuates, i.e. $T_\varepsilon \ge T_0$, for any $\varepsilon > 0$. If u^ε is the solution of the integral equation given by the Duhamel formula (2.38) with initial datum $u_0 \in \mathcal{X}$, then

$$\begin{aligned}
\|u^\varepsilon(t)\|_{\mathcal{X}} &\le \|u_0\|_{\mathcal{X}} + \int_0^t \|F^\varepsilon(u^\varepsilon)u^\varepsilon(s)\|_{\mathcal{X}}\,ds\\
&\le \|u_0\|_{\mathcal{X}} + C\int_0^t \|u^\varepsilon(s)\|^3_{\mathcal{X}}\,ds,
\end{aligned} \tag{2.39}$$

where we applied (2.14). Hence, we get

$$\|u^\varepsilon\|_{\mathcal{C}([0,t],\mathcal{X})} \le \|u_0\|_{\mathcal{X}} + Ct\|u^\varepsilon\|^3_{\mathcal{C}([0,t],\mathcal{X})}.$$

Therefore, by a bootstrap argument (see, for example, Lemma 5.5 in Ref. 7), we prove the existence of such a T_0, and we obtain

$$\forall\, t \in (0,\, T_0), \quad \|u^\varepsilon(t,\cdot)\|_{\mathcal{X}} \le C\|u_0\|_{\mathcal{X}}, \tag{2.40}$$

where $C > 0$ does not depend on ε.

In the case where $u_0 \in \mathcal{Y}$ and $F^\varepsilon(u^\varepsilon) = W(|u^\varepsilon|^2)$, the previous work ensures that (2.40) still holds true. Let $t \in (0,\, T_0)$ be fixed. Using the Duhamel formula (2.38), the tame estimate (2.18) at hand on $W(|u|^2)u$ in $\mathcal{Y}$ and (2.40) leads to

$$\begin{aligned}\|u^\varepsilon(t)\|_{\mathcal{Y}} &\le \|u_0\|_{\mathcal{Y}} + C\int_0^t \|u^\varepsilon(s)\|_{\mathcal{X}}^2 \|u^\varepsilon(s)\|_{\mathcal{Y}}\, ds \\ &\le \|u_0\|_{\mathcal{Y}} + C\|u_0\|_{\mathcal{X}}^2 \int_0^t \|u^\varepsilon(s)\|_{\mathcal{Y}}\, ds.\end{aligned}$$

The Gronwall lemma allows us to conclude that there exists a common existence time $T_1 < T_0$ that only depends on $\|u_0\|_{\mathcal{X}}$ and that

$$\forall\, t \in (0,\, T_1), \quad \|u^\varepsilon(t,\cdot)\|_{\mathcal{Y}} \le C\|u_0\|_{\mathcal{Y}},$$

where C does not depend on ε. Both estimates therefore also provide us with a common bound $M > 0$ that only depends on $\|\psi_0\|_{\mathcal{X}}$ and $\|\tilde{\psi}_0\|_{\mathcal{Y}}$ and end the proof of (2.33).

In order to prove the convergence result (2.34), we set $\psi_0 \in \mathcal{X}$ and $\tilde{\psi}_0 \in \mathcal{Y}$. Consider the initial Schrödinger–Poisson system (1.1)–(1.5) and the intermediate system (1.9)–(1.12), with initial datum ψ_0 and $\tilde{\psi}_0$, respectively. We have already proved the existence and uniqueness of their respective solutions denoted by $\psi^\varepsilon \in \mathcal{C}([0,\, T_0], \mathcal{X})$ and $\tilde{\psi}^\varepsilon \in \mathcal{C}([0,\, T_0], \mathcal{Y})$. The difference $\omega^\varepsilon := \psi^\varepsilon - \tilde{\psi}^\varepsilon$ satisfies the following equation:

$$\begin{aligned}&i\partial_t \omega^\varepsilon = H^\varepsilon \omega^\varepsilon + V^\varepsilon(|\psi^\varepsilon|^2)\psi^\varepsilon - V(|\tilde{\psi}^\varepsilon|^2)\tilde{\psi}^\varepsilon + (V(|\tilde{\psi}^\varepsilon|^2) - W(|\tilde{\psi}^\varepsilon|^2))\tilde{\psi}^\varepsilon, \\ &\omega^\varepsilon(0,x,z) = \psi_0(x,z) - \tilde{\psi}_0(x,z), \quad (x,z) \in \Omega.\end{aligned}$$

According to the Duhamel formula,

$$\omega^\varepsilon(t) = e^{-itH^\varepsilon}(\psi_0 - \tilde{\psi}_0) - i\int_0^t e^{-i(t-s)H^\varepsilon}(V^\varepsilon(|\psi^\varepsilon|^2)\psi^\varepsilon - V^\varepsilon(|\tilde{\psi}^\varepsilon|^2)\tilde{\psi}^\varepsilon)(s) + f^\varepsilon(s)\, ds,$$

where

$$f^\varepsilon(s) := (V(|\tilde{\psi}^\varepsilon|^2) - W(|\tilde{\psi}^\varepsilon|^2))\tilde{\psi}^\varepsilon.$$

Since e^{-itH^ε} is unitary on $\mathcal{X}$ (by the fact that H_x and H_z commute with H^ε) and $(\mathcal{X}, \|\ \|_{\mathcal{X}})$ is an algebra, then we have

$$\begin{aligned}\|\omega^\varepsilon(t)\|_{\mathcal{X}} &\le \|\psi_0 - \tilde{\psi}_0\|_{\mathcal{X}} + C\int_0^t \|V^\varepsilon(|\psi^\varepsilon|^2)\psi^\varepsilon \\ &\quad - V^\varepsilon(|\tilde{\psi}^\varepsilon|^2)\tilde{\psi}^\varepsilon\|_{\mathcal{X}}\, ds + \int_0^t \|f^\varepsilon(s)\|_{\mathcal{X}}\, ds \\ &\le \|\psi_0 - \tilde{\psi}_0\|_{\mathcal{X}} + CM^2\int_0^t \|\omega^\varepsilon(s)\|_{\mathcal{X}}\, ds + \int_0^t \|f^\varepsilon(s)\|_{\mathcal{X}}\, ds,\end{aligned} \tag{2.41}$$

where we used (2.15) and the uniform bound M given by (2.33).

In order to estimate $\|f^\varepsilon(s)\|_{\mathcal{X}}$, we write

$$\|f^\varepsilon(s)\|_{\mathcal{X}} \leq \|V^\varepsilon(|\tilde{\psi}^\varepsilon|^2)(s) - W(|\tilde{\psi}^\varepsilon|^2)(s)\|_{\mathcal{X}}\|\tilde{\psi}^\varepsilon(s)\|_{\mathcal{X}}. \tag{2.42}$$

Note that $v^\varepsilon(|\tilde{\psi}^\varepsilon|^2) := V^\varepsilon(|\tilde{\psi}^\varepsilon|^2)(s) - W(|\tilde{\psi}^\varepsilon|^2)(s)$ satisfies

$$-\partial_z^2 v^\varepsilon(|\tilde{\psi}^\varepsilon|^2) - \varepsilon^2\Delta_x v^\varepsilon(|\tilde{\psi}^\varepsilon|^2) = \varepsilon^2\Delta_x W(|\tilde{\psi}^\varepsilon|^2), \quad t \in (0, T_0),\ (x, z) \in \Omega, \tag{2.43}$$

$$v^\varepsilon(|\tilde{\psi}^\varepsilon|^2)(t, x, 0) = v^\varepsilon(|\tilde{\psi}^\varepsilon|^2)(t, x, 1) = 0, \quad t \in (0, T_0),\ x \in \mathbb{R}^2 \tag{2.44}$$

and therefore applying Lemma 2.4 to Eqs. (2.43)–(2.44) gives, for all $t \in (0, T_0)$,

$$\begin{aligned}\|v^\varepsilon(|\tilde{\psi}^\varepsilon|^2)(t)\|_{\mathcal{X}} &\leq C\varepsilon^2\|\Delta_x W(|\tilde{\psi}^\varepsilon|^2)(t)\|_{\mathcal{X}} \leq C\varepsilon^2\|W(|\tilde{\psi}^\varepsilon|^2)(t)\|_{\mathcal{Y}} \\ &\leq C\varepsilon^2\|\tilde{\psi}^\varepsilon(t)\|_{\mathcal{Y}}^2 \leq CM^2\varepsilon^2,\end{aligned} \tag{2.45}$$

where we used (2.16) and the uniform bound M given by (2.33). Finally, combining (2.41) with (2.42), (2.45) and (2.33) leads to

$$\|\omega^\varepsilon(t)\|_{\mathcal{X}} \leq \|\psi_0 - \tilde{\psi}_0\|_{\mathcal{X}} + C\varepsilon^2 + C\int_0^t \|\omega^\varepsilon(s)\|_{\mathcal{X}}\, ds,$$

which by the Gronwall lemma concludes the proof of (2.34). □

3. Time Averaging of the Intermediate System

In this section, we focus on the intermediate system (1.9)–(1.10). In order to filter out the time oscillations, we denote by ϕ^ε the filtered wave function as in (1.30), which solves system (1.33)–(1.35). In order to state almost-periodicity properties for the nonlinearity G defined by (1.32), we first recall various known facts about almost-periodic functions (in time) with values in $\mathcal{Z}$ (in space) that will be generally denoted as $\Theta(\tau)$. The key fact is the existence of their long-time averaging

$$\Theta_{\mathrm{av}} := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \Theta(\tau)\, d\tau,$$

and the point is, no small divisor estimate is needed to define these long-time averages, as recalled in Proposition 3.2. Then, Proposition 3.3 states several key facts about the nonlinearity G, in particular, we state some tame estimates on G (see (3.1) and (3.2)) and we prove that G is almost periodic in time with values in $\mathcal{Z}$ (see Proposition 3.3(i)). It therefore has a long-time average G_{av} (see (3.3)), computed in (3.4) which inherits the tame estimates at hand for G (see (3.5) and (3.6)).

This proposition therefore allows us to state existence, uniqueness and regularity results for both intermediate and averaged systems in Corollary 3.1. Finally, Proposition 3.4 states the convergence of the filtered intermediate model towards the limit model in $\mathcal{X}$ with additional $\mathcal{Y}$ regularity assumptions on the initial datum. This point will be solved in the next section by a regularization procedure. Let us begin with the following definition, borrowed from Refs. 1 and 20.

Definition 3.1. Let $\mathcal{Z}$ denote either $\mathcal{X}$ or $\mathcal{Y}$. A function $\Theta : \tau \in \mathbb{R} \mapsto \Theta(\tau)$, with $\Theta \in \mathcal{C}(\mathbb{R}, \mathcal{Z})$, is said to be almost-periodic, and we note $\Theta \in \mathrm{AP}(\mathbb{R}, \mathcal{Z})$, whenever the set of translates

$$\{\tau \mapsto \Theta(\tau + h),\ \ h \in \mathbb{R}\}$$

has compact closure in the norm $L^\infty(\mathbb{R}, \mathcal{Z})$.

This definition using the precompactness criterion is usually referred to as Bochner's criterion for almost-periodicity. It is proved, for example, in Ref. 20 and recalled in Ref. 1, that this definition is equivalent to a criterion based on the approximation by trigonometric polynomials.

Proposition 3.1. *Equivalently*, $\Theta \in \mathrm{AP}(\mathbb{R}, \mathcal{Z})$ *if and only if* $\Theta(\tau)$ *is the strong limit of trigonometric polynomials*, *i.e. for any* $\delta > 0$, *there exists a trigonometric polynomial*

$$\Theta^\delta(\tau) := \sum_{n=1}^{N_\delta} \theta_{n,\delta} e^{i\lambda_{n,\delta}\tau} \quad \textit{such that} \ \sup_{\tau \in \mathbb{R}} \|\Theta(\tau) - \Theta^\delta(\tau)\|_{\mathcal{Z}} \le \delta,$$

where the $\theta_{n,\delta}$*'s belong to* $\mathcal{Z}$, *the* $\lambda_{n,\delta}$*'s belong to* $\mathbb{R}$ *and* N_δ *is some finite integer.*

With this definition, it turns out that one may be willing to do some kind of Fourier analysis on almost-periodic functions, and, in particular, the long-time averaging (that stands for the mean mode in the Fourier analysis) is well-defined as is stated in the following proposition borrowed from Refs. 1 and 20.

Proposition 3.2. *Consider* $\Theta \in \mathrm{AP}(\mathbb{R}, \mathcal{Z})$. *Then*, *the following strong limit exists in* $\mathcal{Z}$,

$$\Theta_{\mathrm{av}} := \lim_{T \to \infty} \int_0^T \frac{1}{T} \Theta(\tau)\, d\tau.$$

Moreover, *for any* $\lambda \in \mathbb{R}$, *the Fourier-like coefficient*

$$\widehat{\Theta}(\lambda) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \Theta(\tau) e^{-i\lambda\tau}\, d\tau,$$

is well-defined as a limit in $\mathcal{Z}$. *Last*, *the following Bessel-like inequality holds*: *for any sequence* $\{\lambda_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, *we have*

$$\sum_{n\in\mathbb{N}} \|\widehat{\Theta}(\lambda_n)\|_{\mathcal{Z}}^2 \le \lim_{T \to +\infty} \frac{1}{T} \int_0^T \|\Theta(\tau)\|_{\mathcal{Z}}^2\, d\tau \le \|\Theta\|_{L^\infty(\mathbb{R},\mathcal{Z})}^2.$$

Note that a simple particular case of almost-periodic functions is given by *quasi-periodic* functions, that is, functions which, for any given finite-dimensional frequency vector $\omega = (\omega_1, \ldots, \omega_N)$ whose components are assumed to be pairwise rationally independent, can be written as the finite sum of N trigonometric monomials.

Let us now state a few consequences of Propositions 3.1 and 3.2 that will be of great use in the study of the to-be-averaged nonlinearity G.

Proposition 3.3. *Consider any function $u \in \mathcal{Z}$, then the following conditions hold*:

(i) *G is in* $\mathrm{AP}(\mathbb{R}, \mathcal{Z})$: *If $W(|e^{-i\tau H_z}u|^2)$ is defined by* (1.11)–(1.12), *then*

$$\tau \mapsto e^{i\tau H_z} W(|e^{-i\tau H_z}u|^2)e^{-i\tau H_z}u := G(\tau, u)$$

belongs to $\mathrm{AP}(\mathbb{R}, \mathcal{Z})$.

(ii) *Tame estimate for G in $\mathcal{Z}$: $u \mapsto G(\tau, u)$ is locally Lipschitz in $\mathcal{Z}$ and satisfies the following tame estimates*:

$$\forall\, u \in \mathcal{Z},\ \forall\, \tau > 0, \quad \|G(\tau, u)\|_{\mathcal{Z}} \leq C\|u\|_{\mathcal{X}}^2 \|u\|_{\mathcal{Z}} \tag{3.1}$$

and,

$$\forall\, u, v \in \mathcal{Z},\ \forall\, \tau > 0,$$

$$\|G(\tau, u) - G(\tau, v)\|_{\mathcal{Z}} \leq C(\|u\|_{\mathcal{Z}}^2 + \|v\|_{\mathcal{Z}}^2)\|u - v\|_{\mathcal{Z}}, \tag{3.2}$$

where C is a positive constant that only depends on the nonlinearity G.

(iii) *Long-time averaging for $G(\tau, u)$ in $\mathcal{Z}$: one may define its long-time averaging as the strong limit in $\mathcal{Z}$,*

$$G_{\mathrm{av}}(u) := \lim_{T\to\infty} \frac{1}{T} \int_0^T G(\tau, u)\, d\tau. \tag{3.3}$$

Moreover, $G_{\mathrm{av}}(u)$ is given by

$$G_{\mathrm{av}}(u) = \sum_{p\geq 0} \sum \sum_{(q,r,s)\in\Lambda_p} \sum \alpha_{pqrs} \overline{u_q}\, u_r u_s \chi_p \tag{3.4}$$

where, for all $k \geq 0$, $u_k := \langle u, \chi_k\rangle_{L^2(0,1)}$, $\Lambda_p := \{(q, r, s) \in \mathbb{N}^3,\ E_p + E_q = E_r + E_s\}$ and

$$\alpha_{pqrs} = \int_0^1 \int_0^1 K(z, z')\chi_r(z')\chi_q(z')\chi_s(z)\chi_p(z)\, dz'\, dz.$$

(iv) *Tame estimate for G_{av} in $\mathcal{Z}$: the function $u \in \mathcal{Z} \mapsto G_{\mathrm{av}}(u)$ is locally Lipschitz in $\mathcal{Z}$ and satisfies the following tame estimates*:

$$\forall\, u \in \mathcal{Z}, \quad \|G_{\mathrm{av}}(u)\|_{\mathcal{Z}} \leq C\|u\|_{\mathcal{X}}^2 \|u\|_{\mathcal{Z}} \tag{3.5}$$

and

$$\forall\, u, v \in \mathcal{Z}, \quad \|G_{\mathrm{av}}(u) - G_{\mathrm{av}}(v)\|_{\mathcal{Z}} \leq C(\|u\|_{\mathcal{Z}}^2 + \|v\|_{\mathcal{Z}}^2)\|u - v\|_{\mathcal{Z}}, \tag{3.6}$$

where C is a positive constant that only depends on the nonlinearity G.

Gathering these properties on the nonlinearities G and G_{av} now allows us to state the following corollary that proves the existence, uniqueness and smoothness results for both filtered intermediate and averaged system.

Corollary 3.1. *Let $\varepsilon > 0$ be fixed and consider any function ψ_0 in $\mathcal{Z}$. There exists $T_0 > 0$ that only depends on $\|\psi_0\|_{\mathcal{X}}$ such that the filtered intermediate system*

(1.32)–(1.35) *and the averaged system*,

$$i\partial_t\phi(t) = H_x\phi(t) + G_{\rm av}(\phi(t)), \quad t > 0, \ (x,z) \in \Omega, \tag{3.7}$$

$$\phi(t,x,0) = \phi(t,x,1) = 0, \quad t > 0, \ x \in \mathbb{R}^2, \tag{3.8}$$

$$\phi(0,x,z) = \psi_0(x,z), \quad (x,z) \in \Omega, \tag{3.9}$$

both admit a unique solution in $\mathcal{C}([0,T_0],\mathcal{Z})$, *respectively, denoted by* ϕ^ε *and* ϕ. *Moreover, there exists* $M > 0$ *depending only on* $\|\psi_0\|_{\mathcal{Z}}$ *such that*

$$\sup_{0<\varepsilon<1} (\|\phi^\varepsilon\|_{\mathcal{C}^0([0,T_0],\mathcal{Z})} + \|\phi\|_{\mathcal{C}^0([0,T_0],\mathcal{Z})}) \le M. \tag{3.10}$$

Proof of Proposition 3.3. (i) and (iii) We first claim that, given any $\Theta \in \mathrm{AP}(\mathbb{R},\mathcal{Z})$, the function $\tau \mapsto e^{\pm i\tau H_z}\Theta$ also belongs to $\mathrm{AP}(\mathbb{R},\mathcal{Z})$. In that view, applying Proposition 3.1, we use the characterization of almost-periodic functions as the strong limit in $\mathcal{H}^\ell$ of trigonometric polynomials. Fix a small $\delta > 0$ and $\Theta \in \mathrm{AP}(\mathbb{R},\mathcal{Z})$, we may find a trigonometric polynomial

$$\Theta^\delta(\tau) = \sum_{n=1}^{N_\delta} \theta_{n,\delta} e^{i\lambda_{n,\delta}\tau} \quad \text{such that } \|\Theta - \Theta^\delta\|_{L^\infty(\mathbb{R},\mathcal{Z})} \le \delta, \tag{3.11}$$

where the $\theta_{n,\delta}$ belong to $\mathcal{Z}$ and the $\lambda_{n,\delta}$ are real numbers. As the operator $e^{\pm i\tau H_z}$ preserves the $\mathcal{Z}$ norm, we obtain

$$\|e^{\pm i\tau H_z}\Theta - e^{\pm i\tau H_z}\Theta^\delta\|_{L^\infty(\mathbb{R},\mathcal{Z})} \le \delta.$$

Now, since the $\theta_{n,\delta}$'s coincide with the Fourier-like coefficients $\widehat{\Theta^\delta}(\lambda_{n,\delta})$ defined in Proposition 3.2, the Bessel-like inequality (3.1) reads

$$\sum_{n=1}^{N_\delta} \|\theta_{n,\delta}\|_{\mathcal{X}}^2 \le \|\Theta\|_{L^\infty(\mathbb{R},\mathcal{Z})}.$$

Besides, for δ small enough, we clearly have from (3.11) the uniform in time following bound:

$$\|\Theta^\delta\|_{L^\infty(\mathbb{R},\mathcal{Z})} \le C, \tag{3.12}$$

where C does not depend on δ. By the equivalence stated in Lemma 2.1 and Remark 2.1 and to definition (2.3), estimate (3.12) reads

$$\sum_{n=1}^{N_\delta}\sum_{p\ge 0}(1+E_p^{2+\alpha})\|\langle\theta_{n,\delta},\chi_p\rangle\|_{L^2(\mathbb{R}^2)}^2 + \sum_{n=1}^{N_\delta}\sum_{p\ge 0}(1+E_p^2)\|\langle H_x\theta_{n,\delta},\chi_p\rangle\|_{L^2(\mathbb{R}^2)}^2$$
$$+ \sum_{n=1}^{N_\delta}\sum_{p\ge 0}\|\langle H_x^2\theta_{n,\delta},\chi_p\rangle\|_{L^2(\mathbb{R}^2)}^2 \le C. \tag{3.13}$$

Let us now approximate the infinite sum $\Theta^\delta = \sum_{p\geq 0}\langle\Theta^\delta, \chi_p\rangle\chi_p$ by a finite sum. In that view, one may find $P_\delta \in \mathbb{N}$ such that

$$\sum_{n=1}^{N_\delta}\sum_{p>P_\delta}(1+E_p^{2+\alpha})\|\langle\theta_{n,\delta},\chi_p\rangle\|^2_{L^2(\mathbb{R}^2)} + \sum_{n=1}^{N_\delta}\sum_{p>P_\delta}(1+E_p^2)\|\langle H_x\theta_{n,\delta},\chi_p\rangle\|^2_{L^2(\mathbb{R}^2)} + \sum_{n=1}^{N_\delta}\sum_{p>P_\delta}\|\langle H_x^2\theta_{n,\delta},\chi_p\rangle\|^2_{L^2(\mathbb{R}^2)} \leq \delta.$$

In particular, with this choice for P_δ, and by (2.6) and (2.5), we recover the estimate

$$\sup_{\tau\in\mathbb{R}}\left\|\Theta^\delta(\tau) - \sum_{p=0}^{P_\delta}\langle\Theta^\delta(\tau),\chi_p\rangle\chi_p\right\|_{\mathcal{Z}} \leq \delta,$$

which leads to

$$\sup_{\tau\in\mathbb{R}}\left\|e^{\pm i\tau H_z}\Theta^\delta(\tau) - \sum_{p=0}^{P_\delta}e^{\pm i\tau E_p}\langle\Theta^\delta,\chi_p\rangle\chi_p\right\|_{\mathcal{Z}} \leq \delta.$$

Finally, the function

$$\sum_{p=0}^{P_\delta}e^{\pm i\tau E_p}\langle\Theta^\delta,\chi_p\rangle\chi_p = \sum_{n=0}^{N_\delta}\sum_{p=0}^{P_\delta}e^{\pm i\tau(E_p+\lambda_{n,\delta})}\langle\theta_{n,\delta},\chi_p\rangle\chi_p$$

provides us with a trigonometric polynomial with coefficients in $\mathcal{Z}$ that is a good approximation of $e^{\pm i\tau H_z}\Theta(\tau)$ in $\mathcal{Z}$. Indeed, it satisfies the estimate

$$\sup_{\tau\in\mathbb{R}}\left\|e^{\pm i\tau H_z}\Theta(\tau) - \sum_{n=0}^{N_\delta}\sum_{p=0}^{P_\delta}, e^{\pm i\tau(E_p+\lambda_{n,\delta})}\langle\theta_{n,\delta},\chi_p\rangle\chi_p\right\|_{\mathcal{Z}} \leq 2\delta.$$

This proves that the function $\tau \mapsto e^{\pm i\tau H_z}\Theta(\tau)$ belongs to $\mathrm{AP}(\mathbb{R},\mathcal{Z})$.

To finish the proof of point (i), we only need to prove that, given $\Theta \in \mathrm{AP}(\mathbb{R},\mathcal{Z})$, then $W(|\Theta|^2)$ still belongs to $\mathrm{AP}(\mathbb{R},\mathcal{Z})$. In that view, we recall the explicit form of $W(|\Theta|^2)$ given in (1.15):

$$W(|\Theta|^2)(t,x,z) = \int_0^1 K(z,z')|\Theta(t,x,z')|^2\,dz'. \tag{3.14}$$

Fix $\tau\in\mathbb{R}$, then, in order to approximate $|\Theta|^2$ by a trigonometric polynomial with coefficients in $\mathcal{H}^\ell$, we use (3.11):

$$\begin{aligned}\||\Theta(\tau)|^2 - |\Theta^\delta(\tau)|^2\|_{\mathcal{Z}} &\leq (\|\Theta(\tau)\|_{\mathcal{Z}} + \|\Theta^\delta(\tau)\|_{\mathcal{Z}})\|\Theta(\tau)-\Theta^\delta(\tau)\|_{\mathcal{Z}}\\ &\leq \big(\|\Theta\|_{L^\infty(\mathbb{R},\mathcal{Z})} + \|\Theta^\delta\|_{L^\infty(\mathbb{R},\mathcal{Z})}\big)\|\Theta-\Theta^\delta\|_{L^\infty(\mathbb{R},\mathcal{Z})} \leq (2C+\delta)\delta,\end{aligned}$$

where we applied (3.11) and (3.12), and where $\|\Theta\|_{L^\infty(\mathbb{R},\mathcal{Z})} \leq C$. Moreover, as $\mathcal{Z}$ is an algebra, then it is obvious that $|\Theta^\delta(\tau)|^2$ is a trigonometric polynomial in time with coefficients in $\mathcal{Z}$. To conclude, by (3.14), it is clear that

$$\int_0^1 K(z,z')|\Theta^\delta(\tau)|^2\,dz'$$

is a trigonometric polynomial with coefficients in $\mathcal{Z}$ that approaches $W(|\Theta(\tau)|^2)(t,x,z)$ in $L^\infty(\mathbb{R},\mathcal{Z})$ as δ tends to zero. Finally, since $\Theta \in \mathrm{AP}(\mathbb{R},\mathcal{Z})$, Θ is the limit in $L^\infty(\mathbb{R},\mathcal{Z})$ of trigonometric polynomials $\Theta^\delta(\tau)$ as in (3.11), using the fact that $\mathcal{Z}$ is an algebra, for each $\delta > 0$, the function

$$W(|\Theta^\delta(\tau)|^2)\Theta^\delta(\tau)$$

is a trigonometric polynomial (in time) with coefficient in $\mathcal{Z}$ that approaches $W(|\Theta(\tau)|^2)\Theta(\tau)$ as $\delta \to 0$ in the space $L^\infty(\mathbb{R},\mathcal{Z})$. This ends the proof of (i).

Combining this point (i) and Proposition 3.2, one may define the long-time averaging of G in $\mathcal{Z}$, which proves the first part of point (iii). In order to prove (3.4), consider $u \in \mathcal{Z}$. Then, if $\Theta(\tau)$ denotes $\Theta(\tau) := e^{-i\tau H_z}u \in \mathcal{Z}$, then $\Theta \in \mathrm{AP}(\mathbb{R},\mathcal{Z})$. Indeed, as $u \in \mathcal{Z}$ and $(\chi_p)_p$ is an Hilbertian basis of $L^2(0,1)$, then

$$u(x,z) := \sum_{p\ge 0}\langle u(x,\cdot),\chi_p\rangle\chi_p(z) := \sum_{p\ge 0} u_p(x)\chi_p(z),$$

where the limit holds in $\mathcal{Z}$. As a consequence, $\Theta(\tau) = \sum_{p\ge 0} u_p(x)e^{-i\tau E_p}\chi_p(z)$ can be approached in $\mathcal{Z}$, for any fixed $\delta > 0$ by a finite sum

$$\Theta^\delta(\tau) := \sum_{p=0}^{P_\delta} u_p e^{-iE_p\tau}\chi_p. \tag{3.15}$$

Therefore, we are here in a simple case where the frequencies of the approaching sequence of trigonometric polynomials do not depend on δ.

Computing $e^{i\tau H_z}W(|\Theta^\delta|^2)\Theta^\delta$ with (3.15) leads to

$$\sum_{p=0}^{\infty}\ \sum\sum\sum_{0\le q,r,s\le P_\delta} u_r(x)u_s(x)\overline{u_q}(x)e^{-i(E_r+E_s-E_q-E_p)\tau}\alpha_{pqrs}\chi_p(z), \tag{3.16}$$

which can be approached uniformly in time, choosing as previously an appropriate truncation P_δ for the first sum in (3.16), by

$$\sum\sum\sum\sum_{0\le p,q,r,s\le P_\delta} u_r(x)u_s(x)\overline{u_q}(x)e^{-i(E_r+E_s-E_q-E_p)\tau}\alpha_{pqrs}\chi_p(z).$$

Now, its long-time average is given by

$$\sum\sum\sum\sum_{\substack{0\le p,q,r,s\le P_\delta\\ (q,r,s)\in\Lambda_p}} u_r(x)u_s(x)\overline{u_q}(x)\alpha_{pqrs}\chi_p(z)$$

where α_{pqrs} and Λ_p are defined by

$$\alpha_{pqrs} := \int_0^1\int_0^1 K(z,z')\chi_r(z')\chi_q(z')\chi_s(z)\chi_p(z)\,dz'\,dz$$

and

$$\forall\, p \geq 0, \quad \Lambda_p := \{(q,r,s) \in \mathbb{N}^3,\ E_p + E_q = E_r + E_s\}.$$

As the convergence of $e^{i\tau H_z} W(|\Theta^\delta|^2)\Theta^\delta$ toward $e^{i\tau H_z} W(|\Theta|^2)\Theta$ is uniform in time, the limit $\delta \to 0$ and the average procedure can easily be interverted, which leads to formula (3.4) and ends the proof of (iii).

(ii) and (iv) Consider any $u \in \mathcal{Z}$, and fix $\tau > 0$. Then, since $e^{-i\tau H_z}$ is unitary in the $\mathcal{Z}$-norm, we have

$$\|G(\tau,u)\|_{\mathcal{Z}} \leq \|W(|e^{-i\tau H_z}u|^2)e^{-i\tau H_z}u\|_{\mathcal{Z}}$$

and, using (2.18), we get

$$\|G(\tau,u)\|_{\mathcal{Z}} \leq C\|e^{-i\tau H_z}u\|_{\mathcal{X}}^2\|e^{-i\tau H_z}u\|_{\mathcal{Z}} \leq C\|u\|_{\mathcal{X}}^2\|u\|_{\mathcal{Z}}.$$

Similarly, consider $u, v \in \mathcal{Z}$ and fix $\tau > 0$, then

$$\|G(\tau,u) - G(\tau,v)\|_{\mathcal{Z}} \leq \|W(|e^{-i\tau H_z}u|^2)e^{-i\tau H_z}u - W(|e^{-i\tau H_z}v|^2)e^{-i\tau H_z}v\|_{\mathcal{Z}}.$$

Now, applying (2.19) leads to

$$\|G(\tau,u) - G(\tau,v)\|_{\mathcal{Z}} \leq C(\|u\|_{\mathcal{Z}}^2 + \|v\|_{\mathcal{Z}}^2)\|u-v\|_{\mathcal{Z}}$$

which ends the proof of point (ii). Let us now prove that these tame estimates hold true when we average the nonlinearity G. Consider any $u \in \mathcal{Z}$. Since G_{av} is defined in (3.3) as a strong limit in $\mathcal{Z}$, then

$$\|G_{\mathrm{av}}(u)\|_{\mathcal{Z}} = \left\|\lim_{T\to\infty}\frac{1}{T}\int_0^T G(\tau,u)\,d\tau\right\|_{\mathcal{Z}} \leq \sup_{\tau\in\mathbb{R}}\|G(\tau,u)\|_{\mathcal{Z}} \leq C\|u\|_{\mathcal{X}}^2\|u\|_{\mathcal{Z}}$$

which proves (3.5). A similar computation gives

$$\begin{aligned}\|G_{\mathrm{av}}(u) - G_{\mathrm{av}}(v)\|_{\mathcal{Z}} &\leq \|G(\tau,u) - G(\tau,v)\|_{L^\infty(\mathbb{R},\mathcal{Z})}\\ &\leq C(\|u\|_{\mathcal{Z}}^2 + \|v\|_{\mathcal{Z}}^2)\|u-v\|_{\mathcal{Z}}.\end{aligned}$$

This ends the proof of point (iv). □

Proof of Corollary 3.1. The existence, regularity and uniqueness result is an easy task. Indeed, the existence of a common existence time for the solutions to the intermediate system has already been established in Proposition 2.1. We obtain the associated solution of the filtered intermediate system by filtering out the time oscillations due to the operator H_z as in (1.30).

Now, as far as the averaged system is concerned, the result of existence, uniqueness and regularity is an immediate corollary of Proposition 3.3. Indeed, as already seen in the proof of Proposition 2.1 the key ingredients in order to prove the existence and uniqueness of a local-in-time solution to the nonlinear Schrödinger equation (3.7) is the fact that the mapping

$$u \in \mathcal{Z} \mapsto G_{\mathrm{av}}(u) \in \mathcal{Z}$$

is locally Lipschitz, which is the case here by the estimates (3.5) and (3.6), given by Proposition 3.3, combined with the fact that the propagator $e^{-it\Delta_x}$ is unitary in $\mathcal{Z}$. We again refer to Ref. 11 on these matters. □

Gathering all these properties allows us to perform the standard nonlinear analysis of the equation obtained by averaging in time the filtered equation (1.33)–(1.34), and we state the following proposition.

Proposition 3.4. *Let $\varepsilon > 0$ be fixed and consider any $\psi_0 \in \mathcal{Y}$. If ϕ^ε and ϕ denote the respective solutions to both filtered intermediate system* (1.32)–(1.35) *and averaged system* (3.7)–(3.9) *in $\mathcal{C}^0([0, T_0], \mathcal{Y})$, defined by Corollary* 3.1, *then the following convergence holds*:

$$\|\phi^\varepsilon - \phi\|_{\mathcal{C}([0,T_0],\mathcal{X})} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Remark 3.1. First, note that though the solutions of Proposition 3.1 have the smoothness $\mathcal{C}^0([0, T_0], \mathcal{Y})$, the convergence of the solution ϕ^ε of the filtered intermediate system toward the solution ϕ of the averaged equation only holds in the weaker space $\mathcal{C}^0([0, T_0], \mathcal{X})$. The fact that $\phi^\varepsilon \to \phi$ in $\mathcal{C}^0([0, T_0], \mathcal{X})$ provided ψ_0 only belongs to $\mathcal{X}$ is proven in the next section.

Proof of Proposition 3.4. In order to prove this convergence result, we follow the same lines as in Ref. 1. In that view, let us introduce a "large time" that we denote $T(\varepsilon)$ in order to approach the averaged nonlinearity $G_{\mathrm{av}}(\Psi) := \lim_{T\to\infty}(1/T)\int_0^T G(\tau,\Psi)\,d\tau$ by

$$\tilde{G}_\varepsilon(t,\Psi) := \frac{1}{T(\varepsilon)}\int_t^{t+T(\varepsilon)} G(\tau,\Psi)\,d\tau. \tag{3.17}$$

As a first step, we define the auxiliary solution $\tilde{\phi}^\varepsilon$ to

$$i\partial_t\tilde{\phi}^\varepsilon = H_x\tilde{\phi}^\varepsilon + \tilde{G}_\varepsilon\left(\frac{t}{\varepsilon^2}, \tilde{\phi}^\varepsilon\right), \tag{3.18}$$

$$\tilde{\phi}^\varepsilon(x,0) = \tilde{\phi}^\varepsilon(x,1) = 0 \quad \tilde{\phi}^\varepsilon(0) = \psi_0 \tag{3.19}$$

and find some preliminary bounds.

Step 1. Some preliminary bounds:

Fix $\Theta \in \mathcal{Y}$ and $\tau > 0$. From (3.1), there exists $C > 0$, independent of ε such that

$$\|\tilde{G}_\varepsilon(\tau,\Theta)\|_{\mathcal{Y}} \le \frac{1}{T(\varepsilon)}\int_\tau^{\tau+T(\varepsilon)} \|G(s,\Theta)\|_{\mathcal{Y}}\,ds \le C\|\Theta\|_{\mathcal{X}}^2\|\Theta\|_{\mathcal{Y}}.$$

Using the exact same arguments than in the proof of Proposition 2.1, there exist a common existence time, still denoted by T_0, independent of ε to the intermediate system (1.33)–(1.35), the auxiliary system (3.18)–(3.19) and the limit system (3.7)–(3.9) and a common upper-bound $M > 0$ in $\mathcal{Y}$:

$$\sup_{0<\varepsilon<1}\left[\|\phi^\varepsilon\|_{\mathcal{C}((0,T_0),\mathcal{Y})} + \|\tilde{\phi}^\varepsilon\|_{\mathcal{C}((0,T_0),\mathcal{Y})} + \|\phi\|_{\mathcal{C}((0,T_0),\mathcal{Y})}\right] \le M. \tag{3.20}$$

Moreover, the following uniform Lipschitz property may be stated:

$$\sup_{0<\varepsilon<1} \sup_{0\le\tau\le\frac{T_0}{\varepsilon^2}} \sup_{\substack{\|u\|_{\mathcal{Y}}\le M \\ \|v\|_{\mathcal{Y}}\le M}} [\| G(\tau,u) - G(\tau,v)\|_{\mathcal{Z}} + \|\tilde{G}_\varepsilon(\tau,u) - \tilde{G}_\varepsilon(\tau,v)\|_{\mathcal{Z}}$$

$$+ \| G_{\text{av}}(\tau,u) - G_{\text{av}}(\tau,v)\|_{\mathcal{Z}}] \le CM^2 \|u-v\|_{\mathcal{Z}}, \tag{3.21}$$

where $C > 0$ does not depend on ε.

Step 2: Estimating $\tilde{\phi}^\varepsilon - \phi$ in $\mathcal{X}$:

In order to estimate the difference $\|\tilde{\phi}^\varepsilon - \phi\|_{\mathcal{C}((0,T_0),\mathcal{X})}$, we first estimate, for any given $u \in \mathcal{Y}$, the difference $\|\tilde{G}_\varepsilon(\frac{t}{\varepsilon^2}, u) - G_{\text{av}}(u)\|_{\mathcal{C}((0,T_0),\mathcal{X})}$. In that view, we follow the third and fourth steps of Ref. 1 and, for any u fixed in $\mathcal{Y}$, let us introduce the following convergence rate:

$$\delta(\varepsilon,u) := \sup_{0\le\tau\le 2T_0/\varepsilon^2} \left\| \frac{\varepsilon^2}{2T_0} \int_0^\tau [G(\sigma,u) - G_{\text{av}}(u)]\, d\sigma \right\|_{\mathcal{X}}. \tag{3.22}$$

Inspired by Lemma 4.3 in Ref. 1, we state the following lemma.

Lemma 3.1. (i) *For any given $u \in \mathcal{Y}$, we have $\delta_2(\varepsilon,u) \underset{\varepsilon\to 0}{\longrightarrow} 0$.*
(ii) *Fix $M > 0$ as in* (3.20), *and introduce the uniform convergence rate*

$$\delta_M(\varepsilon) := \sup_{\|v\|_{\mathcal{Y}}\le M} \delta(\varepsilon,v), \qquad \text{then } \delta_M(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.23}$$

(iii) *Assume that, for ε small enough, $\varepsilon^2 T(\varepsilon) \le T_0$, if $M > 0$ is fixed as in* (3.20), *then*

$$\sup_{\|u\|_{\mathcal{Y}}\le M} \left\| \tilde{G}_\varepsilon\left(\frac{t}{\varepsilon^2}, u\right) - G_{\text{av}}(u) \right\|_{\mathcal{C}^0((0,T_0),\mathcal{X})} \le 2T_0 \frac{\delta_M(\varepsilon)}{\varepsilon^2 T(\varepsilon)}. \tag{3.24}$$

Remark 3.2. (a) As the proof of this lemma follows the one of Lemma 4.3 in Ref. 1, we refer the reader to this reference. The key argument appears in the proof of point (ii). It indeed lies on the compactness of the embedding $\mathcal{Y} \subset \mathcal{X}$. The need of this compact embedding, as well as the loss of two derivative, motivates the choice of the regularization space $\mathcal{Y}$.

(b) The right-hand side term in (3.24) does not necessarily tend to zero with ε. It provides us with a necessary condition for the choice of $T(\varepsilon)$ in order for it to tend to zero. In fact, we will choose $T(\varepsilon)$ such that $\varepsilon^2 T(\varepsilon) \to 0$ as $\varepsilon \to 0$ (note that $\varepsilon^2 T(\varepsilon) = \sqrt{\delta_{2,M}}$ will do).

Now, for any $t \in (0, T_0)$, $\omega^\varepsilon(t) := \tilde{\phi}^\varepsilon(t) - \phi(t)$ satisfies

$$i\partial_t \omega^\varepsilon(t) = H_x \omega^\varepsilon(t) + \tilde{G}_\varepsilon\left(\frac{t}{\varepsilon^2}, \tilde{\phi}^\varepsilon(t)\right) - G_{\text{av}}(\phi(t)),$$

$$\omega^\varepsilon(x,0) = \omega^\varepsilon(x,1) = 0, \quad \omega^\varepsilon(0,x,z) = 0$$

and therefore, for all $t \in (0, T_0)$,

$$\|\omega^\varepsilon(t)\|_{\mathcal{X}} \le \int_0^t \left\| \tilde{G}_\varepsilon\Big(\frac{s}{\varepsilon^2}, \tilde{\phi}^\varepsilon(s)\Big) - G_{\text{av}}(\phi(s)) \right\|_{\mathcal{X}} ds.$$

Inserting $\tilde{G}_\varepsilon(\frac{s}{\varepsilon^2}, \phi(s))$ in the difference and combining the Lipschitz estimate (3.21) with the uniform bound (3.20) at hand for ϕ and finally applying point (ii) of Lemma 3.1, we recover

$$\|\omega^\varepsilon(t)\|_{\mathcal{X}} \leq CM^2 \int_0^t \|\omega^\varepsilon(s)\|_{\mathcal{X}} ds + 2T_0^2 \frac{\delta_M}{\varepsilon^2 T(\varepsilon)}.$$

By Gronwall lemma, there finally exists $C(T_0, M) > 0$ such that

$$\forall\, 0 \leq t \leq T_0, \quad \|\tilde{\phi}^\varepsilon(t) - \phi(t)\|_{\mathcal{X}} \leq C \frac{\delta_M}{\varepsilon^2 T(\varepsilon)}. \tag{3.25}$$

Step 3. Estimating $\phi^\varepsilon - \tilde{\phi}^\varepsilon$:

This estimate is more delicate to handle with than the previous one as it relies on an appropriate integration by part in time. Let us fix $0 < T < T_0$, here, T is meant to be close to T_0: we need to have $T + \varepsilon^2 T(\varepsilon) < T_0$ which holds true for ε small enough. The difference $\tilde{\omega}^\varepsilon(t) := \phi^\varepsilon(t) - \tilde{\phi}^\varepsilon(t)$ satisfies

$$i\partial_t \tilde{\omega}^\varepsilon(t) = H_x \tilde{\omega}^\varepsilon + G\left(\frac{t}{\varepsilon^2}, \phi^\varepsilon(t)\right) - \tilde{G}_\varepsilon\left(\frac{t}{\varepsilon^2}, \tilde{\phi}^\varepsilon(t)\right), \quad \tilde{\omega}^\varepsilon(0) = 0.$$

The Duhamel formula, together with (3.21) yields, for all $t \in [0, T]$,

$$\begin{aligned} \|\tilde{\omega}^\varepsilon(t)\|_{\mathcal{X}} \leq CM^2 &\int_0^t \|\tilde{\omega}^\varepsilon(s)\|_{\mathcal{X}} ds \\ &+ \left\| \int_0^t e^{i(t-s)H_x} \left[G\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right) - \tilde{G}_\varepsilon\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right)\right] ds \right\|_{\mathcal{X}}. \end{aligned} \tag{3.26}$$

Following the same arguments as in Ref. 1, we get

$$\int_0^t e^{i(t-s)H_x} \left(\tilde{G}_\varepsilon\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right) - G\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon\right)\right) ds = R_1^\varepsilon + R_2^\varepsilon + R_3^\varepsilon, \tag{3.27}$$

where the remainders R_1^ε, R_2^ε and R_3^ε are to be estimated.

First, R_1^ε is given by

$$\begin{aligned} R_1^\varepsilon := &\int_0^1 \int_0^t e^{i(t-s)H_x} \\ &\cdot \left[-G\left(\frac{s + \varepsilon^2 T(\varepsilon) u}{\varepsilon^2}, \phi^\varepsilon(s + \varepsilon^2 T(\varepsilon) u)\right) + G\left(\frac{s + \varepsilon^2 T(\varepsilon) u}{\varepsilon^2}, \phi^\varepsilon(s)\right)\right], \end{aligned}$$

which by estimate (3.2) gives

$$\begin{aligned} \|R_1^\varepsilon\|_{\mathcal{X}} &\leq CM^2 \int_0^1 \int_0^t \|\phi^\varepsilon(s + \varepsilon^2 T(\varepsilon) u) - \phi^\varepsilon(s)\|_{\mathcal{X}} du ds \\ &\leq CM^2 \varepsilon^2 T(\varepsilon) \|\partial_t \phi^\varepsilon\|_{\mathcal{C}([0, T + \varepsilon^2 T(\varepsilon)], \mathcal{X})}. \end{aligned}$$

Yet, Eq. (1.33), together with the bounds at hand for ϕ^ε in $\mathcal{C}([0, T + \varepsilon^2 T(\varepsilon)], \mathcal{Y})$ and the uniform Lipschitz property (3.1) satisfied by $G(s/\varepsilon^2, \cdot)$, implies

$$\|\partial_t \phi^\varepsilon\|_{\mathcal{C}^0([0, T + \varepsilon^2 T(\varepsilon)], \mathcal{X})} \leq C$$

for some $C>0$ that does not depend on ε, which finally provides

$$\|R_1^\varepsilon\|_{\mathcal{X}} \leq C\varepsilon^2 T(\varepsilon). \tag{3.28}$$

Now, R_2^ε is defined by

$$R_2^\varepsilon := \int_0^1 \int_{\varepsilon^2 T(\varepsilon)u}^{t+\varepsilon^2 T(\varepsilon)u} \left[e^{i(t-s)H_x} - e^{i(t-s+\varepsilon^2 T(\varepsilon)u)H_x}\right] G\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right) dsdu. \tag{3.29}$$

Note that

$$R_2^\varepsilon = \int_0^1 \int_{\varepsilon^2 T(\varepsilon)u}^{t+\varepsilon^2 T(\varepsilon)u} \int_0^{\varepsilon^2 T(\varepsilon)u} \frac{d}{d\sigma}\left(e^{i(t-s)\sigma H_x} G\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right)\right) d\sigma dsdu$$

and therefore, by the uniform Lipschitz estimate (3.21) and the uniform bound (3.20) at hand for ϕ^ε,

$$\begin{aligned}
\|R_2^\varepsilon\|_{\mathcal{X}} &\leq (T+\varepsilon^2 T(\varepsilon))\varepsilon^2 T(\varepsilon) \left\| H_x G\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right)\right\|_{\mathcal{C}^0([0,T+\varepsilon^2 T(\varepsilon)],\mathcal{X})} \\
&\leq \varepsilon^2 T(\varepsilon)(T+\varepsilon^2 T(\varepsilon)) \left\| G\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right)\right\|_{\mathcal{C}^0([0,T+\varepsilon^2 T(\varepsilon)]),\mathcal{Y})} \\
&\leq CM^3 \varepsilon^2 T(\varepsilon).
\end{aligned} \tag{3.30}$$

Finally,

$$\begin{aligned}
R_3^\varepsilon := &-\int_0^1 \int_0^{\varepsilon^2 T(\varepsilon)u} e^{i(t-s)H_x} G\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right) dsdu \\
&+\int_0^1 \int_t^{t+\varepsilon^2 T(\varepsilon)u} e^{i(t-s)H_x} G\left(\frac{s}{\varepsilon^2}, \phi^\varepsilon(s)\right) dsdu,
\end{aligned}$$

and again applying (3.21) and (3.20) gives

$$\|R_3^\varepsilon\|_{\mathcal{X}} \leq C\varepsilon^2 T(\varepsilon) M^3. \tag{3.31}$$

Combining (3.26) with (3.27), (3.28), (3.30) and (3.31), and applying the Gronwall lemma gives

$$\forall\, 0 \leq t \leq T, \quad \|\phi^\varepsilon(t) - \tilde{\phi}^\varepsilon(t)\|_{\mathcal{X}} \leq C\varepsilon^2 T(\varepsilon), \tag{3.32}$$

for some $C>0$, which does not depend on ε.

Step 4. Conclusion:

Gathering the estimates (3.25) in Step 3 and (3.32) in Step 4, we recover

$$\forall\, 0 \leq t \leq T, \quad \|\phi^\varepsilon(t) - \phi(t)\|_{\mathcal{X}} \leq C\left(\varepsilon^2 T(\varepsilon) + \frac{\delta_{2,M}(\varepsilon)}{\varepsilon^2 T(\varepsilon)}\right).$$

According to the choice for $T(\varepsilon)$ made in Remark 3.2, we conclude that

$$\|\phi^\varepsilon - \phi\|_{\mathcal{C}([0,T_0],\mathcal{X})} \leq C\sqrt{\delta_{2,M}(\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.33}$$

□

4. Proof of the Main Theorem

Proposition 4.1. *Consider any $\psi_0 \in \mathcal{Y}$, then there exists $T_0 > 0$ depending only on $\|\psi_0\|_{\mathcal{X}}$ such that ψ^ε and ϕ, the respective solutions to both initial Schrödinger–Poisson system (1.1)–(1.5) and averaged system (3.7)–(3.9) with initial datum ψ_0 exist and are unique in $\mathcal{C}([0, T_0], \mathcal{Y})$. Moreover, the following convergence holds*:

$$\|\psi^\varepsilon - e^{-itH_z/\varepsilon^2}\phi\|_{\mathcal{C}([0,T_0],\mathcal{X})} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Proof. This proposition is easily proved gathering the convergence results that we obtained in Secs. 2 and 3. Indeed, by Proposition 2.1, there exists $T_1 > 0$ depending only on $\|\psi_0\|_{\mathcal{X}}$ such that both initial Schrödinger–Poisson system (1.1)–(1.5) and intermediate system (1.9)–(1.12) with initial datum ψ_0 possess unique solutions, respectively, denoted by $\psi^\varepsilon \in \mathcal{C}([0, T_1], \mathcal{X})$ and $\tilde{\psi}^\varepsilon$ in $\mathcal{C}([0, T_1], \mathcal{Y})$. Moreover, the following holds:

$$\|\psi^\varepsilon - \tilde{\psi}^\varepsilon\|_{\mathcal{C}([0,T_1],\mathcal{X})} \leq C\varepsilon^2,$$

where $C > 0$ does not depend on ε.

Moreover, applying Corollary 3.1 with the initial datum ψ_0, there exists $T_2 > 0$ that only depends on $\|\psi_0\|_{\mathcal{X}}$ such that both filtered intermediate system (1.33)–(1.35) and averaged system (3.7)–(3.9) admit unique solutions in $\mathcal{C}([0, T_2], \mathcal{Y})$ that we denote by ϕ^ε and ϕ. Now, as $e^{-itH_z/\varepsilon^2}\psi^\varepsilon$ satisfies (1.33)–(1.35), if $T_0 = \min(T_1, T_2)$, then, using the unicity of ϕ^ε, we get

$$\forall\, t \in [0, T_0], \quad \tilde{\psi}^\varepsilon(t) = e^{-itH_z/\varepsilon^2}\phi^\varepsilon(t).$$

Moreover, by Proposition 3.4, we get

$$\forall\, t \in [0, T_2], \quad \|\phi^\varepsilon(t) - \phi(t)\|_{\mathcal{X}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Therefore, for all $t \in [0, T_0]$,

$$\|\psi^\varepsilon(t) - e^{-itH_z/\varepsilon^2}\phi(t)\|_{\mathcal{X}} \leq \|\psi^\varepsilon(t) - e^{-itH_z/\varepsilon^2}\phi^\varepsilon(t)\|_{\mathcal{X}} + \|e^{-itH_z/\varepsilon^2}\phi^\varepsilon(t) - e^{-itH_z/\varepsilon^2}\phi(t)\|_{\mathcal{X}}$$

and finally

$$\|\psi^\varepsilon - e^{-itH_z/\varepsilon^2}\phi\|_{\mathcal{C}([0,T_0],\mathcal{X})} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

which ends the proof. □

Proof of the main theorem. Take $\psi_0 \in \mathcal{X}$ and fix a small $\delta > 0$. Now, pick a regularization $\psi_{0,\delta} \in \mathcal{Y}$ of ψ_0 such that

$$\|\psi_0 - \psi_{0,\delta}\|_{\mathcal{X}} \leq \delta. \tag{4.1}$$

Associated to the initial datum $\psi_{0,\delta}$, let us define the functions ψ_δ^ε and $\phi_\delta(t)$ that, respectively, solve the initial Schrödinger–Poisson system (1.1)–(1.5) and the averaged equation (3.7)–(3.8) with initial datum $\psi_{0,\delta}$. Similarly, associated to the initial datum ψ_0, let us define ψ^ε, $\tilde{\psi}^\varepsilon$, ϕ^ε and ϕ that, respectively, solve the initial Schrödinger–Poisson system (1.1)–(1.5) the intermediate system (1.9)–(1.12), the intermediate filtered equation (1.33)–(1.34) and the averaged equation (3.7)–(3.8).

Indeed, we already know that, by Proposition 2.1 and Corollary 3.1, there exists $T_1 > 0$ that only depends on $\|\psi_0\|_{\mathcal{X}}$ such that ψ^ε and $\tilde{\psi}^\varepsilon$ exist and are unique in $\mathcal{C}([0,T_1],\mathcal{X})$, and T_2 depending only on $\|\psi_0\|_{\mathcal{X}}$ such that $\phi^\varepsilon(t)$ and $\phi(t)$ exist and are unique in $\mathcal{C}([0,T_2],\mathcal{X})$. If $T_0 := \min(T_1, T_2)$, then we, moreover, know that they belong to $\mathcal{C}^0([0,T_0],\mathcal{X})$ uniformly in ε by the estimates (2.33) and (3.10).

Applying Proposition 4.1, we also know that, for each $\delta > 0$, there exists $T_{0,\delta} > 0$ depending only on $\|\psi_{0,\delta}\|_{\mathcal{X}}$ such that ψ^ε_δ and ϕ_δ belong to $\mathcal{C}^0([0,T_{0,\delta}],\mathcal{Y})$ uniformly in ε. Since $\|\psi_{0,\delta}\|_{\mathcal{X}} \le \|\psi_0\|_{\mathcal{X}} + \delta$, we may ensure that $\|\psi_{0,\delta}\|_{\mathcal{X}}$ is as close as we wish to $\|\psi_0\|_{\mathcal{X}}$, so that $T_{0,\delta}$ may in turn be assumed as close as needed to T_0. For this reason, we may safely assume for the remaining part of the argument that all the functions $\psi^\varepsilon, \phi, \psi^\varepsilon_\delta$ and ϕ_δ are defined on the *same* time interval $[0, T_0]$. Similarly, according to (2.33) and (3.10), we may safely assume that they are bounded by a common $M > 0$ depending only on $\|\psi_0\|_{\mathcal{X}}$:

$$\sup_{0<\varepsilon<1}\sup_{0<\delta<1}(\|\psi^\varepsilon\|_{\mathcal{C}^0([0,T_0],\mathcal{X})} + \|\psi^\varepsilon_\delta\|_{\mathcal{C}^0([0,T_0],\mathcal{X})} + \|\phi\|_{\mathcal{C}^0([0,T_0],\mathcal{X})} + \|\phi_\delta\|_{\mathcal{C}^0([0,T_0],\mathcal{X})}) \le M. \tag{4.2}$$

We have

$$\|\psi^\varepsilon - e^{-itH_z/\varepsilon^2}\phi\|_{\mathcal{C}([0,T_0],\mathcal{X})} \le \|\psi^\varepsilon - \psi^\varepsilon_\delta\|_{\mathcal{C}([0,T_0],\mathcal{X})} + \|\psi^\varepsilon_\delta - e^{-itH_z/\varepsilon^2}\phi_\delta\|_{\mathcal{C}([0,T_0],\mathcal{X})} + \|\phi_\delta - \phi\|_{\mathcal{C}([0,T_0],\mathcal{X})}. \tag{4.3}$$

On the one hand, Proposition 4.1 asserts that

$$\|\psi^\varepsilon_\delta - e^{-itH_z/\varepsilon^2}\phi_\delta\|_{\mathcal{C}^0([0,T_0],\mathcal{X})} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.4}$$

On the other hand, $\psi^\varepsilon - \psi^\varepsilon_\delta$ satisfies equation

$$i\partial_t(\psi^\varepsilon - \psi^\varepsilon_\delta) = H^\varepsilon(\psi^\varepsilon - \psi^\varepsilon_\delta) + V^\varepsilon(|\psi^\varepsilon|^2)\psi^\varepsilon - V^\varepsilon(|\psi^\varepsilon_\delta|^2)\psi^\varepsilon_\delta$$

with the initial datum $\psi_0 - \psi_{0,\delta}$. Therefore, for all $t \in (0, T_0)$,

$$\|\psi^\varepsilon(t) - \psi^\varepsilon_\delta(t)\|_{\mathcal{X}} \le \|\psi_0 - \psi_{0,\delta}\|_{\mathcal{X}} + \int_0^t \|V^\varepsilon(|\psi^\varepsilon|^2)\psi^\varepsilon(s) - V^\varepsilon(|\psi^\varepsilon_\delta|^2)\psi^\varepsilon_\delta(s)\|_{\mathcal{X}}ds.$$

Applying (2.15) (in the case where $F(u)$ denotes $V(|u|^2)$) and (4.2) gives

$$\|V^\varepsilon(|\psi^\varepsilon|^2)\psi^\varepsilon(s) - V^\varepsilon(|\psi^\varepsilon_\delta|^2)\psi^\varepsilon_\delta(s)\|_{\mathcal{X}} \le CM^2\|\psi^\varepsilon(s) - \psi^\varepsilon_\delta(s)\|_{\mathcal{X}}.$$

Finally, by the Gronwall lemma, we have

$$\|\psi^\varepsilon - \psi^\varepsilon_\delta\|_{\mathcal{C}([0,T_0],\mathcal{X})} \le e^{CM^2T_0}\delta. \tag{4.5}$$

Similarly, as ϕ and ϕ_δ both solve the equation

$$i\partial_t u = -\Delta_x u + G_{\mathrm{av}}(u)$$

with initial data ψ_0 and $\psi_{0,\delta}$, respectively, $\phi - \phi_\delta$ satisfies

$$i\partial_t(\phi - \phi_\delta) = -\Delta_x(\phi - \phi_\delta) + G_{\rm av}(\phi) - G_{\rm av}(\phi_\delta)$$

with initial datum $\psi_0 - \psi_{0,\delta}$. Using the Duhamel formula and the tame estimate at hand on $G_{\rm av}$ given by (3.6) leads to

$$\|\phi^\delta - \phi\|_{\mathcal{C}([0,T_0],\mathcal{X})} \le \delta e^{CM^2 T_0}. \tag{4.6}$$

Finally, combining (4.3)–(4.6), having ε tend to zero, and then choosing δ small enough allows us to conclude this regularization procedure. The main theorem can be deduced applying the identification of $G_{\rm av}$ given in (3.4) and projecting on the pth eigenmode of the operator H_z. □

Appendix. Proof of Lemma 2.4

The existence of a unique solution in $L^2(\Omega)$ comes straightforward as (2.9)–(2.10) is an elliptic equation when $\varepsilon > 0$, and, when $\varepsilon = 0$, the unique solution is explicit.

Let us prove the regularity results. In that view, we denote by $\hat{u}(\xi, z)$ the Fourier transform in the $x \in \mathbb{R}^2$ directions of function $u(\cdot, z)$, z being fixed in $(0, 1)$. We apply this longitudinal Fourier transform to Eqs. (2.9)–(2.10) and we get

$$-\partial_z^2\hat{u}(\xi, z) + \varepsilon^2|\xi|^2\hat{u}(\xi, z) = \hat{f}(\xi, z), \quad (\xi, z) \in \Omega, \tag{A.1}$$

$$\hat{u}(\xi, 0) = \hat{u}(\xi, 1) = 0, \quad \xi \in \mathbb{R}^2. \tag{A.2}$$

Multiplying Eq. (A.1) by $\overline{\hat{u}(\xi, z)}$ and integrating along the z variable over (0, 1) gives

$$\|\partial_z\hat{u}(\xi,\cdot)\|^2_{L^2_z} + \varepsilon^2|\xi|^2\|\hat{u}(\xi,\cdot)\|^2_{L^2_z} \le \|\hat{f}(\xi,\cdot)\|_{L^2_z}\|\hat{u}(\xi,\cdot)\|_{L^2_z}.$$

We combine with the Poincaré inequality (as $\hat{u}$ satisfies (A.2)), and, thus, there exists $C > 0$, which does not depend on ε such that, a.e. in $\xi \in \mathbb{R}^2$,

$$\|\hat{u}(\xi,\cdot)\|_{L^2_z} + \varepsilon^2|\xi|^2\|\hat{u}(\xi,\cdot)\|_{L^2_z} \le C\|\hat{f}(\xi,\cdot)\|_{L^2_z}. \tag{A.3}$$

We therefore get

$$\int_\Omega (1 + |\xi|^2)|\hat{u}(\xi, z)|^2 d\xi dz \le C\int_\Omega (1 + |\xi|^2)|\hat{f}(\xi, z)|^2 d\xi dz \le C\|f\|^2_{\mathcal{X}}.$$

Combining (A.1) with (A.3) also leads to

$$\|\partial_z^2\hat{u}(\xi,\cdot)\|_{L^2_z} \le \|\hat{f}(\xi,\cdot)\|_{L^2_z} + \varepsilon^2|\xi|^2\|\hat{u}(\xi,\cdot)\|_{L^2_z} \le C\|\hat{f}(\xi,\cdot)\|_{L^2_z},$$

which gives

$$\int_\Omega (1 + |\xi|^2)|\partial_z^2\hat{u}(\xi, z)|^2 d\xi dz \le C\|f\|^2_{\mathcal{X}}$$

and

$$\varepsilon^2 \| |\xi|^2 \partial_z^2 \hat{u} \|_{L^2(\Omega)} \le C \| |\xi|^2 \hat{f} \|_{L^2(\Omega)} \| \le C \|f\|_{\mathcal{X}}. \tag{A.4}$$

Finally, let us derive twice Eq. (A.1) with respect to the z variable

$$\partial_z^4 \hat{u}(\xi, z) = -\partial_z^2 \hat{f}(\xi, z) + \varepsilon^2 |\xi|^2 \partial_z^2 \hat{u}(\xi, z).$$

Combined with (A.4), it leads to

$$\|\partial_z^4 \hat{u}\|_{L^2(\Omega)} \le C \|f\|_{\mathcal{X}}.$$

Now that we have obtained the H^2-estimate for u and $\partial_z^2 u$, let us now consider $\omega := V_1 u$. We need to prove independent of ε L^2-estimates for ω and $\partial_z^2 \omega$. In that prospect, a simple computation gives

$$\begin{aligned} -\partial_z^2 \omega - \varepsilon^2 \Delta_x \omega = V_1 f - 2\varepsilon^2 \nabla_x \omega \cdot \frac{\nabla_x V_1}{V_1} \\ + \varepsilon^2 \left(\frac{|2\nabla V_1|^2}{V_1^2} - \frac{\Delta_x V_1}{V_1} \right) \omega, \quad (x, z) \in \Omega. \end{aligned} \tag{A.5}$$

According to Assumption 1.2, $\nabla_x V_1 / V_1$, $|\nabla V_1|^2 / V_1^2$ and $\Delta_x V_1 / V_1$ are bounded on Ω by a positive constant A. Multiplying (A.5) by $\overline{\omega}$ and integrating over Ω yields

$$\begin{aligned} \|\partial_z \omega\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla_x \omega\|_{L^2(\Omega)}^2 \le \|V_1 f\|_{L^2(\Omega)} \|\omega\|_{L^2(\Omega)} + 4A\varepsilon^2 \|\omega\|_{L^2(\Omega)}^2 \\ + 2A\varepsilon^2 \|\nabla_x \omega\|_{L^2(\Omega)} \|\omega\|_{L^2(\Omega)}. \end{aligned}$$

Noticing that

$$2\varepsilon^2 \|\nabla_x \omega\|_{L^2(\Omega)} \|\omega\|_{L^2(\Omega)} \le \varepsilon^3 \|\nabla_x \omega\|_{L^2(\Omega)}^2 + \varepsilon \|\omega\|_{L^2(\Omega)}$$

and combining with Poincaré inequality, we finally get

$$(1 - 4A\varepsilon^2 - A\varepsilon) \|\omega\|_{L^2(\Omega)}^2 + (1 - A\varepsilon) \varepsilon^2 \|\nabla_x \omega\|_{L^2(\Omega)}^2 \le \|V_1 f\|_{L^2(\Omega)} \|\omega\|_{L^2(\Omega)}.$$

Therefore, there exists $C > 0$ such that, for ε small enough,

$$\|\omega\|_{L^2(\Omega)} \le C \|f\|_{\mathcal{X}} \quad \text{and} \quad \varepsilon \|\nabla_x \omega\|_{L^2(\Omega)} \le C \|f\|_{\mathcal{X}}. \tag{A.6}$$

Now, multiplying (A.5) by $\overline{-\partial_z^2 \omega}$ and integrating over Ω easily leads to

$$\begin{aligned} \|\partial_z^2 \omega\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\partial_z \nabla_x \omega\|_{L^2(\Omega)}^2 \le C \|f\|_{\mathcal{X}} \|\partial_z^2 \omega\|_{L^2(\Omega)} + 2A\varepsilon^2 \|\nabla_x \omega\|_{L^2(\Omega)} \|\partial_z^2 \omega\|_{L^2(\Omega)} \\ + 3A\varepsilon^2 \|\omega\|_{L^2(\Omega)} \|\partial_z^2 \omega\|_{L^2(\Omega)}. \end{aligned}$$

Combined with (A.6), this allows us to conclude that there exists $C > 0$, for ε small enough, such that

$$\|\partial_z^2 \omega\|_{L^2(\Omega)} \le C \|f\|_{\mathcal{X}},$$

which ends the proof. ☐

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