Master Eco Stat & Magistère UT1
S7 - Advanced Analysis

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Chapitre 1

Metric spaces

We briefly present in this chapter some basics on topology in order to introduce limit and continuity of functions.

1.1 Metric spaces

The elementary way to introduce limit and continuity is first to describe how we can measure quantitatively the space between two points of the space. In this view, it is natural to define a *distance.*

1.1.1 Distance

We briefly remind the definition of distance on a set E.

Definition 1.1.1 (Distance) *A distance d on a set E is a map from $E \times E$ to $\mathbb{R}^+$ such that*
Symmetric property $d(x, y) = d(y, x)$.
Separation $x \neq y \Longleftrightarrow d(x, y) > 0$.
Triangle inequality $d(x, y) + d(y, z) \geq d(x, z)$.

Remark that the separation property is equivalent to

$$x = y \Longleftrightarrow d(x, y) = 0.$$

Definition 1.1.2 (Metric space) *A metric space is a couple (E, d) where E is a set and d is a distance on the set X.*

1.1.2 Elementary properties

Proposition 1.1.1 *Let be given (E, d) a metric space, the following property holds :*

$$\forall (x, y) \in E^2 \qquad d(x, y) \geq 0.$$

Moreover, the distance between distances is smaller than the distance, meaning that

$$\forall (x, y, z) \in E^3 \qquad |d(x, y) - d(x, z)| \leq d(y, z).$$

<u>*Proof :*</u> To establish the second point, we write the triangle inequality :

$$d(x, y) \leq d(x, z) + d(z, y) \Longleftrightarrow d(x, y) - d(x, z) \leq d(z, y).$$

We also have

$$d(x,z) \leq d(x,y) + d(y,z) \Longleftrightarrow d(x,z) - d(x,y) \leq d(z,y).$$

The two equations above lead to

$$|d(x,y) - d(x,z)| \leq d(y,z).$$

□

1.1.3 Balls in a metric space

Definition 1.1.3 (Balls) *Let be given (E,d) a metric space and a point $x \in E$. For any $r > 0$, the open ball $B(x,r)$ is defined as*

$$B(x,r) = \{y \in E \quad | \quad d(x,y) < r\}.$$

The closed ball $\bar{B}(x,r)$ is defined as

$$\bar{B}(x,r) = \{y \in E \quad | \quad d(x,y) \leq r\}.$$

The following inclusions are obvious :

$$\forall 0 < r < r' \qquad B(x,r) \subset B(x,r').$$

These inclusions are generally strict (but not always).

1.1.4 Bounded sets

Definition 1.1.4 (Bounded sets) *A subset A of a metric space (E,d) is bounded if a closed ball $\bar{B}(x_0,r)$ exists such that $A \subset \bar{B}(x_0,r)$, meaning that :*

$$\forall x \in A \qquad d(x,x_0) \leq r.$$

Of course, we can replace in the definition of bounded sets the closed ball by an open one. At last, the choice of x_0 is not important owing to the triangle inequality.

Definition 1.1.5 (Bounded function) *Let be given a metric space (E,d) and a function $f : X \longmapsto E$, f is bounded on X if $f(X)$ is a bounded subset of E.*

1.1.5 Finite dimensional examples

You will find below a short list of typical examples.

1. For any set E, the function defined by $d(x,y) = \delta_{x,y}$ where δ is the Kronecker symbol (equals to 1 if $x = y$ or 0 if $x \neq y$) is obviously a distance. The ball $B(x,r)$ for any $r \in (0,1)$ is the singleton $\{x\}$.
2. In $\mathbb{R}$, define $d(x,y) = |x-y|$. The balls are intervals of $\mathbb{R}$:

$$B(x,r) =]x-r, x+r[\qquad \text{and} \qquad B(x,r) = [x-r, x+r].$$

3. In $\mathbb{C}$, replace the open (resp. closed) intervals by a true geometrical open (resp. closed) ball.

4. It is an easy exercise to check that

$$\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \qquad d_1(x,y) := \sum_{i=1}^n |x_i - y_i|$$

is a distance on $\mathbb{R}^n$, whatever the integer n is. This distance is called the ℓ_1 distance.

5. It is an easy exercise to check that

$$\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \qquad d_2(x,y) := \sqrt{\sum_{i=1}^n (x_i - y_i)^2}.$$

is a distance on $\mathbb{R}^n$, whatever the integer n is. This distance is called the ℓ_2 distance. We can also speak about the Euclidean distance.

6. It is an easy exercise to check that

$$\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \qquad d_\infty(x,y) := \max_{1 \leq i \leq n} |x_i - y_i|$$

is a distance on $\mathbb{R}^n$, whatever the integer n is. This distance is called the ℓ_∞ distance. We can also speak about the supremum distance.

Figure 1.1 below provide several typical examples.

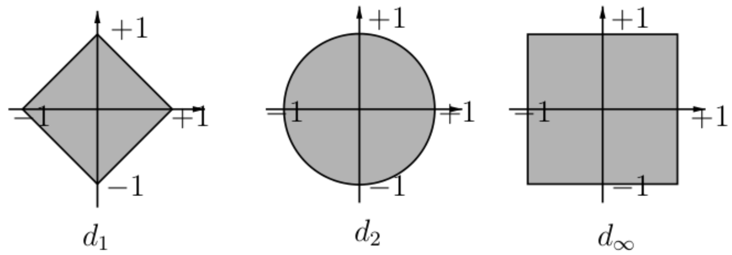


FIGURE 1.1: Several examples of unit balls in $\mathbb{R}^2$.

we will indeed establish the Minkowski theorem that generalises these results for any power between 0 and $+\infty$.

$$\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \qquad d_p(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{1/p}$$

7. From a finite number of metric spaces (E_i, d_i), it is easy to build a product space $E = E_1 \times \ldots \times E_n$ associated to the metric d defined by

$$\forall (x,y) \in E \times E \qquad d(x,y) = \sum_{i=1}^n d_i(x_i, y_i).$$

8. If (E,d) is a metric space, and if φ is an increasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that

$$\varphi(u+v) \leq \varphi(u) + \varphi(v) \qquad \text{and} \qquad \varphi(u) = 0 \Longleftrightarrow u = 0$$

then $\tilde{d} := \varphi \circ d$ is also a distance on E so that $(E, \tilde{d})$ is also a metric space. The proof is quite obvious by using the subadditivity property. Interestingly, we can apply this result with the function

$$\varphi(t) = \frac{t}{1+t} \qquad \text{or} \qquad \varphi(t) = 1 \wedge t,$$

which are bounded functions. Hence, starting from an initial metric space (E, d), it is always possible to build a new metric space $(E, \tilde{d})$ such that $\tilde{d}$ is also a bounded distance on E.

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \qquad \text{or} \qquad \tilde{d}(x, y) = 1 \wedge d(x, y).$$

1.1.6 Infinite dimensional examples

We can also introduce a few examples of metric spaces in an infinite dimensional settings.

1. Let be given E a set and (F, d_F) a metric space, we denote $\mathcal{F}_b(E, F)$ the set of bounded functions from E to F. According to the following distance, $\mathcal{F}_b(E, F)$ becomes a metric space :
$$\forall (f, g) \in \mathcal{F}_b(E, F)^2 \qquad d_\infty(f, g) := \sup_{x \in F} d_F(f(x), g(x)).$$
This distance is illustrated by Figure 1.2 : as shown in Figure 1.2, the balls associated to this distance are "tubes" around a function.

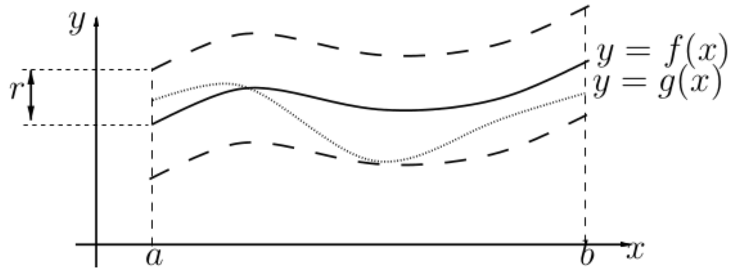


FIGURE 1.2: The function g is at a distance r of the function f. The ball centered on f of radius r is the set of functions that stay in between the upper and lower dot lines.

2. A famous result, that may be seen as an extension of the ℓ^p distance seen in the paragraph above, is given by the Minkowski theorem on L^p spaces. In this view, let us recall that for any measure μ on an interval I, the real space of functions $L^p(I, \mu)$ is
$$L^p(I, \mu) := \left\{ f : I \longmapsto \mathbb{R} \quad | \quad \int_I |f|^p d\mu < \infty \right\}$$
For the sake of simplicity, we will simply consider the Lebesgue measure on I and will establish that
$$\forall (f, g) \in L^p(I) \qquad d_p(f, g) := \left(\int_I |f(x) - g(x)|^p dx \right)^{1/p}$$
defines a distance on $L^p(I)$. The non trivial point comes from the triangle inequality.

Theorem 1.1.1 (Minkowski's Theorem) *For any (f, g) in $L^p(I)$ with $I = [0, 1]$ and $p \geq 1$*
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

<u>*Proof :*</u> First, we admit the Hölder inequality, that assumes :

$$\text{If } f \in L_p(I) \text{ and } g \in L_{p'}(I) \text{ with } \quad \frac{1}{p} + \frac{1}{p'} = 1 \Longrightarrow \|fg\|_1 \leq \|f\|_p \|g\|_{p'} \tag{1.1}$$

We then consider f and g in $L_p(I)$, the triangle inequality leads to :

$$\|f+g\|_p^p = \int_0^1 |f+g|^p(x)dx \leq \int_0^1 (|f|+|g|)(x)|f+g|(x)^{p-1}dx.$$

We expand the term above and obtain :

$$\|f+g\|_p^p \leq \int_0^1 |f|(x)|f+g|(x)^{p-1}dx + \int_0^1 |g|(x)|f+g|(x)^{p-1}dx$$

We shall now apply the Hölder inequality (1.1) to $f \in L^p$ and $(f+g)^{p-1} \in L^{p'}$.

— Therefore, we need to show that $(f+g)^{p-1} \in L_{p'}$. Since $1/p + 1/p' = 1$, we deduce that $\iff p' = \frac{p}{p-1}$ and then

$$\left(|f+g|^{p-1}\right)^{p'} \leq \left(2^{p-1}\left(|f|^{p-1}+|g|^{p-1}\right)\right)^{p'} \leq 2^{(p-1)p'}\left(|f|^{(p-1)p'}+|g|^{(p-1)p'}\right)$$

The two terms of the right hand side are equal to $|f|^p$ and $|g|^p$, because of the relationship between p and p'. Hence, $(f+g)^{p-1}$ belongs to $L_{p'}(I)$.

— We apply the Hölder inequality (1.1) and obtain that

$$\int \|f+g\|_p^p \leq \|f\|_p \|(f+g)^{p-1}\|_{p'} + \|g\|_p \|(f+g)^{p-1}\|_{p'}$$

In the same time, we calculate :

$$\|(f+g)^{p-1}\|_{p'} = \left(\int_0^1 |f+g|^{(p-1)p'}\right)^{1/p'} \quad (1.2)$$

$$= \left(\int_0^1 |f+g|^p\right)^{1-1/p}. \quad (1.3)$$

Hence, we have

$$\|f+g\|_p^p \leq \left(\|f\|_p + \|g\|_p\right) \frac{\|f+g\|_p^p}{\|f+g\|_p}.$$

We then obtain the desired upper bound while dividing by $\|f+g\|_p^p$.

□

To complete the proof, we now come back to the proof of the Hölder inequality, which is shown in the next Lemma.

Lemma 1.1.1 *Assume that* $f \in L_p(I)$ *and* $g \in L_{p'}(I)$ *with* $1/p + 1/p' = 1$, *then* $fg \in L_1(I)$ *and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_{p'}.$$

<u>*Proof :*</u> We can normalize each function and consider the situation where $\|f\|_p = \|g\|_q = 1$ while dividing by $\|f\|_p$ and $\|g\|_{p'}$. We consider $\alpha = 1/p$ and $\beta = 1/p'$. The convex arithmetico-geometric inequality yields

$$\forall (u,v) \in \mathbb{R}_+^2 \qquad u^\alpha v^\beta \leq \alpha u + \beta v.$$

To show this inequality, we can take for example the logarithm of the two expressions above and use the concavity of the logarithm.

Then, we consider $u = |f(x)^p|$ an $v = |g(x)|^{p'}$, and we obtain

$$\int_0^1 |f(x)^p|^{1/p} |g(x)^{p'}|^{1/p'} dx \leq \int_0^1 \frac{1}{p}|f(x)|^p + \frac{1}{p'}|g(x)|^{1/p'}dx = \alpha + \beta = 1.$$

□

1.2 Topology of metric spaces

1.2.1 Open and closed sets

Definition 1.2.1 (Open sets) *Let be given a metric space (E, d). A set $\mathcal{O}$ is open in E if*

$$\forall x \in \mathcal{O} \quad \exists r > 0 \qquad : B(x, r) \subset \mathcal{O}$$

Conversely, it is possible to define the closed sets as follows.

Definition 1.2.2 (Closed sets) *Let be given a metric space (E, d). A set $\mathcal{F}$ is closed in E if $\mathcal{F}^c$ is open in E, meaning that*

$$\forall x \notin \mathcal{F} \quad \exists r > 0 \qquad : B(x, r) \subset \mathcal{F}^c$$

In the metric space $(\mathbb{R}, |.|)$, all the intervals $[a, b]$ with $-\infty < a \leq b < \infty$ are closed. The interval $[a, +\infty[$ is closed since the complementary set is $]-\infty, a[$ is open. Conversely, $[0, 1[$ is not closed and not open.

Definition 1.2.3 (Closure/Adherence) *Let be given a set A of a metric space (E, d), the closure of A is defined as*

$$\bar{A} := \{x \in E \quad | \quad \forall r > 0 : B(x, r) \cap A \neq \emptyset\}.$$

A point x is an accumulation point of A if

$$\forall r > 0 \quad \exists y \in B(x, r) \cap A, x \neq y.$$

Oppositely, x is isolated in A if

$$\exists r > 0 \qquad B(x, r_0) \cap A = \{x\}.$$

For example, consider the set

$$A := \left\{ \frac{1}{n+1}, n \in \mathbb{N} \right\}.$$

The point $1/2$ is isolated in A and is not an accumulation point of A. Oppositely, 0 is an accumulation point of A and does not belong to A.

It is easier to understand $\bar{A}$ according to the next proposition.

Proposition 1.2.1 *$\bar{A}$ is the smallest set (for the inclusion sense) that contains A.*

The proof is left to the reader, or may be found in many lecture notes of topology. An immediate consequence is

Corollary 1.2.1 *A is closed if and only if $\bar{A} = A$.*

The closure operation satisfies standard properties.

Proposition 1.2.2 *If A and B are two subsets of a metric space (E, d), then :*

$$A \subset \bar{A} \qquad \bar{\bar{A}} = \bar{A} \qquad \overline{A \cup B} = \bar{A} \cup \bar{B} \qquad \overline{A \cap B} \subset \bar{A} \cap \bar{B}$$

Note that we do not have necessarily $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$: consider for example $A =]-\infty; 0[$ and $B =]0; +\infty[$, we can check that $\bar{A} \cap \bar{B} = \{0\}$ but $\overline{A \cap B} = \emptyset$.

A last natural definition of this paragraph refers to the notion of density.

Definition 1.2.4 (Dense subset) *We will say that A is a dense subset of (E, d) if*

$$\bar{A} = E.$$

Note that from the definition of the closure of A, it is equivalent to understand a dense subset as follows :

$$\forall x \in E \quad \forall r > 0 \qquad \exists a \in A \cap B(x, r).$$

For example, in $\mathbb{R}$, the set of rational numbers $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. Indeed, since $\mathbb{Q}$ is countable (in bijection with $\mathbb{N}$), we can say that $\mathbb{R}$ is *separable* :

Definition 1.2.5 (Separable metric spaces) *A metric space (E, d) is separable if there exists a countable subset A such that*

$$\bar{A} = E.$$

This definition will be particularly useful when dealing with Hilbert spaces.

1.2.2 Limits in metric spaces and continuity

The metric spaces introduced above permit to extend the usual definition of limiting values to general spaces.

Definition 1.2.6 (Limits) *A sequence $(x_n)_{n\geq 0}$ of a metric space (E, d) converges to x according to the distance d iff*

$$\forall \epsilon > 0 \quad \exists n_0 \in \mathbb{N} \qquad n \geq n_0 \Longrightarrow d(x_n, x) \leq \epsilon.$$

We will use the notation $\lim_n x_n = x$ without recalling the metric used above when no ambiguity is possible. It is immediate to check that limits can be used to characterise the closure of a set.

Proposition 1.2.3 *For any subset A of a metric space (E, d), the two assertions are equivalent :*
- *$x \in \bar{A}$*
- *A sequence $(x_n)_{n\geq 0}$ in $A^{\mathbb{N}}$ exists such that $\lim_n x_n = x$.*

Of course, the definition of the limit introduced above for any metric space (E, d) permit to extend the definition of limit of functions.

Definition 1.2.7 (Pointwise limit of a function) *Let be given $f : E \longmapsto F$ where (E, d) and (F, d') are two metric spaces. We will say that*

$$\lim_{x \longrightarrow x_0} f(x) = \ell \in F$$

if and only if

$$\forall \epsilon > 0 \quad \exists \eta > 0 \qquad d(x, x_0) \leq \eta \Longrightarrow d'(f(x), f(x_0)) \leq \epsilon.$$

We will see later on that some limiting value existence may defer according to the choice of the metric used for the involved metric spaces. Note that the standard properties of limits are still true in the general case of metric spaces (uniqueness, composition, addition . . .).

Definition 1.2.8 (Continuity) *A function $f : E \longmapsto F$ where (E, d) and (F, d') are two metric spaces is continuous iff*

$$\forall x \in E \qquad \lim_{y \longmapsto x} f(y) = f(x).$$

Theorem 1.2.1 (Continuity and topology) *Let be given two metric spaces (E, d) and (F, d'). Consider a function $f : E \longrightarrow F$, then the three following assertions are equivalent :*

— *f is continuous on E.*
— *For any open set $O' \subset F$, $f^{-1}(O')$ is an open set of E.*
— *For any closed set $F' \subset F$, $f^{-1}(F')$ is a closed set of E.*

It is important to note that the theorem above tells something about reciprocal images, but nothing is known about the direct image of closed or open sets. In particular, consider $f(x) = x^2$ and the open interval $I =]-1;1[$. It is immediate to check that $f(I) = [0,1[$, which is not open or closed.

1.2.3 Uniform continuity

We briefly remind the definition of the uniform continuity of a function f.

Definition 1.2.9 *$f : E \longrightarrow F$ is uniformly continuous iff*

$$\forall \epsilon > 0 \exists \eta > 0 \quad \forall (x,y) \in E^2 \qquad d(x,y) \leq \eta \Longrightarrow d'(f(x), f(y)) \leq \epsilon.$$

Some particularly nice functions are Lipschitz ones.

Definition 1.2.10 (Lipschitz functions) *$f : E \longrightarrow F$ is L-Lipschitz iff*

$$\forall (x,y) \in E^2 \qquad d'(f(x), f(y)) \leq L d(x,y).$$

Lipschitz functions are automatically uniformly continuous.

Proposition 1.2.4 *If f is L-Lipschitz, then f is uniformly continuous.*

<u>*Proof :*</u> Consider any $\epsilon > 0$ and remark that if $\eta = \epsilon / L$, then

$$\forall (x,y) \in E^2 \quad d(x,y) \leq \eta \Longrightarrow d'(f(x), f(y)) \leq L d(x,y) \leq L \times \eta \leq \epsilon.$$

□

— Note that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$ and is not Lipschitzian :

$$d(f(x), f(y)) = |x^2 - y^2| = |x - y|\,|x + y|,$$

and in the expression above $|x + y|$ can be arbitrarily large.
— Oppositely, $f : x \longrightarrow \arctan x$ is Lipschitz since

$$f(x) - f(y) = f'(c_{x,y}) \times (x - y) = \frac{1}{1 + c_{x,y}^2}(x - y).$$

As a consequence,

$$|f(x) - f(y)| \leq |x - y|,$$

and f is 1-Lipschitz and thus uniformly continuous.

Definition 1.2.11 (Isometry) *A function $f : E \to F$ between two metric spaces is an isometry iff it is a bijective function such that*

$$\forall (x,y) \in E^2 \qquad d'(f(x), f(y)) = d(x,y).$$

We will refer to an isometric embedding if we do not assume f to be one to one. A linear isometric map will be indeed bijective, and two metric spaces E et F are called isometric iff an isometry between E and F exists.

1.3 Compact sets

The definition of compactness of sets is at the cornerstone of many fundamental results of probability, analysis, PDE, economics .... Even if it is formalized as an abstract topological notion, it is often a very practical tool for an efficient analysis of many concrete problem.

1.3.1 Definition

Definition 1.3.1 *A metric space (E, d) is compact if the following assertion hold :*

— *From any covering of E by open sets $(O_i, i \in I)$, we can extract a finite set J that still covers E :*

$$X = \cup_{i\in I} O_i \Longrightarrow \exists\, J finite : X = \cup_{j\in J} O_j$$

It is easy to see that a compact set E if and only if for any family of closed sets $(F_i, i \in I)$:

$$\cap_{i\in I} F_i = \emptyset \Longrightarrow \exists\, J\text{finite} : \cap_{j\in J} F_j = \emptyset.$$

We can provide here very simple examples of compact sets :

1. The empty set $\emptyset$ is compact.
2. Every finite set is compact, whatever the distance is.
3. The real line $\mathbb{R}$ with the $|.|$ distance is not compact, since :

$$\mathbb{R} = \cup_{n\in\mathbb{Z}}]n, n+2[.$$

The intervals above are all open and it is impossible to extract a finite covering of $\mathbb{R}$.

1.3.2 Bolzano-Weierstrass property

The main result

Theorem 1.3.1 (Bolzano-Weierstrass) *In a metric space (E, d), the three following assertions are equivalent :*

i) A is a compact set.
ii) Every infinite part of A countains an accumulation point.
iii) For each sequence $(x_n)_{n\in\mathbb{N}}$ in $A^{\mathbb{N}}$, we can extract a convergent (in A) subsequence.

We will admit the proof of this famous result, that can be found in many standard lecture notes, level L3. Remark that an immediate consequence of this former result is that every metric compact set is separable (contains a dense countable subset). It can be easily seen since for any $n \in \mathbb{N}$:

$$A = \cup_{x\in A} B(x, 2^{-n}).$$

It is thus immediate to check that

$$A \subset \cup_{1\leq k\leq N_n} B(x_{k,n}, 2^{-n}).$$

An easy consequence is that the set defined by

$$\tilde{A} := \{x_{k,n}\, 1 \leq k \leq N_n, n \in \mathbb{N}\},$$

is dense in A and countable

Consequences The sequential caracterisation of compact sets is very useful, and permits to exhibit many compact sets.

Theorem 1.3.2 (Heine-Borel-Lebesgue) *Every bounded closed real interval is compact.*

Proof : We check that we can apply $iii)$ of Theorem 1.3.1. Let be given a sequence $(x_n)_{n\in\mathbb{N}}$ in an interval $[a,b]$. We build a subsequence as follows : $a_0 = a$ and $b_0 = b$. For any integer $k \geq 0$, define $c_k = (a_k + b_k)/2$ and if $[a_k, c_k]$ contains an infinite number of values of $(x_n)_{n\in\mathbb{N}}$: choose $a_{k+1} = a_k$ and $b_{k+1} = c_k$. Otherwise, define $a_{k+1} = c_k$ and $b_{k+1} = b_k$.

It is immediate to check that for any k, $[a_k, b_k]$ possesses an infinite number of values of $(x_n)_{n\in\mathbb{N}}$. Furthermore, the sequences (a_k) and (b_k) are adjacent and converges through the same limit. Therefore, this limit is also a limit of an extraction of $(x_n)_{n\geq 0}$. $\square$

We can push further the description of compact sets of $\mathbb{R}^d$ and prove the corollary :

Corollary 1.3.1 *In $\mathbb{R}^d$, the compact sets are the bounded closed subsets of $\mathbb{R}^d$.*

Other elementary properties of compact sets We provide other properties of compact sets :

Proposition 1.3.1 *Let be given any metric space (E,d), the following assertions hold :*

i) Every compact set of E is closed and bounded

ii) A finite union of compact sets is compact

iii) Any intersection of compact sets is compact

iv) Any finite or countable product of compact sets is compact (for the distance associated to the cartesian product)

The last point is clearly the most difficult to handle. In the case of countable product of compact sets, it is known as the Tychonov Theorem. The proof can be found in many L3 lecture notes.

1.3.3 Continuous functions on compact sets

Continuous functions act very specifically on compact sets, as pointed by the next result.

Proposition 1.3.2 *For any two metric spaces (E,d) and (F,d') with f a continuous function from E to F, if U is a compact subset of E, then $f(U)$ is a compact subset of F.*

Proof : If U is compact, we consider any covering $(O'_i)_{i\in I}$ of $f(U)$ and see that $\cup_{i\in I} f^{-1}(O'_i)$ is a covering of U, which is a compact set. It is then possible to extract a finite covering of U : $\cup_{1\leq i\leq N_I} f^{-1}(O'_i)$. We then have $f(U) \subset \cup_{1\leq i\leq N_I} O'_i$ and $f(U)$ is compact. $\square$ The famous corollary of this proposition is as follows.

Corollary 1.3.2 *A continuous real function on a compact set attains its lower and upper values.*

Conversely, it is false that a reverse image of a compact set by a continuous function is compact. For example, consider the orthogonal projection on the x-axis of $\mathbb{R}^2$. We have $\pi^{-1}([0,1]) = [0,1] \times \mathbb{R}$, which is not compact.

Theorem 1.3.3 *If f is a continuous function from (E,d) to (F,d') with E compact, then f is uniformly continuous on E.*

1.4 Linear maps

1.4.1 Norms

An important notion related to the distance definition relies on the definition of a norm on a vector space. This norm is not necessarily obtained through a scalar product (see below). In a formal way, a norm is defined as follows.

Definition 1.4.1 (Norm) *A norm on a vector space is a map* $\|.\|$ *from* E *to* $\mathbb{R}^+$ *such that*

— $\|x\| = 0 \Longleftrightarrow x = 0$

— $\|\lambda x\| = |\lambda|\|x\|$

— $\|x + y\| \leq \|x\| + \|y\|$.

It is possible to obtain a distance from a norm through

$$d(x, y) := \|x - y\|.$$

In the opposite, it is not always true that a distance is always derived from a norm : for example $d(x, y) = \delta_{x,y}$ and some other counter-examples exist with intricate functional spaces.

We can provide a short list of examples :

—
$$\|x\|_p = \left(\sum_{i=1}^{p} |x_i|^p\right)^{1/p}.$$

— If we consider the set of continuous functions on $[0, 1]$, the norm

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|,$$

is the norm of the uniform convergence : it means that $f_n \longmapsto f$ unifomly iff

$$\|f_n - f\|_\infty \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow +\infty.$$

— We can also handle

$$N_k(f) := \max_{0\leq i\leq k} \|f^{(i)}\|_\infty \qquad \text{or} \qquad \tilde{N}_k(f) := \left(\sum_{i=1}^{k} \|f^{(i)}\|_\infty^p\right)^{1/p}$$

A second fundamental definition relies on the equivalence between norms.

Definition 1.4.2 (Norm equivalence) *We will say that two norms* N_1 *and* N_2 *are equivalent iff two constants* (c, C) *exist such that*

$$cN_1 \leq N_2 \leq CN_1.$$

It is important to have in mind that the norm equivalence is not an innocent definition. For example, define a sequence of functions $(f_n)_{n\geq 0}$ as illustrated in Figure 1.3 :

We can check that $\|f_n\|_\infty := 1$ and $\|f_n\|_1 := \frac{1}{n+1}$, so that

$$\frac{\|f_n\|_\infty}{\|f_n\|_1} \longrightarrow +\infty \qquad \text{as} \qquad n \longrightarrow +\infty.$$

It means that $\|.\|_\infty$ and $\|.\|_1$ are not equivalent.

You will find many results on this topic in the L3 lectures available on my website. You can also study the book of G. Skandalis : "Topologie et analyse fonctionnelle, 3ème année" or the reference book of Rudin "Analyse fonctionnelle, Principes d'analyse mathématique"

Mainly, we will have in mind the famous result (stated below)

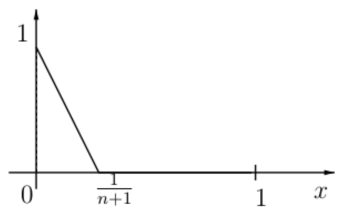


FIGURE 1.3: A triangle function $(f_n)_{n\geq 0}$.

Theorem 1.4.1 (Norm equivalence) *Consider a K-vector spaced E, with K a complete metric space. If E is finite dimensional, then the norms are all pairwise equivalents.*

Remark that the Theorem above is not true when K is not yet a complete body. For example, consider the 2 dimensional space $\mathbb{Q}$-vector space :

$$E = \mathbb{Q}[\sqrt{2}] = \left\{ a + b\sqrt{2} \, : \, (a, b) \in \mathbb{Z} \right\}.$$

It is immediate to check that the two following norms

$$\forall a + b\sqrt{2} \in E \qquad N_1(a + b\sqrt{2}) := |a| \vee |b| \qquad \text{and} \qquad N_2(a + b\sqrt{2}) := |a + b\sqrt{2}|$$

are not equivalent. Consider for example the sequence $u_n = (1 - \sqrt{2})^n$: the whole sequence belongs to E and

$$\forall n \in \mathbb{N} \qquad N_1(u_n) \geq 1 \qquad \text{althouth} \qquad N_2(u_n) \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow +\infty.$$

1.4.2 Continuity of linear maps

Let be given two normed vector spaces (E, N_1) and (F, N_2) and consider a **linear** map $f : E \to F$, we define the norm of the application f as

$$\|f\| = \sup_{x \in E \setminus \{0\}} \frac{N_2(f(x))}{N_1(x)}.$$

Since f is linear, it is obvious to check that we also have :

$$\|f\| = \sup_{N_1(x)=1} N_2(f(x)) = \sup_{N_1(x)\leq 1} N_2(f(x)).$$

This last point can be deduced from

$$\frac{N_2(f(x))}{N_1(x)} = N_2\left(\frac{f(x)}{N_1(x)}\right) = N_2\left(f\left(\frac{1}{N_1(x)}x\right)\right) = N_2(f(y)),$$

with $y = x/N_1(x)$, whose norm is 1 in E.

Say differently, the norm of the linear map $\|f\|$ is the smallest $M > 0$ such that

$$\forall x \in E \qquad N_2(f(x)) \leq mN_1(x).$$

It is quite easy to fully caracterise the continuity of linear maps between normed vector spaces through the following proposition.

Proposition 1.4.1 (Linear continuous maps) *For any linear map $f : E \longrightarrow F$, the three following assertions are equivalent*

1. *f is continuous at 0_E.*
2. *f is continuous.*
3. *$\|f\|$ is finite.*

— We find below a simple computation of a linear map. Let be given a squared matrix A :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We consider f the linear map from $(\mathbb{R}^2, \|.\|_1)$ to $(\mathbb{R}^2, \|.\|_2)$ given by

$$f(x, y) = A(x, y)^t.$$

We can check that

$$\|f(x,y)\|_2^2 = (ax+by)^2 + (cx+dy)^2 \leq \left(|x|\sqrt{a^2+c^2} + |y|\sqrt{b^2+d^2}\right)^2,$$

and the last term is upper-bounded by

$$|x|\sqrt{a^2+c^2} + |y|\sqrt{b^2+d^2} \leq (|x|+|y|)\max\{\sqrt{a^2+c^2}; \sqrt{b^2+d^2}\}.$$

If we consider now a vector (x, y) in the ℓ_1 ball of $\mathbb{R}^2$, we then have $\|(x, y)\|_1 \leq 1$ and

$$\|f(x,y)\|_2 \leq \max\{\sqrt{a^2+c^2}; \sqrt{b^2+d^2}\}.$$

We will easily check that this last upper bound is attained either at point $(1, 0)$, or at point $(0, 1)$. We have thus proved

$$\|f\| = \max\{\sqrt{a^2+c^2}; \sqrt{b^2+d^2}\}.$$

— The impact of the norm choice is very important! For example, consider $E = \mathbb{R}[X]$ and consider

$$\varphi(P) = P(2).$$

If we consider on E the suppremum norm $\|.\|_{\infty,[0,1]}$, then $|\varphi(P)| \leq \|P\|_\infty$ and φ is of course continuous. Nevertheless, consider now the norm given by the maximum of the absolute values of the coefficients of the polynomial, then φ *is not* continuous : $P_n(X) = X^n$ is a normed 1 vector and $\varphi(P_n) = 2^n \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

Sometimes, the term "continuous" is replaced by "bounded" for linear maps between vector spaces mainly because of the equivalence between (1) and (3). In what follows, we denote $\mathcal{L}(E, F)$ the set of continuous and linear maps between E and F. The next proposition is immediate

Proposition 1.4.2 1. *The application from $\mathcal{L}(E, F)$ to $\mathbb{R}$ such that $f \longmapsto \|f\|$ is a norm called operator norm.*

2. *If $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{M}(F, G)$, then*

$$\|g \circ f\| \leq \|g\|.\|f\|.$$

3. *If E is not reduced to $\{0\}$, then*

$$\|Id\| = 1.$$

Proof : The proof relies on three main ingredients and the definition of the operator norm.

1. First, remark that if $f \in \mathcal{L}(E,F)$, then

$$\|f\| = 0 \Longrightarrow \sup_{\|x\|\neq 0} \|f(x)\|/\|x\| = 0 \Longrightarrow \forall x \in E \qquad f(x) = 0.$$

The homogeneity property is obvious. The triangle inequality and the definition of $\|f\|$ and $\|g\|$ implies that

$$\forall x \in E \quad \|x\| \leq 1 \qquad \|(f+g)(x)\| \leq \|f(x)\| + \|g(x)\| \leq \|f\| + \|g\|.$$

Taking now the suppremum in the right hand side, we then deduce that

$$\|f+g\| \leq \|f+g\|.$$

2. The second point come from the following inequalities :

$$\forall x \|x\| \leq 1 \qquad \|g \circ f(x)\| = \|f(x)\| \left\| g\left(\frac{f(x)}{\|f\|(x)}\right) \right\| \leq \|f(x)\| \sup_{\|y\|\leq 1} \|g(y)\| \leq \|f\| \times \|g\|.$$

3. The last point is immediate since $\|Id\| = \sup_x \|f(x)\|/\|x\| = \sup_x \|x\|/\|x\| = 1$.

□

The next corollary is a simple consequence of the proposition above.

Corollary 1.4.1 *For any $f \in \mathcal{L}(E,E)$, we get*

$$\|f^n\| \leq \|f\|^n.$$

You will find below some kind (or not) examples that may deserve attention.

— Consider $\phi : \mathbb{R}[X] \mapsto \mathbb{R}$ given by

$$\phi(P) = P(2).$$

Consider the norm given by the supremum of the absolute value of the coefficients of P : $N(P) = \sup_i |a_i|$ with $P = \sum a_i X^i$.
We then consider the particular case $P_n(X) = X^n$ and we can check that $N(P_n) = 1$ but $\phi(P_n) = 2^n \longrightarrow +\infty$. A simple consequence is that ϕ is not continuous since $\|\phi\| = \infty$.

— In a same way on $\mathbb{R}[X]$ with the suppreum norm and introduce the new norm N_1 defined by

$$N_1\left(\sum a_i X^i\right) = \sum_i |a_i|,$$

We consider the identity map $\psi(P) = P$, considered from the normed vector space $(\mathbb{R}[X], N)$ into the *different* vector spaced $(\mathbb{R}[X], N)$. We remark that $P_n = 1 + X + \ldots + X^{n-1}$ satisfies $N(P_n) = 1$ although $N_1(P_n) = n$. Hence, $\psi = Id$ is not continuous between these two normed vector spaces.

1.4.3 Compactness and consequences in normed vector spaces

We will show that an essential difference exists between finite and infinite dimensional space. We first show a preliminary lemma.

Lemma 1.4.1 *If E, d is a finite dimensional metric space, embedded with a basis $(e_1, \dots, e_n)$. Define $\|x\|_\infty$ as the supremum norm in this basis :*

$$\textit{if} \quad x = \sum_{i=1}^{n} x_i e_i \quad \textit{define} \quad \|x\|_\infty := \max_{1 \le i \le n} |x_i|,$$

then the unit ball $\bar{B}_{\|.\|_\infty}(0,1)$ is compact.

Proof : It is an easy exercise to check that $\phi : (x_1, \dots, x_n) \in \mathbb{R}^n \longmapsto \sum_{i=1}^{n} x_i e_i$ is a continuous isomorphism. Moreover, $\bar{B}_{\|.\|_\infty}(0,1) = \phi(\bar{B}(0,1))$ and $\bar{B}(0,1)$ is compact. $\square$

It is then possible to state the main result of this paragraph.

Theorem 1.4.2 *In a finite dimensional real valued vector space, all the norms are equivalent.*

Proof : Consider any norm $\|.\|$ on E and a basis of E denoted $(e_1, \dots, e_n)$. We can build the corresponding norm $\|.\|_\infty$ introduced in Lemma 1.4.1 and write

$$\|x\| = \left\| \sum_{i=1}^{n} x_i e_i \right\| \le \sum_{i=1}^{n} |x_i| \|e_i\| \le \|x\|_\infty \left(\sum_{i=1}^{n} \|e_i\| \right)$$

This inequality being true for any x in E, we deduce that

$$\|.\| \le \left(\sum_{i=1}^{n} \|e_i\| \right) \|.\|_\infty \tag{1.4}$$

In the meantime, (1.4) says that $Id : (E, \|.\|_\infty) \longrightarrow (E, \|.\|)$ is a linear continuous function. Since the sphere $\mathcal{S}_{\|.\|_\infty}(0,1)$ is a compact set and $\|\|$ is continuous (a norm is always a continuous function), then $\|\|$ attains its minimal value on $\mathcal{S}_{\|.\|_\infty}(0,1)$:

$$\exists x^* \in E \,\text{s.t.}\, \|x^*\|_\infty = 1 \quad \text{and} \quad \|x^*\| = \inf_{\|x\|_\infty = 1} \|x\|.$$

x^* belongs to the unit sphere $\mathcal{S}_{\|.\|_\infty}(0,1)$ so that $x^* \neq 0$ and $\|x^*\| = \delta > 0$. Consequently, for any $x \in E$:

$$\|x\| = \|x\|_\infty \left\| \frac{x}{\|x\|_\infty} \right\| \ge \delta \|x\|_\infty. \tag{1.5}$$

Now, (1.4) and (1.5) permits to deduce that $\|.\|$ and $\|.\|_\infty$ are equivalent. This last result is true for any norm on E, which ends the proof. $\square$

Chapitre 2

Banach spaces

Notations

— H Hilbert Space
— $\langle , \rangle$ the inner (scalar) product on H
— $\|.\|_2$ the L_2 norm derived from the inner product
— ...

2.1 Banach spaces

2.1.1 Cauchy Sequences

Definition 2.1.1 (Cauchy Sequences) *A sequence $(u_n)_{n\geq 0}$ in a metric space(E, d) satisfies the Cauchy property iff :*

$$\forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall (p, q) \geq N \qquad d(u_p, u_q) \leq \epsilon.$$

You will find below a short list of nice properties of Cauchy sequences.

Proposition 2.1.1 — *All extraction of a Cauchy sequence is a Cauchy sequence.*
— *A Cauchy sequence is always a bounded sequence.*
— *A sequence converges iff it is a Cauchy sequence with a convergent extraction.*
— *A uniformly continuous function transforms a Cauchy sequence in a Cauchy sequence.*

Proof : Left for the reader as an exercice. See the lecture notes of L3. □

Remark that of course, the second point is only a one side implication and some trivial counter examples can be found : consider for example $u_n = (-1)^n$, which is a bounded sequence but does not satisfy the Cauchy criterion. It should also be noted that a Cauchy sequence does not always converge : $u_n = 1/n$ on $(0, 1]$ is a Cauchy sequence but does not converge in $(0, 1]$ (this sequence converges to $0 \notin (0, 1]$).

2.1.2 Complete spaces, Banach spaces

Definition 2.1.2 (Complete spaces) *A metric space (E, d) is complete iff all the Cauchy sequences are convergent in E.*

The next proposition is naturally used for building several complete spaces.

Proposition 2.1.2 — *If (E, N_E) and (F, N_F) are complete, then the product $E \times F$ is complete for the norm $N = N_E + N_F$.*

— *If E is complete and F closed in E, then F is complete.*

Proof : Left for the reader as an exercice, see L3 Lecture Notes. □

At last, let us briefly state the next important result.

Theorem 2.1.1 *The vector space $\mathbb{R}^p$ with the euclidean distance is complete : every Cauchy sequence of $\mathbb{R}^p$ converges.*

Proof : Not so trivial, see L3 lectures. □

Definition 2.1.3 (Banach spaces) *E is a Banach space if it is a complete normed vector space.*

You will find below some examples.

Finite dimensional examples

1. With the usual distance on $\mathbb{R}$, $\mathbb{Q}$ is not complete but $\mathbb{R}$ is complete.
   For example, consider in $\mathbb{Q}$ the sequence $u_0 = 1$, $u_1 = 1.4$, $u_2 = 1.41$, $u_3 = 1.414$, ... It is easy to see that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges in $\mathbb{R}$. It converges towards $\sqrt{2} \notin \mathbb{Q}$ and thus $\mathbb{Q}$ is not complete.
2. Fortunately, this example is quite intricate and many situations are much more simpler according to the next result.

   Proposition 2.1.3 *Every finite dimensional $\mathbb{R}$ vector space is a Banach space.*

Infinite dimensional example Infinite dimensional examples are more complex (set of sequences, functions, ...). A first important example :

Proposition 2.1.4 *The set of continuous function $\mathcal{C}([0,1],\mathbb{R})$ is a Banach space when the norm used is $\|.\|_\infty$.*

Proof :

Considérons pour cela une suite de Cauchy de $\mathcal{C}([0,1],\mathbb{R})$, notée $(f_n)_{n\geq 1}$ et remarquons que pour tout x de $[0,1]$, la suite $(f_n(x))_{n\geq 1}$ est de Cauchy dans $\mathbb{R}$ qui est complet. Aussi, $(f_n(x))_{n\geq 1}$ est convergente vers une limite que nous noterons $f(x)$.

On sait aussi que

$$\forall \epsilon > 0 \quad \exists\, n_0 \quad n \geq n_0, p \geq 0 \Longrightarrow \sup_{x\in[0,1]} |f_n(x) - f_{n+p}(x)| \leq \epsilon.$$

En passant à la limite en p, on a alors que

$$\forall \epsilon > 0 \quad \exists\, n_0 \quad n \geq n_0 \Longrightarrow |f_n(x) - f(x)| \leq \epsilon.$$

Autrement dit, on a :

$$N_\infty(f_n, f) \Longrightarrow 0 \quad \text{lorsque} \quad n \longrightarrow +\infty.$$

Il reste ensuite à démontrer que la fonction f est continue. Ce dernier point provient du fait que l'ensemble des fonctions continues est un fermé pour la norme infinie. Si $f_n \longrightarrow f$ en norme infinie, alors

$$\forall \epsilon > 0 \qquad \exists n_\epsilon \quad n \geq n_\epsilon \quad \Longrightarrow N_\infty(f_n - f) \leq \epsilon.$$

Fixons alors $x \in [0,1]$ et $n \geq n_\epsilon$, on sait alors que f_n est continue en x, donc il existe $\eta > 0$ pour lequel

$$|x-y| \leq \eta \Longrightarrow |f_n(x) - f_n(y)| \leq \epsilon.$$

Puis,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq 3\epsilon.$$

□

Again, the norm used is very important : consider the same vector space with the L^1 norm $N_1(f) = \int_0^1 |f|$, this set is not complete now. In particular, consider the sequence f_n equals to 1 between 0 and $1/2 - 1/n$, to 0 between $1/2 + 1/n$ and 1, and affine between $1/2 - 1/n$ and $1/2 + 1/n$. We get easily that it is a Cauchy sequence :

$$N_1(f_{n+p} - f_n)) \leq \frac{1}{2n}.$$

Nevertheless, the limit of f_n is not continuous near 1/2, showing that $(\mathcal{C}([0,1], \mathbb{R}), N_1)$ is not complete.

Proposition 2.1.5 *The space ℓ^p for $p \geq 1$ is a Banach space where ℓ^p refers to the set of sequences $(u_n)_{n\geq 0}$ such that $\sum (u_n)^p < \infty$.*

Preuve : Là encore, la preuve se déroule en découpant les ϵ. On considère une suite de Cauchy $(u^n)_{n\geq 1}$ dans ℓ^p. Chaque coordonnée de la suite est donc une suite de Cauchy de $\mathbb{R}$, donc convergente vers une quantité telle que

$$u_k^n \longrightarrow u_k.$$

Il faut alors démontrer que u_k est bien un élément de ℓ^p. Cela se démontre en suivant encore le schéma de la preuve précédente. Enfin, il reste à prouver que $\|u^n - u\|_p \longrightarrow 0$ lorsque n tend vers l'infini. On consultera un manuel d'analyse de licence pour une démonstration complète. □

We end the enumeration of examples with maybe the most important complete set for the beginning of the lecture on functional analysis. Let be given I a closed bounded interval of $\mathbb{R}$ and denote $\mathbb{L}^p(I)$ the functions $f : I \longrightarrow \mathbb{R}$ such that $\int_I f^p < \infty$, we can state the next famous result.

Theorem 2.1.2 *$\mathbb{L}^p(I)$ with $\|f\|_p = \left(\int_I |f|^p\right)^{1/p}$ is complete.*

Pour une preuve de ce résultat, on pourra par exemple consulter le livre d'H. Brezis.

The next proposition permits to build a large number of Banach spaces and can be compared to the Proposition 2.1.2.

Proposition 2.1.6
- *All closed space in a Banach space is Banach.*
- *A cartesian product of spaces $E \times F$ is Banach iff E and F are Banach spaces.*

2.1.3 Series of vectors

Definition 2.1.4 (Convergent series) *For any countable set of vectors $(u_k)_k$ in a normed space E, we say that $\sum u_k$ converges in E iff*

$$\lim_{K \longrightarrow +\infty} \sum_{j=1}^{K} u_j \qquad \textit{exists}$$

and belongs to E. If such a limit exists, we will refer to series of $(u_k)_k$.

Of course, we should have in mind that this definition highly depends on the norm on the vector space E since the convergence depends on this definition.

Definition 2.1.5 (Absolutely convergent series) *For any countable set of vectors $(u_k)_k$ in a normed space E, we say that $\sum u_k$ absolutely converges in E iff*

$$\sum_{j=1}^{\infty} \|u_j\| < \infty.$$

Of course, absolutely convergent series permit to obtain Cauchy sequences : if we denote

$$r_n = \sum_{j=1}^{n} \|u\|_j \qquad \text{and} \qquad S_n = \sum_{j=1}^{n} u_j$$

we see that r_n is an increasing and bounded sequence, thus convergent in $\mathbb{R}$. A consequence is that the sequence S_n is a Cauchy sequence :

$$S_{n+p} - S_n = \sum_{n+1}^{n+p} u_j,$$

so that the triangle inequality yields

$$\|S_{n+p} - S_n\| \leq r_{n+p} - r_n \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow +\infty.$$

As a consequence, when E is a Banach space, the series $\sum u_k$ converges as soon as the series is absolutely convergent. Indeed, we even have the characterization of completeness :

Proposition 2.1.7 *Let E be a normed vector space. E is a Banach space iff for any series of vector (u_k)*

$$\sum_{j=1}^{\infty} \|u_j\| < \infty \Longrightarrow \sum_{j=1}^{\infty} u_j \quad \textit{converges in} \quad E.$$

<u>*Proof :*</u> We have already seen that if E is a Banach space, then absolutely convergent series is stronger than convergent series. We thus study the reverse implication and consider $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence in E : for any $k \in \mathbb{N}$, we can find N_k such that

$$\forall m \geq n \geq N_k \qquad \|x_m - x_n\| \leq 2^{-k}.$$

Without loss of generality, we can assume that $(N_k)_{k\geq 0}$ is increasing and we then define $u_1 = x_{N_1}$ and

$$u_{k+1} = x_{N_{k+1}} - x_{N_k}.$$

We can check that $(u_j)_{j\geq 1}$ is an absolutely convergent sequence since $\|u_k\| \leq 2^{-k}$ and a consequence of our assumption is that it is a convergent series in E. But remark also that the partial sums are

$$\sum_{k=1}^{K} u_j = x_{N_{K+1}}.$$

Consequently, we have found an extraction of $(x_n)_{n\geq 0}$ that converges. This proves that the whole Cauchy sequence (x_n) converges in E. □

It is now possible to associate some Banach spaces with linear maps. In particular, let be given E and F two normed vector spaces such that F is a Banach space, we have the following result.

Theorem 2.1.3 *$\mathcal{L}(E,F)$ is a Banach space as soon as F is a Banach space.*

Proof : The main idea is to use the result of Proposition 2.1.7. We then consider an abolutely convergent sequence $(u_k)_{k\geq 1}$ of $\mathcal{L}(E,F)$, meaning that

$$\sum_{k=1}^{\infty} |||u_k||| < \infty.$$

For any vector x in E, we then have

$$\|u_k(x)\|_F \leq |||u_k||| \|x\|_E,$$

so that

$$\sum_k \|u_k(x)\|_F \leq \left(\sum_{k=1}^{\infty} |||u_k|||\right) \|x\|.$$

Hence, for any x, $(u_k(x))$ is an absolutely convergent series, and since F is a Banach space we know that $u_k(x)$ is a convergent series in F. Let us denote

$$S(x) = \sum_{k=1}^{\infty} u_k(x).$$

We know that S is linear since each u_k are linear, and

$$\|S(x)\| \leq \left(\sum_{k=1}^{\infty} |||u_k|||\right) \|x\| \Longrightarrow |||S||| \leq \sum_{k=1}^{\infty} |||u_k|||.$$

Thus S is a linear continuous map between E and F. It remains to show that the partial sums $S_n = \sum_{k=1}^{n} u_k$ converges to S. It simply relies on the triangle inequality since

$$(S - S_n)(x) = \sum_{k=n+1}^{\infty} u_k(x),$$

and

$$|||S - S_n||| \leq \sum_{k=n+1}^{\infty} |||u_k||| \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow +\infty.$$

□

2.2 Dual space

Definition 2.2.1 (Dual) *Let E be a normed vector space, we define E^* the dual space of E as*

$$E^* = \mathcal{L}(E, \mathbb{R}).$$

In the definition above, we may define the dual space with $\mathbb{C}$ instead of $\mathbb{R}$. As pointed by Theorem 2.1.3, the dual E^* is always a Banach space since $\mathbb{R}$ (or $\mathbb{C}$) is complete.

We will see along this lecture several examples, and we start with the following one.

Remark 2.2.1 *If $E = c_0$ is the set of real sequences (y_n) such that*

$$\lim_{n \longrightarrow +\infty} y_n = 0.$$

E is normed with the $\|.\|_\infty$ norm. We consider a sequence $u \in \ell_1$ (such that $\|u\|_1 < \infty$), then we define f_u as follows :

$$\forall y \in E \qquad f_u(y) := \sum_{k=1}^{\infty} y_k x_k.$$

We can see that f_u is linear, mapping E to $\mathbb{R}$. Moreover, we have

$$f_u(y) \leq \|y\|_\infty \|u\|_{\ell^1}.$$

Hence, f_u is also continuous such that $|||f_u||| \leq \|u\|_{\ell^1}$ and $f_u \in E^*$. By considering the sequence y^N equals to ± 1 from the integer 1 to N and null after N, we can see that

$$f_u(y^N) = \sum_{j=1}^{N} |u_j| \longrightarrow \|u\|_{\ell_1} \qquad \text{as} \qquad N \longrightarrow +\infty.$$

An easy consequence is that $|||f_u||| = \|u\|_{\ell^1}$ so that $u \longmapsto f_u$ is an isometry from ℓ^1 to E^*. We will see later that in fact, the dual space E^* can be identified by ℓ^1.

The dual space E^* can be thought itself as a normed vector space : for this purpose, we can define the dual norm

$$\forall X \in E^* \qquad \|X\|_* := \sup_{u \in E : \|u\|=1} |X(u)|.$$

Hence, the norm of X is simply the operatorial norm defined for linear continuous map above.

A very nice space for the dual computation is the finite dimensional $\mathbb{R}$ vector space $\mathbb{R}^n$ as pointed by the next result.

Theorem 2.2.1 *Assume $E = \mathbb{R}^n$ with the euclidean norm, then E^* is a n dimensional real vector space. As a consequence, the dual of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ (embedded with the euclidean norm).*

Proof : Let us consider $\varphi \in \mathcal{L}(E, \mathbb{R})$ and denote $(e_1, \ldots, e_n)$ the canonical base of E. Since a linear map is determined through the image of the canonical base, we can check that the dual canonical family $(e_j^*)_{1 \leq j \leq n}$ is a base of E^*, where

$$e_j^*(e_i) = \delta_{i,j} \qquad \text{the so called Kronecker symbol, whose value is 1 if j=i, 0 otherwise.}$$

First, each e_j^*, for any j between 1 and n is a linear map in E^*. Moreover, it is a continuous map and an easy computation yields $\|e_j^*\|_* = 1$.

Now, we can show that it is a generative family of E^* since

$$\forall x \in E \qquad \phi(x) = \phi(x_1 e_1 + \ldots + x_n e_n) = x_1 \phi(e_1) + \ldots + x_n \phi(e_n) = e_1^*(x)\phi(e_1) + \ldots + e_n^*(x)\phi(e_n).$$

As a consequence, we obtain :

$$\phi = \phi(e_1) e_1^* + \ldots + \phi(e_n) e_n^*.$$

It is also easy to check that (e_j^*) is an independent family of vectors : assume that a n uple $(a_1, \ldots, a_n)$ exists such that the linear combination of elements of E^* satisfies

$$a_1 e_1^* + \ldots + a_n e_n^* = 0.$$

Then, we get

$$0 = \left(\sum_{i=1}^{n} a_i e_i^*\right)(e_j) = a_j.$$

This ends the proof of the independency of the family, and shows that $\{\mathbb{R}^n\}^* \simeq \mathbb{R}^n$ $\square$

2.3 Ascoli's Theorem

2.3.1 Theoretical result

We introduce a supplementary definition on metric spaces.

Definition 2.3.1 (Equicontinuity) . *Let be given a family of applications $\mathcal{F}$ from (E,d) to (E',d'). The family $\mathcal{F}$ is equicontinuous if*

$$\forall \epsilon > 0 \quad \forall x \in E \quad \exists \delta > 0 \quad \forall f \in \mathcal{F} \qquad d(x,y) \leq \delta \Longrightarrow d'(f(x), f(y)) \leq \epsilon, \forall f \in \mathcal{F}.$$

Sometimes, the equicontinuity assumption is replaced by the uniform equicontinuity given as follows.

Definition 2.3.2 (Uniform equicontinuity) *Let be given a family of applications $\mathcal{F}$ from (E,d) to (E',d'). The family $\mathcal{F}$ is uniformly equicontinuous if*

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall f \in \mathcal{F} \quad \forall (x,y) \in E^2 \qquad d(x,y) \leq \delta \Longrightarrow d'(f(x), f(y)) \leq \epsilon, \forall f \in \mathcal{F}.$$

We can enumerate several examples :
— A finite family of continuous functions on compact spaces is always equicontinuous.
— A family of L-Lipschitz functions is equicontinuous.

Indeed, on compact spaces, the definition of equicontinuity and uniform equicontinuity are equivalent (as it is the case with the definition of continuity and uniform continuity on compact spaces). The proof is rather simple.

Proposition 2.3.1 *If X is a compact space of a metric space (E,d) and if $\mathcal{F}$ is equicontinuous from X to (E',d'), the $\mathcal{F}$ is uniformly equicontinuous.*

Proof : Consider $\epsilon > 0$ and for any x in X, we consider $\delta_x > 0$ introduced in Definition 2.3.1. We can build a simple covering of X :

$$X = \cup_{x \in X}(B(x, \delta_x) \cup X).$$

Now, extract a finite covering :

$$X = \cup_{1 \leq i \leq n_\epsilon}(B(x_i, \delta_{x_i}) \cup X)$$

and define

$$\delta^* := \min_{1 \leq i \leq n_\epsilon} \delta_{x_i}.$$

We then check that $\forall (x,y) \in E^2$, if $d(x,y) \leq \delta^*$, then $d'(f(x), f(y)) \leq \epsilon$, whatever f is. $\square$

We can now state the most important result of the paragraph.

Theorem 2.3.1 (Arzela-Ascoli's Theorem) *Consider a compact metric space X and $\mathcal{F} \subset \mathcal{C}(K, \mathbb{R})$. The set $(\mathcal{F}, \|.\|_\infty)$ is compact if and only if $\mathcal{F}$ is closed, bounded and equicontinuous.*

Proof :

$\Leftarrow$ Let us first assume that $\mathcal{F}$ is equicontinuous, closed and bounded and consider a sequence of $\mathcal{F}$ denoted $(f_n)_{n\in\mathbb{N}}$. We want to build an extraction of $(f_n)_{n\in\mathbb{N}}$ that converges in $\mathcal{F}$ with respect to $\|.\|_\infty$. In this view, since X is a compact set, we can build a countable dense sequence $(x_k)_{k\geq 0}$:

$$\overline{(x_k)_{k\geq 0}} = X.$$

We will then extract a sequence of $(f_n)_{n\in\mathbb{N}}$ suitably (through the Cantor diagonal process).

— $((f_n)(x_1))_{n\geq 0}$ is a bounded sequence of $\mathbb{R}$, meaning that we can extract a subsequence through φ_1 such that

$$f_{\varphi_1(n)}(x_1) \longrightarrow l_1 \qquad \text{as} \qquad n \longrightarrow +\infty.$$

— Consider now $((f_{\varphi_1}(n))(x_2))_{n\geq 0}$, which is a bounded set. Hence, there exists φ_2 such that

$$f_{\varphi_1\circ\varphi_2(n)}(x_2) \longrightarrow l_2 \qquad \text{as} \qquad n \longrightarrow +\infty.$$

— Sequentially, we build φ_p such that

$$f_{\varphi_1\circ\varphi_2\circ\varphi_p(n)}(x_p) \longrightarrow l_p \qquad \text{as} \qquad n \longrightarrow +\infty.$$

Now, we define

$$\psi(p) = \varphi_1 \circ \ldots \circ \varphi_p(p),$$

and we can check that every sequence $(f_{\psi(p)}(x_k))_{p\geq 0}$ converges to l_k, whatever k is.

We then establish that $(f_{\psi(p)})_{p\geq 1}$ is a Cauchy sequence of the Banach space $\mathcal{C}(X, E')$. Fix $\epsilon > 0$: a small $\delta > 0$ exists such that

$$\forall (x,y) \in X^2 \quad d(x,y) \leq \delta \Longrightarrow d'(f_{\phi(p)}(x), f_{\phi(p)}(y)) \leq \epsilon.$$

Moreover, since the sequence of points $(x_k)_{k\geq 0}$ is dense and X is compact, we can cover with a finite number of balls with a radius δ the set X :

$$X \subset \cup_{i=1}^{I} B(x_{k_i}, \delta).$$

As a convergent sequence, $(f_{\psi(p)}(x_{k_i}))_{p\geq 0}$ is a Cauchy sequence for every i. We conclude since

$$\forall x \in X \qquad \exists x_i \quad d(x, x_i) \leq \delta.$$

Then, forall x in X, we have

$$d'(f_{\psi(p)}(x), f_{\psi(q)}(x)) \leq d'(f_{\psi(p)}(x), f_{\psi(p)}(x_i)) + d'(f_{\psi(p)}(x_i), f_{\psi(q)}(x_i)) + d'(f_{\psi(q)}(x_i), f_{\psi(q)}(x)) \leq 3\epsilon$$

if p and q are chosen large enough, whatever x is. We have shown that $(f_{\psi(p)})_{p\geq 0}$ is a Cauchy sequence of $(\mathcal{C}(X, E'), \|.\|_\infty)$, which is a Banach space. Consequently, $(\mathcal{F}, \|.\|_\infty)$ is compact.

$\Rightarrow$ We assume that $\mathcal{F}$ is compact and consider $\epsilon > 0$. We can find $(f_1, \ldots, f_p)$ in $\mathcal{F}$ such that $\mathcal{F} \subset \cup_{i=1}^{p} B_{\|.\|_\infty}(f_i, \epsilon)$. As a finite set of functions on a compact set, it is easy to see that the family $(f_1, \ldots, f_p)$ is equicontinuous :

$$\forall x \in X \quad \exists \alpha > 0 \quad \forall y \in X \qquad d(x,y) \leq \alpha \Longrightarrow \forall i \in \{1 \ldots p\} \quad d(f_i(x), f_i(y)) \leq \epsilon.$$

For such a α, we then immediately obtain

$$\forall f \in \mathcal{F} \qquad d(f(x), f(y)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(y)) + d(f_i(y), f(y)) \leq 3\epsilon.$$

This ends the proof. $\square$

2.3.2 Applications to ODE

We briefly sketch the standard result of Peano, which is an almost direct consequence of the Ascoli Theorem.

Theorem 2.3.2 (Peano's Theorem) *Consider a continuous and bounded real valued function H on $[a,b]\times\mathbb{R}$. For any $(x_0,y_0)\in[a,b]\times\mathbb{R}$, a $\delta>0$ exists such that we can find a differentiable map f such that*

$$\forall x\in[x_0-\delta,x_0+\delta] \qquad f'(x)=H(x,f(x)) \qquad \textit{with} \qquad y_0=f(x_0).$$

Proof : Since we will need the Schauder result joint with the Ascoli Theorem, the proof is postponed to the last section.

The usage of Ascoli's result should be though about as follows : it permits to exhibit some functional compact spaces. This result in turn can be exploited with some fixed point theorem (see Section 4) to solve some implicit equations, in the sense of exhibit the existence of a fixed point of an operator.

For example, in the Peano Theorem above, we are looking for

$$f=\Phi(f) \qquad \text{with} \qquad \Phi(f)(x)=y_0+\int_{x_0}^{x}H(u,f(u))du,$$

which drives us naturally to the introduction of the operator Φ.

Chapitre 3

Hilbert spaces

3.1 Inner product

3.1.1 Sesquilinear forms

In what follows, E, F will denote two (complex) $\mathbb{C}$-vector spaces.

Definition 3.1.1 (Conjugate linear map) *$f : E \to F$ is conjugate (antilinear/semilinear) iff*

$$\forall \lambda \in \mathbb{C} \quad \forall (x, y) \in E^2 \qquad f(x + y) = f(x) + f(y) \qquad \textit{and} \qquad f(\lambda x) = \bar{\lambda} f(x).$$

We provide next a rather trivial example : consider $E = F = \mathbb{C}^n$ with a canonical base $(e_1, \ldots, e_n)$,

$$f\left(\sum_{i=1}^{n} x_i e_i\right) := \sum_{i=1}^{n} \bar{x}_i e_i$$

is a semilinear transformation of E.

Note that the composition of two semilinear transformations is a linear transformation.

Definition 3.1.2 (Bilinear form) *A map $B : E \times E \longrightarrow \mathbb{C}$ is a sesquilinear formr iff :*
— *for any y, $B(., y)$ is linear*
— *for any x, $B(x, .)$ is semilinear*

If E is a *real* linear space, B is symmetric iff

$$\forall (x, y) \in E^2 \qquad B(x, y) = B(y, x).$$

Proposition 3.1.1 (Polarization) *We can state important equalities related to linear forms*
— *If E is a complex vector space and B is sesquilinear, then*

$$\forall (x, y) \in E^2 \qquad 4B(x, y) = B(x+y, x+y) - B(x-y, x-y) + i\left[B(x + iy, x + iy) - B(x - iy, x - iy)\right]$$

— *If E is a real vector space and B is a symmetric bilinear form, then*

$$\forall (x, y) \in E^2 \qquad 4B(x, y) = B(x + y, x + y) - B(x - y, x - y)$$

These identities are important since they prove that it is enough to know the value of the sesquilinear form (or of the bilinear symmetric form) on the *diagonal*, *i.e.* to know the value of $B(t, t)$ for any $t \in E$.

Corollary 3.1.1 *If E is a complex linear space and B a sesquilinear form, then the following assertions are equivalent*

i) For all $(x, y) \in E^2 \qquad B(x, y) = \overline{B(y, x)}$

ii) For all $x \in E \qquad B(x, x) \in \mathbb{R}$.

Definition 3.1.3 (Hermitian form) *For any complex vector space E, a form B is Hermitian if conditions i) or ii) of the corollary above are fulfilled. In particular, we will note that B is an Hermitian form iff*

— *For all $y \in E$ the map $B(., y)$ is linear*

— *For all $(x, y) \in E^2 \qquad B(x, y) = \overline{B(y, x)}$.*

3.1.2 Scalar (inner) product

We will say that a Hermitian form B is *positive* if

$$\forall x \in E^2 \qquad B(x, x) \geq 0.$$

We then get the fundamental definition.

Definition 3.1.4 (Inner product) *$\langle , \rangle$ is an Inner product iff it is a Hermitian positive form (on a complex space) or a symmetric positive bilinear form (on a real space).*

We can state now the famous Cauchy-Schwarz inequality.

Theorem 3.1.1 (Cauchy Schwarz inequality) *An inner product on E satisfies*

$$\forall (x, y) \in E^2 \qquad |\langle x, y\rangle| \leq |\langle x, x\rangle| \times |\langle y, y\rangle|$$

<u>*Proof :*</u> Consider a pair $(x, y) \in E^2$ and a complex number u (of modulus 1) such that $u\langle x, y\rangle = |\langle x, y\rangle|$. Since the inner product is a positive form, we have

$$\forall t \in \mathbb{R} \qquad \langle ux + ty, ux + ty\rangle \geq 0.$$

Now, remark that

$$P(t) = \langle ux + ty, ux + ty\rangle = \langle ux, ux\rangle + 2tRe(\langle ux, y\rangle)t^2\langle y, y\rangle = \langle x, x\rangle + 2t|\langle x, y\rangle| + t^2\langle y, y\rangle.$$

We can see that P is a second order real polynomial, which is always nonnegative. A direct consequence is that

$$\Delta = 4|\langle x, y\rangle|^2 - 4\langle x, x\rangle\langle y, y\rangle \leq 0$$

We then obtain the desired inequality. □

Proposition 3.1.2 (Minkowski's inequality) *If E is vector space with an inner produc $\langle , \rangle$, then*

$$x \longmapsto \sqrt{\langle x, x\rangle}$$

is a semi-norm on E (satisfies the triangle inequality).

Proof : The demonstration is rather simple : consider a pair $(x, y) \in E^2$ and write

$$\begin{aligned}
\langle x+y, x+y\rangle &= \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \overline{\langle x,y\rangle} \\
&\leq \langle x,x\rangle + \langle y,y\rangle + 2|\langle x,y\rangle| \\
&\leq \left(\sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle}\right)^2
\end{aligned}$$

We deduce the triangle inequality. □

This last property is sometimes referred to as the parallelogram inequality. It is important since it permits to build a large number of norms from a scalar product.

Proposition 3.1.3 *If $\langle , \rangle$ is a scalar product on E and if $\langle x, x\rangle \neq 0$ for $x \neq 0_E$, then $\sqrt{\langle , \rangle}$ is a norm on E denoted $\|.\|$. We can recover the scalar product from this norm with*

$$2\langle x, y\rangle = \|x+y\|^2 - \|x\|^2 - \|y\|^2$$

3.2 Basic properties of Hilbert spaces

3.2.1 Definition

In what follows, we will consider the vector space E with its inner product $\langle , \rangle$ and the corresponding norm is denoted $\|.\|$. We first state some basic properties.

Proposition 3.2.1 *For all $y \in E$, the linear form $\ell_y : x \longmapsto \langle x, y\rangle$ is continuous from $(E, \|.\|)$ to $\mathbb{C}$. Moreover, the map $y \longmapsto \ell_y$ is semilinear and isometric from E to E^*.*

Proof : The first point comes from the Cauchy-Schwarz inequality :

$$|\ell_y(x)| \leq \|y\|\|x\|.$$

Since ℓ_y is linear and satisfies the inequality above, ℓ_y continuous and we have

$$|||\ell_y||| \leq \|y\|.$$

Regarding now the second point, we have to understand that it is a result on the dual space E^* of E. We denote $\ell : y \longmapsto \ell_y$ and since ℓ_y is continuous, we see that $\ell : E \longrightarrow E^*$. Moreover, since $|||\ell_y||| \leq \|y\|$, it is immediate to see that

$$|||\ell||| := \sup_{y\in E} \frac{|||\ell_y|||}{\|y\|} \leq 1.$$

But we also check that for all $y \in E$:

$$\|y\|^2 = \langle y, y\rangle = \ell_y(y),$$

leading to the simple observation that

$$|||\ell_y||| = \|y\|.$$

Hence, ℓ is an isometry from E to E^*.

Finally, it is easy to check that ℓ is semilinear :

$$\ell_{y_1+y_2}(\xi) = \langle \xi, y_1+y_2\rangle = \langle \xi, y_1\rangle + \langle \xi, y_2\rangle = \ell_{y_1}(\xi) + \ell_{y_2}(\xi),$$

and in a same way $\ell_{\lambda y}(\xi) = \bar{\lambda}\ell_y(\xi)$. □

Definition 3.2.1 (Hilbert space) *Let be given a real (or complex) vector space H with an inner product $\langle , \rangle$ that produces a norm $\|x\| := \sqrt{\langle x, x\rangle}$. If H is **complete** for $\|.\|$, then H is an Hilbert space.*

In particular, the Cauchy Schwarz inequality holds in H : $|\langle x, y\rangle| \leq \|x\|.\|y\|$.

Remark 3.2.1 (The example $\mathbb{L}^2(\Omega, \mu)$) *A famous example is the Hilbert space of squared integrable functions with respect to a reference measure μ. In particular, the inner product is given by*

$$\forall (f, g) \in \mathbb{L}^2(\Omega, \mu) \times \mathbb{L}^2(\Omega, \mu) \qquad \langle f, g\rangle := \int_\Omega f(x)g(x)d\mu(x)$$

When $\Omega = \mathbb{R}$ and μ is the Lebesgue measure, we recover the standard $L^2(\mathbb{R})$ space. If now $\Omega = \mathbb{N}$ and μ is the couting measure on $\mathbb{N}$, we recover the set ℓ^2 of sequences such that

$$\sum u_n^2 < \infty.$$

In the example above, we have used the fact that $\mathbb{L}^2$ is a complete space. Note that it is not so obvious and deserve a short proof (even if it is a classical exercise).

Theorem 3.2.1 (Fisher-Riesz Theorem) *Let be given a measured space (Ω, μ), for any $p \in [1; +\infty]$, $L^p(\Omega, \mu)$ is a Banach space.*

Proof : We only give a proof for $1 \leq p < +\infty$. The case $p = \infty$ deserves a special attention and can be found in [Brezis,chapter IV].

Let be given a Cauchy sequence $(f_n)_{n\geq 1}$ of L^p and we aim to show that (f_n) converges in L^p and as it is commonly used, it is enough to show that a subsequence of (f_n) converges. From the Cauchy criterion, we can find (n_1, n_2) such that

$$\|f_{n_2} - f_{n_1}\|_p \leq 1/2.$$

A recursive construction permits to find an increasing sequence of integer $n_{k+1} > n_k$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}.$$

Let us denote

$$g_k(x) = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|.$$

It is an easy exercice to check that $\|g_k\|_p \leq 1$, for all integer $k \in \mathbb{N}$. Moreover, by construction, we can see that $(g_k)_{k\in\mathbb{N}}$ is an increasing sequence of functions. The monotone convergence theorem yields the convergence of g_k towards g in L^p and we thus define

$$g(x) = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|(x).$$

We have $\lim_{k\longrightarrow+\infty} g_k = g$ almost surely and the triangle inequality leads to

$$|f_{n_{K+1}}(x) - f_{n_k}(x)| \leq |f_{n_{K+1}}(x) - f_{n_K}(x)| + \ldots + |f_{n_{k+1}}(x) - f_{n_k}(x)| \leq g(x) - g_{n_k}(x). \quad (3.1)$$

We thus have that $f_{n_k}(x)$ is almost surely a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a complete space, we then deduce that $f_{n_k}(x)$ almost surely converges towards a limiting value denoted $f(x)$.

It remains to show that $f \in L^p$ and $\|f - f_{n_k}\|_p \longrightarrow 0$ as $k \longrightarrow +\infty$.

— The first point is immediate by taking the limit $K \longrightarrow +\infty$ in (3.1), which yields

$$|f - f_{n_k}|(x) \leq g(x) \quad \forall x \in \Omega a.s.$$

Thus, $\|f - f_{n_k}\|_p \leq \|g\|_p$ and $\|f\|_p \leq \|f - f_{n_k}\|_p + \|f_{n_k}\|_p < \infty$.

— The second point comes from the fact that $|f(x) - f_{n_k}|^p \longrightarrow 0$ almost surely and

$$|f - f_{n_k}|^p \leq g^p,$$

that is an integrable upper bound. The Lebesgue dominated convergence permits to conclude.

□

Proposition 3.2.2 *If E is a closed set in an Hilbert space H, then E is an Hilbert space.*

3.2.2 Orthogonality and Projection in Hilbert spaces

Definition 3.2.2 (Orthogonality) *Let H an Hilbert space, we say that $x, y \in H$ are orthogonal iff $\langle x, y\rangle = 0$ and following the geometric standards, we will also use the notation $x \perp y$.*
For any family of vectors $(x_n)_{n\geq 1}$, we say the family to be orthogonal iff

$$\forall m \neq n \qquad \langle x_m, x_n\rangle = 0.$$

Furthermore, it is an orthonormal family if one has $\|x_m\| = 1$ for all m.

It is easy to establish in this context the Pythagorean relation

Proposition 3.2.3 (Pythagore)

$$x \perp y \Longleftrightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

We can moreover extend this relation to a family of orthogonal vectors $(x_1, \ldots, x_n)$:

$$\|x_1 + \ldots + x_n\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Proof : Easy with the remark that $\|x_1 + \ldots + x_n\|^2 = \langle x_1 + ldots + x_n, x_1 + ldots + x_n\rangle$. □

It is easy to check that if $x \perp y_j$ for j varying between 1 and n, then x is orthogonal to any vector of the linear span of $(y_j)_{1\leq j\leq n}$, which will be written as

$$x \perp Vect(y_1, \ldots, y_n).$$

The next proposition is a fundamental tool of Hilbert spaces.

Proposition 3.2.4 (Orthogonal projection) *Let be given a finite orthonormal family of vectors $(e_1, \ldots, e_n)$ of an Hilbert space H and define $F = \mathrm{Span}(e_1, \ldots, e_n)$. For any vector $x \in H$, the vector*

$$y = \sum_{i=1}^{n} \langle x, e_i\rangle e_i$$

is the orthogonal projection of x on F :

$$\forall z \in F \qquad (x - y) \perp z.$$

Proof : It is easy to check that $y \in \text{Span}(e_1, \ldots, e_n) = F$. Moreover, $x - y$ is orthogonal to each vector (e_i) and thus orthogonal since

$$\langle x - y, e_i \rangle = \langle x, e_i \rangle - \sum_{j=1}^{n} \langle x, e_j \rangle \langle e_i, e_j \rangle = \langle x, e_i \rangle - \langle x, e_i \rangle = 0.$$

As a consequence, $x - y$ is orthogonal to a generative family and thus to the space F. □

We can also state a famous result, useful in the framework of Fourier analysis, the so-called Bessel inequality.

Proposition 3.2.5 (Bessel's inequality) *Let be given an Hilbert space H and an orthonormal family $(e_n)_{n\geq 1}$ in H. For all $x \in H$, one has*

$$\sum_{k\geq 1} \langle x, e_k \rangle^2 \leq \|x\|^2$$

Proof : It is enough to show this inequality for a finite family since the conclusion can be extended to the infinite case by passing through the limit. We then consider $(e_1, \ldots, e_n)$ an orthonormal family and $y = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$. We define $F = \text{Span}(e_1, \ldots, e_n)$ and from the proposition above, we have

$$x - y \perp F.$$

Hence, the Pythagorean theorem implies

$$\|x\|^2 = \|(x - y) + y\|^2 = \|x - y\|^2 + \|y\|^2 \geq \|y\|^2 \geq \sum_{i=1}^{n} \langle x, e_i \rangle^2$$

We then obtain the result. □

We will see later on that this kind of inequality can be expressed in terms of Hilbert basis.

3.3 Projection on a closed convex space

3.3.1 Main result

The next result is at the cornerstone of the Hilbert analysis, and has various applications, in analysis (Fourier Analysis), PDE (Sobolev spaces), in optimisation (linear minimization problem), probability and statistics (conditional expectation), economics (existence of Pareto optima), and game theory (von Neuman equilibrium). We will provide some various applications at the end of the chapter. You will find in Figure 3.1 a very simple geometric illustration of the result. In particular, the result of *ii*) becomes clearer when looking at Figure 3.1.

Let us begin by the statement and the proof of the main result.

Theorem 3.3.1 *Let H be an Hilbert space and F a nonempty closed convex subset of H.*

*i) Then, for all $x \in H$, there exists a **unique** point in F, called the orthogonal projection of x onto F, such that*

$$d(x, F) = \inf_{z\in F} \|x - z\| = \|x - y\|.$$

ii) This point is denoted in what follows $P_F(x)$ and is fully characterized by the property :

$$\forall z \in F \qquad \Re\left(\langle x - y, z - y \rangle\right) \leq 0,$$

where $\Re(\xi)$ denotes the real part of ξ.

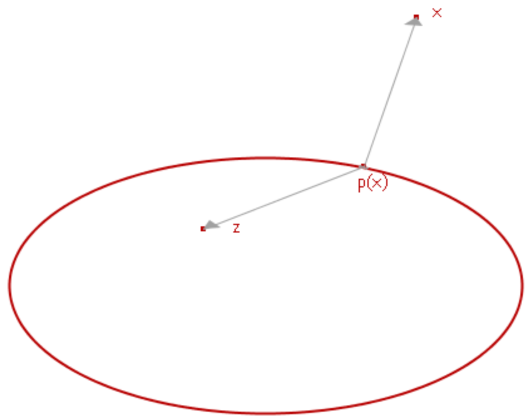


FIGURE 3.1: Projection of x on the convex closed set C.

Remark 3.3.1 *A brief reminder : a set C is convex iff*

$$\forall (x, y) \in C^2 \quad \forall t \in [0, 1] \qquad tx + (1-t)y \in C.$$

Proof : If $x \in F$, there is nothing to prove since $y = x$ is the unique point of F that minimises the distance to x (the minimal value is 0). Moreover, assume that y exists such that

$$\forall z \in F \qquad \Re\left(\langle x - y, z - y\rangle\right) \leq 0,$$

and apply this property to $z = x \in F$, we get

$$\|x - y\| \leq 0 \Longrightarrow x = y.$$

Proof of $i)$: We now study the generic case where $x \notin F$ and we define δ as the minimal distance between x and F :

$$\delta := \inf\{\|x - y\|, y \in F\} > 0.$$

In the same time, we also consider for all integer $n \in \mathbb{N}$ the following set :

$$C_n := \left\{y \in F : \|x - y\|^2 \leq \delta^2 + 2^{-2n}\right\}$$

It is an easy exercice to check that C_n is a nonempty closed set included in F, which is decreasing :

$$\forall n \in \mathbb{N} \qquad C_{n+1} \subset C_n.$$

Consider $(y_1, y_2) \in C_n^2$, we know that $u = (y_1 + y_2)/2$ is in C_n since F is convex. Thus, we have

$$\|x - u\| \geq \delta.$$

Moreover, the parallelogram formula yields

$$\|x - y_1\|^2 + \|x - y_2\|^2 = 2\|x - u\|^2 + \frac{1}{2}\|y_1 - y_2\|^2 \tag{3.2}$$

First, note that the right hand side of (3.2) can be lower bounded as follows :

$$2\|x - u\|^2 + \frac{1}{2}\|y_1 - y_2\|^2 \geq 2\delta^2 + \frac{1}{2}\|y_1 - y_2\|^2.$$

Second, the left hand side of (3.2) is upper bounded easily through the definition of C_n :

$$\|x-y_1\|^2+\|x-y_2\|^2 \leq 2\delta^2+22^{-n}$$

Taking together the last equalities, we obtain

$$\|y_1-y_2\|^2 \leq 42^{-n} \Longrightarrow \|y_1-y_2\| \leq 2^{1-n}.$$

To sum up, (C_n) is a sequence of decreasing set whose diameter is going to 0, we can apply Lemma 3.3.1 to conclude that a unique point y exists such that

$$\cap_n C_n = \{y\} \in F.$$

Hence, y is the unique point of F such that $\|x-y\| = d(x,F)$.

Proof of $ii)$: Let consider $z \in F$ and a real value $t \in [0,1]$. F is convex so that $z_t := tz+(1-t)y \in F$. Hence we deduce that

$$\begin{array}{llll} \forall t \in [0,1] & \delta^2 \leq \|x-z_t\|^2 & = & \langle x-z_t, x-z_t\rangle \\ & & = & \|x-y\|^2+t^2\|z-y\|^2-2t\Re\left(\langle x-y, z-y\rangle\right) \\ & & = & \delta^2+t^2\|z-y\|^2-2t\Re\left(\langle x-y, z-y\rangle\right) \end{array}$$

Substracting δ^2 and simplifying by t, we obtain that

$$\forall t \in [0,1] \qquad 2\Re\left(\langle x-y, x-y\rangle\right) \leq t\|z-y\|^2.$$

Since this inequality is true for any t, we can take t arbitrarily small and deduce that

$$\Re\left(\langle x-y, z-y\rangle\right) \leq 0.$$

Conversely, consider $y' \in F$ such that

$$\forall z \in F \qquad \Re\left(\langle x-y', z-y'\rangle\right) \leq 0,$$

and apply this relation for $y=z$, to get

$$\begin{array}{lll} 0 & \geq & \Re\left(\langle x-y', y-y'\rangle\right) \\ & \geq & \Re\left(\langle x-y, y-y'\rangle\right)+\Re\left(\langle y-y', y-y'\rangle\right) \\ & = & \|y-y'\|^2-\underbrace{\Re\left(\langle x-y, y'-y\rangle\right)}_{\textbf{negative}} \geq \|y-y'\|^2. \end{array}$$

Thus, we obtain $y=y'$. □

It is necessary to establish the following technical lemma to complete the proof of the projection Theorem.

Lemma 3.3.1 *Let E be a complete metric space and C_n a sequence of nonempty closed subspace, decreasing (with respect to the inclusion) with an asymptotic vanishing diameter, then*

$$\exists\,!x \in E \qquad \{x\} = \cap_n C_n$$

Proof : This result follows a typical argument of complete metric spaces. Remark that each C_n is non-empty. Hence, we can consider a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} \qquad x_n \in C_n.$$

Since $C_{n+2} \subset C_{n+1} \subset C_n \ldots \subset C_1$, it is easy to see that

$$\forall m \geq n \qquad d(x_m, x_n) \leq d_n,$$

where d_n denotes the diameter of C_n. Now, our assumption states that $d_n \longmapsto 0$ as $n \longmapsto +\infty$ and we deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of E. It implies in turn that $(x_n)_{n\in\mathbb{N}}$ converges towards x^* and of course

$$\forall n \in \mathbb{N} \quad \forall m \geq n \qquad x_m \in C_n.$$

Passing to the limit on m, we have shown that

$$\forall n \in \mathbb{N} \qquad x^* \in C_n,$$

meaning that

$$x^* \in \cap_n C_n.$$

The uniqueness of x^* is obvious : consider two points x^* and y^* in the intersection. We deduce that

$$\forall n \in \mathbb{N} \qquad d_n \geq d(x^*, y^*).$$

Since d_n is going to 0, we can conclude that $d(x^*, y^*) = 0$ and $x^* = y^*$. $\square$

A useful application of Theorem 3.3.1 is commonly encountered when F is a linear subspace of an Hilbert space H. This is stated as a corollary of Theorem 3.3.1.

Corollary 3.3.1 *If F is a closed linear subspace of an Hilbert space H, there exists a unique projection $P_F : H \longmapsto F$ such that*

$$\|x - P_F(x)\| = d(x, F) := \inf_{z\in F} \|x - z\|.$$

Moreover, P_F is linear, $x - P_F(x) \perp F$ and P_F is 1-Lipschitz.

Proof : Let $x \in H$ and $z \in F$, we know that

$$\Re\left(x - P_F(x), z - P_F(x)\right) \leq 0.$$

F being a linear subspace, we deduce that

$$\forall w \in F \qquad \Re\left(x - P_F(x), w\right) \leq 0.$$

Now, replace w by $-w$ and deduce that $\langle x - P_F(x), w\rangle = 0$ for all w.

We now study the Lipschitz constant of P_F. For x_1, x_2 in H, we write :

$$\begin{aligned} \|x_1 - x_2\|^2 &= \|[x_1 - P_F(x_1)] - [x_2 - P_F(x_2)] + (P_F(x_1) - P_F(x_2))\|^2 \\ &= \|[x_1 - P_F(x_1)] - [x_2 - P_F(x_2)]\|^2 + \|P_F(x_1) - P_F(x_2)\|^2 \\ &\geq \|P_F(x_1) - P_F(x_2)\|^2. \end{aligned}$$

We can read the result with the extremity of the computation above. $\square$

When the linear subspace F is generated by a finite family of **orthonormal** vectors $(e_1, \dots, e_n)$, this projection is then explicit and we can check that

$$P_F(x) = \sum_{i=1}^{n} \langle x, e_i \rangle e_i.$$

This comes from the fact that $x - P_F(x) \perp F$ as pointed in a paragraph above.

Another algebraic and topological consequence of Theorem 3.3.1 is the decomposition of the Hilbert space as a direct sum. It comes from the decomposition $x = [x - P_F(x)] + P_F(x)$ and the trivial intersection $F \cap F^{\perp} = \{0\}$.

Theorem 3.3.2 *If F is a closed linear subspace of H, we then have*

$$H = F \oplus F^{\perp}$$

3.3.2 Some topological difficulties

We state now a proposition, which will be useful for deriving density results in Hilbert spaces.

Proposition 3.3.1 *If there exists $x \in H$ and a subset $A \subset H$ such that*

$$\forall a \in A \qquad \langle x, a \rangle = 0,$$

then $x \perp \bar{A}$ (where $\bar{A}$ denotes the adherence of the set A).

Proof : We state the proof, which is quite simple, to handle topological results and orthogonality. We begin by showing that for any $x \in H$, the map $\varphi_x : a \longmapsto \langle x, a \rangle$ is continuous. It is easy to check that φ_x is linear. Moreover, the Cauchy Schwarz inequality implies that

$$|\varphi_x(a)| \leq \|x\| \|a\|,$$

so that φ_x is linear and continuous, with an operatorial norm bounded by $\|x\|$.

Consider $a \in \bar{A}$, then one can find a sequence $(a_n)_{n \geq 1}$ of elements in A such that $\|a - a_n\| \longrightarrow 0$. Now, we know from our assumption that $x \perp a_n$, meaning that

$$\forall n \in \mathbb{N} \qquad \langle x, a_n \rangle = 0.$$

We then deduce that

$$\lim_{n \longmapsto +\infty} \langle x, a_n \rangle = 0.$$

But the left hand side of the equality above is $\langle x, a \rangle$ since $a \longmapsto \langle x, a \rangle$ is continuous. This ends the proof. $\square$

It is important to understand that even if in a finite dimensional settings, things are generally easy, it is no longer the case when dealing with the infinite dimensional case.

Remark 3.3.2 *As an example, remark that in a finite dimensional case, all subspace are necessarily closed, which is not the case if the dimension is infinite. For example, consider the set ℓ_c of sequences in ℓ^2 that vanishes from a certain integer. We can see that ℓ_c is a dense subset of ℓ^2 since for any $u \in \ell^2$ and for any $\epsilon > 0$, we can find an integer n_ϵ such that*

$$\sum_{k \geq n_\epsilon} u_k^2 \leq \epsilon^2.$$

Hence, the sequence $\bar{u}$ defined as

$$\bar{u}_k = u_k \mathbf{1}_{k \leq n_\epsilon}$$

is such that $\|\bar{u} - u\|_2 \leq \epsilon$ and $\bar{u} \in \ell_c$. Consequently, we see that ℓ_c is a vector space such that

$$\overline{\ell_c} = \ell^2,$$

and ℓ_c is not closed.

Remark 3.3.3 *A similar phenomenon occurs when dealing with $\mathcal{C}([0,1],\mathbb{R})$ since*

$$\overline{\mathcal{C}([0,1],\mathbb{R})} = \mathbb{L}^2([0,1]).$$

Definition 3.3.1 (Orthogonal of a set) *For any F a linear subspace of H, we define*

$$F^\perp := \{x \in H : \forall a \in F, \langle x, a \rangle = 0\}$$

We can derive the next result

Proposition 3.3.2 *Let F be a linear subspace of H, then*
- *$F^\perp$ is a closed set.*
- *If $G \subset F$, then $F^\perp \subset G^\perp$.*
- *The following equality is true :*
$$F^\perp = \overline{F}^\perp$$
- *If F is a closed vector space, then*
$$F^{\perp\perp} = F$$
- *If F is a closed vector space, then*
$$E = F \oplus F^\perp.$$

<u>*Proof :*</u> The first point comes from the Proposition 3.3.1.
The second point is an exercise.
To obtain the third point, we remark that $F \subset \bar{F}$ so that $\bar{F}^\perp \subset F^\perp$ and we have the first inclusion. Consider now an element of $y \in F^\perp$, we want to show that $y \in \bar{F}^\perp$. In this view, consider any vector $x \in \bar{F}$: there exists a sequence $(x_n)_{n\geq 1}$ of F such that $x_n \longrightarrow x$. Since $y \in F^\perp$, we obtain $\langle y, x_n \rangle = 0$ for all n, and taking the limit we deduce that $\langle y, x \rangle = 0$. We have shown that $y \in \bar{F}^\perp$, which is true for any y so that $F^\perp \subset \bar{F}^\perp$.

The third point comes from the projection theorem. First remark that $F \subset F^{\perp\perp}$ is always true since $x \in F \Longrightarrow \forall y \in F^\perp \langle x, y \rangle = 0 \Longrightarrow x \in \{F^\perp\}^\perp$. Conversely, assume that $x \in F^{\perp\perp} \notin F$. Since F is closed, we can find $p_F(x) \in F$ such that $x - p_F(x) \perp F$. It means that $x - p_F(x) \in F^\perp$. We know that $x \in F^{\perp\perp}$ so that $\forall y \in F^\perp \langle x, y \rangle = 0$. In the same time, we also have $\langle p_F(x), y \rangle = 0$. Hence, $x - p_F(x)$ is orthogonal to $F^\perp$ and belongs to $F^\perp$, thus $x - p_F(x) = 0$, which is a contradiction.

For the last point, we define the projection P_F on the convex set F and of course

$$x = P_F(x) + x - P_F(x),$$

showing that $E = F + F^\perp$. The sum is obviously a direct sum. □

We end this paragraph with this important criterion on density in Hilbert spaces.

Theorem 3.3.3 *A linear subspace A of H is dense iff $A^\perp = \{0\}$.*

Proof : We show the two implications separately. Assume that A is dense in H, then

$$\bar{A} = H.$$

As pointed in the third point of Proposition 3.3.2, we have

$$A^\perp = \bar{A}^\perp = H^\perp.$$

It is now easy to check that $H^\perp = \{0\}$: if $x \neq 0$, then $\|x\|^2 = \langle x, x\rangle \neq 0$ and $x \notin H^\perp$.

Conversely, we assume that $A^\perp = \{0\}$. Simple inclusions show that $A \subset \bar{A} \Longrightarrow \bar{A}^\perp \subset A^\perp \Longrightarrow A^{\perp\perp} \subset \bar{A}^{\perp\perp} = \bar{A}$ since $A^\perp$ is closed (see Proposition 3.3.2). We have obtained that $\{0\}^\perp \subset \bar{A}$, which in turn implies that

$$H = \{0\}^\perp = \bar{A}.$$

This ends the proof. $\square$

3.4 Hilbert basis

Definition 3.4.1 (Hilbert basis) *In an Hilbert space H, a system of vectors $(e_i)_{i\in I}$ is an Hilbert basis of H if $(e_i)_{i\in I}$ is an orthonormal family such that*

$$\overline{(e_i)_{i\in I}} = H.$$

Note that $\overline{(e_i)_{i\in I}}$ refers to any limit of elements

$$\sum_{i\in I} \alpha_i e_i.$$

If I is a finite set, it denotes a simple sum, but when I is infinite, it refers to a limit of a sequence of finite sums, for example

$$\lim_{N\longrightarrow+\infty} \sum_{i=1}^{N} \alpha_i e_i.$$

We then obtain the following theorem.

Theorem 3.4.1 (Structure of separable Hilbert spaces) *Let be given H an Hilbert separable space, then one has :*

1. *there exists a countable Hilbert basis*
2. *If $(e_i)_{i\in\mathbb{N}}$ denotes the Hilbert basis, then*

$$H = \left\{ \sum_{i=1}^{\infty} x_i e_i \quad | \quad \sum_{i=1}^{+\infty} |x_i|^2 < \infty \right\}.$$

Moreover, for any $x \in H$, one has

$$x = \sum_{i=1}^{\infty} \langle x, e_i\rangle e_i.$$

3. The Pythagore theorem holds :

$$\|x\|_2^2 = \sum_{i=1}^{\infty} |x_i|^2,$$

as well as the Parseval equality :

$$\langle x, y\rangle = \sum_{i=1}^{\infty} \bar{x}_i y_i.$$

Proof : We first establish the first point. We assume that H is separable and consider a dense sequence $(x_n)_{n\in\mathbb{N}}$. It is easy to build a second sequence $(e_n)_{n\in\mathbb{N}}$, which is orthonormal w.r.t. $\langle , \rangle$ through the Gram-Schmidt algorithm and such that the vector space spanned by $(x_n)_{n\leq N}$ and the vector space spanned by $ex_n)_{n\leq N}$ are the same. Since the sequence $(x_n)_{n\leq N}$ is dense in H, then we obtain that

$$\overline{(e_i)_{i\in\mathbb{N}}} = H.$$

We now study the second point. Define for any x in H the partial sum

$$S_N(x) = \sum_{i=1}^{N} \underbrace{\langle x, e_i\rangle}_{:=x_i} e_i.$$

We have

$$\|x - S_N(x)\|_2^2 = \|x\|_2^2 + \sum_{i=1}^{N} |x_i|^2 - 2\Re\langle x, \sum_{i=1}^{N} x_i e_i\rangle = \|x\|_2^2 - \sum_{i=1}^{n} |x_i|^2,$$

where the last equality is obtained by expanding the partial sum and using the orthonormality of the Hilbert basis. We then deduce that

$$\sum_{i=1}^{N} |x_i|^2 \leq \|x\|_2^2 < \infty$$

It means that the sequence of coordinate is in $\ell^2(\mathbb{N})$. Moreover, the partial sums form a Cauchy sequence (using the Pythagore equality in the finite case) and thus $(S_N)_{N\geq 0}$ is a Cauchy sequence in H. Hence, $S_N \longrightarrow w \in H$ as $N \longrightarrow +\infty$. It is then easy to check that $w - x$ is orthogonal to every vector e_i of the Hilbert basis, and then orthogonal to the whole space H. Hence, $w = x$. The last points are easy consequences of this formula. □

3.5 Applications of the Projection Theorem and Hilbert spaces

3.5.1 Conditional expectation

Consider a measured space $(\Omega, \mathcal{A}, \mu)$ and the set of measurable functions $\mathbb{L}^2(\mu)$. We consider two measurable random variables X and Y and the σ algebra generated by the events X-measurable is denoted $\mathcal{A}_X$. This σ-algebra permits to define $\mathbb{L}^2(X)$, which is the set of squared integrable function $\mathcal{A}_X$ measurable. As already pointed in the beginning of the chapter, both $\mathbb{L}^2(\mu)$ and $\mathbb{L}^2(X)$ are complete. Indeed, $\mathbb{L}^2(\mu)$ is an Hilbert space *via* the inner product

$$\langle f, g\rangle_\mu = \int_\Omega f(w)g(w)d\mu(\omega) = \mathbb{E}[fg]$$

Moreover, $\mathbb{L}^2(X)$ is closed in $\mathbb{L}^2(\mu)$ and we can then define the projection of $\mathbb{L}^2(\mu)$ onto $\mathbb{L}^2(X)$.

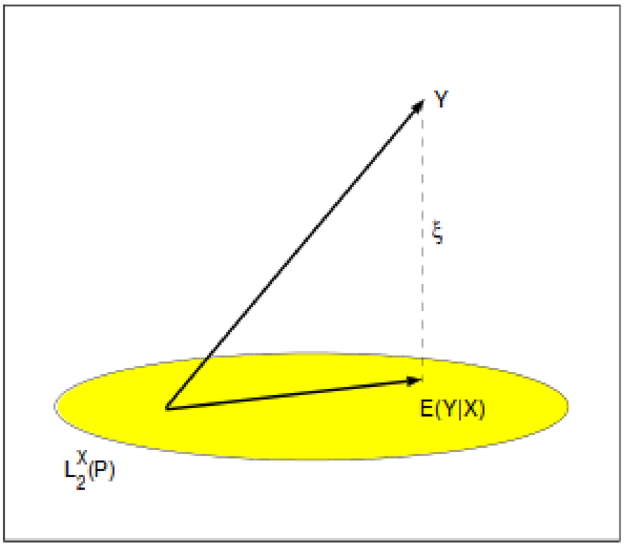


FIGURE 3.2: Projection of Y on $\mathbb{L}^2(X)$ and conditionnal expectation $\mathbb{E}[Y|X]$.

A simple consequence of this abstract construction is that for any $\mathcal{A}$-measurable random variable Y, there exists a projection of Y that belongs to $\mathbb{L}^2(X)$. This random variable is a function of X (as an element of $\mathbb{L}^2(X)$), which is denoted

$$\mathbb{E}[Y|X] := P_{\mathbb{L}^2(X)}(Y).$$

This definition is illustrated in Figure 3.2.

The random variable $\mathbb{E}[Y|X]$ satisfies all the properties of the projection written before. In particular, we have

$$Y - \mathbb{E}[Y|X] \perp \mathbb{L}^2(X),$$

meaning that

$$\forall f \in \mathbb{L}^2(X) \qquad \mathbb{E}\left[(Y - \mathbb{E}[Y|X])f(X)\right] = 0. \tag{3.3}$$

In particular, to compute the $\mathbb{E}[Y|X]$, if we can find g such that for any $f \in \mathbb{L}^2(X)$, we have

$$\mathbb{E}[Yf(X)] = \mathbb{E}[g(X)f(X)],$$

then Equation (3.3) shows that $\mathbb{E}[Y|X] = g(X)$.

Another important interpretation is that

$$\mathbb{E}\left[|Y - \mathbb{E}[Y|X]|^2\right] := \inf_{Z \in \mathbb{L}^2(X)} \mathbb{E}[|Z - X|^2],$$

which signifies that $\mathbb{E}[Y|X]$ is the best approximation of Y in $\mathbb{L}^2(X)$ and therefore $\mathbb{E}[Y|X]$ is the random variable that minimises the variance of prediction when X is known.

Note that the purpose of this paragraph is not to provide an exhaustive description of the difficult notion of conditionnal expectation. Just remark that the « projection »formulation permit to obtain simple properties such as

$$\mathbb{E}[\mathbb{E}[Y|X_1]|X_2] = \mathbb{E}[\mathbb{E}[Y|X_2]|X_1] = \mathbb{E}[\mathbb{E}[Y|(X_1, X_2)].$$

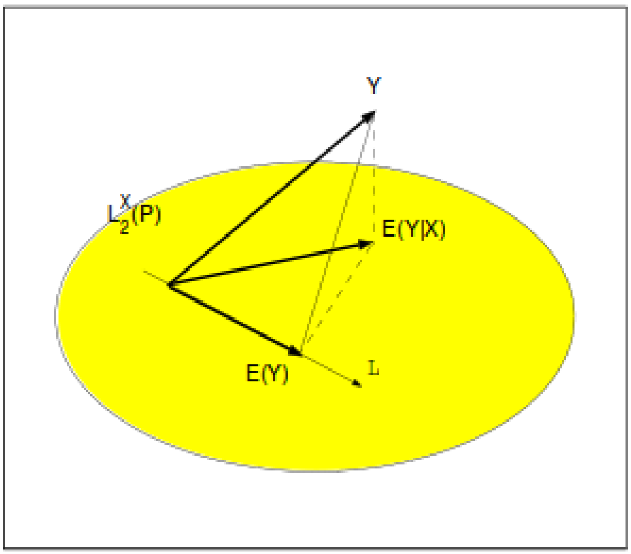


FIGURE 3.3: Decomposition of the variance of Y with $\mathbb{L}^2(X)$.

We can find at another example by considering the variance decomposition formula that relies on Figure 3.3 (and is just the Pythagorean relation in $\mathbb{L}^2(\mu)$).

$$\mathbb{E}\left(Y - E(Y)\right)^2 = \mathbb{E}\left(\mathbb{E}[Y|X] - E(Y)\right)^2 + \mathbb{E}\left([Y - \mathbb{E}[Y|X]\right)^2.$$

This relation can also be written as

$$Var(Y) = Var(\mathbb{E}[Y|X]) + \mathbb{E}[Var(Y|X)].$$

Note that the conditionnal expectation will play a very important role in the definition of Markov chains and Markov processes (memoryless random evolutions), and in the definition of Martingales.

3.5.2 Fourier Analysis : the $\mathbb{L}^2$ theory

3.5.2.1 On the use of Hilbert basis

Hilbert spaces make it possible to use the fundamental structure of Hilbert basis to obtain a good approximation of infinite dimensional objects through the computations of a finite number of inner products.

Theorem 3.5.1 *Let be given an Hilbert basis $\{e_i, i \in I\}$ of an Hilbert space H, then for any $f \in H$ and for any subset J of I, we have for any set of coefficients $(a_j)_{j\in J}$:*

$$\left\| f - \sum_{j\in J} \langle f, e_j\rangle e_j \right\| \leq \left\| f - \sum_{j\in J} a_j e_j \right\|$$

Proof : The proof is rather simple. Write

$$\begin{aligned}\left\|f-\sum_{j\in J}a_je_j\right\|^2 &= \left\|f-\sum_{j\in J}\langle f,e_j\rangle e_j+\sum_{j\in J}(\langle f,e_j\rangle-a_j)e_j\right\|^2\\ &= \left\|f-\sum_{j\in J}\langle f,e_j\rangle e_j\right\|^2+\left\|\sum_{j\in J}(\langle f,e_j\rangle-a_j)e_j\right\|^2\\ &\quad +2\langle f-\sum_{j\in J}\langle f,e_j\rangle e_j,\sum_{j\in J}(\langle f,e_j\rangle-a_j)e_j\rangle\end{aligned}$$

The last term of the right hand side may transformed into :

$$\langle f-\sum_{j\in J}\langle f,e_j\rangle e_j,\sum_{j\in J}(\langle f,e_j\rangle-a_j)e_j\rangle=\sum_{k\in J}(\langle f,e_k\rangle-a_k)\underbrace{\langle e_k,\langle f-\sum_{j\in J}\langle f,e_j\rangle e_j\rangle}_{:=0}$$

The basis being orthonormal, the last inner product vanishes. □

The geometrical interpretation is easy : denote $\pi_J(f)$ the following "projection" :

$$\pi_J(f):=\sum_{j\in J}\langle f,e_j\rangle e_j.$$

We note that $\pi_J(f)$ is the orthogonal projection of f into the set spanned by $\{e_j,j\in J\}$, using the inner product $\langle,\rangle$.

We can deduce the following corollary :

Corollary 3.5.1 *Let $K\subset J$, then*

$$\|f-\pi_J(f)\|\le\|f-\pi_K(f)\|.$$

This inequality is more or less a generalization of the Bessel inequality

Theorem 3.5.2 (Bessel Inequality) *If $\{e_i,i\in I\}$ is an Hilbert basis of H and if $J\subset I$:*

$$\forall f\in H\qquad\sum_{i\in J}|\langle f,e_i\rangle|^2\le\|f\|^2.$$

Moreover, the following equality holds in H (in the sense of the norm $\|.\|$) :

$$\|f\|^2=\sum_{i\in I}|\langle f,e_i\rangle|^2$$

Proof : We follow the same strategy : Consider J any subset of I, we have :

$$\langle f-\pi_J(f),\pi_J(f)\rangle=0.$$

The Pythagore theorem yields :

$$\|f\|^2=\|f-\pi_J(f)\|^2+\|\pi_J(f)\|^2.$$

The norm $\|\pi_J(f)\|^2$ may be simplified according to the orthonormality relationships among $(e_j)_{j\in J)}$:

$$\|\pi_J(f)\|^2=\sum_{j\in J}|\langle f,e_j\rangle|^2.$$

Therefore, we obtained :

$$\sum_{j \in J} |\langle f, e_j \rangle|^2 \leq \|f\|^2.$$

The second identity is the so-called Plancherel equality. We use the important fact that the Hilbert basis $(e_i)_{i \in I}$ is a total family of H, and then the conitnuity of the norm.

In particular, consider the following series :

$$g = \sum_{i \in I} \langle f, e_i \rangle e_i.$$

The series rely on a Cauchy sequence of H : since I is at the most countable, we can define

$$g_n = \sum i \leq n \langle f, e_i \rangle e_i$$

and remark with the Pythagore theorem that :

$$\|g_{n+p} - g_n\|^2 \leq \sum_{k \geq n+1} \langle f, e_i \rangle^2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

Consequently, $(g_n)_{n \geq 1}$ converges in H towards g. Now, it is important to remark that

$$\forall i \in I \qquad \langle f - g, e_i \rangle = \langle f, e_i \rangle - \langle \sum_{j \in I} \langle f, e_j \rangle e_j, e_i \rangle = 0.$$

Hence, $f - g$ is orthogonal to the whole Hilbert basis $(e_i)_{i \in I}$, and thus is 0 in H. We conclude that

$$f = \sum_{i \in I} \langle f, e_i \rangle e_i.$$

□

Definition 3.5.1 *The series above is the expansion of f in the Hilbert basis $(e_i)_{i \in I}$:*

$$\sum_{i \in I} \langle f, e_i \rangle e_i.$$

This definition $\sum_{i \in I} \langle f, e_i \rangle e_i$ should be understood as a limit when j takes all the possible values in I. We have pointed that $\|.\|$ has a tight relationship with $\langle , \rangle$. Hence, if we handle now a second norm (for example the supremum norm on functions), nothing is guaranteed for the equality :

$$\|f - \sum_{i \in I} \langle f, e_i \rangle e_i\|_\infty = 0$$

In other words, the following equation may not hold :

$$\sum_{i=1}^{\infty} \langle f, e_i \rangle e_i \longmapsto f \qquad \text{lorsque} \qquad n \longmapsto +\infty \qquad \text{for the supremum norm} \qquad \|.\|_\infty.$$

3.5.2.2 Fourier series

We denote $\mathcal{C}(\mathcal{T})$ the set of continuous functions on $[0, 2\pi]$, with real or complex values such that $f(0) = f(2\pi)$. We set $\mathcal{T} = [0, 2\pi]$. If f does not satisfy the boundary condition, we can extend f on a larger interval : we define $\forall t \in T \quad f(4\pi - t) = f(t)$ and then obtain a 4π periodic function.

We denote μ the normalized Lebesgue measure on $\mathcal{T}$: $\mu = \lambda/(2\pi)$. We recall the important result on $\mathbb{L}^p(\mathcal{T})$:

Theorem 3.5.3 *$\mathbb{L}^p(\mathcal{T}, \mu)$ is complete for $p \geq 1$.*

Below, we only deal with the case $p = 2$, mainly because of the existence of a natural inner product on $\mathbb{L}^2(\mathcal{T})$, which is an Hilbert space :

$$\langle f, g \rangle = \int_{\mathcal{T}} f\bar{g}d\mu.$$

Definition 3.5.2 (Fourier coefficients) *We denote $E = \{e_n : \mathbb{R} \mapsto \mathbb{C}\}$ with $e_n(t) = e^{\mathrm{int}}$. Then, the Fourier coefficients of f are given by*

$$\forall n \in \mathbb{Z} \qquad c_n(f) = \langle e_n, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-\mathrm{int}}dt$$

We will study the decomposition of f in Fourier series with the help of the Hilbert space theory described above.

Proposition 3.5.1 *$E = \{e_n : \mathbb{R} \mapsto \mathbb{C}\}$ is an orthonormal family of $\mathbb{L}^2(\mathcal{T}, \mu)$.*

This result is rather simple and comes from the remark : $\bar{e_n} = e_{-n}$ and $e_n e_p = e_{n+p}$. In fact, E is even more important than just being an orthonormal family of vectors in $\mathbb{L}^2(\mathcal{T}, \mu)$. This fact is illustrated by the next theorem.

Theorem 3.5.4 *$E = \{e_n : \mathbb{R} \mapsto \mathbb{C}\}$ is an Hilbert basis of $\mathbb{L}^2(\mathcal{T})$. Therefore, we have*

$$\forall f \in \mathbb{L}^2(\mathcal{T}) \qquad f = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle e_n$$

with a convergence of the series in $\mathbb{L}^2(\mathcal{T})$.

The proof of this important result relies on a density result of *Span*(E) in $\mathbb{L}^2(\mathcal{T})$ and on a uniqueness of the Fourier decomposition. We first state a preliminary lemma.

Lemma 3.5.1 *If f is **continuous** on $\mathcal{T}$, then*

$$\forall n \in \mathbb{Z} \qquad \langle f, e_n \rangle = 0 \qquad \Longleftrightarrow \qquad f = 0.$$

Proof : The other side implication is obvious, and we only focus our attention on the direct implication. We consider f such that $\forall n \in \mathbb{Z} \quad \langle f, e_n \rangle = 0$ and we assume that f is not zero everywhere. We assumed that f is periodic, and therefore up to a shift on the x coordinate, we can find an interval $[-h, h]$ where f is non zero. This shift does not modify the sequence of Fourier coefficients of f, since they are all zero. (A translation of f with a quantity τ multiplies each coefficient by $e^{n\tau}$).

We have assumed that f is orthogonal to E : $\langle f, e_n \rangle = 0$ for any integer n. Hence, f is orthogonal to any trigonometric polynomial of the form :

$$P_n(x) = (1 + \cos x - \cos h)^n,$$

because P_n can be expressed as a linear combinations of $e_k, -n \leq k \leq n$. Moreover, for any $x \in]-h, h[$

$$\cos x \geq \cos h.$$

Therefore, we deduce that

$$\lim_{n \longmapsto +\infty} P_n(x) = +\infty \qquad \text{when} \qquad x \in]-h, h[.$$

But we also know that

$$\forall x \in]-\pi, \pi[\backslash [-h, h] : \qquad 0 < 1 + \cos x - \cos h < 1.$$

Consequently, we get :

$$\lim_{n \longmapsto +\infty} P_n(x) = 0 \qquad \text{pour} \qquad x \in]-\pi, \pi[\backslash [-h, h].$$

Using that

$$\langle f, P_n \rangle = 0 \qquad \forall n \in \mathbb{Z},$$

and the Chasles relationship :

$$\int_{-\pi}^{\pi} f P_n = \int_{[-\pi, -h[\cup]h, \pi]} f P_n + \int_{-h}^{h} f P_n,$$

the dominated convergence theorem implies that

$$\int_{[-\pi, -h[\cup]h, \pi]} f P_n \longmapsto 0 \qquad \text{when} \qquad n \longmapsto +\infty.$$

At last, the Fatou lemma states that

$$\int_{-h}^{h} f P_n = \pm\infty.$$

and we obtain a contradiction. The sign + or − above depends on the sign of f on $[-h, h]$. □

We now show our main result on the Fourier series of $\mathbb{L}^2$ functions.

Proof of the theorem : We will show that E is an Hilbert basis. The lemma implies that any continuous function orthogonal to E is necessary zero. This result may be extended to general functions of $\mathbb{L}^2$.

Indeed, consider f in $\mathbb{L}^2(\mathcal{T})$ such that $\langle f, e_n \rangle = 0$, for any n. We consider Φ defined by

$$\Phi(x) = \int_{-\pi}^{x} f(t) dt.$$

The Fubini theorem makes it possible to show that

$$\langle \Phi, e_n \rangle = 0, \qquad \forall n \in \mathbb{Z}.$$

Moreover, Φ is a continuous function orthogonal to E and therefore ϕ is zero. We then deduce that f is zero almost everywhere because it is the derivative of Φ. This point concludes the demonstration of the density of $Span(E)$ in $\mathbb{L}^2(\mathcal{T}, \mu)$.

We deduce that E is an Hilbert basis and the Plancherel identity then leads to (the equality holds in $\mathbb{L}^2(\mathcal{T})$) :

$$f = \sum_{n \in \mathbb{Z}} \langle f, e_n \rangle e_n.$$

□

We can also states the famous following corollaries.

Corollary 3.5.2 (Bessel, Plancherel, Parseval) *For any $(f,g) \in \mathbb{L}^2(\mathcal{T})$, we have :*

— *Bessel inequality :*

$$\sum_{k \in J} |c_k(f)|^2 \leq \|f\|_2^2 = \frac{1}{2\pi} \int_{\mathcal{T}} |f^2|(t)dt$$

— *Plancherel identity :*

$$\sum_{k \in \mathbb{Z}} |c_k(f)|^2 = \|f\|_2^2$$

— *Parseval identity :*

$$\sum_{k \in \mathbb{Z}} c_k(f)\overline{c_k(g)} = \langle f, g \rangle$$

We should mention that a pointwise theory of Fourier series exist, this approach will not be dealt with in this Lecture. We highlight the fact that the results stated in these lecture notes are sufficient to obtain nice formula with series, and will imply important results on non parametric statistics.

3.5.3 Non-parametric statistics

3.5.3.1 Smoothness class

We have seen the important result in the paragraph above : for any $f \in \mathbb{L}^2([0,1])$, we can reconstruct f through its Fourier series :

$$f = \sum_{k \in \mathbb{Z}} c_k(f) e_k.$$

We push our analysis further by introducing some more stringent functional spaces. Such spaces include the knowledge of smoothness of the function f.

We will say that f is s-smooth if and only if f belongs to $\mathcal{H}_s(\mathbb{R})$ where :

$$\mathcal{H}_s(R) := \left\{ f \in \mathbb{L}^2([0,1]) \, : \, f^{(s)} \in \mathbb{L}^2 \quad \text{with} \quad \|f^{(s)}\| \leq R \right\}.$$

Above, $f^{(s)}$ refers to the s-th derivative of f. In particular, s is implicitely assumed to be an integer (this assumption may be relaxed with additional tecnicalicities).

This smoothness assumption leads to a nice decreasing approximation property, given by the next proposition.

Proposition 3.5.2 *For any $f \in \mathcal{H}_s(R)$, one has*

$$\|f - \sum_{-K \leq k \leq K} c_k(f) e_k\|_2^2 \leq R^2 K^{-2s}$$

Proof : We write the Parseval identity and obtain that

$$\|f - \sum_{-K\leq k\leq K} c_k(f)e_k\|_2^2 = \left\| \sum_{|k|>K} c_k(f)e_k \right\|_2^2 = \sum_{|k|>K} |c_k(f)|^2.$$

We use the simple trick :

$$\sum_{|k|>K} |c_k(f)|^2 = \sum_{|k|>K} |k|^{2s}|c_k(f)|^2|k|^{-2s},$$

and we should note that $|c_k(f')| = |k|c_k(f)$. We can iterate this relationship and obtain

$$|c_k(f^{(s)}| = |k|^s|c_k(f)|.$$

Coming back to the tail series we have to upper bound, we have :

$$\sum_{|k|>K} |c_k(f)|^2 = \sum_{|k|>K} |c_k(f^{(s)})|^2|k|^{-2s} \leq R^2K^{-2s}.$$

This ends the proof of the proposition. □

3.5.3.2 White noise model as a standard benchmark

We are interested in the "simplest" non-parametric estimation problem : we observe a sequence of n noisy functions

$$\forall j \in \{1\dots n\} \quad \forall x \in [0,1] \qquad f_j(x) = f(x) + \sigma w_j(x). \tag{3.4}$$

We are then interested in recovering f when we assumed that $(w_j(x))_{1\leq j\leq n}$ is a Gaussian white noise model. The objective of this lecture is not to do a Here, as in many other statistical problems, there axe two distinct roles for the Gaussian model :

- As a "continuous-time" model of interest in its own right.
- As a "canonical limiting-problem" appearing in connection with many other discrete-time models involving non-parametric estimation of a function. We observe indeed a high frequency data of each curve :

  $$f_j(x_t) = f(x) + \sigma\xi_t^j,$$

  where t lives in a high frequency grid of $[0,1]$ and $(\xi_t^j)_{t,j}$ stands for a sequence of i.i.d. standard Gaussian random variable.

In particular, the last approach makes it possible to pass to the limit, when the grid in t becomes asymptotically the whole segment $[0,1]$.

We will keep in mind that (3.4) is nothing more than a model equivalent to

$$\forall \psi \in \mathbb{L}^2([0,1]) \qquad \langle \psi, X_i\rangle \sim \mathcal{N}(<\psi, f\rangle, \sigma^2\|\psi\|_2^2)$$

with the covariance structure :

$$Cov(\langle \psi, X_i\rangle, \langle \phi, X_i\rangle) = \langle \psi, \phi\rangle.$$

In particular, we shall apply this last remark to the Fourier basis $(e_k)_{k\in\mathbb{Z}}$. We obtain an important feature of the white noise model :

$$\langle e_k, X_i\rangle \sim \mathcal{N}(c_k(f), \sigma^2\|e_k\|^2) = \mathcal{N}(c_k(f), \sigma^2).$$

Moreover, the covariance of two randomized Fourier coefficients is

$$Cov(\langle e_k, X_i\rangle, \langle e_j, X_i\rangle) = \langle e_k, e_j\rangle = \mathbf{1}_{k=j}.$$

Therefore, the empirical Fourier coefficients of one curve form an infinite family of Gaussian random variables, with a covariance $\sigma^2 Id$. In other words, if we denote

$$\theta_k^i = c_k(X_i) = \langle e_k, X_i\rangle,$$

then θ_k^i and θ_k^i are Gaussian and independent when $k \neq j$.

3.5.3.3 Non-parametric estimation

The main idea consists in estimating the Fourier coefficient sequence of the function f. Of course, we only have in our hands n observations $X_1, \ldots, X_n$. Therefore, there is no miracle : we cannot expect to infer the value of an infinite sequence with a finite number of observations n. Nevertheless, it is possible to use the smoothness assumption $\mathcal{H}_s(R)$ and deduce an efficient method of estimation for f.

Estimation of a fixed Fourier coefficient This estimation problem is standard. It is enough to define

$$\hat{\theta}_k = \frac{1}{n}\sum_{i=1}^n \theta_k^i = \frac{1}{n}\sum_{i=1}^n \langle e_k, X_i\rangle.$$

We have seen that each θ_k^i are distributed as a $\mathcal{N}\left(c_k(f), \sigma^2\right)$ random variable. The curve $(X_i)_{1\leq i\leq n}$ being independent, we can deduce that :

$$\hat{\theta}_k \sim \mathcal{N}\left(c_k(f), \frac{\sigma^2}{n}\right).$$

Note that if the random variables θ_k^i follow another law (different from a Gaussian one), the conclusion of the remark above should be slightly modified. But the main remark is that $\hat{\theta}_k$ is an unbiased estimation of $c_k(f)$ with a variance reduced by a factor n.

Consequently, the mean squared error of estimating $c_k(f)$ with $\hat{\theta}_k$ is

$$\mathbb{E}[|\hat{\theta}_k - c_k(f)|^2] = \frac{\sigma^2}{n}.$$

Finally, we should note that the estimation above is optimal : the Cramer-Rao lower bound yields : for any estimator $\tilde{\theta}_k$ of $c_k(f)$:

$$\mathbb{E}[|\tilde{\theta}_k - c_k(f)|^2] = \frac{\sigma^2}{n}.$$

Therefore, there is nothing more to do than the LLN for the estimation of each Fourier coefficient $c_k(f)$.

Estimation of f We have seen that each estimation of each Fourier coefficient will induce a M.S.E. of at least $\sigma^2 n^{-1}$. Consequently, if we manage to estimate f through its description by the infinite sequence $(c_k(f))_{k\in\mathbb{Z}}$, we will certainly obtain an infinite M.S.E., which is not satisfactory ;).

A concurrent idea relies on a hard-thresholding strategy approach : we will estimate the low-frequency coefficients and use the approximation result of Proposition 3.5.2. We introduce the following estimator of the infinite sequence :

$$\hat{\theta_k}^d = \hat{\theta}_k \mathbf{1}_{|k|\leq d}.$$

This estimator satisfies :

$$\forall k \in \mathbb{Z} \qquad \mathbb{E}[\hat{\theta_k}^d - c_k(f)|^2] \leq \mathbf{1}_{|k|\leq d}\frac{\sigma^2}{n} + \mathbf{1}_{|k|>d}|c_k(f)|^2.$$

Hence, the estimator $\hat{f}$ defined by :

$$\hat{f} = \sum_{|k|\leq d} \hat{\theta_k}^d e_k,$$

satisfies

$$\mathbb{E}\left[\int_0^1 (\hat{f}-f)^2\right] = \mathbb{E}[\|\hat{f}-f\|_2^2 = \sum_{k\in\mathbb{Z}} \mathbb{E}[\hat{\theta_k}^d - c_k(f)]^2.$$

The last equation is a consequence of the Parseval identity. We then obtain the following decomposition :

$$\mathbb{E}\left[\int_0^1 (\hat{f}-f)^2\right] = \sum_{|k|\leq d} \mathbb{E}[\hat{\theta_k} - c_k(f)]^2 + \sum_{|k|>d} |c_k(f)|^2.$$

This decomposition is nothing more than the standard bias-variance decomposition. In particular, the variance term is equal to $\sigma^2\frac{d}{n}$ while the bias is what is left over in large frequencies. This bias is upper bounded by $R^2 d^{-2s}$, according to Proposition 3.5.2. We deduce that

$$\mathbb{E}\left[\int_0^1 (\hat{f}-f)^2\right] \leq \sigma^2\frac{d}{n} + R^2 d^{-2s}.$$

It remains to do a very simple optimization of the upper bound above with respect to d. We easily see that the optimal choice for d is $d \sim n^{\frac{1}{2s+1}}$. We then obtain the *rate of convergence* of the estimator $\hat{f}$:

$$\mathbb{E}\left[\int_0^1 (\hat{f}-f)^2\right] \leq C n^{-2s/(2s+1)}. \tag{3.5}$$

We should note that :

- This rate is significantly slower than the parametric rate of convergence of estimators in regular finite dimensional model. In such cases, the optimal estimators have a M.S.E. of the order n^{-1}. Here, regardless the value of s, we always have $2s/(2s+1) < 1$.
- When s becomes large, the rate becomes close to n^{-1}. Therefore, an "infinite" smoothness parameter s may be seen as a fully parametric situation.
- It is possible to obtain a sharper result where C explicitely depends on R^2 and σ^2, but we have omitted this detail for the sake of simplicity.
- The rate obtained above is optimal with respect to n : it is possible to show a kind of Cramer-Rao lower bound in this infinite dimensional case $\mathcal{H}_s(R)$. Such a lower bound matches the upper bound and says that any estimator of f in $\mathcal{H}_s(R)$ will have a M.S.E. greater than $n^{-2s/(2s+1)}$.

3.5.4 Construction of the Brownian Motion

3.5.4.1 The Haar basis of $(\mathcal{L}^2([0,1],\|.\|_2)$

We propose a construction of the Brownian Motion (B.M. for short in what follows). This construction is not the unique way to build the theoretical object. Nevertheless, it is a good illustration of the theory of Hilbert spaces.

Consider $H = L^2([0,1])$ equipped with the inner product

$$\forall (f,g) \in H^2 \qquad \langle f,g\rangle = \int_0^1 f(t)g(t)dt.$$

For any integer $n \in \mathbb{N}$, which will be a “frequency", we denote :

$$\forall k \in \{0,\ldots,n-1\} \qquad D_{n,k} = [k2^{-n},(k+1)2^{-n}[,$$

which is simply the k-th interval of size 2^{-n} in $[0,1]$. We also denote

$$\mathcal{D}_n := \left\{ f \in H \quad | \quad f(x) = \sum_{k=0}^{n-1} \alpha_k \mathbf{1}_{D_{n,k}}(x) \right\}$$

Functions in $\mathcal{D}_n$ are constant on each dyadic interval $D_{n,k}$. It is easy to see that

$$\mathcal{D}_n \subset \mathcal{D}_{n+1}.$$

We can establish the first proposition.

Proposition 3.5.3 *The set $\cup_{n\in\mathbb{N}}\mathcal{D}_n$ is dense in $(H,\|.\|_\infty)$ and in $(H,\|.\|_2)$.*

<u>*Proof :*</u> Remark that the set of continuous functions on $[0,1]$ is dense in $(H,\|.\|_\infty)$. Hence, given any function f in $(H,\|.\|_\infty)$ and for any $\epsilon > 0$, we can find $g \in \mathcal{C}([0,1])$ such that

$$\|f-g\|_\infty \leq \epsilon.$$

We introduce ω as the continuity modulus of g :

$$\omega_g(h) := \sup_{|x-y|\leq h} |g(x)-g(y)|.$$

Since g is continuous on $[0,1]$, which is compact, we then deduce that g is uniformly continuous on $[0,1]$ and thus

$$\lim_{h\longrightarrow 0} \omega_g(h) = 0.$$

Now, we define

$$\forall n \in \mathbb{N} \quad g_n(x) = f(2^{-n}\lfloor 2^n x\rfloor),$$

which is a stepwise constant function on the 2^n dyadic intervals of size 2^{-n}. Moreover, we have the obvious bound

$$\|g-g_n\|_\infty \leq w_g(2^{-n}) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

Thus, we can find n large enough such that

$$\|g_n - f\|_2 \leq \|g_n - f\|_\infty \leq 2\epsilon.$$

□

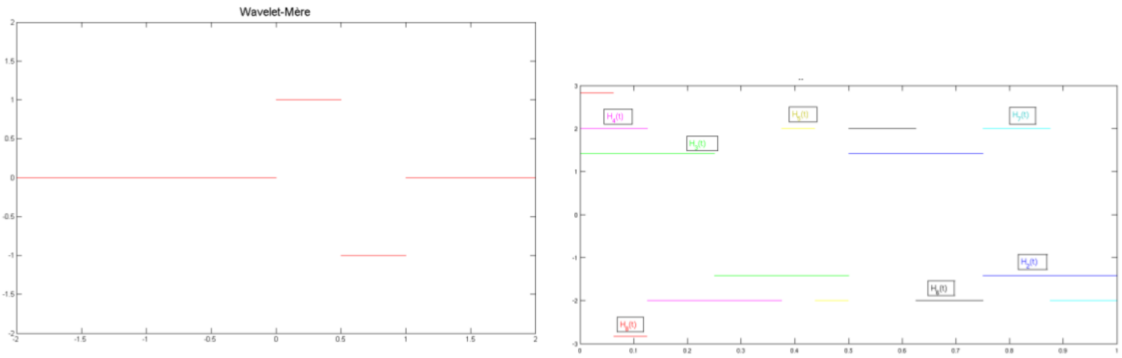


FIGURE 3.4: Left : the “mother" Haar wavelet u_1. Right : examples of elements taken in the Haar basis of $L^2([0,1])$.

The set $\mathcal{D}$ is useful to build an orthonormal Hilbert basis of H. The construction could follow the Gram-Schmidt procedure starting from the set $(\mathbf{1}_{D_{n,k}}, n \in \mathbb{N}, 1 \leq k \leq 2^n)$. But here, this procedure can be suitably skipped, in favour of a visual construction. We start with the constant function on $[0,1]$, which is necessarily equal to 1 to obtain a normed vector of H :

$$u_0(t) := \mathbf{1}_{[0,1]}(t)$$

For a fixed frequency n, we define

$$\forall n \geq 0 \quad \forall k \in \{0,\ldots,2^n-1\} \qquad u_{2^n+k} = 2^{n/2}\left[\mathbf{1}_{D_{n+1,2k}} - \mathbf{1}_{D_{n+1,2k+1}}\right]. \tag{3.6}$$

Proposition 3.5.4 (Haar's Basis) *The sequence $(u_{2^n+k}, n \in \mathbb{N}, 0 \leq k \leq 2^n-1)$ is an Hilbert basis of $L^2([0,1], \|.\|_2)$.*

Proof :

Orthonormal system : We provide here a nice picture, that should be understood as a proof of this result. It is easy to see that the sequence $(u_p)_{p\geq 0}$ forms an orthonormal basis of $\mathcal{D}$ as illustrated in Figure 3.4.

Note that the orthogonality of the sequence comes from the oscillation of a function at frequency n as compared to other functions with a strictly lower frequency (red and pink functions in Figure 3.4), or comes from the disjoint supports when the two elements share the same frequency (green and blue functions in Figure 3.4). At last, the fact that each element has an L^2 norm equals to 1 comes from the normalization by $2^{n/2}$ in Equation (3.6).

Density : It remains to show that the sequence $(u_p)_{p\geq 0}$ forms a dense sequence of $L^2([0,1])$. In this view, for any $n \in \mathbb{N}$, we denote $m = 2^n - 1$ and the set of vectors $\mathcal{U}_m := Vect\,(u_p)_{0\leq p\leq m}$ represents the set of functions whose frequency is lower or equal to n. Moreover, as an orthonormal system, we have

$$\dim(\mathcal{U}_m) = m+1$$

If we consider also

$$\mathcal{D}_n := Vect\left\{\mathbf{1}_{I_{n,k}}, 0 \leq k \leq 2^n-1\right\},$$

we have

$$\mathcal{D}_n \subset \mathcal{U}_m,$$

and $\dim \mathcal{D}_n = 2^n$, so that $\mathcal{U}_m = \mathcal{D}_n$. From Proposition 3.5.3, we obtain the desired result. $\square$

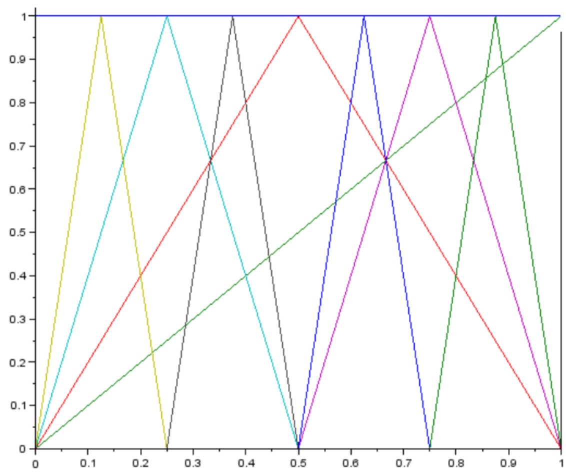


FIGURE 3.5: Some functions taken from the Schauder basis.

3.5.4.2 Explicit construction of the Brownian Motion

We use now the Haar basis to mimic the famous oscillations of the B.M. From the Haar basis, we build the Schauder basis of the Cameron-Martin space :

$$\forall p \in \mathbb{N} \qquad \psi_p(t) := \int_0^t u_p(x)dx.$$

ψ_p is the triangle basis as shown in Figure 3.5 except for ψ_0, that is

$$\psi_0(t) = t.$$

We consider now an infinite countable sequence of i.i.d. random Gaussian variables $\xi(\omega) := (\xi_k(\omega))_{k\geq 0}$, where for any integer k : $\xi_k \sim \mathcal{N}(0,1)$ and the associated process to the event ω is

$$\forall n \in \mathbb{N} \qquad B_t^n(\omega) := \sum_{k=0}^{2^n-1} \psi_k(t)\xi_k(\omega)$$

Definition 3.5.3 *The Brownian motion is the "limit" random variable B_t^n, when $n \longrightarrow +\infty$.*

The important part of this definition corresponds to the proof of the existence of such a limit. This is the purpose of the next result.

Theorem 3.5.5 (Construction of the Brownian motion) *The random variable $B_t(\omega) := \lim_{n\longrightarrow+\infty} B_t^n(\omega)$ satisfies*

1. *$B_t(\omega)$ is defined a.s. and continuous*
2. *B_t is a centered Gaussian process, whose covariance is*

$$Cov(B_t, B_s) = s \wedge t.$$

3. *$(B_t)_{t\in[0,1]}$ is a Markov process : $B_{t+s} - B_t$ is a Brownian motion independent from $(B_u)_{0\leq u\leq t}$.*

To sum up, the Brownian motion is a continuous Gaussian process, which satisfies the Markov property. Moreover, $B_t \sim \mathcal{N}(0,t)$ and the conditional laws are fully characterised through the property

$$\forall (t,s) \in \mathbb{R}_+^2 \qquad \mathcal{L}\left(B_{t+s} | (B_u)_{u \leq t}\right) \sim \mathcal{N}(B_t, s).$$

Proof : Point 1) : We aim to show that the sequence B_t^n converges in L^2 almost surely.

From one frequency to another, we consider the auxiliary process

$$\forall n \in \mathbb{N} \qquad X_t^n(\omega) = B_t^{n+1}(\omega) - B_t^n(\omega).$$

We can write the simple bound :

$$\sup_{t\in[0,1]} X_t^n(\omega) = \sup_{t\in[0,1]} \sum_{k=2^n}^{2^{n+1}-1} \psi_k(t)\xi_k(\omega) \leq \sup_{2^n \leq k \leq 2^{n+1}-1} |\xi_k(\omega)| \times \sup_{t\in[0,1]} \psi_k(t)$$

since in the sum above, at the most one term is not equal to 0 (by considering the support of the triangle Schauder functions). Furthermore, the length of the support of the triangle function is 2^{-n} so that :

$$\forall k \in \left\{2^n, \ldots, 2^{n+1}-1\right\} \quad \forall t \in [0,1] \qquad |\psi_k(t)| \leq 2^{n/2} \int_0^t u_k(t)dt \leq 2^{n/2} 2^{-n} \leq 2^{-n/2}.$$

It remains to understand the size of maximal value of 2^n i.i.d. gaussian random variables. Using the independence, we have

$$\mathbb{P}\left(\max_{0\leq k\leq 2^n-1} |\xi_k| \geq \sqrt{2n}\right) = 1 - \mathbb{P}\left(\max_{0\leq k\leq 2^n-1} |\xi_k| \geq \sqrt{2n}\right) = 1 - \mathbb{P}(|\xi_0| \geq \sqrt{2n})^{2^n}.$$

For any $a \in (0,1)$, we have $1 - na \leq (1-a)^n$ so that $1-(1-a)^n \leq na$. It yields

$$\begin{aligned}
\mathbb{P}\left(\max_{0\leq k\leq 2^n-1} |\xi_k| \geq \sqrt{2n}\right) &\leq 2^n \mathbb{P}(|\xi_0| \geq \sqrt{2n}) \leq \frac{2^{n+1}\int_{\sqrt{2n}}^{+\infty} e^{-u^2/2}du}{\sqrt{2\pi}} \\
&\leq \frac{2^{n+1}}{\sqrt{2\pi}} \int_{\sqrt{2n}}^{+\infty} \frac{u}{\sqrt{2n}} e^{-u^2/2}du \\
&\leq \frac{1}{\sqrt{\pi n}} \left(\frac{2}{e}\right)^n.
\end{aligned}$$

Consequently, if $A_n := \left\{\omega \,:\, \sup_{t\in[0,1]} |X_t^n| > \sqrt{2n}2^{-n/2}\right\}$, the Borel-Cantelli Theorem leads to

$$\mathbb{P}(\limsup A_n) = 0,$$

meaning that for almost surely ω, there exists n_ω such that we obtain that

$$\forall n \geq n_\omega \qquad \sup_{t\in[0,1]} |X_t^n| \leq \sqrt{2n}2^{-n/2}.$$

It implies the almost surely uniform convergence of B_t^n toward its limit on $[0,1]$. From this uniform convergence, we also conclude that $t \longmapsto B_t$ is a continuous process on $[0,1]$ with $B_0 = 0$.

Point 2) : For any time $t > 0$, $(B_t)_{t\in[0,1]}$ is a limit of some Gaussian processes since $(B_t^n)_{t\in[0,1]}$ are all continuous Gaussian processes. It is a classical result that the limit is still a Gaussian process (see Dachuna-Castelle & Duflo). The argument relies on the multivariate characteristic function of a Gaussian process, which is completely described through its mean and covariance matrix.

The computation of the covariance is more interesting. We begin by an important remark :

$$s \wedge t = \int_0^1 \mathbf{1}_{[0,s]}(u)\mathbf{1}_{[0,t]}(u)du.$$

We use here the Hilbert basis of $L^2([0,1])$ to get

$$\begin{aligned}
\forall (f,g) \in L^2([0,1])^2 \qquad \langle f, g\rangle &= \langle \left(\sum_{p\geq 0} h_p \int_0^1 f(u)h_p(u)du\right), \left(\sum_{p\geq 0} h_p \int_0^1 g(u)h_p(u)du\right)\rangle \\
&= \sum_{p\geq 0} \int_0^1 f(u)h_p(u)du \times \int_0^1 g(u)h_p(u)du
\end{aligned}$$

Apply now this relation to $f = \mathbf{1}_{[0,s]}$ and $g = \mathbf{1}_{[0,t]}$ to obtain

$$s \wedge t = \langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]}\rangle = \sum_{p\geq 0} \int_0^s h_p(u)du \int_0^t h_p(u)du = \sum_{p\geq 0} \psi_p(s)\psi_p(t). \tag{3.7}$$

Now, consider $(s,t) \in [0,1]^2$, and compute

$$\mathbb{E}[B_t^2] = \mathbb{E}\left[\sum_{n\geq 0} X_t^n\right]^2 = \mathbb{E}\sum_{n\geq 0}[X_t^n]^2 = \sum_{n\geq 0}\mathbb{E}[X_t^n]^2,$$

where we use above the a.s. L^2 convergence of the series, and the fact that from one frequency to another, the random variables X_t^n are independent. Hence,

$$\mathbb{E}[B_t^2] = \sum_{n\geq 0} \sum_{2^n \leq k \leq 2^{n+1}-1} \psi_k^2(t) = t$$

where we used (3.7). In the same time, we also have

$$\mathbb{E}[B_t B_s] = \sum_{k\geq 0}\sum_{l\geq 0} \mathbb{E}[\psi_k(t)\psi_l(t)\xi_k\xi_l] = \sum_{k\geq 0} \psi_k(t)^2 = s \wedge t.$$

Point 3) : The last point comes from the following remark.

$$B_{t+s}^n(\omega) - B_t^n(\omega) = \sum_{k=0}^{2^n-1} [\psi_k(t+s) - \psi_k(t)]\xi_k(\omega) = \sum_{k=0}^{2^n-1} \int_t^{t+s} u_k(x)dx\xi_k(\omega) = \sum_{k=0}^{2^n-1} \int_0^s u_k(t+x)dx\xi_k(\omega)$$

Now, from the construction of the Haar basis, the $(u_k)_{k\geq 0}$ are piecewise constant functions, the gaussian random variables are i.i.d. and the law of ξ_k is equal to the law of $-\xi_k$. A direct consequence, is that

$$\mathcal{L}(B_{t+s}^n(\omega) - B_t^n(\omega)) = \mathcal{L}(B_s^n(\omega)).$$

(A picture would be helpful to fully understand the argument above.) □

It is important to point out that many applications involve the Brownian motion : in the mathematical modelling of Financial series through stochastic differential equations and Ito's calculus with the Black-Scholes equations for example, in PDE with the Heat equations, in Kinetic theory, ... Figure 3.6 represents a typical trajectory of the Brownian motion between 0 and 1.

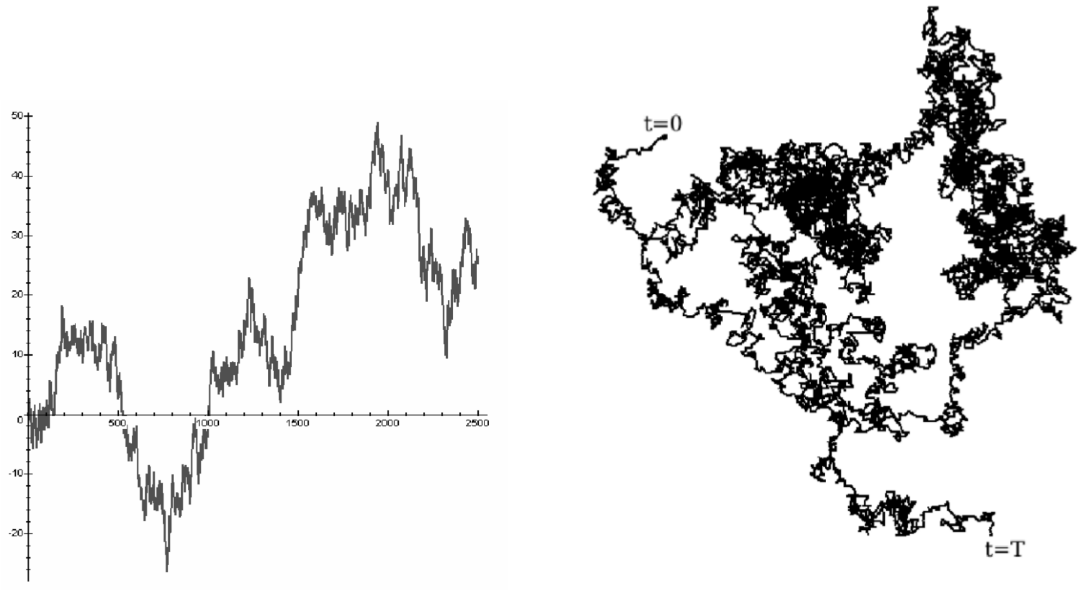


FIGURE 3.6: Some trajectories of the Brownian motion, in a 1 and 2 dimensional space.

Extension to the real line Our construction of the Brownian motion is given for $t \in [0,1]$. It is now easy to complement this construction and define a Brownian motion on $\mathbb{R}_+$. With the third property of Theorem **??**, we can first build a countable sequence of independent Brownian motions, that are denoted $((B_{t,k})_{t\in[0,]})_{k\geq\mathbb{N}}$. Now, we define

$$.\forall t \in \mathbb{R} \qquad B_t = \sum_{k=1}^{n_t} B_{1,k} + B_{t-n_t,n_t+1} \qquad \text{where} \qquad n_t \leq t \leq n_t + 1. \tag{3.8}$$

Since for any integer k, we have $B_{0,k} = 0$, the process $t \longmapsto B_t$ is still continuous on $\mathbb{R}$. It is an easy exercice to check that the process is also Gaussian with a covariance $\mathbb{E}[B_t B_s] = t \wedge s$: consider for example $t \geq s$

$$\begin{aligned} \mathbb{E}B_tB_s &= \sum_{1\leq k\leq n_t} \sum_{1\leq j\leq n_s} \mathbb{E}[B_{1,k}B_{1,j}] + \sum_{k=1}^{n_t} \mathbb{E}[B_{1,k}B_{s-n_s,n_s+1} \\ &\quad + \sum_{j=1}^{n_s} \mathbb{E}[B_{1,j}B_{t-n_t,n_t+1} + \mathbb{E}[B_{t-n_t,n_t+1}B_{s-n_s,n_s+1}]. \end{aligned}$$

We can now study all the situations ($n_t = n_s$ or $n_t > n_s$) and check that

$$\mathbb{E}B_tB_s = t \wedge s.$$

Lastly, the Markov property trivially holds from the construction (3.8).

The multi-dimensional Brownian motion The multi-dimensional Brownian motion is again a stochastic process, indexed by a real time $t \in \mathbb{R}_+$ that belongs to $\mathbb{R}^d$. We still use the notation $B_t := (B_t^1, \ldots, B_t^d)$ where d is the dimension of the space. Each coordinate B_t^j follows the dynamic of a Brownian motion, *independently* of the other coordinates.

3.5.4.3 The Brownian motion and the Laplacian

The purpose of this paragraph is to exhibit the strong link that exists between the Brownian motion and the second order differential operator Laplacian. Let be given a standard Brownian motion $(B_t)_{t\geq 0}$. As a Markov process, it is possible to consider the infenitesimal evolution of B_{t+h} knowing that $B_t = x \in \mathbb{R}^d$. To do so, we consider a test function $f \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R})$ and introduce the linear operator

$$\mathcal{L}_t(f)(x) := \lim_{h\to 0} \frac{\mathbb{E}[f(B_{t+h})|B_t = x] - f(x)}{h}.$$

It is easy to see that the above equality does not depend on t from the Markov property of the Brownian motion (it is indeed the case for any Markov process). We are driven to the next definition :

Definition 3.5.4 (Infinitesimal geenrator) *The infinitesimal generator of the Brownian motion is defined as*

$$\forall f \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}) \qquad \mathcal{L}(f)(x) := \lim_{h\to 0} \frac{\mathbb{E}[f(B_{t+h})|B_t = x] - f(x)}{h}.$$

Theorem 3.5.6 *For any function $f \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R})$, we have*

$$\mathcal{L}(f)(x) = \frac{1}{2}\Delta f(x)$$

<u>*Proof :*</u> We introduce ξ as a standard Gaussian vector of $\mathbb{R}^d$. Then, we have through a simple Taylor formula

$$f(B_{t+h}) = f(B_t + \sqrt{h}\xi) = f(B_t) + \sqrt{h}\langle \xi, \nabla f(B_t)\rangle + h\frac{{}^t\xi D^2 f(B_t)\xi}{2} + \mathcal{O}(h^{3/2}).$$

Now, assume that $f(B_t) = x$ and we obtain

$$\mathbb{E}[f(B_{t+h})|B_t = x] - f(x) = \sqrt{h}\mathbb{E}\langle \xi, \nabla f(x)\rangle + h\mathbb{E}\left(\frac{{}^t\xi D^2 f(x)\xi}{2}\right) + \mathcal{O}(h^{3/2}).$$

The first order term vanishes since $\mathbb{E}[\xi] = 0$ and the second order term can be detailed :

$$\mathbb{E}\left(\frac{{}^t\xi D^2 f(x)\xi}{2}\right) = \sum_{1\leq (i,j)\leq d} \mathbb{E}[\xi_j\xi_i] D^2_{i,j} f(x) = \sum_{i=1}^{d} \mathbb{E}[\xi_i^2] D^2_{i,i} f(x) = \Delta f(x)$$

where we use from line to line that ξ_i and ξ_j are independent when $i \neq j$ and the second moment of a Gaussian r.v. is 1. We finally obtain

$$f(B_{t+h}) = f(B_t + \sqrt{h}\xi) = f(B_t) + h\frac{\Delta f(x)}{2} + \mathcal{O}(h^{3/2}).$$

Passing to the limit, we get

$$\mathcal{L}(f) = \frac{1}{2}\Delta f$$

□

It is valuable to point that the B.M. can be shown to be the only continuous Gaussian stochastic process such that $Cov(X_s, X_t) = s \wedge t$. It is also important to say that the evolution of the Markov process is completely characterized through the infenitesimal generator $\mathcal{L}$ applied on a sufficiently rich class of function (its domain). This is far beyond the scope of this lecture, but many details can be found in the classical book of Ethier & Kurtz (level Masters Degree Lecture Notes).

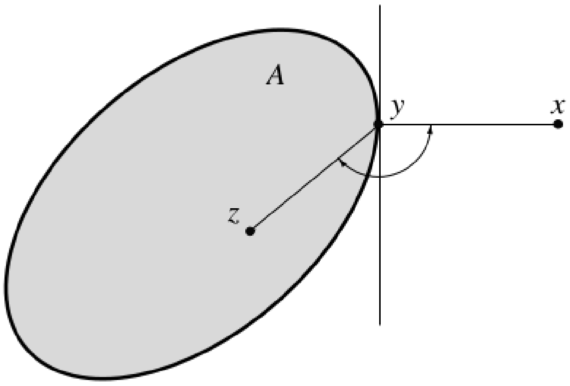


FIGURE 3.7: Projection on a closed convex set A and angle characterisation.

3.5.5 Geometric separation with hyperplanes

An important consequence of the projection theorem is the separation result described below, which is the tool for optimization and economics applications. It is also useful in game theory. Remind first the picture associated to the projection theorem (see Figure 3.7). Denote A the convex set, and $y = P_A(x)$, an important feature is that the angle between $x - y$ and $z - y$ is obtuse, *whatever z is in A*.

This remark implies that the convex set A is entirely included in an half space defined by the point y and the orthogonal direction $x - y$. A trivial consequence is that as soon as $x \notin A$, we can find a hyperplane that separates x and A.

Theorem 3.5.7 *Let be given a closed convex set A of an Hilbert space H and $x \notin A$, then we can find $r \in H$ such that*

$$\sup_{z \in A} \langle r, z\rangle < \langle r, x\rangle.$$

<u>*Proof :*</u> The proof is immediate if we consider $r = x - y \neq 0$ with $y = P_A(x)$ as soon as $x \notin A$. The characterization of the projection is as follows :

$$\forall z \in A \qquad \langle x - y, z - y\rangle \leq 0.$$

But the left hand side can be written as

$$\forall z \in A \qquad \langle r, z - x + x - y\rangle = \langle r, z - x\rangle + \|r\|^2.$$

We then obtain

$$\forall z \in A \qquad \langle r, z\rangle \leq \langle r, x\rangle - \|r\|^2.$$

Since $r \neq 0$, we obtain the result. □

The geometric interpretation is easy : an hyperplane $\mathcal{H}$ is defined *via* a vector r and a constant C such that the set of points ξ in H is in $\mathcal{H}$ iff

$$\langle \xi, r\rangle = C.$$

Moreover, the position of another point α with respect to the hyperplane is obtained through a comparison of $\langle \alpha, r\rangle$ with C. An easy consequence is that a suitable hyperplane is obtained by taking the value of the constant C :

$$C = \|r\|^2/2,$$

so that

$$\langle r, x\rangle = \langle r, r+y\rangle = \|r\|^2 \geq \|r\|^2/2.$$

In the meantime, when $z \in A$, we get

$$\langle r, z\rangle = \underbrace{\langle r, z-y\rangle}_{\leq 0} + \langle r, y\rangle = -\|r\|^2/2.$$

We can even push our analysis further, and consider now two convex sets A_1 and A_2 in an Hilbert space and introduce the separation definition.

Definition 3.5.5 *Two convex sets A_1 and A_2 are strictly separated by the linear form $x \longmapsto \langle r, x\rangle$ iff*

$$\sup_{x_1 \in A_1} \langle r, x_1\rangle < \inf_{x_2 \in A_2} \langle r, x_2\rangle$$

or

$$\sup_{x_2 \in A_2} \langle r, x_2\rangle < \inf_{x_1 \in A_1} \langle r, x_2\rangle.$$

Roughly speaking, the convex sets A_1 and A_2 are separated by each hyperplane defined through the linear form

$$\langle r, x\rangle = \alpha,$$

when α belongs to the interval $[\sup_{x_2 \in A_2} \langle r, x_2\rangle; \inf_{x_1 \in A_1} \langle r, x_2\rangle]$.

A stricking property of convex separation is given below.

Corollary 3.5.3 *If A_1 and A_2 are closed convex sets of an Hilbert space H with A_2 compact. A_1 and A_2 are separated iff they are disjoint.*

Proof : The implication $\Longrightarrow$ is immediate. Conversely, assume that A_1 and A_2 are disjoint with A_2 compact and consider $A_1 - A_2 := \{x_1 - x_2 : x_1 \in A_1, x_2 \in A_2\}$. This set is closed and convex (left as an exercice to the reader). Moreover, $0 \notin A_1 - A_2$ since A_1 and A_2 are disjoint. Hence, we can find $r \in H$ such that

$$\sup_{z \in A_1 - A_2} \langle z, r\rangle < 0.$$

But remark now that

$$\begin{aligned} \sup_{z \in A_1 - A_2} \langle z, r\rangle &= \sup_{x_1 \in A_1, x_2 \in A_2} \langle x_1 - x_2, r\rangle = \sup_{x_1 \in A_1} \sup_{x_2 \in A_2} \langle x_1 - x_2, r\rangle \\ &= \sup_{x_1 \in A_1} \langle x_1, r\rangle + \sup_{x_2 \in A_2} \langle -x_2, r\rangle\rangle = \sup_{x_1 \in A_1} \langle x_1, r\rangle - \inf_{x_2 \in A_2} \langle x_2, r\rangle\rangle. \end{aligned}$$

This ends the proof. $\square$

We could push further the description of separation of convex sets in Hilbert spaces. Deeper results will be encountered in the practical sessions associated to these lecture notes (Farkas lemma, Caratheodory Theorem to name a few).

3.5.6 von Neuman's minimax Theorem

3.5.6.1 Generic case

Let us consider two players P_1 and P_2, whose strategies are described in a set E_1 and E_2. Given two strategies $x \in E_1$ and $y \in E_2$, the **loss of player P_1 is the reward of player** P_2 and is defined in a function $f(x, y)$. Of course, we assume here that no information about the choice of P_1 or the choice of P_2 is available. We are going to show that under appropriate convexity hypotheses :

$$\sup_{y \in E_2} \inf_{x \in E_1} f(x, y) = \inf_{x \in E_1} \sup_{y \in E_2} f(x, y).$$

In what follows, we will use

$$\alpha := \inf_{x \in E} \sup_{y \in F} f(x, y) \qquad \text{and} \qquad \beta := \sup_{y \in F} \inf_{x \in E} f(x, y).$$

Note that without any assumptino on f, the Max-Min inequality is as follows.

Proposition 3.5.5 (Max-Min inequality)

$$\beta \leq \alpha$$

Proof : Define $g : y \longmapsto \inf_{x \in E_1} f(x, y)$ and remark that $g(y) \leq f(x, y), \forall x \in E_1$. Taking the supremum on y, we obtain $\sup_{y \in E_2} g(y) \leq \sup_{y \in E_2} f(x, y), \forall x \in E_1$. We thus obtain

$$\sup_{y \in E_2} \inf_{x \in E_1} f(x, y) \leq \sup_{y \in E_2} f(x, y), \forall x \in E_1.$$

Taking the infimum on the right hand side, it leads to

$$\sup_{y \in E_2} \inf_{x \in E_1} f(x, y) \leq \inf_{x \in E_1} \sup_{y \in E_2} f(x, y).$$

□

Let us briefly discuss on the assumptions needed to obtain the minimax result.

— $\mathbf{H_E}$: The sets E_1 and E_2 are compact and convex.

— $\mathbf{H_f}$: The function f satisfies the following properties :

$$\forall y \in E_2 \quad x \longmapsto f(x, y) \quad \text{is convex and continuous.}$$

$$\forall x \in E_1 \qquad y \longmapsto f(x, y) \quad \text{is concave and continuous.}$$

We can slightly weaken the assumption on the continuity of f and just handle semi-continuity of f. For the sake of simplicity, we only deal here with the continuous case. We can simply illustrate the payoff that players P_1 and P_2 could obtain in Figure 3.8.

Interpretation : The von Neumann result stands for the existence of an equilibrium (x^*, y^*) in this two-players zero-sum game in the following sense : if the player controlling the strategy x modifies his strategy when the player plays y^*, he increases his loss (what he is not supposed to like!) : it is thus his interest to play x^*. Similarly, if the player controlling the strategy y modifies his strategy when the first player plays x^*, the player diminishes his gain. This property of equilibrium of saddle points justifies their use as a reasonable solution in a two-person zero-sum game.

According to the assumptions above, we can state the general von Neumann theorem, whose proof is omitted here and can be found in [J.P. Aubin, Applied functional analysis, Chapter 2].

Theorem 3.5.8 (von Neumann's Theorem, Generic case) *Under* $\mathbf{H_E}$ *and* $\mathbf{H_f}$*, we get*

$$\beta = \sup_{y \in E_2} \inf_{x \in E_1} f(x, y) = \inf_{x \in E_1} \sup_{y \in E_2} f(x, y) = \alpha$$

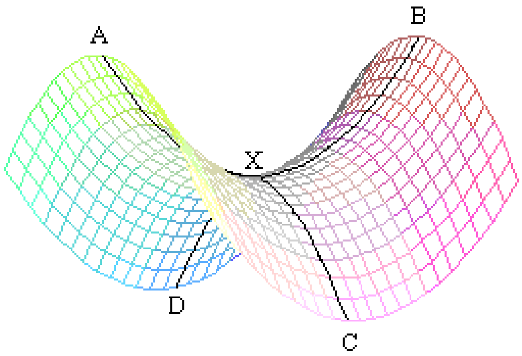


FIGURE 3.8: Illustration of the payoff f : convex when y is fixed, concave when x is fixed.

3.5.6.2 Mixed strategies

We first describe the subcase of mixed strategies of the von Neumann result. We now imagine that two persons are playing a zero-sum game with loss $-f(i,j)$ for player 1 and reward $f(i,j)$ for player 2 and the set of strategies E_1 and E_2 is **finite** (and no longer convex). In this case, the Max-Min inequality is always true

$$\max_j \min_i f(i,j) \leq \min_i \max_j f(i,j),$$

but the converse inequality is false in general. For example, imagine that f is given as follows

P1 / P2	Strategy 1	Strategy 2	Strategy 3
Strategy 1	0	1	-1
Strategy 2	-1	0	1
Strategy 3	1	-1	0

It is an easy exercice to check that

$$\max_j \min_i f(ij) = -1 \qquad \text{although} \qquad \min_i \max_j f(ij) = 1.$$

Hence, in this example, we can remark that there does not necessarily exist an optimal strategy for the second player : $\max_j \min_i f(i,j)$ is the best reward the second player could expect if the first player discover its choice. The mixed situation occurs when now the choice of each player becomes randomized with a probability distribution chosen by each player. Let us denote p_1 and p_2 two probability distributions for the players 1 and 2. The expected loss of the first player is then

$$\mathbb{E}_{I\sim p_1, J\sim p_2} f(I,J)$$

where I and J are *independent* random variables sampled with p_1 and p_2. We can make the definition of the mean reward more explicit :

$$F(p_1,p_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_1(i) p_2(j) f(i,j). \tag{3.9}$$

We will show below the von Neumann minimax Theorem, that proves the existence of an equilibrium in the mixed strategies chosen by the two players, which is a saddle point of F. The two players must then choose each one a precise couple of strategies (p_1^*, p_2^*) if they do not want to "loose something".

Theorem 3.5.9 (von Neumann mixed strategy minimax Theorem) *For any loss/reward function in a zero-sum two-player games, we have*

$$\max_{p_2} \min_{p_1} F(p_1, p_2) = \min_{p_1} \max_{p_2} F(p_1, p_2) := F(p_1^*, p_2^*)$$

We present here a proof of Jean Ville (1938), that greatly simplifies the initial proof of von Neumann and uses the separation theorem of convex sets, in the context of mixed strategies. The cornerstone of the proof is Corollary 3.5.3, which may be view as an application of the Hahn-Banach Theorem. We begin by the following lemma.

Lemma 3.5.2 *Let $(f_i)_{1 \leq i \leq p}$ be linear forms on $\mathbb{R}^n$ such that*

$$\forall x \geq 0 \quad \exists i \in \{1, \ldots, p\} \qquad f_i(x) := \langle f_i, x \rangle \geq 0.$$

Then, we can find a convex combination of the linear forms $(f_i)_{1 \leq i \leq p}$ such that

$$\forall x \geq 0 \qquad f(x) := \langle f, x \rangle \geq 0.$$

Note that a convex combination is

$$f = \sum_{i=1}^{p} \lambda_i f_i \qquad \text{such that} \qquad \sum_{i=1}^{p} \lambda_i = 1 \qquad \text{with} \qquad \lambda_i \geq 0, \forall i \in \{1, \ldots, p\}.$$

At last, remind that a linear form is simply an element of the dual of $\mathbb{R}^n$. In our simple case here, a linear form $x \longmapsto f_i(x)$ is just described by a vector of $\mathbb{R}^n$, still denoted f_i here.
Proof : We use a contradictory proof and assume that such a linear combination does not exist. We define C_1 as the convex enveloppe of the elements $(f_i)_{1 \leq i \leq p}$.

$$C_1 := Conv((f_i)_{1 \leq i \leq p}) := \left\{ \sum_{i=1}^{p} \lambda_i f_i : \lambda_i \geq 0, \forall i \in \{1, \ldots, p\} \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i = 1 \right\}$$

In the meantime, define also C_2 as

$$C_2 := \{ f \in \mathbb{R}^n : \forall x \geq 0 \quad \langle f, x \rangle \geq 0 \}.$$

It is immediate to check that C_1 and C_2 are closed convex sets of $\mathbb{R}^n$. If the conclusion of the lemma is false, then C_1 and C_2 are disjoints and Corollary 3.5.3 then implies that we can find a linear form that separates C_1 and C_2, meaning that a vector $x^* \in \mathbb{R}^n$ exists such that

$$\forall f \in C_1 \quad \langle f, x^* \rangle < \alpha \quad \text{and} \quad \forall f \in C_2 \quad \langle f, x^* \rangle > \alpha.$$

Since $\{0_{\mathbb{R}^n}\} \in C_2$, we see that $\alpha < 0$. Moreover, for all integers i, the vector λe_i (whose value is 0 at each coordinate, except at the i-th one with value λ) is in C_2 as soon as $\lambda \geq 0$.

Now, for any vector $y \notin \{\mathbb{R}_+\}^n$ and an integer i such that $e_i(y) = y_i < 0$, we can find a sufficiently large λ such that $\lambda e_i(y) < \alpha$, meaning that necessarily $x^* \in \{\mathbb{R}_+\}^n$. We then have found x^* such that

$$x^* \in \{\mathbb{R}_+\}^n \quad \text{and} \quad \forall f \in C_1 \qquad f(x) < \alpha < 0.$$

But of course, the initial linear forms $(f_i)_{1 \leq i \leq p}$ belongs to C_1 (as a trivial convex combination) and we have obtained

$$\forall j \in \{1 \leq i \leq p\} \qquad f_j(x^*) < \alpha < 0.$$

This is a contradiction with our assumption, and the Lemma is shown. $\square$

We can state an easy consequence by applying the Lemma above to $f_i - \phi$ and obtain the next result.

Proposition 3.5.6 *Let be given a linear form $\phi : \mathbb{R}^n \longrightarrow +\infty$ and a family of p linear forms $(f_i)_{1\leq i\leq p}$ such that*

$$\forall x \geq 0 \quad \exists i_x : f_i(x) \geq \phi(x),$$

then a convex combination $\psi = \lambda_1 f_1 + \ldots + \lambda_p f_p$ exists such that $\psi \geq \phi$ (everywhere).

It is now time to prove the main result of this paragraph.

Proof : [of Theorem 3.5.9] We prove the result with a potential number of choices for the two players that may be different : $n_1 \neq n_2$. Define $\mathcal{P}_1$ the set of probability distributions on the finite set $\{1, \ldots, n_1\}$ and $\mathcal{P}_2$ the set of probability distributions on the finite set $\{1, \ldots, n_2\}$. These two sets are closed and convex.

Recall the of F given in (3.9) by

$$\forall p_1 \in \mathcal{P}_1 \quad \forall p_2 \in \mathcal{P}_2 \qquad F(p_1, p_2) = \sum_{i=1}^{n}\sum_{j=1}^{n} p_1(i)p_2(j)f(i,j)$$

and denote $g_j(p_1) = \sum_{i=1}^{n} p_1(i)f(i,j)$. $g_j(p_1)$ refers to the expected loss of the first player, following a strategy given by p_1, when the second player chooses j. The important fact is that g_j is a linear form from $\mathcal{P}_1$ to $\mathbb{R}$.

The Fubini theorem leads to

$$F(p_1, p_2) = \sum_{j=1}^{n} p_2(j)g_j(p_1).$$

It is immediate to check that

$$\max_{p_2\in\mathcal{P}_2} F(p_1, p_2) = \max_{1\leq j\leq n} g_j(p_1), \tag{3.10}$$

which means that if the player 2 knew the strategy p_1 of the first player, it is optimal to choose a "Dirac" measure as a strategy. If $\alpha = \min_{p_1\in\mathcal{P}_1} \max_{p_2\in\mathcal{P}_2} F(p_1, p_2)$, Equation (3.10) shows that α satisfies :

$$\forall p_1 \in \mathcal{P}_1 \quad \alpha \leq \max_{p_2\in\mathcal{P}_2} F(p_1, p_2) = \max_{1\leq j\leq n} g_j(p_1).$$

Therefore, we have

$$\forall p_1 \in \mathcal{P}_1 \quad \exists j \in \{1 \ldots n_2\} \quad \text{such that} \quad g_j(p_1) \geq \alpha = \alpha \sum_{i=1}^{n} p_1(i) = \alpha\phi(p_1),$$

where ϕ is the linear form of $\mathcal{P}_1$ defined by $\phi(p) = p(1) + \ldots + p(n_1)$. We can apply Proposition 3.5.6 and obtain the existence of $\lambda_1, \ldots, \lambda_{n_2}$ such that

$$\lambda_1 g_1 + \ldots + \lambda_{n_2} g_{n_2} \geq \phi.$$

As a convex set of coefficients, we identify $p_2^* := (\lambda_1, \ldots, \lambda_{n_2})$ as a probability distribution on $\mathcal{P}_2$. We have shown

$$F(p_1, p_2^*) \geq \alpha.$$

This inequality being true whatever p_1 is, we deduce that

$$\min_{p_1\in\mathcal{P}_1} F(p_1, p_2^*) \geq \alpha,$$

which in turn implies that

$$\max_{p_2 \in \mathcal{P}_2} \min_{p_1 \in \mathcal{P}_1} F(p_1, p_2^*) \geq \alpha.$$

This ends the proof. $\square$

The minimax von Neumann theorem can also be demonstrated with the help of some fixed point theorem or duality of linear optimization. It should be understood as the existence of a uniquer saddle point of the function f with gap-free duality. We have chosen to present the proof with a convex separation approach instead.

3.5.7 Characterisation of the Pareto optima

Consider now an n-person situation described when we are given a set of available strategies X for the n-uple of players, and some loss functions of the player $i : x \in X \longmapsto f_i(x) \in \mathbb{R}, i = 1 \ldots, n$ associated to each strategy $x \in X$. These loss functions permit to define a partial pre-ordering for an n-uple :

$$\forall (x, y) \in X \times X \qquad x \lesssim y \Longleftrightarrow \forall i \in \{1 \ldots n\} \quad f_i(x) \leq f_i(y)$$

The purpose of Pareto's optima is to distinguish the minimal elements for this pre-ordering.

Definition 3.5.6 (Pareto's minimum) *We shall say that x^* is a weak Pareto minimum if there exists no other strategy $y \in X$ such that*

$$y \lesssim x^*.$$

Here, the convex aggregation of each loss function will play a central role. It is possible to select a Pareto minimum by minimizing on X a convex combination f_λ of the initial loss functions :

$$f_\lambda(x) := \sum_{i=1}^{n} \lambda_i f_i(x).$$

We want to decipher the relationship between weak Pareto minimizer and minimizer of a convex combination of loss functions. A first link is given below.

Proposition 3.5.7 *If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \{\mathbb{R}_+^*\}^n$ is a set of convex coefficients, then the following assertion holds :*

$$x^* \in \arg\min f_\lambda \Longrightarrow x^* \quad \textit{is weakly Pareto optimal}$$

Proof : Consider a convex set of coefficients $\lambda = (\lambda_1, \ldots, \lambda_n)$ and its associated convex aggregation f_λ. We assume that x^* minimizes f_λ that is not a weak Pareto minimum. From Definition 3.5.6, it means that a strategy y exists such that

$$\forall i \in \{1 \ldots n\} \quad f_i(y) < f_i(x^*).$$

We deduce that in this case, $f_\lambda(y) < f_\lambda(x^*)$, which contradicts the definition of x^*. $\square$

The reverse implication would be much more interesting. We will show that appropriate convexity hypotheses imply the reverse implication, namely, that every weak Pareto minimum can be obtained by minimizing a suitable loss function on X.

Theorem 3.5.10 *Assume that*

— *X is a convex subset of a vector space H*
— *each loss functions f_i is convex.*

Then

$$x^* \quad \textit{is weakly Pareto optimal} \Longrightarrow \exists \lambda \in Conv(\{\mathbb{R}_+^*\}^n) \quad x^* \in \arg\min f_\lambda.$$

<u>Proof :</u> The proof is splitted in three parts.

— Let us define $\phi : x \in X \longmapsto \{f_1(x), \dots, f_n(x)\}$ and the set of vectors

$$\vec{\phi}(X)^+ := \phi(X) + \{\mathbb{R}_+^*\}^n := \left\{\phi(x) + u \,:\, x \in X, u \in \{\mathbb{R}_+^*\}^n\right\}$$

It is easy to check that $\phi(y) \in \vec{\phi}(X)^+$ if and only if there exists $a \in X$ such that $f_i(y) > f_i(a)$, forall i, if and only if y is not weakly Pareto optimal.
Hence, if x^* is weakly Pareto optimal, then $\phi(x^*) \notin \vec{\phi}(X)^+$.

— The set $\vec{\phi}(X)^+$ is convex : consider two elements y_1, y_2 in $\vec{\phi}(X)^+$ and $\alpha \in [0,1]$, then

$$\exists (x_1, x_2) \in X^2,\ \exists (u_1, u_2) \in \{\mathbb{R}_+^*\}^n \times \{\mathbb{R}_+^*\}^n \qquad y_1 = \phi(x_1) + u_1 \qquad \text{and} \qquad y_2 = \phi(x_2) + u_2$$

Then, we obtain that

$$\alpha y_1 + (1-\alpha) y_2 = \alpha \phi(x_1) + (1-\alpha)\phi(x_2) + \alpha u_1 + (1-\alpha) u_2.$$

Since $\phi = (f_1, \dots, f_n)$, each coordinate of ϕ is a convex function. It leads to

$$\phi\left[\alpha x_1 + (1-\alpha) x_2\right] \lesssim \alpha \phi(x_1) + (1-\alpha)\phi(x_2)$$

A simple consequence is

$$\begin{aligned}
\alpha y_1 + (1-\alpha) y_2 \quad &= \quad \phi\left[\alpha x_1 + (1-\alpha) x_2\right] \\
& \quad + \underbrace{\underbrace{(\alpha u_1 + (1-\alpha) u_2)}_{\text{belongs to}\{\mathbb{R}_+^*\}^n} + \underbrace{\left[\alpha \phi(x_1) + (1-\alpha)\phi(x_2) - \phi\left[\alpha x_1 + (1-\alpha) x_2\right]\right]}_{\text{belongs to}\{\mathbb{R}_+^*\}^n}}_{\text{belongs to}\{\mathbb{R}_+^*\}^n}
\end{aligned}$$

This proves that $\alpha y_1 + (1-\alpha) y_2 \in \vec{\phi}(X)^+$, and $\vec{\phi}(X)^+$ is thus a convex set.

— We will show that a vector λ^* exists such that the conclusion of the Theorem 3.5.10 holds. Remark that the two first points show that $\phi(x^*)$ does not belong to the convex set $\vec{\phi}(X)^+$. The separation result given in Theorem 2.1.3 applied to $x = \phi(x^*)$ and $A = \vec{\phi}(X)^+$ states that a linear form given by $\langle \lambda^*, . \rangle$ exists such that

$$\begin{aligned}
\langle \lambda^*, \phi(x^*) \rangle \quad &\leq \quad \inf_{v \in \vec{\phi}(X)^+} \langle \lambda^*, v \rangle && (3.11) \\
&= \quad \inf_{x \in X, u \in \{\mathbb{R}_+^*\}^n} \left[\langle \lambda^*, \phi(x) \rangle + \langle \lambda^*, u \rangle\right] \\
&= \quad \inf_{x \in X} \langle \lambda^*, \phi(x) \rangle + \inf_{u \in \{\mathbb{R}_+^*\}^n} \langle \lambda^*, u \rangle && (3.12)
\end{aligned}$$

Since $\inf_{x \in X} \langle \lambda^*, \phi(x) \rangle \leq \langle \lambda^*, \phi(x^*) \rangle$, Inequality (3.12) implies that

$$\inf_{u \in \{\mathbb{R}_+^*\}^n} \langle \lambda^*, u \rangle \geq 0$$

But we can also write $\inf_{u \in \{\mathbb{R}_+^*\}^n} \langle \lambda^*, u \rangle$ as

$$\inf_{u \in \{\mathbb{R}_+^*\}^n} \langle \lambda^*, u \rangle = \inf_{u_1 > 0 \dots, u_n > 0} \left(\sum_{i=1}^n \lambda_i^* u_i\right) = \sum_{i=1}^n \inf_{u_i > 0} \lambda_i^* u_i \geq 0$$

Hence, all real values λ_i^* are all nonnegative (otherwise the infimum would be $-\infty$) and

$$\inf_{u \in \{\mathbb{R}_+^*\}^n} \langle \lambda^*, u \rangle = 0.$$

From the inequality (3.11), we obtain that $\lambda^* \neq 0_{\mathbb{R}^n}$ and thus

$$\|\lambda^*\|_1 = \sum_{i=1}^{n} |\lambda_i^*| = \sum_{i=1}^{n} \lambda_i^* > 0.$$

We now consider the vector of $\mathbb{R}^n$:

$$\tilde{\lambda} := \frac{\lambda^*}{\|\lambda^*\|_1}.$$

The great advantage of this normalized vector is that $\tilde{\lambda} \in \{\mathbb{R}_+\}^n$ and

$$\sum_{i=1}^{n} \tilde{\lambda}_i = \sum_{i=1}^{n} \frac{\lambda_i^*}{\|\lambda^*\|_1} = \frac{\|\lambda^*\|_1}{\|\lambda^*\|_1} = 1,$$

meaning that one can build a convex combination with the help of the set of coefficients $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$. We use again (3.12) to obtain

$$\begin{array}{rll} & \langle \lambda^*, \phi(x^*) \rangle & \leq \inf_{x \in X} \langle \lambda^*, \phi(x) \rangle \\ \iff & \langle \tilde{\lambda}, \phi(x^*) \rangle & \leq \inf_{x \in X} \langle \tilde{\lambda}*, \phi(x) \rangle \\ \iff & \sum_{i=1}^{n} \tilde{\lambda}_i f_i(x^*) & \leq \inf_{x \in X} \displaystyle\sum_{i=1}^{n} \tilde{\lambda}_i f_i(x) \\ \iff & x^* \in \arg\min f_{\tilde{\lambda}}. & \end{array}$$

This last point ends the proof of the theorem. □

Let us briefly make two remarks. The cooperarice concepts of a solution in n-person games consist of defining selection processes for Pareto minima. We shall briefly describe a procedure for selecting a Pareto minimum. Let us denote by

$$\alpha_i := \inf_{x \in X} f_i(x),$$

which represents the minimal loss of the $i - th$ player (when he is the only one who plays). Suppose that we are given a strategy $x_0 \in X$ such that

$$f_i(x_0) > \alpha_i, \forall i \in \{1 \ldots, n\}.$$

The following quantity measures the maximum size of relative losses of the players yielded by the strategy x :

$$u(x) := \max_{1 \leq i \leq n} \frac{f_i(x) - \alpha_i}{f_i(x_0) - \alpha_i}.$$

We can therefore build a weak Pareto minimum with the help of the next proposition.

Proposition 3.5.8 *Let $d := \inf_{x \in X} u(x)$, then x^* minimizes u on X iff*

$$\forall i \in \{1, \ldots, n\} \quad f_i(x^*) \leq (1 - d)\alpha_i + d f_i(x_0).$$

Moreover, x^* is a weak Pareto minimum if X is compact and the functions $(f_i)_{1 \leq i \leq n}$ are continuous.

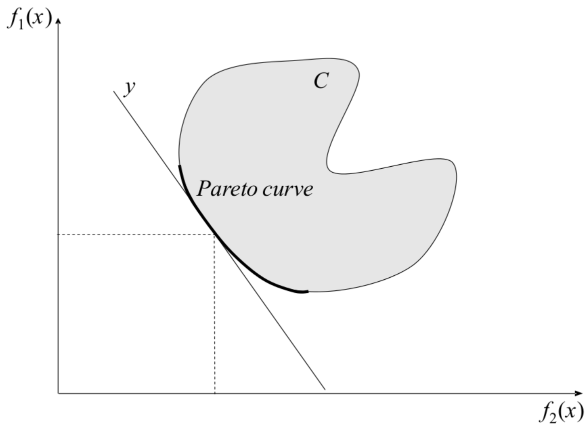


FIGURE 3.9: Illustration of a possible set of weak Pareto minima (as a Pareto curve in bold line). The point selected in bottom left is more or less the one built through the proof of Theorem 3.5.10.

Proof : Left to the reader. □

Of course, this selection depends (only) on the initial strategy x_0 chosen in X. Remark that in all this paragraph, nothing has been said about the uniqueness of Pareto weak minimum, which is generally false. The set of Pareto weak minima is often a curve in X, as examplified by Figure 3.9

Chapitre 4

Fixed point theorems

The chapter is splitted in two parts. Section 4.1 presents the theoretical fixed point theorems and pays a particular attention to the several assumptions needed to obtain these results (finite dimensional space, compactness, ...). Section 4.2 is maybe much more interesting and proposes some examples in several fields (game theory, economics, probability theory, Markov chains or ordinary differential equation).

4.1 Fixed point theorems

The following results are stated for functions that act on a normed vector space $f : E \longrightarrow E$.

4.1.1 Banach-Picard's Theorem

Theorem 4.1.1 (Banach-Picard) *Let be given a Banach space* $(E, \|.\|)$, *A a closed set of E and f a contracting map of A, meaning that :*

$$\exists k \in (0,1) \quad \forall (x,y) \in A^2 \qquad \|f(x) - f(y)\| \leq k\|x - y\|,$$

Then f admits a unique fixed point in A :

$$\exists !\, x \in E \qquad f(x) = x.$$

The assumptions above are absolutely necessary : for example consider $A = (0,1)$ and $f(x) = x/2$, the function f is $1/2$ Lipschitz and thus is a contracting map of A. But the unique fixed point of f is $0 \notin (0,1)$. Similarly, if $A = [0, +\infty[$ define $f(x) = \sqrt{1+x^2}$, we then have

$$|f(x) - f(y)| \leq k_{x,y}|x - y| \qquad \text{with} \qquad 0 \leq k_{x,y} \leq 1.$$

It means that the contracting coefficient $k_{x,y}$ is each time stricly lower than 1 but tends to 1 as x and y are going to infinity. Hence, this coefficient cannot be upper bounded by a number strictly lower than 1 : the function f is not k-Lipschitz with $k < 1$ *independent* from x and y. In that case, the Banach-Picard Theorem does not hold and we can check easily that indeed f does not possess any fixed point :

$$x^* = \sqrt{1 + \{x^*\}^2} \Longrightarrow \{x^*\}^2 = 1 + \{x^*\}^2 \Longrightarrow 0 = 1 \ldots$$

We are now driven to the proof of Theorem 4.1.1.
Proof : Existence : Let be given any $x_0 \in A$, we build the recursive sequence

$$\forall n \in \mathbb{N} \qquad x_{n+1} = f(x_n).$$

Since $f : A \longrightarrow A$, it is immediate to check that x_n belongs to A for all n. Moreover, we can write

$$\|x_n - x_{n+1}\| = \|f(x_{n-1}) - f(x_n)\| \leq k\|x_{n-1} - x_n\| \ldots \leq k^n\|x_0 - x_1\|.$$

Consequently, the triangle inequality yields

$$\begin{aligned}
\forall (n,p) \in \mathbb{N} \qquad \|x_n - x_{n+p}\| &\leq \sum_{k=0}^{p-1} \|x_{n+k} - x_{n+k+1}\| \\
&\leq \left(\sum_{k=0}^{p-1} k^{n+j}\right) \|x_0 - x_1\| \\
&= \frac{1-k^p}{1-k}\|x_0, x_1\| \\
&\leq \frac{k^n}{1-k}\|x_0 - x_1\|
\end{aligned}$$

This last upper bound trivially implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and since A is a Banach space, we deduce the convergence of this sequence towards x^* in A.

Now, f is a continuous map (as a contracting map) and we obtain

$$\lim_{n\longrightarrow+\infty} \underbrace{f(x_n)}_{:=x_{n+1}} = f(x^*),$$

leading to

$$x^* = f(x^*).$$

We conclude that one fixed point of f exists in A.
Uniqueness : Assume that two points (x_1^*, x_2^*) are fixed points of f. We can write :

$$\|x_1^* - x_2^*\| = \|f(x_1^*) - f(x_2^*)\| \leq k\|x_1^* - x_2^*\|,$$

meaning that

$$(1-k)\|x_1^* - x_2^*\| \leq 0.$$

Since $k \in (0,1)$, it yields $x_1^* = x_2^*$.

We now study the uniqueness property. □

Many generalizations of this theorem exist, some of them are useful in the Probability theory field. In particular, we state the next consequence of the Banach-Picard Theorem. In what follows, the notation f^p refers to $\underbrace{f \circ f \circ \ldots \circ f}_{p\,\text{times}}$

Corollary 4.1.1 *Let be given A a closed subset of a Banach space $(E, \|.\|)$ and $f : A \to A$ such that an integer p exists with f^p is a contracting map. Then f admits a unique fixed point.*

Proof : We first apply the result of the Banach-Picard to the application f^p, which is a contraction. Hence, a unique fixed point x^* exists for f^p.

Then, remark that any fixed point of f is necessarily a fixed point of f^p and thus if a fixed point of f exists, he is necessarily unique. Let us denote $y = f(x^*)$, we have

$$f^p(y) = f^{p+1}(x^*) = f(f^p(x^*)) = f(x^*) = y,$$

which shows that y is also a fixed point of f^p. From the uniqueness property of the fixed point of f^p, we conclude that $y = x^*$, *i.e.*

$$x^* = f(x^*).$$

□

4.1.2 Brouwer's Theorem

The Brouwer Theorem is a very strong result of Functional Analysis, the assumption are clearly weaker than the assumptions of the Banach-Picard Theorem (the involved map is not yet a contracting map). The important loss of the result is the uniqueness, which is not true in the Brouwer Theorem. This theorem possesses many applications, among them the famous Nash Theorem in Games Theory, the Perron-Frobenius Theorem in linear algebra and Markov chains, some funny examples (Salop and Hotelling models rely on simple applications of Brouwer's result). We refer to Section 4.4.2 for examples.

We begin by the statement of the main result. In what follows, we will use the notation B_n for the unit euclidean ball and S^{n-1} for the unit euclidean sphere of $\mathbb{R}^n$.

Theorem 4.1.2 (Brouwer's Theorem - Unit ball) *Every continuous function $f : B_n \longrightarrow B_n$ has a fixed point.*

We can illustrate the Brouwer Theorem through the historical example that made it possible for Luitzen Egbertus Jan Brouwer to suspect the result to be true. The example related to the cup of coffee movement is shown in Figure 4.1. [1]

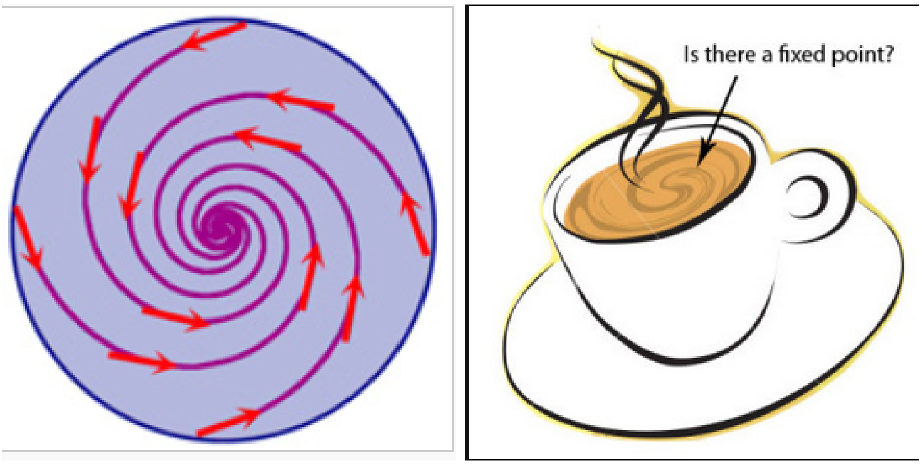


FIGURE 4.1: The cup of coffee example of function f of the unit ball with a fixed point. When turning the coffee with a spoon, one point at the least seems immobile..

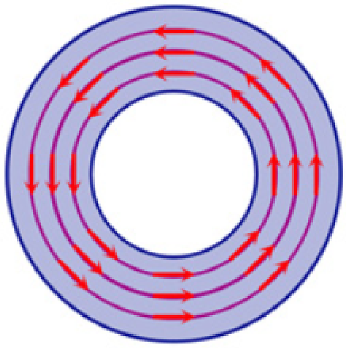
Of course, the geometry of the ball is important, and the result is no longer true for other examples of domains as pointed by Figure 4.2.

But this result can be extended to various domains of $\mathbb{R}_n$ as follows.

Corollary 4.1.2 (Brouwer's Theorem - Convex compact set) *For any convex compact set of $\mathbb{R}_n$ denoted C, if $f : C \longrightarrow C$ is a continuous map, then f admits a fixed point.*

Proof : [of Corollary 4.1.2] First, remark that C is bounded and can be included in an euclidean ball with a radius sufficiently large $R > 0$. Then, considering the convex compact set $C^R := \{x/R : x \in C\}$ and the auxiliary map $g_R : x \in C^R \longmapsto g(Rx)/R \in C^R$, we see that a fixed point of g_R leads to a fixed point for g. Hence, without loss of generality, we can assume that $R = 1$, *i.e.* $C \subset B_n$.

1. L.E.J. Brouwer, proving Alfred Renyi's statement, "A mathematician is a machine for turning coffee into theorems".



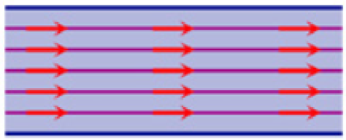


FIGURE 4.2: The Brouwer Theorem does not hold if the domain is unbounded or possesses some hole..

Let be given a continuous map $g : C \longrightarrow C$ and define π as the projection of $\mathbb{R}^n$ on the euclidean ball B_n. Since C is a convex compact set, then π is a continuous map and $f = g \circ \pi$ is continuous from B_n to B_n. Hence, Theorem 4.1.2 implies that f has a fixed point :

$$\exists x^* \in B_n \qquad f(x^*) = x^*.$$

Now, remark that $\pi(x^*) \in C$ and $g(\pi(x^*)) \in C$ since $g : C \longrightarrow C$. An easy consequence of the equality $f(x^*) = x^*$ is that in fact

$$x^* \in C,$$

meaning that $\pi(x^*) = x^*$. Hence, $g(x^*) = x^*$ and x^* is a fixed point of g. □

Before the proof of Theorem 4.1.2, we first show that it is enough to establish the Brouwer Theorem for smooth functions of $\mathcal{C}^\infty(B_n, B_n)$. This is given by the next Lemma.

Lemma 4.1.1 *Assume that every function of $\mathcal{C}^\infty(B_n, B_n)$ has a fixed point, then any function of $\mathcal{C}^0(B_n, B_n)$ has also a fixed point.*

Proof : We consider a function $g \in \mathcal{C}^0(B_n, B_n)$ and approximate g by a sequence of polynomials such that

$$\forall k \in \mathbb{N} \qquad \|g - P_k\|_\infty \leq \frac{1}{k}$$

If $g : B_n \longrightarrow B_n$, it is not true that $P_k : B_n \longmapsto B_n$. However, we can easily modify P_k to do so by considering :

$$Q_k = (1 + 1/k)^{-1} P_k,$$

since

$$\begin{array}{lcl} \forall x \in B_n \qquad |Q_k(x)| & = & (1 + 1/k)^{-1} |P_k(x)| \\ & \leq & (1 + 1/k)^{-1} \left[|g(x)| + |P_k(x) - g(x)| \right] \\ & \leq & (1 + 1/k)^{-1} \left[1 + 1/k \right] \leq 1 \end{array}$$

Moreover, Q_k is still an approximation of g :

$$\begin{aligned} \|Q_k - g\|_\infty &= \left\| \frac{k}{k+1} P_k - g \right\|_\infty \leq \left\| \frac{k(P_k - g)}{k+1} - \frac{g}{k+1} \right\|_\infty \\ &\leq \frac{k}{k+1} \|P_k - g\|_\infty + \|g\|_\infty (k+1)^{-1} \\ &\leq \frac{k}{k+1} \times \frac{1}{k} + \frac{1}{k+1} = \frac{2}{k+1}. \end{aligned}$$

We can apply now the assumption to the map $Q_k : B_n \longrightarrow B_n$ and show that a sequence $(x_k)_{k\in\mathbb{N}}$ exists such that

$$Q_k(x_k) = x_k.$$

We then obtain

$$|g(x_k) - x_k| \leq \frac{2}{k+1},$$

Consider now an extraction and a sequence $(x_{\varphi(k)})_{k\in\mathbb{N}}$ that converges to a limit point x^* in B_n (which is compact), it leads to

$$\exists\, x^* \in B_n \quad g(x^*) = x^*.$$

□

A second important ingredient for the proof of Theorem 4.1.2 is the no-retractation Lemma, which may be easily illustrated in dimension 2. The result seems quite logical : it is impossible to press the whole ball B_n on its boundary S^{n-1} in a differentiable way, without creating a "hole". Even if intuitive, the proof of the no-retractation of the ball to the sphere is the main difficulty of the proof of the Brouwer theorem.

Lemma 4.1.2 (No-retractation Lemma) *There is no $\mathcal{C}^1$ function $\phi : B_n \longrightarrow S^{n-1}$ such that $f(x) = x, \forall x \in S^{n-1}$.*

Proof :

Introduction of an auxiliary family of functions Assume that such a function f exists and define $(f_t)_{t\in[0,1]}$ as

$$\forall t \in [0,1] \quad f_t = tf + (1-t)Id_{B_n}.$$

This function f_t is differentiable for any $t \in [0,1]$, and denoting df_t denotes its derivative, we then know that $df_t(x)$ is the tangent linear map of f_t at point x.

Note that $f_0 = Id_{B_n}$, which is an invertible application. Moreover, a simple computation yields

$$\forall x \in B_n \qquad \|df_t(x) - Id_{B_n}(x)\| = t\|df(x) - Id_{B_n}(x)\| \leq t\left[\|df\|_\infty + 1\right],$$

where $\|df\|_\infty = \sup_{x\in B_n} \|df(x)\| < \infty$ since f is $\mathcal{C}^1$. Consequently, we have

$$\|df_t - Id\| \leq \frac{1}{2} \qquad \text{as soon as} \qquad 0 \leq t \leq \frac{1}{2(\|df\|_\infty + 1)}$$

Writing now

$$df_t = Id_{B_n} + df_t - Id_{B_n},$$

it is easy to check that df_t is invertible for $t \in \left[0, \frac{1}{2(\|df\|_\infty+1)}\right]$ and as a continuous function of t (for any $x \in B_n$) the determinant of $df_t(x)$ cannot change of sign (otherwise it would vanish and then df_t would be non invertible). Consequently, we have

$$\forall x \in B_n \quad \forall t \in \left[0, \frac{1}{2(\|df\|_\infty + 1)}\right] \qquad \det(df_t(x)) > 0$$

Diffeomorphisms of the unit ball The family of functions $(f_t)_{t\in[0,1]}$ are indeed smooth diffeomorhisms of the unit ball when t is small enough. This can be seen as follows.

Pick $t \in \left[0, \frac{1}{2(\|df\|_\infty+1)}\right]$ and assume that $f_t(x) = f_t(y)$ and write that

$$\forall (x,y) \in B_n^2 \qquad \|x-y\| \leq \|(f_t(x)-x)-(f_t(y)-y)\| \leq \frac{1}{2}\|x-y\|,$$

where the last inequality comes from the mean-value Theorem applied to $x \longmapsto f_t - Id_{B_n}$ which is differentiable and whose derivative has a norm lower than $1/2$ when $t \in \left[0, \frac{1}{2(\|df\|_\infty+1)}\right]$. This shows that $x = y$ and f_t is injective when $t \in \left[0, \frac{1}{2(\|df\|_\infty+1)}\right]$.

To show that f_t is surjective, we study $f_t(\mathring{B}_n)$. First, $f_t(\mathring{B}_n)$ is an open set of $\mathring{B}_n$ (since f_t is locally invertible). The important point is that $f_t(\mathring{B}_n)$ is also closed in $\mathring{B}_n$. Consider $(y_n)_{n\in\mathbb{N}} \in f_t(\mathring{B}_n)$ such that $y_n \longrightarrow y^* \in \mathring{B}_n$, we need to show that $y^* \in f_t(\mathring{B}_n)$.

A sequence $(x_n)_{n\in\mathbb{N}}$ exists such that $y_n = f_t(x_n)$. Since $x_n \in \mathring{B}_n \subset B_n$, we can extract a convergent subsequence $x_{\varphi(n)} \longrightarrow x^*$. Obviously, we have

$$y^* = f_t(x^*).$$

It remains to show that $x^* \in \mathring{B}_n$, otherwise $x^* \in S^{n-1}$ and $f_t(x^*) = tf(x^*) + (1-t)x^* = x^* \in S^{n-1}$. It is impossible since $y^* \notin S^{n-1}$. Consequently, $x^* \in \mathring{B}_n$ and $y^* \in f_t(\mathring{B}_n)$.

As a non empty closed an open set of $\mathring{B}_n$, we then deduce that $f_t(\mathring{B}_n) = \mathring{B}_n$ and f_t is a surjection of $\mathring{B}_n$. Since $f_t(x) = x$ for any $x \in S^{n-1}$, we conclude that $f_t(B_n) = B_n$.

Conclusion We introduce

$$\forall t \in [0,1] \quad k(t) = \int_{B_n} \det(df_t(x))dx.$$

Since $t \longmapsto f_t$ is a degree 1 polynomial in t (in the space of applications), it is easy to see that it is still the case for $t \longmapsto df_t(x)$ (in the space of $n \times n$ matrices), whenever x is fixed in B_n. Hence, while integrating over B_n, we deduce that the application $t \longmapsto k(t)$ is a degree n real polynomial.

Moreover, f_t is a $\mathcal{C}^1$ diffeomorphism of the unit ball B_n and a simple change of variable $x \longmapsto f_t(x)$ leads to

$$Vol(B_n) = \int_{B_n} \mathbf{1}dy = \int_{f_t^{-1}(B_n)} \mathbf{1}(f_t(x))|\det(df_t(x))|dx = \int_{B_n} |\det(df_t(x))|dx = k(t),$$

since the determinant is always positive.

We can now conclude that k is a real polynomial function of t, which is constant on the interval $\left[0, \frac{1}{2(\|df\|_\infty+1)}\right]$. Hence, k is a constant function of t, whatever t is. In particular

$$k(0) = k(1) = \int_{B_n} \det(df(x))dx$$

but our baseline assumption on f yields that

$$\forall x \in B_n \qquad \|f(x)\|^2 = 1.$$

Differentiating this relation with respect to x shows that

$$\forall x \in B_n \qquad df(x)(f(x)) = 0 \Longrightarrow \det(df(x)) = 0.$$

Hence, $k(1) = 0 \neq k(0)$. We conclude that a such retractation does not exist. $\square$

We are now driven to the proof of the Brouwer theorem.

Proof : [Of Theorem 4.1.2] The idea is relatively simple! From Lemma 4.1.1, we it is enough to establish the result for any function f of $\mathcal{C}^\infty(B_n)$. Hence, consider $f \in \mathcal{C}^\infty(B_n)$ that does not possess any fixed point and build F as shown in Figure 4.3. Since $x \neq f(x)$ for all x, we can define the line $\mathcal{D}_x$ passing by x and $f(x)$ and $F(x)$ is the closest point of x on the line $\mathcal{D}_x$ that intersect the sphere S^{n-1}.

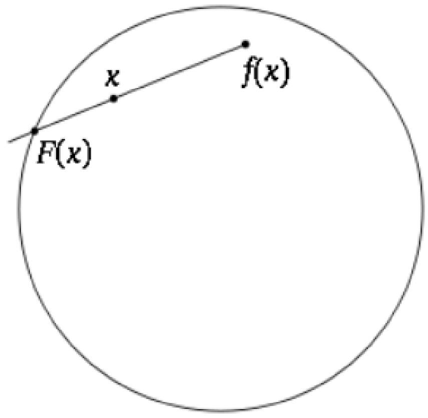


FIGURE 4.3: The construction of F from the continuous function f.

It is an easy exercice to check that F is a $\mathcal{C}^1$ application from B_n to S^{n-1} and

$$\forall x \in S^{n-1} \qquad F(x) = x.$$

Hence, F is a smooth retractation of the unit ball to the unit sphere. Such a function does not exist, meaning that f possesses a fixed point. $\square$

4.1.3 Schauder's Theorem

The Schauder Theorem extends the result of the Brouwer fixed point Theorem to the situation of infinite dimensional space. As usual, a common way to generalize a result from the finite dimensional case to the infinite dimensional situation is to handle convex compact sets in Banach spaces.

Theorem 4.1.3 (Schauder's Fixed Point Theorem) *Let E a Banach space and $K \subset E$ a convex and compact set. Then, every continuous function $f : K \longrightarrow K$ has a fixed point.*

Proof : First step : Compact covering

As a compact set, we can deduce that f is uniformly continuous on K (and not only continuous). Consequently, if we consider $\epsilon > 0$, a $\delta > 0$ exists such that

$$d(x, y) \leq \delta \Longrightarrow d(f(x), f(y)) \leq \epsilon.$$

In the meantime, we can apply the Borel Lebesgue property for this radius ϵ to the set K and find a finite covering :

$$K \subset \cup_{j=1}^k B(x_j, \delta)$$

In particular, we can introduce the vector space

$$L = span(f(x_j), 1 \leq j \leq k),$$

which is finite dimensional. This permits to consider the $\tilde{K} = K \cap L$, which is now a convex compact set included in a finite dimensional space.

Second step : Reduction to the finite dimensional case We aim to apply Brouwer's result to deduce the Schauder Theorem. This is not immediate and we have to first build some kind of "triangle" (or Schauder) function. These functions are quite similar to the ones introduced for the construction of the Brownian Motion, from a geometrical point of view at least. They are built as follows :

$$\forall j \in \{1, \ldots, k\} \qquad \phi_j(x) = \left[1 - \frac{\|x - x_j\|}{\delta}\right]_+,$$

where $[.]_+$ denotes the positive part symbol. Remark that in particular, each ϕ_j satisfies

$$\forall x \in B(x_j, \delta) \qquad \phi_j(x) > 0.$$

Since $(B(x_j, \delta))_{1 \leq j \leq k}$ is a δ-covering of K, we then deduce that

$$\forall x \in K \qquad \sum_{j=1}^{k} \phi_j(x) > 0,$$

and an artificial normalisation permits to define

$$\psi_j(x) = \frac{\phi_j(x)}{\sum_{i=1}^{k} \phi_i(x)},$$

so that

$$\forall x \in K \qquad \sum_{j=1}^{k} \psi_j(x) = 1.$$

We are naturally driven to approximate the initial function f using the normalized Schauder triangle basis :

$$g(x) := \sum_{j=1}^{k} \psi_j(x) f(x_j).$$

It is obvious to see that g is a continuous map. Moreover, each $f(x_j) \in \tilde{K}$ and for any $x \in K$, the set $(\psi_j(x))_{1 \leq j \leq k}$ forms a convex set of coefficients. From the convexity of $\tilde{K}$, we deduce that

$$\forall x \in K \qquad g(x) \in \tilde{K}.$$

Third step : Application of Brouwer's result We then apply the fixed point result of Brouwer to the continuous map $g_{|\tilde{K}}$ (the function g restricted to the set $\tilde{K}$). The set $\tilde{K}$ being convex, compact and included in a finite dimensional space, we then deduce that

$$\exists\, y \in \tilde{K} \qquad g(y) = y.$$

Now, we can compute

$$\begin{aligned} f(y) - y &= f(y) - g(y) \\ &= f(y) \left(\sum_{j=1}^{k} \psi_j(y) \right) - \left(\sum_{j=1}^{k} \psi_j(x) f(x_j) \right) \\ &= \sum_{j=1}^{k} \psi_j(y) \left[f(y) - f(x_j) \right] \end{aligned}$$

But remark that if $\psi_j(y) \neq 0$, then necessarily $y \in B(x_j, \delta)$, and the uniform continuity of f leads to

$$\psi_j(y) \neq 0 \Longrightarrow |f(y) - f(x_j)| \leq \epsilon.$$

Consequently, we get

$$|f(y) - y| \leq \sum_{j=1}^{k} \psi_j(y)|f(y) - f(x_j)| \leq \epsilon \sum_{j=1}^{k} \psi_j(y) = \epsilon.$$

Hence, for any $m \in \mathbb{N}$, we have found y_m in K such that

$$|f(y_m) - y_m| \leq 2^{-m}.$$

The compactness of K permits to extract a subsequence of $(y_m)_{m\geq 1}$ that converges towards $y^* \in K$, and the continuity of f then implies :

$$f(y^*) = y^*.$$

□

4.1.4 Kakutani's Theorem

This paragraph is now dedicated to the study of the existence of a fixed point for a *family* of applications (and not only one). Of course, to deal with this additional complexity, we have to impose a restriction on the set of admissible applications we will consider. The Kakutani Theorem involves a set of linear and continuous maps of a non-empty compact and convex set K of any normed vector space (not necessarily Banach). All the structural hypothesis is brought by the set K and the linearity assumption.

Theorem 4.1.4 (Kakutani's fixed point theorem) *Let E be a normed vector space and K a non-empty compact convex subset of E. Consider $F \subset \mathcal{L}_c(E)$ a subset of continuous and linear map of E and assume that F satisfies :*

— $\forall f \in F \qquad f(K) \subset K$

— $\forall (f,g) \in F^2 \qquad f \circ g = g \circ f$

Then, there exists a common fixed point for all the applications of F :

$$\exists\, x^* \in K \quad \forall f \in F \qquad f(x^*) = x^*.$$

Proof : We split the proof in three parts.

First part : singleton. We consider the case when F is reduced to a single application (denoted f in what follows). In that simple case, pick any $x_0 \in K$ and define the sequence

$$u_n(x_0) = \frac{1}{n} \sum_{k=0}^{n-1} f^k(x_0).$$

Remark that $u_n(x_0)$ is a convex combination of the points $f^k(x_0)$. Moreover, $f(K) \subset K$, which in turn implies that

$$\forall k \in \mathbb{N} \qquad f^k(x_0) \in K.$$

We thus deduce that $u_n(x_0) \in K$ since K is a convex set. Moreover, the linearity of f implies that

$$\|f(u_n(x_0)) - u_n(x_0)\| = \frac{\|f^n(x_0) - x_0\|}{n} \leq \frac{\delta(K)}{n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

As a compact set, K has in particular a finite diameter $\delta(K)$, justifying the above limit. Lastly, we can extract a convergent subsequence from $(u_n(x_0))_{n\geq 0}$, which necessarily converges towards a fixed point of f since f is continuous.
Second part : finite number of linear maps. We establish the result using an induction argument on the number of elements in F. Note that the first point shows that the result holds when $n = 1$. Assuming that the result is true until integer $n-1$, we consider the case where $F = \{f_1, \dots, f_n\}$. If we denote

$$\tilde{K} := \{x \in K : \forall j < n \, f_j(x) = x\} ,$$

we know that $\tilde{K}$ is compact (as a closed set of the compact set K). Moreover, $\tilde{K}$ is also convex owing to the linearity of the applications $(f_j)_{1\leq j\leq n-1}$.

A key remark is that $\tilde{K}$ is stable through the action of f_n, it is implied by the following equality :

$$\forall j \in \{1 \dots, n-1\} \quad \forall x \in \tilde{K} \qquad f_n(x) = f_n(f_j(x)) = f_n \circ f_j(x) = f_j \circ f_n(x),$$

leading to the fact that $f_n(x)$ is a fixed point of each f_j, meaning that $f_n(x) \in \tilde{K}$. Applying now the first point to the linear map f_n on the compact convex set $\tilde{K}$, we know that one point $x^* \in \tilde{K}$ exists such that $f_n(x^*) = x^*$. It yields

$$\exists x^* \in K \quad \forall j \in \{1, \dots, n\} \qquad f_j(x^*) = x^*.$$

Third part : extension to any family of linear maps.

For any $f \in F$, we introduce the notation

$$K_f := \{x \in K \, : f(x) = x\} .$$

We know that K_f is a closed convex set of K and the second point leads to

$$\forall F' \subset F \quad |F'| < \infty \Longrightarrow \cap_{f \in F'} K_f \neq \emptyset,$$

where $|F'|$ refers to the number of elements in the set F'.

But we know that K is compact, meaning that from every covering with open sets, we can extract a finite covering with open sets. Traducing this property with the complementary sets, it means that for any family of closed sets $G_i, i \in I$ whose intersection is empty, we have :

$$\cap_{i\in I} G_i = \emptyset \Longrightarrow \exists I' \subset I \, |I'| < \infty \quad \text{and} \quad \cap_{i \in I'} G_i = \emptyset.$$

Conversely, it implies that

$$\left\{\forall I' \subset I \, : \, |I'| < +\infty \, \Rightarrow \, \cap_{i\in I'} G_i \neq \emptyset\right\} \quad \Longrightarrow \quad \cap_{i\in I} G_i \neq \emptyset \tag{4.1}$$

Applying (4.1) to the closed sets K_f, we then deduce that

$$\cap_{f\in F} K_f \neq \emptyset$$

It then permits to find x in K such that $x \in K_f, \forall f \in F$. □

4.2 Applications to Ordinary Differential Equations

Many applications exist in applied mathematics and economics of the fixed point theorems stated above. We have chosen to detail some of them in O.D.E., P.D.E. and probability. Further details will be provided in the developments of the Game Theory lecture (next semester) by Jérôme Renault.

4.2.1 The Cauchy-Lipschitz Theorem

The first applications is concerned by the existence and uniqueness of solution of ordinary differential equations described by a Cauchy problem. Let be given a Banach space $(E, \|.\|)$ and a continuous function $f : \mathbb{R} \times E \longrightarrow E$, we are interested in the solutions of

$$(\mathcal{P}) : \qquad \dot{x}(t) = f(t, x(t)), \forall t \geq t_0 \qquad \text{with} \qquad x(t_0) = x_0 \in E.$$

We first make the following assumption on the smoothness of f.

Definition 4.2.1 (Local Lipschitz application) *A function $f(t, u)$ is local Lipschitz with respect to the variable u if and only if for all couple (t_0, u_0), a neighborhood V exists and a constant $k > 0$ such that*

$$\forall (t, u_1) \in V \forall (t, u_2) \in V \qquad \|f(t, u_1) - f(t, u_2)\| \leq k\|u_1 - u_2\|.$$

We can state the Cauchy-Lipschitz Theorem, which is an application of the Banach-Picard fixed point result.

Theorem 4.2.1 (Cauchy-Lipschitz Theorem) *Assume that f is local Lipschitz w.r.t. x and continuous, then $(\mathcal{P})$ has a unique local solution.*

Proof : Introduction of a key application φ.
We will see all along the proof the meaning of *unique local solution.* Without loss of generality, we can assume that $t_0 = 0$. Consider $x_0 \in E$, we know that an open neighborhood U of $(0, x_0)$ exists such that

$$\forall (t, x), (x, y) \in U^2 \qquad \|f(t, x) - f(t, y)\| \leq \|f(t, x) - f(t, x_0)\| + \|f(t, x_0) - f(t, y)\| \leq 2k\beta := M,$$

where β is the diameter of the second component of U and $M = 2k\beta$.

From now on, we consider a small parameter $h > 0$, that will be chosen later and the time window J refers to $]-h, h[$. Similarly, B will denote the open ball of E centered at point x_0 with radius β. We consider $\mathcal{F} := \mathcal{C}^0(J, \bar{B})$. It is easy to check that $(\mathcal{F}, \|.\|_\infty)$ is a metric space, where

$$\forall f \in \mathcal{F} \qquad \|f\|_\infty := \sup_{t \in J} \|f(t)\|.$$

Moreover, we have already seen that this kind of functional space is a Banach space (every Cauchy sequence of $\mathcal{F}$ converges in $\mathcal{F}$).

If we consider now the linear map ϕ such that

$$\forall f \in \mathcal{F} \qquad \phi(f) = f(0) \in E,$$

it is immediate to check that ψ is continuous so that $\mathcal{G} := \psi^{-1}(\{x_0\})$ is closed in $\mathcal{F}$ and thus complete.

We introduce now the key application $\varphi : u \in \mathcal{G} \longmapsto \varphi(u) \in \mathcal{G}$, defined as

$$\forall t \in J \qquad \varphi(u)(t) := x_0 + \int_0^t f(s, u(s))ds.$$

It is rather evident that for any choice of $u \in \mathcal{G}$, $\varphi(u)$ is a continuous function of t when $t \in J$. Moreover, $\varphi(u)(0) = x_0$. It remains to remark that $\forall t \in J$

$$\|\varphi(u)(t) - x_0\| \leq \int_0^t \|f(s, u(s))\|ds \leq |t - t_0| \times \sup_{(s,x) \in J \times \bar{B}} \|f(s, x)\| \leq h \times [\sup_{|s| \leq h} \|f(s, x_0)\| + M].$$

It means that if h is chosen so that $h \times [\sup_{|s|\leq h} \|f(s,x_0)\| + M] \leq \beta$, then $\varphi(u) \in \mathcal{G}$ as soon as $u \in \mathcal{G}$. Consequently, we can find h small enough such that $\mathcal{G}$ is stable through the action of φ.

Application of the Banach-Picard Theorem.

We are turned to the application of a fixed point result. In this view, remark that φ can be seen as a contraction, $\forall (u,v) \in \mathcal{G}^2$, we have

$$\|\varphi(u)-\varphi(v)\|_\infty = \sup_{t\in J}\left\| \int_0^t f(s,u(s)) - f(s,v(s)ds \right\| \leq \int_0^h \|f(s,u(s))-f(s,v(s)\| ds \leq kh\|u-v\|_\infty.$$

Consequently, if h is chosen so that $kh < 1$ (say 1/2 for example), then φ is a contracting map of $(\mathcal{G}, \|.\|_\infty)$. The Banach-Picard Theorem then implies that a unique fixed point of φ exists in $\mathcal{G}$. And it is easy to check that such a fixed point satisfies $(\mathcal{P})$! $\square$

Remark 4.2.1 *We omit in this paragraph many interesting stuff on the theory of maximal solutions. Let's just remark that indeed, the Cauchy-Lipschitz Theorem states the existence and uniqueness of solution locally around each initial conditions of a Cauchy problem. Nothing is said about the existence of a global solution on a given interval time $[0,T]$. To extend a local solution to a global one, some non-explosion criterion are generally needed to obtain a such existence.*

At last, remark that the local Lipschitz condition is unavoidable to obtain the local uniqueness of solution : consider for example the problem

$$\dot{x}(t) = \sqrt{x(t)} \qquad \textit{with} \qquad x(0) = 0,$$

has a constant solution $x(t) = 0$, for all time $t \geq 0$. But another solution exists around $t = 0$ since we can formally write

$$\frac{\dot{x}(t)}{\sqrt{x(t)}} = 1 \Longrightarrow \left(\sqrt{x(t)}\right)' = \frac{1}{2},$$

so that $x(t) = \frac{t^2}{4}$ is also solution around 0.

4.3 Applications to Markov chains

4.3.1 The Perron-Frobenius Theorem

Let us now describe important applications in the field of Markov chains : we start with a general result of Linear Algebra with first “restrictive assumptions".

Theorem 4.3.1 *Consider a matrix $M \in \mathcal{M}_{n,n}(\mathbb{R})$ such that*

$$\forall (i,j) \in \{1\ldots,n\}^2 \qquad M_{i,j} > 0.$$

Then M has at least one strictly positive eigenvalue.

Proof : We consider the set

$$S = \{x \in \mathbb{R}^n \quad \forall i : x_i \geq 0 \,\text{and}\, \|x\|_1 = 1\},$$

and we define the application

$$f : x \in S \longrightarrow \frac{Mu}{\|Mu\|_1} \in S$$

It is possible to check that f is well defined and continuous on S since

$$\|Mu\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} M_{i,j} u_j \right| \geq \|u\|_1 \times m,$$

where $m = \inf_{i,j} M_{i,j}$. Moreover, S is a closed compact set of $\mathbb{R}^n$. It is also convex since it is an intersection of convex sets. At last, we have

$$\forall u \in S \qquad f(u) = v \quad \text{satisfies} \quad \forall i \, v_i \geq 0 \quad \text{and} \quad \|v\|_1 = 1.$$

Hence, the Brouwer Theorem implies that f possesses a fixed point in S. It means that a vector $u \in S$ exists such that

$$f(u) = u \Longleftrightarrow Mu = \|Mu\|_1 u.$$

Written in a different way, u is an eigenvector of M whose eigenvalue is positive with all coordinate positive. ☐

4.3.2 Finite state Markov chains

Notation 4.3.1 (State space) *We denote E a finite state space, of size N in what follows.*

Definition 4.3.1 (Markov chain with transition matrix Q) *A sequence of r. v. $(X_n)_n$ is an homogeneous Markov chain, valued in E and of transition matrix Q if $\mathcal{L}(X_{n+1}|\mathcal{F}_n) = \mathcal{L}(X_{n+1}|X_n)$ and*

$$\mathbb{P}[X_{n+1} = j | X_n = i] := Q_{i,j}.$$

You can observe an example of a graph representation of a Markov chain in Figure 4.4.

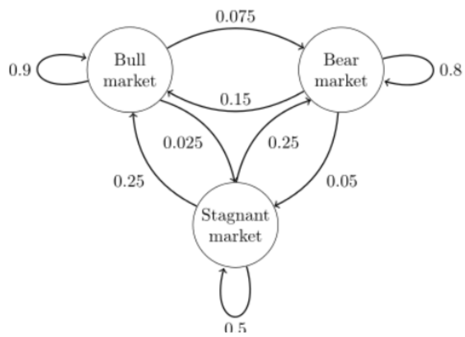


FIGURE 4.4: An example of Markov chain representation with a state space of size 3. Each edge of the directed graph is weighted by the probability transition obtained in the transition matrix.

In what follows, we will assume that $(X_n)_n$ **is an irreducible Markov chain**, meaning that

$$\forall (x, y) \in E \qquad \exists \, n_{x,y} \qquad | \qquad Q_{x,y}^{n_{x,y}} > 0.$$

Notation 4.3.2 ($\mathbb{P}_\mu$ and $\mathbb{E}_\mu$) *We will denote as $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ the probability and conditional expectations given the fact that X_0 is sampled according to the distribution μ.*

Notation 4.3.3 (μg) *For any real function g defined on E, μg is the mean of g according to μ :*

$$\mu g := \sum_{x \in E} \mu(x) g(x) = \mathbb{E}_{Y \sim \mu} g(Y).$$

Notation 4.3.4 (μQ) *The left multiplication of the matrix Q by a vector μ of $\mathbb{R}^N$ corresponds to*

$$\forall i \in [1, N] \qquad (\mu Q)(j) = \sum_{i=1}^{N} \mu(i) Q_{i,j}$$

This notation makes sense in particular when μ is a probability distribution : indeed we can show that μQ is the distribution of the random variable X_1 when $X_0 \sim \mu$ and $(X_n)_{n\in\mathbb{N}}$ is a Markov chain of transition Q.

Notation 4.3.5 (Qf) *When f is a function of E, we denote Qf the right multiplication of Q by f :*

$$\forall x \in E \qquad (Qf)(x) = \sum_{j=1}^{n} Q_{x,j} f(j)$$

This corresponds to the action of the transition matrix Q on f and we obtain in $Qf(x)$ the average value of $f(X_1)$ when $X_0 = x$.

In this lecture, we will be interested in the long time behaviour of the chain $(X_n)_n$. This asymptotic behaviour is described by the evolution of the distribution of X_n at time n, which is denoted μ_n. As pointed above, $\mu_n := \mu_{n-1} Q = \ldots = \mu Q^n$ where $\mu = \mathcal{L}(X_0)$.

Definition 4.3.2 (Invariant measure π) *We will say that π is an invariant measure of $(X_n)_{n\in\mathbb{N}}$ if it satisfies the fixed point equation*

$$\pi Q = \pi.$$

In the sequel, we will be interested to the situations where π is not uniformly zero, and can be normalized in a probability distribution.

When E is finite and irreducible, we will show that a unique invariant measure exists for the Markov chain. Such a result could be generalized in many different settings (continuous state space, continuous time Markov chain). At last, the stability of the Markov chain $(X_n)_{n\in\mathbb{N}}$ consists in looking at the evolution of a distance between the law at time n and this invariant measure π when $n \longrightarrow +\infty$.

4.4 Invariant measures (E fini)

Lemma 4.4.1 (Existence of invariant measure, finite case) *When E is finite, there exists at least one invariant measure of the homogeneous Markov chain with transition matrix Q.*

Proof : The proof is easy : consider the application $f : \mu \longmapsto \mu Q$, where μ belongs to the simplex of probability distribution over E, which is denoted S. This simplex S is closed, convex and compact (as a bounded set of a finite dimensional space). Moreover, $f(S) \subset S$ and the Brouwer Theorem states that f has a fixed point on S. This fixed point is an invariant measure for Q. □

Another important point is about the positivity of all invariant measures of irreducible chains.

Lemma 4.4.2 *If Q is irreducible, then for any invariant measure π we have $\pi(x) > 0, \forall x \in E$.*

Proof : As a probability distribution, one point $x_0 \in E$ exists such that $\pi(x_0 > 0$. Now, we can link any point $y \in E$ with a path starting at x_0 and ending at y within a finite number of steps since the chain is irreducible. We know that $\pi = \pi Q^k$, for any integer k. Hence

$$\pi(y) = (\pi Q^{n_{x_0,y}})(y) \geq \pi(x_0) Q^{n_{x_0,y}}(x_0, y) > 0.$$

This concludes the proof. □

Definition 4.4.1 (Dirichlet forms) *Let be given a measure π, we define the Dirichlet form associated to the chain Q as*

$$\mathcal{E}(f) := \frac{1}{2} \sum_{(x,y)\in E^2} \pi(x)Q(x,y)[f(x)-f(y)]^2 = \frac{1}{2}\mathbb{E}_\pi\left([f(X_1)-f(X_0)]^2\right).$$

Lemma 4.4.3 *If f is an harmonic function (i.e. satisfying $Qf = f$) and if Q is irreducible, then f is constant over E.*

Proof : We know that the invariant measure $\pi > 0$ over E since the chain is irreducible. We compute the Dirichlet form and obtain :

$$\begin{aligned} 2\mathcal{E}(f) &= \mathbb{E}_\pi f^2(X_1) + \mathbb{E}_\pi f^2(X_0) - 2\mathbb{E}_\pi f(X_1)f(X_0) \\ &= 2[\pi f^2 - \pi(f(Qf))] \\ &= 2[\pi f^2 - \pi(f^2)] \\ &= 0 \end{aligned}$$

Hence, an Harmonic function f has a null Dirichlet form and from the initial expression, we see that

$$Q(x,y) > 0 \Longrightarrow f(x) = f(y).$$

Moreover, we can find a path that link x to y since Q is irreducible, and this path has a positive probability. It means that f is constant. □

Using this Lemma, we can prove now the important baseline Theorem of irreducible Markov chains on finite state space models.

Theorem 4.4.1 *If E is finite and Q irreducible, the there exists a unique invariant measure for the Markov chain Q.*

Proof : We know that one invariant distribution exists from Lemma 4.4.1. Moreover, every invariant measure satisfies $\pi = \pi Q$ and π^t is an eigenvector of Q^t, whose eigenvalue is equal to 1.

Moreover, Lemma 4.4.3 shows that the eigenspace of Q associated to the eivenvalue 1 has a dimension equal to 1 (and only contains the constant functions). Since Q and Q^t possess the same spectral structure (same dimension of eigenspaces), we can deduce that Q^t has an eigenspace associated to the eigenvalue 1 of dimension 1. In such a case, there exists only one invariant distribution. □

Small illustrations :

Exercice Consider the chain over $E = \{0,1\}$ such that

$$Q = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}.$$

1/ Identify the invariant measure.
2/ We fixed $X_0 \sim \mu$. Prove that

$$\mathbb{P}(X_n = 0) = \frac{b}{a+b} + (1-a-b)^n \left(\mu(0) - \frac{b}{a+b}\right).$$

3/ Recover the expression of the invariant distribution.

Many deep developments can be obtained following this introduction to the Markov evolutions. We advise interested readers to continue in a Master 2 in applied mathematics...

4.4.1 Partial Differential Equations and Schauder's result

Many applications not shown here, may be encountered in an Master 2 lecture in applied mathematics...

4.4.2 Application to game theory : the Nash equilibrium Theorem

4.4.2.1 Description

In game theory, the Nash equilibrium definition refers to a situation of non-cooperative game. It refers to a situation of n players with $n \geq 2$. A Nash equilibrium is a kind of equilibrium in the game where each player has chosen a strategy and no player can benefit from a modification of his strategy while the others keep their strategy unchanged.

Say differently, in the 2 players situations, A and B are in Nash equilibrium if A is taking the best decision he can knowing the strategy of B and in the same way B is taking the best decision he can knowing the strategy of A. In the case of a number $n > 2$ of players, a situation is in a Nash equilibrium if for any player $i \in \{1, \ldots, n\}$, the player P_i is choosing the best action he can knowing the other choices of the other player.

One famous example of Nash equilibrium is the prisoner's dilemma : two criminals are arrested by the police and each one is isolated in a room. The police asks each prisoner to choose a strategy :

— betray the other criminal by saying that the other criminal has committed the crime
— cooperate and keep silent

The "outcome" of each prisoner is decided as follows :

— If the 2 criminals betray, then they are charged by 2 years of prison
— If the 2 criminals keep silent, then they are charged by 1 year of prison
— If one criminal (A) keeps silent and the other B betrays A, then B will be set free while A will be charged of 3 years.

It is easy to see that a Nash equilibrium of such a game exists : a mutual betrayal is the "optimal" choice for the two criminals. The dilemma then is that mutual cooperation yields a better outcome than mutual defection but it is not the rational outcome because from a self-interested perspective, the choice to cooperate, at the individual level, is irrational.

4.4.2.2 Mathematical approach

We now provide a mathematical definition of the problem and its resolution. We denote by (S, f) a couple of $S = (S_1, \ldots, S_n)$ set of n strategies, S_i corresponding to the possible choices offered to player i and f is an application that computes the n rewards associated to a set of strategies chosen by the n players :

$$\forall x = (x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n : f(x) = (f_1(x), \ldots, f_n(x)).$$

Each value $f_i(x)$ corresponds to the reward (outcome) obtained by player i with the set of strategies $x = (x_1, \ldots, x_n)$.

Definition 4.4.2 (Nash Equilibrium) *A Nash Equilibrium $x^\star$ is a n couples $x^\star = (x_1^\star, \ldots, x_n^\star)$ such that*

$$\forall i \in \{1 \ldots n\} \quad \forall x_i \in S_i \qquad f_i(x^\star) \geq f_i(x_i, x_{-i}^\star).$$

Remark 4.4.1 *Note that the Nash equilibrium is different from a Pareto optima. You can for example consider the prisoner dilemma example : the Pareto optimal of the game is to cooperate for the 2 criminals while the Nash equilibrium corresponds to the radical inverse choice : mutual* *betrayal. The difference comes from the cooperation involved in the Pareto situation while the Nash equilibrium refers to non-cooperative games.*

Several games are possible, some of them are of the type pure-strategy (one strategy in S_i is chosen by player i). Some other games are related to mixed-strategy : player i chooses a strategy in S_i randomly, according to a probability distribution π_i on S_i. The global strategy is then $\pi = \pi_1 \times \ldots \times \pi_n$ (tensor product corresponding to an **independent** choice of each player). The mean outcome of the game for player i is

$$\mathbb{E}_{X \sim \pi}[f_i(X)],$$

where X is a n-valued random variable sampled according to π.

4.4.2.3 Nash's Theorem

The famous Nash result dates back from 1950 (PhD) and 1951 (the original famous paper in Annals of Maths). We can state this result as follows.

Theorem 4.4.2 (Nash theorem) *If each S_i is finite, every n-persons game with mixed strategies has as least one Nash equilibrium.*

Proof : First claim, notations : We will establish the result using the Brouwer theorem (as it was done in the original paper of Nash). We introduce the notation : $x = (x_i, x_{-i})$, where :
— x is a global choice of the n players
— x_i is the choice of player i
— x_{-i} refers to the choice of the players except i
Since we deal now with mixed-strategies, the strategy of player i is denoted by π_i. Similarly, the strategy of the other players is denoted by π_{-i}.
Second claim, compactness : From the finiteness of each set of pure strategy S_i, we know that $\Pi(S_i)$ is a compact set, where $\Pi(S_i)$ refers to the set of probability distribution over S_i.
Third claim, gain function : The gain function measures the increase of efficiency from a mixed strategy to a pure one :

$$\forall a \in S_i \qquad r_i(\pi, a) := 0 \wedge u_i(a, \pi_{-i}) - u_i(\pi),$$

when the player i decides to switch his decision from π_i to a. The function $g(\pi) = (g_1(\pi), \ldots, g_n(\pi))$ defined by

$$\forall a \in S_i \qquad g_i(\pi)(a) := \pi_i(a) + r_i(\pi, a),$$

satisfies :

$$\sum_{a \in S_i} g_i(\pi)(a) = \sum_{a \in S_i} \pi_i(a) + \sum_{a \in S_i} r_i(\pi, a) = 1 + \sum_{a \in S_i} r_i(\pi, a).$$

Since each gain function is positive, we conclude that

$$\sum_{a \in S_i} g_i(\pi)(a) \geq 1 > 0.$$

Fourth step, fixed point argument : We define the function $f = (f_1, \ldots, f_n) : \Pi \longrightarrow \Pi$ by

$$\forall a \in S_i \qquad f_i(\pi)(a) = \frac{g_i(\pi)(a)}{\sum_{a \in S_i} g_i(\pi)(a)}$$

It is easy to check that $f_i(\pi)(.)$ defines a probability distribution on S_i and thus induces a mixed strategy for player i. Moreover, f_i is a continuous function of π because $\sum_{a\in S_i} g_i(\pi)(a) \geq 1 > 0$.

At last, Π is a compact convex set. Thus, the Brouwer theorem may be applied : a fixed point $\pi^\star$ of f exists and satisfies

$$f(\pi^\star) = \pi^\star.$$

Fifth step, key relationship : We now show that $\pi^\star$ is a Nash equilibrium. For this purpose, it is enough to show that each gain function is uniformly zero :

$$\forall i \in \{1 \dots n\} \quad \forall a \in S_i \qquad r_i(\pi^\star, a) = 0,$$

meaning that each player has no benefit from moving its strategy from $\pi_i^\star$ to a pure strategy. Conversely, we assume that the gain functions are not uniformly zero :

$$\exists i_0 \in \{1, \dots, n\} \quad \exists a \in S_{i_0} \qquad r_{i_0}(\pi^\star, a) > 0.$$

It means that

$$K := \sum_{a\in S_{i_0}} g_{i_0}(\pi^\star)(a) > 1.$$

Now, we use the fixed point caracterisation :

$$\begin{aligned} \pi^\star = f(\pi^\star) &\implies \pi_{i_0}^\star = f_{i_0}(\pi^\star) \\ &\implies \pi_{i_0}^\star = \frac{g_{i_0}(\pi^\star)}{K} \\ &\implies \pi_{i_0}^\star = \frac{\pi_{i_0}^\star + r_{i_0}(\pi^\star, .)}{K} \\ &\implies \pi^s tar_{i_0} = \frac{1}{K-1} r_{i_0}(\pi^\star, .). \end{aligned}$$

Hence, $\pi_{i_0}^\star$ is proportional to the gain $r_{i_0}(\pi^\star, .)$.

In particular, for a pure strategy $a \in S_{i_0}$, if $r_{i_0}(\pi^\star, a) > 0$, then

$$r_{i_0}(\pi^\star, a) = u_{i_0}(a, \pi_{-i_0}^\star) - u_{i_0}(\pi^\star)$$

and in that case

$$\pi_{i_0}^\star(a) r_{i_0}(\pi^\star, a) = \pi_{i_0}^s tar(a) \left[u_{i_0}(a, \pi_{-i_0}^\star) - u_{i_0}(\pi^\star)\right]$$

Assume now that $r_{i_0}(\pi^\star, a) = 0$, since $\pi_{i_0}^\star$ is proportional to the gain $r_{i_0}(\pi^\star, .)$, the relationship

$$\pi_{i_0}^\star(a) r_{i_0}(\pi^\star, a) = \pi_{i_0}^\star(a) \left[u_{i_0}(a, \pi_{-i_0}^\star) - u_{i_0}(\pi^\star)\right]$$

still holds because $\pi_{i_0}^\star(a) = 0$. In all cases, we deduce that

$$\forall a \in S_{i_0} \qquad \pi_{i_0}^\star(a) r_{i_0}(\pi^\star, a) = \pi_{i_0}^\star(a) \left[u_{i_0}(a, \pi_{-i_0}^\star) - u_{i_0}(\pi^\star)\right]$$

Sixth step, Nash equilibrium : We obtain a contradiction as follows. Using that the average value of u_{i_0} is the sum over all the strategies in S_{i_0}, we get :

$$\begin{aligned} 0 &= \left[u_{i_0}(\pi_{i_0}^\star, \pi_{-i_0}^\star) - u_{i_0}(\pi_{i_0}^\star, \pi_{-i_0}^\star)\right] \\ &= \left[\sum_{a\in S_{i_0}} \pi_{i_0}^\star(a) u_{i_0}(a, \pi_{-i_0}^\star)\right] - u_{i_0}(\pi_{i_0}^\star, \pi_{-i_0}^\star) \\ &= \sum_{a\in S_{i_0}} \pi_{i_0}^\star(a) \left[u_{i_0}(a, \pi_{-i_0}^\star) - u_{i_0}(\pi_{i_0}^\star, \pi_{-i_0}^\star)\right]. \end{aligned}$$

Now, the key identity obtained in the fifth point yields

$$0 = \sum_{a \in S_{i_0}} \pi^\star_{i_0}(a) r_{i_0}(\pi^\star, a),$$

and the fact that $r_{i_0}(\pi^\star, .) = (C-1)\pi^\star_{i_0}$ leads to

$$0 = (C-1) \sum_{a \in S_{i_0}} \pi^\star_{i_0}(a)^2.$$

Since $\pi^\star_{i_0}$ is a probability distribution over S_{i_0} and $C > 1$, the above sum cannot be zero. We then obtain our contradiction. $\square$

We should now provide an additional remark : the proof of the Nash theorem relies on the use of a key application from the set of mixed strategy into the same set and on the Brouwer theorem. Indeed, a much more simpler proof is possible using a corollary of the Kakutani theorem, that states a fixed point result for set-valued function.

As a conclusion, we should highlight the fact that the above proof is not yet constructive : it does not provide an algorithmic way to find a Nash equilibrium, even with an approximation argument. However, it is possible to write the equilibrium condition for mixed strategy and then solve these Nash equilibrium conditions by brute force resolutions.