Bull. Sci. math. 157 (2019) 102796

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

# Cubic perturbations of symmetric elliptic Hamiltonians of degree four in a complex domain

Bassem Ben Hamed [a], Ameni Gargouri [b], Lubomir Gavrilov [c,*]

[a] Ecole Nationale d'Electronique et des Télécommunications de Sfax, Route de Tunis km 10, BP 1163, 3021 Sfax, Tunisie
[b] Faculté des Sciences de Sfax, Département de Mathématiques, BP 1171, 3000 Sfax, Tunisie
[c] Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, 31062 Toulouse, France

ARTICLE INFO

*Article history:*
Received 1 July 2019
Available online 20 September 2019

*Keywords:*
Limit cycles
Zeros of elliptic integrals
Duffing oscillator

ABSTRACT

We consider arbitrary one-parameter cubic deformations of the Duffing oscillator $x'' = x - x^3$. In the case when the first Melnikov function $M_1$ vanishes, but $M_2 \neq 0$ we compute the general form of $M_2$ and study its zeros in a suitable complex domain.



## 1. Introduction

Consider the perturbed Duffing oscillator

$$X_\epsilon : \begin{cases} \dot{x} = H_y + \epsilon f(x,y,\epsilon) \\ \dot{y} = -H_x + \epsilon g(x,y,\epsilon) \end{cases} \tag{1}$$

* Corresponding author.
*E-mail addresses:* bassem.benhamed@gmail.com (B.B. Hamed), ameni.gargouri@gmail.com (A. Gargouri), lubomir.gavrilov@math.univ-toulouse.fr (L. Gavrilov).

https://doi.org/10.1016/j.bulsci.2019.102796
0007-4497/

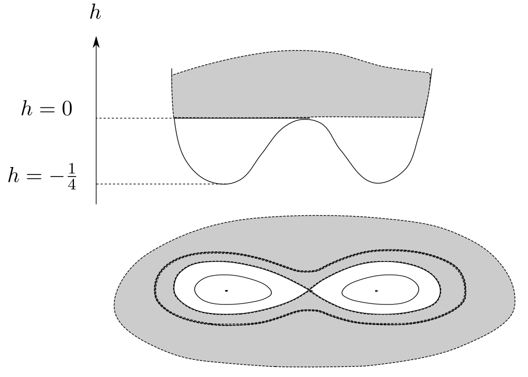


**Fig. 1.** Phase portrait of $X_0$ and the graph of $-\frac{x^2}{2}+\frac{x^4}{4}$.

Where $f(x,y,\epsilon)$, $g(x,y,\epsilon)$ are arbitrary cubic polynomials:

$$f(x,y,\epsilon) = \lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 xy + \lambda_4 x^2 + \lambda_5 y^2 + \lambda_6 x^2 y + \lambda_7 xy^2 + \lambda_8 x^3 + \lambda_9 y^3$$
$$g(x,y,\epsilon) = \gamma_0 + \gamma_1 x + \gamma_2 y + \gamma_3 xy + \gamma_4 x^2 + \gamma_5 y^2 + \gamma_6 x^2 y + \gamma_7 xy^2 + \gamma_8 x^3 + \gamma_9 y^3$$

and the parameters $\lambda_i = \sum_{j\geq 0}\lambda_{i,j}\epsilon^j$, $\lambda_{i,j}\in\mathbb{R}$ and $\gamma_i = \sum_{j\geq 0}\gamma_{i,j}\epsilon^j$, $\gamma_{i,j}\in\mathbb{R}$ are analytic functions of the small parameter $\epsilon$. For $\epsilon = 0$ the system is integrable, with a first integral

$$H = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

and its phase portrait is shown on Fig. 1. The exterior period annulus and the two interior period annuli on Fig. 1 give rise to three displacement maps of $X_\varepsilon$ with power series expansions of the form

$$d(h,\epsilon) = \epsilon^k M_k(h) + \epsilon^{k+1} M_{k+1}(h) + \dots.$$

(where as usual $h$ is the restriction of $H$ on a suitable cross-section to the period annulus). The number of the limit cycles bifurcating from each period annulus is bounded by the number of the zeros of the first non-vanishing Melnikov function $M_k$. According to the Poincaré-Pontryagin formula

$$M_1(h) = \int_{H=h} \omega_0 dx = \int_{H=h} g(x,y,0)dx - f(x,y,0)dy$$

is a complete elliptic integral. Its zeros correspond to limit cycles bifurcating from the corresponding period annulus. It is well known, that in our case the first non-vanishing Melnikov function $M_k$ is a complete elliptic integral, see [7, Corollary 1], and [9,10], and its general form has been established in formula (23) and Theorem 3 of [7], as a linear combination of complete elliptic integrals.

Our first result is an explicit formula for the second Melnikov function $M_2$, under the hypothesis that $M_1$ is identically zero, see Proposition 3 and Proposition 4. The main tool is the Iliev formula for $M_2$ [11]. This formula was proved using the method of [3]. Our second result is an estimate for the number of the zeros of $M_2$, Lemma 3, 4, 5, 6. From this we deduce the maximal cyclicity of the period annuli, when at least $M_1$ or $M_2$ does nor vanish identically.

The paper is organized as follows. In section 2 we compute the Melnikov functions $M_1(h)$ and $M_2(h)$ (when $M_1(h) \equiv 0$) in the interior and exterior eight-loop case see respectively. In section 3 we recall some known Picard-Fuchs equations, which will be used later. Finally, in section 4 we describe the monodromy of the Abelian integrals, based on the classical Picard-Lefschetz theory, and then apply the so called Petrov trick, to obtain estimates to the number of their zeros in a suitable complex domain.

For previous related results in a real or complex domain we refer the reader to [15,17, 18,5,2,14,6].

## 2. Computation of Melnikov functions

Let $\{\gamma(h)\}_h$ be the continuous family of ovals of the non-perturbed system, where

$$\gamma(h) \subset \{H = h\}$$

with $h \in \Sigma = (h_c, h_s)$ in the interior eight-loop case and $h \in \Sigma = (h_s, +\infty)$ in the exterior eight-loop case, where $h_s = 0$, $h_c = -1/4$ are the critical values of $H$.

Consider the complete elliptic integrals

$$I_i(h) = \begin{cases} I_{\widetilde{\omega}_i} = \displaystyle\oint_{\gamma(h)} x^i y dx \ \text{ if } \ \Sigma = (h_c, h_s) \\ I_{\widetilde{\widetilde{\omega}}_i} = \displaystyle\oint_{\gamma(h)} x^i y dx \ \text{ if } \ \Sigma = (h_s, +\infty) \end{cases} \tag{2}$$

The Abelian integrals $I_k$, $k \geq 0$, can be expressed as linear combinations of $I_0, I_1, I_2$, with coefficients in the field $\mathbb{R}(h)$. In the exterior eight-loop case the symmetry $(x, y) \to (\pm x, y)$ transforms the oval $\gamma(h)$ to $-\gamma(h)$ which implies that $I_k(h) \equiv 0$ for odd $k$.

As well known, if we parameterize the displacement map by the Hamiltonian level h, then the following power series expansion holds

$$d(h, \epsilon) = P(h, \epsilon) - h = \epsilon M_1(h) + \epsilon^2 M_2(h) + ...., h \in \Sigma \tag{3}$$

Where $P(h, \epsilon)$ is the first return map, $\Sigma$ is an open interval depending on the case under consideration. Our first goal will be to calculate explicitly the first Melnikov function $M_1$ and then $M_2$ in (3). We use the Iliev formula [11].

We denote:

$$\begin{aligned} f(x, y, 0) &= \lambda_{0,1} + \lambda_{1,1}x + \lambda_{2,1}y + \lambda_{3,1}xy + \lambda_{4,1}x^2 + \lambda_{5,1}y^2 + \lambda_{6,1}x^2y \\ &\quad + \lambda_{7,1}xy^2 + \lambda_{8,1}x^3 + \lambda_{9,1}y^3 \end{aligned} \tag{4}$$

$$\begin{aligned} g(x, y, 0) &= \gamma_{0,1} + \gamma_{1,1}x + \gamma_{2,1}y + \gamma_{3,1}xy + \gamma_{4,1}x^2 + \gamma_{5,1}y^2 + \gamma_{6,1}x^2y \\ &\quad + \gamma_{7,1}xy^2 + \gamma_{8,1}x^3 + \gamma_{9,1}y^3 \end{aligned} \tag{5}$$

$$\begin{aligned} f_\epsilon(x, y, 0) &= \lambda_{0,2} + \lambda_{1,2}x + \lambda_{2,2}y + \lambda_{3,2}xy + \lambda_{4,2}x^2 + \lambda_{5,2}y^2 + \lambda_{6,2}x^2y \\ &\quad + \lambda_{7,2}xy^2 + \lambda_{8,2}x^3 + \lambda_{9,2}y^3 \end{aligned} \tag{6}$$

$$\begin{aligned} g_\epsilon(x, y, 0) &= \gamma_{0,2} + \gamma_{1,2}x + \gamma_{2,2}y + \gamma_{3,2}xy + \gamma_{4,2}x^2 + \gamma_{5,2}y^2 + \gamma_{6,2}x^2y \\ &\quad + \gamma_{7,2}xy^2 + \gamma_{8,2}x^3 + \gamma_{9,2}y^3 \end{aligned} \tag{7}$$

We recall, that non-perturbed Hamiltonian system has two bounded (interior) period annuli and one unbounded (exterior) period annulus.

### *2.1. Computation of $M_1$*

#### *2.1.1. The interior Duffing oscillator*

**Proposition 1.** *The first Melnikov functions $M_1$ for the perturbed interior Duffing oscillator have the form*

$$M_1(h) = \alpha_0(h)I_0 + \alpha_1 I_1 + \alpha_2 I_2 \tag{8}$$

*where*

$$\alpha_0(h) = c_0 + c_1 h, \alpha_1 = 2\lambda_{4,1} + \gamma_{3,1}, \alpha_2 = c_2,$$

*and*

$$c_0 = \lambda_{1,1} + \gamma_{2,1}, c_1 = \frac{4}{7}(\lambda_{7,1} + 3\gamma_{9,1}), c_2 = \gamma_{6,1} + 3\lambda_{8,1} + \frac{1}{7}\lambda_{7,1} + \frac{3}{7}\gamma_{9,1}.$$

**Proof.** It is well known that:

$$M_1(h) = \int\limits_{H=h} \omega_0 dx = \int\limits_{H=h} g(x, y, 0)dx - f(x, y, 0)dy,$$

where

$$\int_{H=h} g(x,y,0)dx = \int_{H=h} [y(\gamma_{2,1}+\gamma_{3,1}x+\gamma_{6,1}x^2)+y^2(\gamma_{5,1}+\gamma_{7,1}x)+\gamma_{9,1}y^3]dx$$

$$-\int_{H=h} f(x,y,0)dy = -\int_{H=h} [\lambda_{1,1}x+\lambda_{3,1}xy+\lambda_{4,1}x^2+\lambda_{6,1}x^2y+\lambda_{7,1}xy^2+\lambda_{8,1}x^3]dy$$

Or

$$xdy = d(xy)-ydx, \qquad xydy = d(x\frac{y^2}{2})-\frac{y^2}{2}dx, \qquad x^2dy = d(x^2y)-2xydx$$

$$x^2ydy = d(x^2\frac{y^2}{2})-xy^2dx, \qquad xy^2dy = d(x\frac{y^3}{3})-\frac{y^3}{3}dx, \qquad x^3dy = d(x^3y)-3x^2ydx$$

Therefore we can rewrite $\int_{H=h}\omega$ in the form $\int_{H=h}\omega = dQ(x,y,0)+yq(x,y,0)dx$ with

$$Q(x,y,0) = \gamma_{0,1}x+\frac{\gamma_{1,1}}{2}x^2+\frac{\gamma_{4,1}}{3}x^3+\frac{\gamma_{8,1}}{4}x^4$$

and

$$\begin{aligned} yq(x,y,0) = {} & [(\lambda_{1,1}+\gamma_{2,1})+(\gamma_{3,1}+2\lambda_{4,1})x+(\gamma_{6,1}+3\lambda_{8,1})x^2]y \\ & +[(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})+(\lambda_{6,1}+\gamma_{7,1})x]y^2+(\gamma_{9,1}+\frac{\lambda_{7,1}}{3})y^3. \end{aligned}$$

Then

$$\begin{aligned} M_1(h) = {} & (\gamma_{2,1}+\lambda_{1,1})I_0+(\gamma_{3,1}+2\lambda_{4,1})I_1+(\gamma_{6,1}+3\lambda_{8,1})I_2 \\ & +(\frac{\lambda_{7,1}}{3}+\gamma_{9,1})\int_{H=h} y^3dx, \end{aligned}$$

and

$$\int_{H=h} y^3dx = \int_{H=h} y(2h+x^2-\frac{x^4}{2}) = 2hI_0+I_2-\frac{I_4}{2} = \frac{12h}{7}I_0+\frac{3}{7}I_2$$

implies (8). □

*2.1.2. The exterior Duffing oscillator*

**Proposition 2.** *The first Melnikov functions $M_1$ for the perturbed exterior Duffing oscillator have the form*

$$M_1(h) = \alpha_0(h)I_0+\alpha_2I_2 \tag{9}$$

*where*

$$\alpha_0(h) = c_0 + c_1 h, \qquad \alpha_2 = c_2$$
$$c_0 = \lambda_{1,1} + \gamma_{2,1}, \qquad c_1 = \frac{4}{7}(\lambda_{7,1} + 3\gamma_{9,1}), \qquad c_2 = \gamma_{6,1} + 3\lambda_{8,1} + \frac{1}{7}\lambda_{7,1} + \frac{3}{7}\gamma_{9,1}.$$

**Proof.** It is similar to the proof in the exterior case, with the only exception that $I_1 = 0$. □

*2.2. Computation of $M_2$*

If $M_1 = 0$, the Iliev formula [11] for the second Melnikov function $M_2(h)$ reads

$$\begin{aligned} M_2(h) = &\int\limits_{H=h} [G_{1h}(x,y)P_2(x,h) - G_1(x,y)P_{2h}(x,h)]dx \\ &- \int\limits_{H=h} \frac{F(x,y)}{y}(f_x(x,y,0) + g_y(x,y,0))dx \\ &+ \int\limits_{H=h} g_\epsilon(x,y,0)dx - f_\epsilon(x,y,0)dy \end{aligned}$$

where

$$F(x,y) = \int_0^y f(x,s,0)ds - \int_0^x g(s,0,0)ds, \qquad G(x,y) = g(x,y,0) + F_x(x,y)$$

and $G_1(x,y)$, $G_2(x,y)$ are the odd and even parts of $G(x,y)$ with respect to $y$. Thus if

$$\begin{aligned} G(x,y) = &\, y[(\lambda_{1,1} + \gamma_{2,1}) + (\gamma_{3,1} + 2\lambda_{4,1})x + (\gamma_{6,1} + 3\lambda_{8,1})x^2 + y^2(\gamma_{9,1} + \frac{\lambda_{7,1}}{3})] \\ &+ y^2[(\gamma_{5,1} + \frac{\lambda_{3,1}}{2}) + (\gamma_{7,1} + \lambda_{6,1})x] \end{aligned}$$

then

$$G(x,y) = G_1(x,y) + G_2(x,y), \qquad G_1(x,y) = yp_1(x,y^2), \qquad G_2(x,y) = p_2(x,y^2)$$

with

$$p_1(x,y^2) = (\lambda_{1,1} + \gamma_{2,1}) + (\gamma_{3,1} + 2\lambda_{4,1})x + (\gamma_{6,1} + 3\lambda_{8,1})x^2 + y^2(\gamma_{9,1} + \frac{\lambda_{7,1}}{3})$$

and

$$p_2(x,y^2) = y^2[(\gamma_{5,1} + \frac{\lambda_{3,1}}{2}) + (\gamma_{7,1} + \lambda_{6,1})x]$$

- $P_2(x,h)$ is the polynomial

$$P_2(x,h)=\int_0^x P_2(s,2h+2U(s))ds=2hx(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})+hx^2(\gamma_{7,1}+\lambda_{6,1})$$
$$+\frac{x^3}{3}(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})+\frac{x^4}{4}(\gamma_{7,1}+\lambda_{6,1})-\frac{x^5}{10}(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})-\frac{x^6}{12}(\gamma_{7,1}+\lambda_{6,1})$$

- We note that

$$\begin{aligned}G_{1h}(x,y)&=G_{1y}(x,y)/y\\&=(\lambda_{1,1}+\gamma_{2,1})/y+(\gamma_{3,1}+2\lambda_{4,1})\frac{x}{y}+(\gamma_{6,1}+3\lambda_{8,1})\frac{x^2}{y}+3y(\gamma_{9,1}+\frac{\lambda_{7,1}}{3}).\end{aligned} \tag{10}$$

- $$\begin{aligned}g(x,y,0)&=\gamma_{0,1}+\gamma_{1,1}x+\gamma_{4,1}x^2+\gamma_{8,1}x^3+y(\gamma_{2,1}+\gamma_{3,1}x+\gamma_{6,1}x^2)\\&\quad+y^2(\gamma_{5,1}+\gamma_{7,1}x)+\gamma_{9,1}y^3\end{aligned}$$

- $$\begin{aligned}F(x,y)&=\int_0^y f(x,s,0)ds-\int_0^x g(s,0,0)ds=\lambda_{0,1}y+\lambda_{1,1}xy-\gamma_{0,1}x\\&\quad+\frac{\lambda_{2,1}}{2}y^2-\frac{\gamma_{1,1}}{2}x^2+\frac{\lambda_{3,1}}{2}xy^2\\&\quad+\lambda_{4,1}x^2y-\frac{\gamma_{4,1}}{3}x^3+\frac{\lambda_{5,1}}{3}y^3+\frac{\lambda_{6,1}}{2}x^2y^2+\frac{\lambda_{7,1}}{3}xy^3\\&\quad+\lambda_{8,1}x^3y+\frac{\lambda_{9,1}}{4}y^4-\frac{\gamma_{8,1}}{4}x^4.\end{aligned}$$

- Then

$$\begin{aligned}-\frac{F(x,y)}{y}&=-\frac{1}{y}(\int_0^y f(x,s,0)ds-\int_0^x g(s,0,0)ds)\\&=-\lambda_{0,1}-\lambda_{1,1}x+\gamma_{0,1}\frac{x}{y}-\frac{\lambda_{2,1}}{2}y+\frac{\gamma_{1,1}}{2}\frac{x^2}{y}-\frac{\lambda_{3,1}}{2}xy\\&\quad-\lambda_{4,1}x^2+\frac{\gamma_{4,1}}{3}\frac{x^3}{y}-\frac{\lambda_{5,1}}{3}y^2-\frac{\lambda_{6,1}}{2}x^2y-\frac{\lambda_{7,1}}{3}xy^2-\lambda_{8,1}x^3\\&\quad-\frac{\lambda_{9,1}}{4}y^3+\frac{\gamma_{8,1}}{4}\frac{x^4}{y}.\end{aligned}$$

In fact:

$$\int_0^y f(x,s,0)ds = \lambda_{0,1}y + \lambda_{1,1}xy + \lambda_{2,1}\frac{y^2}{2} + \lambda_{3,1}x\frac{y^2}{2} + \lambda_{4,1}x^2y$$
$$+ \lambda_{5,1}\frac{y^3}{3} + \lambda_{6,1}x^2\frac{y^2}{2} + \lambda_{7,1}\frac{xy^3}{3} + \lambda_{8,1}x^3y + \lambda_{9,1}\frac{y^4}{4}$$
$$\int_0^x g(s,0,0)ds = \gamma_{0,1}x + \gamma_{1,1}\frac{x^2}{2} + \gamma_{4,1}\frac{x^3}{3} + \gamma_{8,1}\frac{x^4}{4}$$

*2.2.1. The interior Duffing oscillator*

Lemma 2 implies easily the linear independence of the functions $I_0(h)$, $hI_0(h)$, $I_1(h)$ and $I_2(h)$. As $I_1 = c(4h-3)$ then $M_1 = 0$ implies

$$\lambda_{1,1} + \gamma_{2,1} = 0 \tag{11}$$
$$\lambda_{7,1} + 3\gamma_{9,1} = 0 \tag{12}$$
$$2\lambda_{4,1} + \gamma_{3,1} = 0 \tag{13}$$
$$\gamma_{6,1} + 3\lambda_{8,1} = 0 \tag{14}$$

**Proposition 3.** *The function $M_2(h)$ has the follows form:*

$$M_2(h) = (\alpha_0 + 4\alpha_1 h)I_0 + (\beta_0 + 4h\beta_1)I_1 + \rho I_2 \tag{15}$$

*where*

$$\alpha_0 = -\lambda_{0,1}(\lambda_{3,1} + 2\gamma_{5,1}) + \lambda_{1,2} + \gamma_{2,2}$$
$$\alpha_1 = (\lambda_{3,1} + 2\gamma_{5,1})(-\frac{1}{7}\lambda_{8,1} - \lambda_{5,1})$$
$$\beta_0 = -(\lambda_{3,1} + 2\gamma_{5,1})(\lambda_{1,1} - \frac{1}{8}\lambda_{7,1})$$
$$+ 2(\lambda_{6,1} + \gamma_{7,1})(\lambda_{0,1} + 2\lambda_{4,1} - 2\lambda_{7,1}) + 2\lambda_{4,2} + \gamma_{3,2}$$
$$\beta_1 = -\frac{1}{2}\lambda_{7,1}(\lambda_{3,1} + 2\gamma_{5,1}) + 3\lambda_{7,1}(\lambda_{6,1} + \gamma_{7,1})$$
$$\rho = (\lambda_{3,1} + 2\gamma_{5,1})(\lambda_{4,1} - \frac{1}{7}\lambda_{5,1} - \frac{8}{7}\lambda_{8,1}) - 2\lambda_{1,1}(\lambda_{6,1} + \gamma_{7,1}) + \gamma_{6,2}$$
$$+ 3\lambda_{8,2} + \frac{1}{7}\lambda_{7,2} + \frac{3}{7}\gamma_{9,2}.$$

**Proof.** According to the Iliev formula

$$M_2 = \int_{H=h} [G_{1h}(x,y)P_2(x,h) - G_1(x,y)P_{2h}(x,h)]dx$$

$$- \int\limits_{H=h} \frac{F(x,y)}{y}(f_x(x,y,0)+g_y(x,y,0))dx$$
$$+ \int\limits_{H=h} g_\epsilon(x,y,0)dx - f_\epsilon(x,y,0)dy$$

where

$$\int\limits_{H=h} g_\epsilon dx - f_\epsilon dy = [\lambda_{1,2}+\gamma_{2,2}+\frac{4}{7}(\lambda_{7,2}+3\gamma_{9,2})h]I_0 + (2\lambda_{4,2}+\gamma_{3,2})I_1$$
$$+ [\gamma_{6,2}+3\lambda_{8,2}+\frac{1}{7}\lambda_{7,2}+\frac{3}{7}\gamma_{9,2}]I_2$$

By using (11), (13), (14) and (15) we have: $p_1(x,y^2)=0$ then $G_1(x,y)=0$ and (10) becomes zero.

Thus

$$M_2(h) = -\int\limits_{H=h} \frac{F(x,y)}{y}(f_x+g_y)dx + \oint\limits_{H=h} g_\epsilon dx - f_\epsilon dy$$
$$= +[-\lambda_{0,1}(\lambda_{3,1}+2\gamma_{5,1})+\lambda_{1,2}+\gamma_{2,2}]I_0 + [(\lambda_{3,1}+2\gamma_{5,1})(-\frac{1}{7}\lambda_{8,1}-\lambda_{5,1})]4hI_0$$
$$+ [-(\lambda_{3,1}+2\gamma_{5,1})(\lambda_{1,1}-\frac{1}{8}\lambda_{7,1})$$
$$+ 2(\lambda_{6,1}+\gamma_{7,1})(\lambda_{0,1}+2\lambda_{4,1}-2\lambda_{7,1}) + 2\lambda_{4,2}+\gamma_{3,2}]I_1$$
$$+ [-\frac{1}{2}\lambda_{7,1}(\lambda_{3,1}+2\gamma_{5,1}) + 3\lambda_{7,1}(\lambda_{6,1}+\gamma_{7,1})]hI_1$$
$$+ [(\lambda_{3,1}+2\gamma_{5,1})(\lambda_{4,1}-\frac{1}{7}\lambda_{5,1}-\frac{8}{7}\lambda_{8,1}) - 2\lambda_{1,1}(\lambda_{6,1}+\gamma_{7,1}) + \gamma_{6,2}$$
$$+ 3\lambda_{8,2}+\frac{1}{7}\lambda_{7,2}+\frac{3}{7}\gamma_{9,2}]I_2. \quad \square$$

*2.2.2. The exterior Duffing oscillator*

In a way similar to the interior Duffing oscillator, we conclude that if $M_1 = 0$ then

$$\lambda_{1,1}+\gamma_{2,1} = 0 \quad (16)$$
$$\lambda_{7,1}+3\gamma_{9,1} = 0 \quad (17)$$
$$\gamma_{6,1}+3\lambda_{8,1} = 0 \quad (18)$$

**Proposition 4.** *The function $M_2(h)$ has the follows form:*

$$M_2(h) = (4h+1)^{-1}[(\alpha_0+4\alpha_1 h+\alpha_2 h^2)I_0 + (\beta_0+4h\beta_1)I_2] \quad (19)$$

*where*

$$\alpha_0 = -\lambda_{0,1}(\lambda_{3,1}+2\gamma_{5,1}) + \lambda_{1,2} + \gamma_{2,2} - \gamma_{0,1}(2\lambda_{4,1}+\gamma_{3,1})$$

$$\alpha_1 = -\lambda_{0,1}(\lambda_{3,1}+2\gamma_{5,1}) + \lambda_{1,2} + \gamma_{2,2} - \frac{4}{7}(\lambda_{5,1}(\lambda_{3,1}+2\gamma_{5,1})) + \frac{4}{7}(\lambda_{7,2}+3\gamma_{9,2}) - \frac{8}{7}(\lambda_{8,1}(\lambda_{6,1}+\gamma_{7,1})) + \frac{\gamma_{4,1}}{3}(2\lambda_{4,1}+\gamma_{3,1}) + \frac{8}{15}(\gamma_{3,1}+2\lambda_{4,1})(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})$$

$$\alpha_2 = -\frac{4}{7}(\lambda_{5,1}(\lambda_{3,1}+2\gamma_{5,1})) + \frac{4}{7}(\lambda_{7,2}+3\gamma_{9,2}) - \frac{8}{7}(\lambda_{8,1}(\lambda_{6,1}+\gamma_{7,1}))$$

$$\beta_0 = -[2\lambda_{4,1}\lambda_{3,1} + \frac{\lambda_{3,1}\gamma_{3,1}}{2} + 2\lambda_{4,1}\gamma_{5,1} + 2\lambda_{1,1}\lambda_{6,1} + 2\lambda_{1,1}\gamma_{7,1} - \gamma_{6,2} - 3\lambda_{8,2} - \frac{1}{7}\lambda_{7,2} - \frac{3}{7}\gamma_{9,2} + \frac{\lambda_{5,1}}{7}(\lambda_{3,1}+2\gamma_{5,1}) + \frac{16}{7}(\lambda_{8,1}(\lambda_{6,1}+\gamma_{7,1})) - 5(2\lambda_{4,1}+\gamma_{3,1})(\frac{\gamma_{4,1}}{3}+\gamma_{0,1}) + \frac{17}{15}(\gamma_{3,1}+2\lambda_{4,1})(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})]$$

$$\beta_1 = -[2\lambda_{4,1}\lambda_{3,1} + \frac{\lambda_{3,1}\gamma_{3,1}}{2} + 2\lambda_{4,1}\gamma_{5,1} + 2\lambda_{1,1}\lambda_{6,1} + 2\lambda_{1,1}\gamma_{7,1} - \gamma_{6,2} - 3\lambda_{8,2} - \frac{1}{7}\lambda_{7,2} - \frac{3}{7}\gamma_{9,2} + \frac{\lambda_{5,1}}{7}(\lambda_{3,1}+2\gamma_{5,1}) + \frac{16}{7}(\lambda_{8,1}(\lambda_{6,1}+\gamma_{7,1})) - \frac{1}{5}(\gamma_{3,1}+2\lambda_{4,1})(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})]$$

**Proof.** The same way of proof of property 3, we use also the formula of Iliev [11]:

$$M_2(h) = \int_{H=h} [G_{1h}(x,y)P_2(x,h) - G_1(x,y)P_{2h}(x,h)]dx - \int_{H=h} \frac{F(x,y)}{y}(f_x(x,y,0)+g_y(x,y,0))dx + \int_{H=h} g_\epsilon(x,y,0)dx - f_\epsilon(x,y,0)dy$$

By using (16), (18) and (19) we have $p_1(x,y^2) = (\gamma_{3,1}+2\lambda_{4,1})x$ and (10) becomes

$$G_{1h}(x,y) = (\gamma_{3,1}+2\lambda_{4,1})\frac{x}{y}$$

Then

- $\displaystyle\int_{H=h} [G_{1h}(x,y)P_2(x,h) - G_1(x,y)P_{2h}(x,h)]dx = -2(\gamma_{3,1}+2\lambda_{4,1})(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})I_2$

$$+2h(\gamma_{3,1}+2\lambda_{4,1})(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})I_2'+\frac{1}{3}(\gamma_{3,1}+2\lambda_{4,1})(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})I_4'$$
$$-\frac{1}{10}(\gamma_{3,1}+2\lambda_{4,1})(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})I_6'$$

and by using the Picards-Fuchs equations (see for instance [1], for more details) we have

$$\begin{aligned}
I_2' &= (4h+1)^{-1}(5I_2-I_0)\\
I_4' &= (4h+1)^{-1}(4hI_0+5I_2)\\
I_6' &= (4h+1)^{-1}[\frac{4}{3}(4h+1)I_2+\frac{4}{3}h(5I_2-I_0)+\frac{4}{3}(4hI_0+5I_2)].
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{H=h}[G_{1h}(x,y)P_2(x,h)-G_1(x,y)P_{2h}(x,h)]dx\\
&\quad=(4h+1)^{-1}(\gamma_{3,1}+2\lambda_{4,1})(\gamma_{5,1}+\frac{\lambda_{3,1}}{2})[(\frac{4}{5}h-\frac{17}{15})I_2-\frac{32}{15}hI_0]\\
&\int_{H=h} g_\epsilon dx-f_\epsilon dy\\
&\quad=[\lambda_{1,2}+\gamma_{2,2}+\frac{4}{7}(\lambda_{7,2}+3\gamma_{9,2})h]I_0+[\gamma_{6,2}+3\lambda_{8,2}+\frac{1}{7}\lambda_{7,2}+\frac{3}{7}\gamma_{9,2}]I_2
\end{aligned}$$

By using (16) we have also

$$(f_x+g_y)=(2\lambda_{4,1}+\gamma_{3,1})x+(\lambda_{3,1}+2\gamma_{5,1})y+2(\lambda_{6,1}+\gamma_{7,1})xy$$

Then

- $$\begin{aligned}
&-\int_{H=h}\frac{F(x,y)}{y}(f_x+g_y)dx+\oint_{H=h} g_\epsilon dx-f_\epsilon dy\\
&=[-\lambda_{0,1}(\lambda_{3,1}+2\gamma_{5,1})+\lambda_{1,2}+\gamma_{2,2}]I_0\\
&+[-\frac{4}{7}(\lambda_{5,1}(\lambda_{3,1}+2\gamma_{5,1}))+\frac{4}{7}(\lambda_{7,2}+3\gamma_{9,2})-\frac{8}{7}(\lambda_{8,1}(\lambda_{6,1}+\gamma_{7,1}))]hI_0\\
&-[2\lambda_{4,1}\lambda_{3,1}+\frac{\lambda_{3,1}\gamma_{3,1}}{2}+2\lambda_{4,1}\gamma_{5,1}+2\lambda_{1,1}\lambda_{6,1}+2\lambda_{1,1}\gamma_{7,1}-\gamma_{6,2}-3\lambda_{8,2}\\
&-\frac{1}{7}\lambda_{7,2}-\frac{3}{7}\gamma_{9,2}+\frac{\lambda_{5,1}}{7}(\lambda_{3,1}+2\gamma_{5,1})\\
&+\frac{16}{7}(\lambda_{8,1}(\lambda_{6,1}+\gamma_{7,1}))]I_2\\
&+\gamma_{0,1}(2\lambda_{4,1}+\gamma_{3,1})I_2'+\frac{\gamma_{4,1}}{3}(2\lambda_{4,1}+\gamma_{3,1})I_4'
\end{aligned}$$

Or we have $I_2'=(4h+1)^{-1}(5I_2-I_0)$ and $I_4'=(4h+1)^{-1}(4hI_0+5I_2)$.

Then we can obtain by using the above information Proposition 2. □

## 3. Picards-Fuchs equations

The results of this section are known, or can be easily deduced, see [12,13,18].

First we note that the affine complex algebraic curve

$$\Gamma_h = \{(x,y) \in \mathbb{C} : H(x,y) = h\}$$

is smooth for $h \neq 0, -1/4$ and has the topological type of a torus with two removed points $\infty^{\pm}$ (at "infinity"). Its homology group is therefore of rang three, the corresponding De Rham group has for generators the (restrictions of) polynomial differential one-forms

$$ydx,\ xydx,\ x^2ydx$$

which are also generators of the related Brieskorn-Petrov $\mathbb{C}[h]$-module [4].

**Lemma 1.** *The integrals $I_i$, $i = 0, 2$, satisfy the following system of Picard-Fuchs:*

$$I_0(h) = \frac{4}{3}hI_0'(h) + \frac{1}{3}I_2'(h) \tag{20}$$

$$I_2(h) = \frac{4}{15}hI_0'(h) + \left(\frac{4}{5}h + \frac{4}{15}\right)I_2'(h) \tag{21}$$

**Proof.** See proof of lemma 5 of Petrov [16] for details. □

The above equations imply the following asymptotic expansions near $h = 0$ (they agree with the Picard-Lefshetz formula)

**Lemma 2.** *The integrals $I_i$, $i = 0, 2$, have the following asymptotic expansions in the neighborhood of $h = 0$:*

$$I_0(h) = (-h + \frac{3}{8}h^2 - \frac{35}{64}h^3 + ...)\ln h + \frac{4}{3} + a_1h + a_2h^2 + ...$$

$$I_2(h) = (\frac{1}{2}h^2 - \frac{5}{8}h^3 - \frac{315}{256}h^4 ...)\ln h + \frac{16}{15} + 4h + b_2h^2 + ...$$

**Proof.** For proof see [8]. □

## 4. Zeros of Abelian integrals in a complex domain

Our goal will be to find the upper bounds number of the zeroes of the Abelian integrals defined in (8) and (19) on the interval of existence of the ovals $\{\gamma(h)\}$.

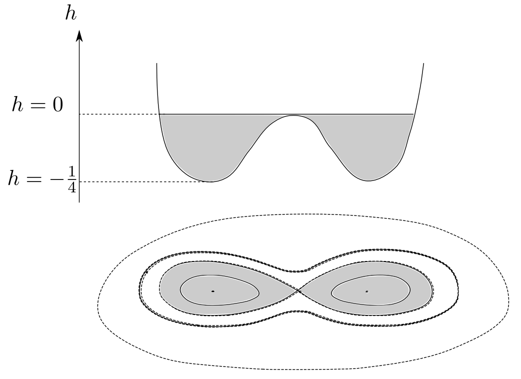


**Fig. 2.** Phase portrait of $X_0$ and the graph of $-\frac{x^2}{2}+\frac{x^4}{4}$.

All families of cycles will depend continuously on a parameter $h$ and will be defined without ambiguity in the complex half-plane $h : Im(h) > 0$. This will allow a continuation on $\mathbb{C}$ along any curve avoiding the real critical values of $H$.

We use the well known Petrov method which is based on the argument principle. This gives an information on the complex limit cycles of the system in the interior and exterior eight-loop, see later Lemmas 3, 5 and 6, respectively.

Our primary motivation was that the complex methods we use, are necessary to understand the bifurcations from the separatrix eight-loop. Another reason is, that the complexity of the bifurcation set of $M_1$, $M_2$ in a complex domain is directly related to the number of the zeros of $M_1$, $M_2$. This observation can be possibly generalized to higher genus curves.

### *4.1. The interior eight-loop case*

In this section, we consider the interior eight-loop case, with period annulus as shown in Fig. 2 (hatched part). Let $\gamma(h) \subset \{H = h\}$ be the continuous family of ovals of the non-perturbed system defined on the maximal open interval $\Sigma = (h_c, h_s)$, where for $h = h_c = -\frac{1}{4}$ the oval degenerates into two centers $\delta_{-1}$, $\delta_1$ respectively at the singular point $(-1, 0)$, $(1, 0)$ and for $h = h_s = 0$ every oval $\delta_{-1}$ or $\delta_1$ becomes a homoclinic loop of the Hamiltonian $dH = 0$.

The family $\delta_h$ represents a continuous family of cycles vanishing at the centers $\delta_{-1}$ and $\delta_1$.

**Theorem 1.** *The maximal cyclicity of the interior period annulus* $\{(x, y) \in \mathbb{R}^2 : H(x, y) > 0\}$ *of* $dH = 0$ *with respect to one-parameter analytic deformation (1) is*

**(i)** *three, if* $M_1 \neq 0$.
**(ii)** *four, if* $M_1 = 0$ *but* $M_2 \neq 0$.

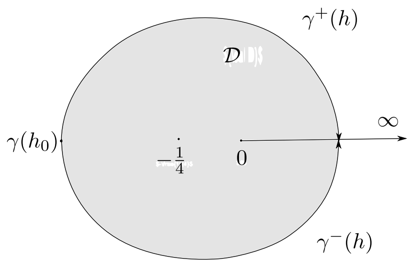


**Fig. 3.** The analytic continuation of a cycle $\gamma(h)$ in the domain $\mathcal{D} = \mathbb{C} \setminus [0, +\infty)$.

*4.1.1. The monodromy of Abelian integrals*

An Abelian integrals $I(h)$ of the form (2) is a multivalued analytic functions in $h \in \mathbb{C}$, single-valued in the complex domain (Fig. 3)

$$\mathcal{D} = \mathbb{C} \setminus [0, +\infty).$$

Moreover, along the segment $[0, +\infty)$ the integral $I(h)$ has a continuous limit when $h \in \mathcal{D}$ tends to a point $h_0 \in [0, +\infty)$. Namely, for $h \in \mathcal{D}$, let $\{\gamma(h)\}_h$ be a continuous family of cycles, vanishing at the saddle point as $h$ tends to $h_s = 0$.

The family $\{\gamma(h)\}_h$ has two analytic complex-conjugate continuations on $(-\infty, 0)$, depending on the way in which the $h$ approaches this segment $[0, +\infty)$. For $h \in (0, +\infty)$ denote $\gamma(h) = \gamma^+(h)$ the limit obtained when $Im(h) > 0$. The cycle $\gamma^-(h)$ is defined in a similar way. It is important to note, the as $I(h)$ is real-analytic on $(-\infty, 0)$, then $\gamma^-(h) = \overline{\gamma^+(h)}$ for $h \in (0, +\infty)$ (as follows also from the Schwarz reflection principle). Finally, the Picard-Lefschetz formula implies

$$\gamma^+(h) = \gamma^-(h) + \delta_0(h)$$

where $\delta_0(h)$ is a continuous family of cycles vanishing at the saddle point as $h \to 0$.

*4.1.2. Zeros of the first return map in a complex domain*

**Lemma 3.** *The first non-vanishing Poincaré-Pontryagin-Melnikov function (8) has at most three zeros in the complex domain $\mathcal{D}$.*

**Lemma 4.** *The second Poincaré-Pontryagin-Melnikov function (15) of the first return map has at most four zeros in the complex domain $\mathcal{D}$.*

**Proof of Lemma 3.** It follows from theorem of Petrov [16]. We sketch the proof:

We denote

$$M_1(h) = \alpha_0(h)I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h) = \oint_{\gamma(h)} \omega = I_\omega(h), h \in \mathcal{D}$$

The monodormy of $I_1$ is $I_1$ on the ray $\{0 < h\}$ (because of symmetry). Then $I_1(h) = a + bh = c(4h + 1)$, where $c \in \mathbb{R}$. Indeed, $I_1(h)$ is univalued, of moderate growth, has no poles, vanishes at $h = -1/4$, and grows no faster that $h$ as $h$ tends to infinity. It follows that

$$M_1(h) = \alpha_0(h)I_0(h) + \alpha_2 I_2(h) + c(4h + 1).$$

We shall use the argument principle for analytic functions in the domain

$$\mathcal{D}_R = \mathcal{D} \cap \{|h| \leq R\}$$

as follows. Consider a contour encircling $\mathcal{D}_R$. The number of zeros of the integral $M_1(h)$ in this domain is the number of rotations of the curve described by $M_1(h)$ about the origin as $h$ describes the contour.

• As $h$ describes the circle $\{|h| = R\}$, for some fixed sufficiently big $R > 0$, the integral $M_1(h)$ behaves as $h^{\frac{7}{4}}$. Thus the increase of the argument of $M_1(h)$ is close to $\frac{7\pi}{2} < 4\pi$.

• Along the cut $[0, \mathbb{R}]$, the number of zeros of $M_1(h)$ about the origin is bounded by the number of zeros of the imaginary part of $M_1$, and

$$Im\, M_1(h) = \int\limits_{\delta_0(h)} \omega, \text{ where } \delta_0 = \gamma^+ - \gamma^-.$$

Therefore

$$Im\, M_1(h) = \alpha_0(h) \oint\limits_{\delta_0(h)} ydx + \alpha_2 \oint\limits_{\delta_0(h)} x^2 ydx, \quad h \in [0, R]$$

and by lemmas 7 and 8 of Petrov [16] cannot exceed 1. We conclude that the total increase of the argument of $M_1$ along the border of $\mathcal{D}_R$ can not exceed three, which proves Lemma 3. □

**Proof of Lemma 4.** We denote

$$M_2(h) = (\alpha_0 + 4\alpha_1 h)I_0 + (\beta_0 + 4h\beta_1)I_1 + \rho I_2 = \oint\limits_{\gamma(h)} w = I_w(h), h \in \mathcal{D},$$

where $\alpha_i$, $\beta_i$ and $\rho$ are defined in (15).

By making use the expression of $I_1 = c(4h + 1)$. Then

$$M_2(h) = \mu(h) + \alpha_0(h)I_0(h) + \alpha_2 I_2(h) + \rho I_2$$

where

$$\alpha_0(h) = \alpha_0 + 4\alpha_1 h$$

$$\mu(h) = 16ch^2(-\frac{1}{2}\lambda_{7,1}(\lambda_{3,1} + 2\gamma_{5,1}) + 3\lambda_{7,1}(\lambda_{6,1} + \gamma_{7,1}))$$
$$+ h[-12c(-\frac{1}{2}\lambda_{7,1}(\lambda_{3,1} + 2\gamma_{5,1}) + 3\lambda_{7,1}(\lambda_{6,1} + \gamma_{7,1})) + 4\beta_0] - 3\beta_0$$

and apply, as in the proof of Lemma 3, the argument principle to $M_2$. The number of zeros of the integral in this domain is the number of rotations of the curve described by $M_2(h)$ about the origin as $h$ describes the border of $\mathcal{D}_R$.

- As $h$ describes the circle $\{|h| = R\}$; the integral $M_2(h)$ behaves as $h^2$ and the increase of the argument of $M_2(h)$ is close to $4\pi$.
- Along the cut $(0, R]$, the number of zeros of $M_2(h)$ about the origin is bounded by the number of zeros of the imaginary part of $M_2(h)$ and

$$Im\, M_2(h) = \alpha_0(h) \oint_{\delta_0(h)} ydx + (\alpha_2 + \rho) \oint_{\delta_0(h)} x^2 ydx, \quad h \in [0, R].$$

Lemmas 7 and 8 of Petrov [16] imply that the number of the zeros of $Im\, M_2(h)$ cannot exceed 1.

Consequently, the total number of circuits cannot exceed four, which implies Lemma 4 and hence Theorem 1. □

### *4.2. The exterior eight-loop case*

In this section we consider the exterior eight-loop case, with period annulus as shown in Fig. 1. Let $\gamma(h)_h$ be the continuous family of exterior ovals of the non-perturbed system defined on the maximal open interval $\Sigma = (0, +\infty)$, where

$$\gamma(h) \subset \{H = h\}.$$

**Theorem 2.** *The maximal cyclicity of the exterior period annulus $\{(x, y) \in \mathbb{R}^2 : H(x, y) > 0\}$ of $dH = 0$ with respect to one-parameter analytic deformation (1) is*

**(i)** *two, if $M_1 \neq 0$.*
**(ii)** *four, if $M_1 = 0$ but $M_2 \neq 0$.*

**Remark 1.** The above Theorem claims that from any compact, contained in the open exterior period annulus $\{(x, y) \in \mathbb{R}^2 : H(x, y) > 0\}$, bifurcate at most four limit cycles (if $M_2 \neq 0$). It says nothing about the limit cycles bifurcating from the separatrix eight-loop or from infinity (i.e. the equator of the Poincaré sphere).

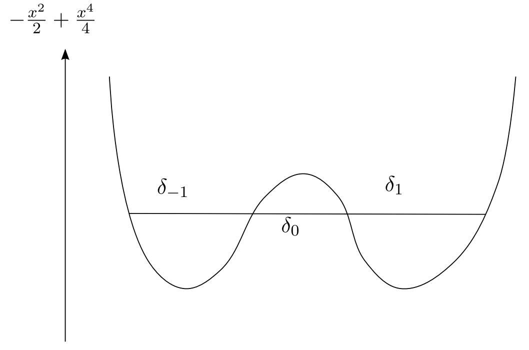


**Fig. 4.** The vanishing cycles $\delta_0(h), \delta_1(h), \delta_{-1}(h)$ for $-\frac{1}{4} < h < 0$.

*4.2.1. The monodromy of Abelian integrals*

The Abelian integrals $I(h)$ of the form (2) are multivalued functions in $h \in \mathbb{C}$ which become single-valued analytic functions in the complex domain

$$\mathcal{D} = \mathbb{C} \setminus [0, -\infty).$$

Along the segment $[0, -\infty)$ the integrals have a continuous limit when $h \in \mathcal{D}$ tends to a point $h_0 \in [0, -\infty)$, depending on the sign of the imaginary part of $h$. Namely, if $Im(h) > 0$ we denote the corresponding limit by $I^+(h)$, and when $Im(h) > 0$ by $I^-(h_0)$. We use a similar notation for the continuous limits of loops $\gamma(h)$ when $h$ tends to the segment $[0, -\infty)$. We have therefore

$$I^{\pm}(h) = \int\limits_{\gamma^{\pm}(h)} \omega$$

where $\omega$ is a polynomial one-form. The monodromy $I^+(h) - I^-(h)$, $h \in [0, -\infty)$ depends therefore on the monodromy of $\gamma(h)$ which is expressed by the Picard-Lefshetz formula. Namely, for $h \in \mathcal{D}$, define the continuous families of closed loops

$$\delta_0(h), \delta_1(h), \delta_{-1}(h)$$

which vanish at the singular points $(0, 0), (0, 1), (0, -1)$ when $h$ tends to 0 or $-1/4$ respectively, and in such a way that $Im(h) > 0$, see Fig. 4. This defines uniquely the homology classes of the loops, up to an orientation. From now on we suppose that the loop $\gamma(h)$ for $h > 0$ is oriented by the vector field $X_0$, and that the orientation of $\delta_0(h), \delta_1(h), \delta_{-1}(h)$ are chosen in such a way that

$$\gamma(h) = \delta_0(h) + \delta_1(h) + \delta_{-1}(h),\ h \in \mathcal{D}.$$

According to the definition of the vanishing cycles

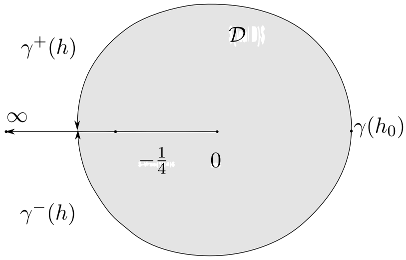


**Fig. 5.** The analytic continuation of a cycle $\gamma(h)$ in the domain $\mathcal{D} = \mathbb{C} \setminus [0, -\infty)$.

$$\gamma^+(h) = \delta_0^+(h) + \delta_1^+(h) + \delta_{-1}^+(h), h \in (-\infty, 0], \tag{22}$$

and the Picard-Lefschetz formula implies

$$\gamma^-(h) = -\delta_0^+(h) + \delta_1^+(h) + \delta_{-1}^+(h), h \in [-1/4, 0] \tag{23}$$

and

$$\gamma^-(h) = -\delta_0^+(h), h \in (-\infty, -1/4] \tag{24}$$

For a further use we note that (Fig. 5)

$$\delta_0^-(h) = \delta_0^+(h), h \in (-1/4, +\infty) \tag{25}$$

$$\delta_1^-(h) = \delta_1^+(h), \delta_{-1}^-(h) = \delta_{-1}^+(h), h \in (-\infty, 0) \tag{26}$$

**Lemma 5.** *The first non-vanishing Poincaré-Pontryagin-Melnikov function (8) has at most two zeros in the complex domain $\mathcal{D}$.*

**Lemma 6.** *The second Poincaré-Pontryagin-Melnikov function (19) of the first return map has at most four zeros in the complex domain $\mathcal{D}$.*

**Lemma 7.** *The Abelian integrals $I_0(h)$ and $I_0'(h)$ do not vanish in $\mathcal{D}$.*

**Proof of Lemma 7.** $I_0'(h)$ is a period of the holomorphic one-form $\frac{dx}{y}$ on the elliptic curve $\Gamma_h$, and therefore does not vanish. For real values of $h$ $I_0'(h)$ represents the period of the orbit $\gamma(h)$ of $dH = 0$, while $I_0(h)$ equals the area of the interior of $\gamma(h)$. It is remarkable, that $I_0(h)$ does not vanish in a complex domain too. Indeed, consider the analytic function

$$F(h) = \frac{I_0(h)}{I_0'(h)}, h \in \mathcal{D}.$$

We shall count its zeros in $\mathcal{D}$ by making use of the argument principle as the proof of previous lemma (see subsection 4.1.2).

*Let $D \subset \mathbb{C}$ be a relatively compact domain, with a piece-wise smooth boundary. We suppose, that $f : D \to \mathbb{C}$ is a continuous function, which is complex-analytic in $D$, except at a finite number of points on the border $\partial D$. We suppose also that $f$ does not vanish on $\partial D$. Denote by $Z_D(f)$ the number of zeros of $f$ in $D$, counted with multiplicity. The increment of the argument $Var_{\partial D}(argf)$ of $f$ along $\partial D$ oriented counter-clockwise is well defined and equals the winding number of the curve $f(\partial D) \subset \mathbb{C}$ about the origin, divided by $2\pi$. The argument principle states then that*

$$2\pi Z_D(f) = Var_{\partial D}(argf) \tag{27}$$

Apply now the formula (27) to the function $F$ in the intersection of a big disc with a radius $R$ and the complex domain $\mathcal{D}$. Along the circle of radius $R$, for $R$ sufficiently big, the decrease of the argument of $F$ is close to $2\pi$, while along the branch cut $(-\infty, 0)$ we have

$$\begin{aligned}2\sqrt{-1}Im(F(h) = F^+(h) - F^-(h) &= \frac{I_0(h)}{I_0'(h)} - \frac{\overline{I_0(h)}}{\overline{I_0'(h)}} \\ &= \frac{\oint_{\gamma^+} ydx}{\oint_{\gamma^+} \frac{dx}{y}} - \frac{\oint_{\gamma^-} ydx}{\oint_{\gamma^-} \frac{dx}{y}} = \frac{W(h)}{|\oint_{\gamma^+} \frac{dx}{y}|^2},\end{aligned}$$

where

$$W(h) = det \begin{pmatrix} \oint_{\gamma^+} ydx & \oint_{\gamma^+} \frac{dx}{y} \\ \oint_{\gamma^-} ydx & \oint_{\gamma^-} \frac{dx}{y} \end{pmatrix}.$$

According to subsection 4.2.1, the function has two different determinations along $(-\infty, -1/4)$ and $(-1/4, 0)$, both of which have no monodromy, and hence are rational in $h$. In fact, (20) implies that $W(h)$ is a non-zero constant. If $W(h) = c$ in $(-\infty, -1/4)$, then it equals $2c$ in $(-1/4, 0)$. Therefore along the branch cut the argument of $F^+$ or $F^-$ increases by at most $\pi$. Summing up the above information, we conclude that $F$ has no zeros in $\mathcal{D}$. □

**Proof of Lemma 5.** We denote

$$F(h) = \frac{M_1(h)}{I_0(h)} = M_1(h) = \alpha_0(h) + \alpha_2 \frac{I_2(h)}{I_0(h)}, h \in \mathcal{D}$$

We apply, as in the proof of Lemma 7, the argument principle to $F$. Along a big circle the increase of the argument of $F$ is close to $\pi$. Along the branch cut $(-\infty, 0]$ we have

$$2\sqrt{-1}Im(F(h)) = F^+(h) - F^-(h) = \alpha_2 \frac{W(h)}{|I_0(h)|^2}$$

where

$$W(h) = \det \begin{pmatrix} \oint_{\gamma^+} yx^2dx & \oint_{\gamma^+} ydx \\ \oint_{\gamma^-} yx^2dx & \oint_{\gamma^-} ydx \end{pmatrix} = ch(4h+1), c = const. \neq 0.$$

Therefore the imaginary part of $F(h)$ along the branch cut $(-\infty, 0)$ vanishes at most once, at $-1/4$ Summing up the above information, we get that $F$ has at most two zeros in the complex domain $\mathcal{D}$. □

**Proof of Lemma 6.** We denote

$$F(h) = (4h+1)\frac{M_2(h)}{I_0(h)}, h \in \mathcal{D},$$

and apply, as in the proof of Lemma 7, the argument principle to $F$. By making use of (19) we have

$$F(h) = \mu(h)\frac{I_2(h)}{I_0(h)} + \lambda(h) \tag{28}$$

where

$$\lambda(h) = \alpha_0 + 4\alpha_1 h + 4\alpha_2 h^2,\ \mu(h) = \beta_0 + 4\beta_1 h. \tag{29}$$

Along a big circle the increase of the argument of $F$ is close to $4\pi$. Along the branch cut $(-\infty, 0]$ we have as before

$$2\sqrt{-1}Im(F(h)) = F^+(h) - F^-(h) = \mu(h)\frac{W(h)}{|I_0(h)|^2}$$

where

$$W(h) = \det \begin{pmatrix} \oint_{\gamma^+} yx^2dx & \oint_{\gamma^+} ydx \\ \oint_{\gamma^-} yx^2dx & \oint_{\gamma^-} ydx \end{pmatrix} = ch(4h+1), c = const. \neq 0.$$

Therefore the imaginary part of $F(h)$ along the branch cut $(-\infty, 0)$ vanishes at most two, at $-1/4$ and at the root of $\mu(h)$. Summing up the above information, we get that $F$ has at most four zeros in the complex domain $\mathcal{D}$. □

## Declaration of competing interest

No competing interest.

## Acknowledgements

We are grateful to the referee for the valuable remarks and corrections. The text was written while the first two authors were visiting the Institute of Mathematics of Toulouse. There are obliged for the hospitality. LG has been partially supported by the Grant No DN 02-5 of the Bulgarian Fund "Scientific Research".

## References

[1] P. Deligne, Équations différentielles à points singuliers réguliers, Lect. Notes Math., vol. 163, Springer-Verlag, 1970.
[2] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four: IV. Figure eight-loop, J. Differ. Equ. 188 (2003) 512–554.
[3] J.P. Françoise, Successive derivates of a first return map, application to the study of quadratic vector field, Ergod. Theory Dyn. Syst. 16 (1996) 87–96.
[4] L. Gavrilov, Petrov modules and zeros of Abelian integrals, Bull. Sci. Math. 122 (8) (1998) 571–584.
[5] L. Gavrilov, The infinitesimal 16th Hilbert problem in the quadratic case, Invent. Math. 143 (3) (2001) 449–497.
[6] L. Gavrilov, On the number of limit cycles which appear by perturbation of Hamiltonian two-saddle cycles of planar vector fields, Bull. Braz. Math. Soc., New Series 42 (1) (2011) 1–23.
[7] L. Gavrilov, I.D. Iliev, The displacement map associated to polynomial unfoldings of planar Hamiltonian vector fields, Am. J. Math. 127 (2005) 1153–1190.
[8] L. Gavrilov, I.D. Iliev, Perturbations of quadratic Hamiltonian two-saddle cycles, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 32 (2015) 307–324.
[9] Lubomir Gavrilov, Abelian integrals related to Morse polynomials and perturbations of plane Hamiltonian vector fields, Ann. Inst. Fourier (Grenoble) 49 (2) (1999) 611–652.
[10] L. Gavrilov, Higher order Poincaré-Pontryagin functions and iterated path integrals, Ann. Fac. Sci. Toulouse Math. (6) 14 (4) (2005) 663–682.
[11] I.D. Iliev, On second order bifurcations of limit cycles, J. Lond. Math. Soc. 58 (2) (1998) 353–366.
[12] I.D. Iliev, L.M. Perko, Higher order bifurcations of limit cycles, J. Differ. Equ. 154 (1999) 339–363.
[13] J.M. Jebrane, H. Żoładek, Abelian integrals in nonsymmetric perturbation of symmetric Hamiltonian vector field, Adv. Appl. Math. 15 (1) (1994) 1–12.
[14] C. Liu, Estimate of the number of zeros of Abelian integrals for an elliptic Hamiltonian with figure-of-eight loop, Nonlinearity 16 (2003) 1151–1163.
[15] G.S. Petrov, Elliptic integrals and their non-oscillatorness, Funct. Anal. Appl. 20 (1986) 46–49.
[16] G.S. Petrov, Complex zeros of an elliptic integral, Funct. Anal. Appl. 21 (1987) 87–88.
[17] G.S. Petrov, Nonoscillatorness of elliptic integrals, Funct. Anal. Appl. 24 (1990) 45–50.
[18] C. Rousseau, H. Żołądek, Zeroes of complete elliptic integrals for 1:2 resonance, J. Differ. Equ. 94 (1991) 41–54.