



Centers of reversible cubic perturbations of the symmetric 8-loop Hamiltonian

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Abstract. We show that the center set of reversible cubic systems, close to the symmetric Hamiltonian system $x' = y, y' = x - x^3$, has two irreducible components of co-dimension two in the parameter space. One of them corresponds to the Hamiltonian stratum, the other to systems which are a polynomial pull back of an appropriate linear system.

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1. Introduction. The paper is a contribution to the study of the center set of plane polynomial cubic vector fields, in a neighbourhood of the symmetric Hamiltonian cubic vector field X_0

$$X_0 : \begin{cases} \dot{x} = y, \\ \dot{y} = x - x^3. \end{cases} \quad (1.1)$$

X_0 is Hamiltonian, and has a first integral

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

Under a small analytic perturbation, the centers of X_0 near $(1, 0)$ and $(-1, 0)$ are either simultaneously destroyed, or simultaneously persistent. The set of cubic vector fields close to X_0 and having a center near $(\pm 1, 0)$ is the *center set* \mathcal{C}_0 . It is known that the center set is an algebraic set in the space of parameters, but even the number of its irreducible components is unknown.

Recently, Iliev, Li, and Yu [4] studied special one-parameter families of perturbations of the form

$$X_\varepsilon : \begin{cases} \dot{x} = y + \varepsilon P(x, y), \\ \dot{y} = x - x^3 + \varepsilon Q(x, y), \end{cases} \quad (1.2)$$

where P, Q are arbitrary *fixed* real cubic polynomials. The displacement function near the singular points $(\pm 1, 0)$ has an analytic expansion

$$d(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \dots \tag{1.3}$$

where as usual h is the restriction of the Hamiltonian H on a cross-section of the vector field. The so called Melnikov functions M_k vanish if and only if the displacement map is the zero map, that is to say, the centers $(1, 0)$ and $(-1, 0)$ are simultaneously persistent. It was shown then in [4, Theorem 1] that if $M_1 = M_2 = M_3 = M_4 = 0$, then the displacement map is identically zero and therefore X_ε has a center near $(1, 0)$ and $(-1, 0)$. The set of such a system is an algebraic set \mathcal{C}^l contained in \mathcal{C} . It turns out that \mathcal{C}^l is a union of vector spaces, and a vector field X_ε which belongs to an irreducible component of \mathcal{C}^l is either Hamiltonian, or y -reversible, or x -reversible. It is clear that when a vector field X_ε is Hamiltonian, or y -reversible, then it has a center near $(\pm 1, 0)$. If, however, the vector field is x -reversible, it does not follow that it has a center near $(\pm 1, 0)$. Therefore, it makes sense to consider the case of x -reversible systems separately.

The purpose of this paper is to give a complete description of the center set \mathcal{C} under the restriction that the vector field is x -reversible, that is to say, the associated foliation by orbits is invariant under the involution $(x, y) \rightarrow (-x, y)$. Our approach is the following. The invariance under $x \rightarrow -x$ suggests to introduce the quotient vector field which is still polynomial and cubic. It turns out that this new vector field is of Liénard type, whose centers were extensively studied. We apply a classical result of Čerkas, revisited recently by Chrystopher, see [1–3]. As a result, we obtain that the center set has two co-dimension two smooth irreducible components which correspond either to Hamiltonian systems, or to systems obtained as a polynomial pull back from linear systems. From this, the result of the paper follows.

2. Statement of the results. Consider the following perturbed cubic system

$$X_\lambda : \begin{cases} \dot{x} = y + P(x, y), \\ \dot{y} = x - x^3 + Q(x, y), \end{cases} \tag{2.1}$$

where

$$P(x, y) = \sum_{i+j \leq 3} a_{ij} x^i y^j, \quad Q(x, y) = \sum_{i+j \leq 3} b_{ij} x^i y^j,$$

$\lambda = \{a_{ij}, b_{ks}\}$ are small parameters. For $\lambda = 0$, the system X_0 has a first integral

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

and two centers at $(x, y) = (\pm 1, 0)$ shown in Fig. 1. The perturbed vector field X_λ has therefore a saddle, close to the origin $(0, 0)$, as well as two anti-saddles (centers or foci) close to $(\pm 1, 0)$. The anti-saddles near $(\pm 1, 0)$ are either simultaneously centers, or saddles¹.

¹this fact is obvious if the vector field X_λ is x -reversible, but holds true for arbitrary analytic perturbations too

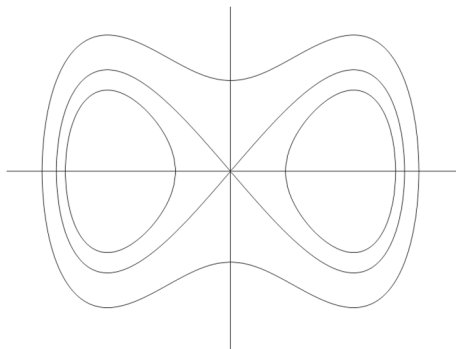


FIGURE 1. The phase portrait of (1.1)

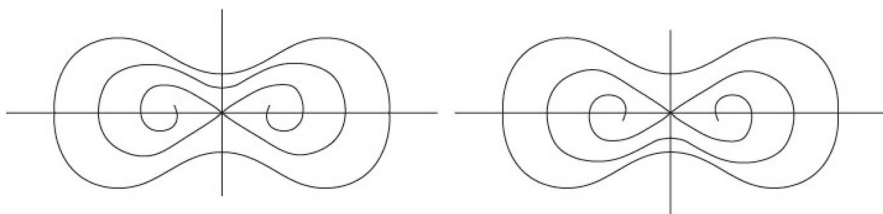


FIGURE 2. The phase portrait of the perturbed reversible system (2.2) with two foci. The exterior period annulus persists

The center set \mathcal{C}_0 of *small* parameters λ for which the vector field has a center near $(\pm 1, 0)$ is a germ of an analytic set in the space of parameters λ . The set \mathcal{C}_0 is in fact algebraic, it is globally defined as the zero set of a finite family of polynomials in λ , but its number of irreducible components is not known in general (Fig. 2).

The purpose of the present paper is to describe \mathcal{C}_0 in the particular case when X_λ is reversible in x . More precisely

Definition 1. The vector field (2.1) is said to be reversible with respect to x provided that the involution $x \mapsto -x$ sends X_λ to $-X_\lambda$, or equivalently $P(-x, y) = P(x, y)$, $Q(-x, y) = -Q(x, y)$. The set of reversible cubic systems (2.2) having a center near $(\pm 1, 0)$ is denoted by \mathcal{C}_0^R .

The set X_λ^R of vector fields reversible in x

$$X_\lambda^R : \begin{cases} \dot{x} = y + P^R(x, y), \\ \dot{y} = x - x^3 + Q^R(x, y), \end{cases} \quad (2.2)$$

is therefore parameterised by the space of special cubic polynomials P^R, Q^R of the form

$$P^R(x, y) = a_{00} + a_{20}x^2 + a_{21}x^2y + a_{01}y + a_{02}y^2 + a_{03}y^3, \quad (2.3)$$

$$Q^R(x, y) = b_{10}x + b_{11}xy + b_{12}xy^2 + b_{30}x^3, \quad (2.4)$$

where $\lambda = \{a_{ij}, b_{ks}\}$ are small real parameters. The possible phase portraits of the perturbed vector field X_λ^R in the finite plane (that is to say, in a disc of finite radius) is shown in Fig. 1, where there is some unknown number of limit cycles. We note that the orbits which belong to the exterior period annulus are always closed because the vector field X_λ^R in consideration is reversible. We have a strict inclusion $\mathcal{C}_0^R \subset \mathcal{C}_0$ and \mathcal{C}_0^R is an algebraic set in the parameter space. We shall prove

Theorem 2.1. *The center set \mathcal{C}_0^R of reversible cubic systems X_λ^R with a center near $(\pm 1, 0)$ has two irreducible components of co-dimension two in the set of polynomials (2.3), (2.4). The components correspond either to Hamiltonian systems or to systems which are obtained as a polynomial pull back from an appropriate linear system.*

To explicitly describe \mathcal{C}_0^R , we shall normalise X_λ first as follows. Note that the affine transformations

$$(x, y) \mapsto (\alpha x, y + \beta)$$

transform a reversible cubic system to a reversible cubic system of the same form, and therefore act on the parameter space and the center set \mathcal{C}_0^R .

Therefore, performing an appropriate affine change of x, y , we may assume that X_λ^R has a singular point at $(1, 0)$ for all sufficiently small λ and by the x -reversibility, it will have another singular point at $(-1, 0)$. The normalised vector field (2.2) takes the form

$$X_\lambda^R : \begin{cases} \dot{x} = y + a_{20}(x^2 - 1) + a_{21}x^2y + a_{01}y + a_{02}y^2 + a_{03}y^3, \\ \dot{y} = x - x^3 + b_{30}(x^3 - x) + b_{11}xy + b_{12}xy^2, \end{cases} \quad (2.5)$$

(we denote the coefficients of this normalised reversible vector field by the same letters a_{ij}). Theorem 2.1 is an obvious consequence of the following

Theorem 2.2. *The system (2.5) has a non-degenerate real center at $(\pm 1, 0)$ if and only if*

$$2a_{20} + b_{11} = 0$$

and either

- $a_{21} + b_{12} = 0$ (Hamiltonian case), or
- $(1 - b_{30})a_{02} = a_{20}(2a_{21} - b_{12})$ (pull back case).

In the second case, the system (2.5) is a polynomial pull back of a linear system under the map

$$(x, y) \rightarrow (x^2 - 1 + P_2(x), y^2), P_2(y) = \frac{1}{b_{30} - 1}(b_{11}y + b_{12}y^2).$$

Note that the trace of X_λ^R at $(\pm 1, 0)$ equals $2a_{20} + b_{11}$. To determine the center conditions of (2.5), we shall use the Cherkas-Christopher theorem which we explain in the next section.

3. The Cherkas-Christopher theorem. Consider the plane cubic differential system

$$\begin{cases} \dot{\xi} = P_3(y) + P_1(y)\xi, \\ \dot{y} = -\xi - P_2(y), \end{cases} \quad (3.1)$$

where

$$P_1(y) = a_0 + a_1y, \quad P_2(y) = b_1y + b_2y^2, \quad P_3(y) = c_1y + c_2y^2 + c_3y^3 \in \mathbb{R}[y]. \quad (3.2)$$

Upon substituting $\xi \rightarrow R(x, y)$ in (3.1), where R is a quadratic polynomial, we get a new cubic system (3.1^{*}), which is a pull back of (3.1) under the polynomial map $(\xi, y) = (R(x, y), y)$. In such a way, centers of (3.1) produce new centers of more general cubic systems.

In this section, we determine the necessary and sufficient conditions so that the cubic system (3.1) has a non-degenerate singular point at the origin of center type. We assume therefore, through this section, that (3.1) already has a linear center at the origin, that is to say,

$$a_0 - b_1 = 0, \quad c_1 - a_0b_1 > 0. \quad (3.3)$$

Substituting $\xi = u - P_2(y)$ in (3.1), we get

$$\begin{aligned} udu + [u((a_0 - b_1) + (a_1 - 2b_2)y) + (c_1 - a_0b_1)y \\ + (c_2 - a_0b_2 - a_1b_1)y^2 + (c_3 - a_1b_2)y^3] dy = 0, \end{aligned}$$

which is the polynomial foliation of the Liénard equation

$$\begin{cases} \dot{y} = u, \\ \dot{u} = -q(y) - up(y), \end{cases} \quad (3.4)$$

where

$$p(y) = a_0 - b_1 + (a_1 - 2b_2)y, \quad (3.5)$$

$$q(y) = (c_1 - a_0b_1)y + (c_2 - a_0b_2 - a_1b_1)y^2 + (c_3 - a_1b_2)y^3. \quad (3.6)$$

The center set of the above Liénard system is well known, see [1, 3]. It has a center at the origin if and only if the primitives $P(y) = \int p(y)dy$ and $Q(y) = \int q(y)dy$ have a common composition factor $W(y)$ with a Morse critical point at the origin. Therefore $a_0 - b_1 = 0$, $W = P$, and there exists a degree two polynomial \tilde{Q} such that $Q(y) = \tilde{Q}(W(y))$. The latter is equivalent to the condition that Q is even when $p \neq 0$. In the case $p = 0$, the system is obviously Hamiltonian. We summarise the result in the following

Theorem 3.1. *The system of Liénard type (3.1) has a non-degenerate center at the origin if and only if*

$$a_0 - b_1 = 0, \quad c_1 - a_0b_1 > 0$$

and either

- $a_1 - 2b_2 = 0$ (Hamiltonian case), or
- $c_2 - a_0(a_1 + b_2) = 0$ (pull back case).

4. Proof of Theorem 2.2. The substitution $\xi = x^2 - 1$ takes the system (2.5)

$$(y + a_{20}(x^2 - 1) + a_{21}x^2y + a_{01}y + a_{02}y^2 + a_{03}y^3)dy - (x - x^3 + b_{30}(x^3 - x) + b_{11}xy + b_{12}xy^2)dx = 0$$

to the Liénard form (3.1)

$$(\xi + P_2(y))d\xi + (P_3(y) + P_1(y)\xi)dy = 0,$$

where

$$\begin{aligned} P_1(y) &= \frac{2}{1 - b_{30}}(a_{20} + a_{21}y), \\ P_2(y) &= \frac{1}{b_{30} - 1}(b_{11}y + b_{12}y^2), \\ P_3(y) &= \frac{2}{1 - b_{30}}((1 + a_{01} + a_{21})y + a_{02}y^2 + a_{03}y^3). \end{aligned}$$

Applying Theorem 3.1 with

$$\begin{aligned} a_0 &= \frac{2a_{20}}{1 - b_{30}}, & a_1 &= \frac{2a_{21}}{1 - b_{30}}, \\ b_1 &= -\frac{b_{11}}{1 - b_{30}}, & b_2 &= -\frac{b_{12}}{1 - b_{30}}, \\ c_1 &= \frac{2(1 + a_{01} + a_{21})}{1 - b_{30}}, & c_2 &= \frac{2a_{02}}{1 - b_{30}}, & c_3 &= \frac{2a_{03}}{1 - b_{30}}, \end{aligned}$$

we obtain the equation for the center set \mathcal{C}_0^R for the normalised reversible vector field (2.5).

Finally, to find the first integral in the logarithmic case, we recall that (2.5) is a polynomial pull back under $u = x^2 - 1 + P_2(y)$ of the Liénard system

$$\begin{cases} \dot{y} = u, \\ \dot{u} = -q(y) - up(y), \end{cases} \tag{4.1}$$

where (in the pull back case)

$$\begin{aligned} p(y) &= (a_1 - 2b_2)y, \\ q(y) &= (c_1 - a_0b_1)y + (c_3 - a_1b_2)y^3. \end{aligned}$$

As (4.1) is also y -reversible, it is a pull back of the following linear system

$$\begin{cases} \dot{v} = 2u, \\ \dot{u} = -(c_1 - a_0b_1) - (c_3 - a_1b_2)v - u(a_1 - 2b_2), \end{cases} \quad (4.2)$$

under the map $y \rightarrow v = y^2$. This completes the proof of Theorem 2.2. We note finally that the system (4.2) (as any non-degenerate linear system) has a logarithmic first integral of the form $l_1^\alpha l_2^\beta$ where l_1, l_2 are linear functions in λ, u and α, β are suitable complex numbers. Although (4.2) has a Darboux type first integral, it has no center, except in the Hamiltonian case $a_1 - 2b_2 = 0$. The reversible vector field X_λ^R also has a first integral of Darboux type (pull back of the Darboux first integral of the linear system), but its centers near $(\pm 1, 0)$ are of pull back type. An explicit computation of this integral in some cases can be found in [4, Section 5]. Thus the centers we have found are both of pull back and Darboux type. In the Żoładek's classification of (known) cubic centers, they appear as $CD_4^{(8)}$ [5, 6]. \square

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