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INVARIANT ASYMPTOTIC STABLE TORI IN THE PERTURBED SINE-GORDON EQUATION

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On the basis of the techniques developed in [9] it is shown, that some perturbed nonlinear equations including the sine-Gordon equation, the nonlinear wave equation and the one-dimensional Klein — Gordon equation possess invariant asymptotic stable finite-dimensional tori in a neighbourhood of the origine in a suitable functional space. Each torus can be considered as obtained after a finite number of Andronov — Hopf bifurcations, i. e. the hypothesis of Landau for the evolution of turbulence [7] is realised.

1. Introduction. As Nikolenko [9] has shown, in the perturbed Korteveg — de (Vries — Bürgers equation

$$u_t + u_{xxx} + uu_x = \nu u_{xx} + \varepsilon u + f(u) \tag{1.1}$$

with boundary condition $u(x+2\pi, t) = u(x, t)$ because of the competition between the viscosity ν and the influx of energy supplied from the term $\varepsilon u + f(u)$, there arise complicated limit modes. In particular, for a certain class of nonlinearities f, the equation possesses invariant asymptotic stable tori whose dimension increases when the viscosity decreases. This corresponds to the hypothesis of Landau for the evolution of turbulence ([7], § 27). An analogical phenomenon is observed in the perturbed Schrödinger equation [9]. Equation (1.1) generalize the Korteveg — de Vries equation in the case of viscous flow. We shall note that because of the fact, that the Schrödinger equation and the Korteveg — de Vries equation can be interpreted as infinite-dimensional Hamiltonian systems, their solutions are fibred on finite or infinite-dimensional tori, i. e. the behaviour after the perturbation is qualitative different. All we said up to now holds (as will be shown) for the sine-Gordon equation:

$$u_{tt} = u_{xx} - \sin u, \quad u(x+2\pi, t) = u(x, t). \tag{1.2}$$

It is considered as ordinary differential equation in a suitable Banach space — 2. The corresponding Cauchy problem is investigated in the Appendix. In 3 the techniques of the normal forms is applied to the perturbed equation (1.2). The following normal form of the perturbed sine-Gordon equation is obtained:

$$\frac{d}{dt} u_n = \lambda_n u_n + u_n \Big(\sum_{k=-N}^{N} \Phi_k^n |u_k|^2 \Big) + \dots \tag{1.3}$$

The dots mean nonlinear terms of order higher than three and $N \in \mathbb{Z}$, $\lambda_n = \alpha\delta/2 - \delta n^2 + (\alpha^2\delta^2/4 - \alpha\delta^2 n^2 - \delta^2 n^4 - 1 - n^2)^{1/2}$. Here δ is a small parameter. If we fix the sufficiently small parameter δ and then move α we see that when α passes through the point $2n^2$, then a pair of eigenvalues intersect the imaginary axis. Consider the truncated equation

$$\frac{d}{dt} u_n = \lambda_n u_n + u_n \Big(\sum_{k=-N}^{N} \Phi_k^n . |u_k|^2 \Big). \tag{1.3'}$$

After a finite number of bifurcations in (1.3)′, we shall obtain a finite-dimensional torus, which is invariant and for suitable δ it is asymptotically stable. The main problem is how to prove that this remains true for equation (1.3). Unfortunately the known methods are applicable only for the first and the second bifurcation. The problem of the third bifurcation is not worked out ([3], p. 231). That is why we shall drop this maybe more natural way. Till the end of the paper α will be fixed and we shall not mention the word "bifurcation". Here the above reasonings may be considered like heuristic, but they will be justified by Theorem 4.1. which is the main result of the paper. The techniques used in the proof is roughly speaking the contraction mappings theorem and the idea of the Andronov — Hopf bifurcation.

The same holds for the nonlinear wave equation and the one-dimensional Klein — Gordon equation, as will appear further.

2. Functional spaces and change of variables. Let us consider the differential equations

$$u_{tt}=u_{xx}-\sin u \qquad \text{— sine-Gordon}$$

$$u_{tt}=u_{xx}-u \qquad \text{— Klein — Gordon}$$

$$u_{tt}=u_{xx}-u+V(u) \qquad \text{— nonlinear wave equation}$$

in the periodic case $u(x+2\pi,\ t)=u(x,\ t)$. As it is well known the above equations are Hamiltonian systems. In this paper we shall investigate the qualitative behaviour of the solution of the following perturbed equation:

$$u_{tt}=u_{xx}-u+\alpha\delta u_t+2\delta u_{txx}-2\delta^2 u_{xxxx}+f(u_t)+g(u), \tag{2.1}$$

$$u(x+2\pi,\ t)=u(x,\ t),\quad \alpha\in \mathbf{R}. \tag{2.2}$$

Here f and g are nonlinear operators described further down, $\delta>0$ is a "small" parameter. We denote by W_k the Banach space of 2π-periodic real functions $u(x)=\Sigma_{n\in \mathbf{Z}}u_n e^{inx}$, $u_n=\overline{u}_{-n}$ such that $d^k u/dx^k\in L_2[0,\ 2\pi]$ with a norm $\|u\|_{W_k}=(\Sigma_{n\in \mathbf{Z}}(1+|n|)^{2k}\cdot|u_n|^2)^{1/2}$ $k\geq 1$. We identify any element of W_k with some infinite sequence $(u_0,\ u_{\pm 1},\ u_{\pm 2},\dots)$, where $u_n\in \mathbf{C}$, $\overline{u}_{-n}=u_n$. If we drop the condition $u_n=\overline{u}_{-n}$, with the same norm we shall obtain the Banach space $W_{k,\mathbf{C}}$. The operation "multiplication" in $W_{k,\mathbf{C}}$ is introduced in the following way: $(uv)_n=\Sigma_{p+q=n}u_p v_q$, $u=(u_0,\ u_{\pm 1},\dots)$, $v=(v_0,\ v_{\pm 1},\dots)$. The number u_n we shall call n's Fourier coefficient of the function u. Henceforth $W_{k,\mathbf{C}}$ will be considered like a real Banach space.

Consider the Banach space $\widetilde{W}^k_s$, consisting of the ordered pairs $\binom{u}{v}$, where $u\in W_k$, $v\in W_s$, with a norm $\left\|\binom{u}{v}\right\|_{\widetilde{W}^k_s}=\|u\|_{W_k}+\|v\|_{W_s}$.

If we denote

$$A=\begin{pmatrix} 0 & 1\\ -1+\frac{d^2}{dx^2}-2\delta^2\frac{d^4}{dx^4}, & \alpha\delta+2\delta\frac{d^2}{dx^2}\end{pmatrix},\quad \Phi\binom{u}{v}=\begin{pmatrix}0\\ g(u)+f(v)\end{pmatrix}$$

the equation (2.1), (2.2) takes a form

$$\frac{dw}{dt}=Aw+\Phi(w) \tag{2.3}$$

— ordinary differential equation in a suitable Banach spase $\widetilde{W}^k_s$. With the help of the multiplication introduced thus:

$$\begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} uu' \\ vv' \end{pmatrix}$$

$\widetilde{W}_s^k$ becomes an algebra. What is more:

Lemma 2.1. *$\widetilde{W}_s^k$ is a Banach algebra.*

The proof is a direct consequence of [5], Theorem 2. 17. p. 555. W_k and $W_{k,C}$ are Banach algebras, too. The space $\widetilde{W}_{s,C}^{k,C} = W_{s,C} \times W_{k,C}$ is defined by analogy.

Denote by $\mathscr{H}_0$ the class of all nonlinear operators f defined in some neighbourhood of the origine in W_1 and such that $f\colon u \to \Sigma_{k=2}^{\infty} f_k . u^k$, where f_k are real numbers and the radius of convergency of the power series $\Sigma_{k=2}^{\infty} f_k . z^k$, $z \in C$ is not a zero. It is clear that the restriction $f|_{W_k}$ maps some neighbourhood of the origin in W_k continuously in W_k for any integer $k \geq 1$. Let S is a neighbourhood of the origin in W_1 and

$$\varphi\colon S \to W_1, \ \varphi(u) = \sum_{k=2}^{\infty} \varphi_k(u)$$

is (Frechet) analytic operator in u, i. e. there the above power series is sumable and $\varphi_k\colon W_1 \to W_1$ are homogeneous of order k, continuous operators such that

$$\varphi_k(u) = \widetilde{\varphi}_k \underbrace{(u, u, \ldots, u)}_{k\text{-times}}$$

for some k — real linear continuous symmetrical operator

$$\widetilde{\varphi}_k\colon \underbrace{W_1 \times W_1 \times \ldots \times W_1}_{k\text{-times}} \to W_1.$$

The n's Fourier coefficient of the function $(\varphi_k(u))(x)$ has a form

$$\varphi_k(u)_n = \frac{1}{2\pi} \cdot \int_0^{2\pi} \varphi_k(u)(x) e^{-inx}\, dx = \sum_{p_1, p_2, \ldots, p_k \in Z} \varphi^n_{p_1, p_2, \ldots, p_k} \cdot u_{p_1}, u_{p_2} \ldots u_{p_k},$$

where $u_p = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ipx}\, dx$ and $\varphi^n_{p_1, p_2, \ldots, p_k}$ are complex numbers. Let φ and ψ are two analytic operators. If $|\varphi^n_{p_1, p_2, \ldots, p_k}| \leq \psi^n_{p_1, p_2, \ldots, p_k} \geq 0$, we shall say that ψ majorizes φ in their common domain of definition (it is clear in this case the domain of φ contains the domain of ψ). The class of operators analytic in some neighbourhood $S_{W_1}(0, r)$ in W_1 and majorized there by operators of $\mathscr{H}_0$ we shall denote by $\mathscr{H}$. If $\varphi \in \mathscr{H}_0$ and $\varphi\colon u \to \Sigma_{k=2}^{\infty} f_k . u^k$, then it is easy to see that $\varphi^n_{p_1, p_2, \ldots, p_k} = 0$ for $\Sigma_{i=1}^k p_i \neq n$ and $\varphi^n_{p_1, \ldots, p_k} = f_k$ for $\Sigma_{i=1}^k p_i = n$. Hence, if $\psi \in \mathscr{H}$, then $\psi^n_{p_1, p_2, \ldots, p_k} = 0$ for $\Sigma_{i=1}^k p_i \neq n$.

Denote by $\mathscr{M}$ the subset of $\mathscr{H}$, consisting of nonlinear homogeneous operators of order three. Henceforth we shall suppose that $f \in \mathscr{M}$, $g \in \mathscr{H}$, $g(u) = \Sigma_{k=3}^{\infty} g_k(u)$, where $g_k(u)$ are homogeneous operators of order k. Now we shall give some general definitions.

Let the domain of definition of the linear operator A is D_A, of the nonlinear operator Φ is D_Φ, S is a neighbourhood of the zero in some other Banach space in which D_A is a dense set. Let $A\colon D_A \to S$ and $D_A \subset D_\Phi$. Roughly speaking, we call a solution of (2.3) classical or generalized according to that, if for each t it lies in D_A or S respectively. Now we proceed to a full definition. Let S is a neighbourhood of the origin in the Banach space B and D_A is dense set in B. Every continuously differentiable (in the sense of B) function of the time $t \in [0, T]$, taking values in D_A and satisfying in the interval $[0, T]$ the equation (2.3), we shall call classical solution. Every continuous function (in the sense of B) $u(t)$ of the time $t\in[0, T]$, taking values in S, we shall call generalized solution of (2.3), if there exists a sequence of classical solutions uniformly converging on the interval $[0, T]$ to $u(t)$ (with respect to the norm of B).

The problem of finding a generalized solution $u(t)$, $t\in[0, T]$ of (2.3), satisfying given initial conditions, we shall call Cauchy problem. The local solvability of the Cauchy problem for (2.3) with nonlinearities $f, g\in\mathscr{H}$ and suitable S and D_A will be given in the Appendix. This will be enough for our purposes. Hence forth we shall drop the adjective "generalized".

Let $H\colon S\to H(S)\subset B$ is a diffeomorphism, such that holds

$$H(S\cap D_A)=H(S)\cap D_A. \tag{2.4}$$

Consider a classical solution $W(t)$ of (2.3), such that $W(t)\in H(S)\cap D_A$ for each $t\in[0, T]$. Then $S\cap D_A$ will contain the set of all values of the continuously differentiable function $v(t)=H^{-1}(W(t))$. For the vector function $v(t)$ holds the equality $\frac{dw}{dt}=\frac{dH}{dv}\frac{dv}{dt}=(A+\Phi)\circ H(v)$, where $\frac{dH}{dv}\colon B\to B$ is the Frechet derivative of the operator $H\colon S\to B$ at the point $v\in S$. The operator $\frac{dH}{dv}$ is inversible and $(\frac{dH}{dv})^{-1}=\frac{d(H^{-1})}{dw}$. Consequently we can write down

$$\frac{dv}{dt}=\left(\frac{dH}{dv}\right)^{-1}\circ(A+\Phi)\circ H(v),\quad t\in[0, T]. \tag{2.5}$$

The equality $w=H(v)$ connect both the classical solutions and the generalized solutions of (2.3) and (2.5). Really if $\{v_n\}$ is a sequence of classical solutions of (2.5) and $v_n\overset{B}{\rightrightarrows}v$, then $H(v_n)\overset{B}{\rightrightarrows}H(v)$ and therefore $w=H(v)$ is generalized solution of (2.3). The converse is true by reason of (2.4). Thus the correlation (2.4) becomes a correctness condition for the change of variables $w=H(v)$ in equation (2.3).

3. A normal form of the perturbed equation. The reducing of the perturbed equation (2.3) to a normal form will be done by two stages. First we shall diagonalize the linear operator A and then we shall cancel some nonresonance coefficients of the nonlinear part. Further down we suppose that $B=\widetilde{W}_1^3$, $D_A=\widetilde{W}_3^5$. The natural immersion $\widetilde{W}_3^5\hookrightarrow\widetilde{W}_1^3$ is continuous and it has a dense image.

3.1. Here we diagonalize the operator A. It means that in Fourier co-ordinates A will has a form of a diagonal matrix. The eigenfunction of the operator are

$$\left\{\begin{pmatrix}0\\ e^{inx}\end{pmatrix}+\frac{1}{\lambda_n}\begin{pmatrix}e^{inx}\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ e^{inx}\end{pmatrix}+\frac{1}{\bar\lambda_n}\begin{pmatrix}e^{inx}\\ 0\end{pmatrix}\colon n\in\mathbf{Z}\right\}.$$

The corresponding eigenvalues are $\{\lambda_n, \bar\lambda_n\colon n\in\mathbf{Z}\}$, where $\lambda_n=\alpha\delta/2-\delta n^2+\sqrt{\alpha^2\delta^2/4-\alpha\delta^2n^2-\delta^2n^4-1-n^2}$. Consider the linear operator defined like this $L\colon \widetilde{W}_n^{n+2}\to\widetilde{W}_{n,C}^{n,C}$, $n\geq 0$, $n\in\mathbf{Z}$,

$$\begin{pmatrix}v\\ \mathring{v}\end{pmatrix}=L\begin{pmatrix}u\\ \mathring{u}\end{pmatrix},\quad u\in W_{n+2},\ \mathring{u}\in W_n,\ v\in W_{n,C},\ \mathring{v}\in W_{n,C}$$

$$v=(v_0, v_{\pm1},\dots),\quad \mathring{v}=(\mathring{v}_0, \mathring{v}_{\pm1},\dots)$$

$$u=(u_0, u_{\pm1},\dots),\quad \mathring{u}=(\mathring{u}_0, \mathring{u}_{\pm1},\dots)$$

$$v_n=\lambda_n u_n+\mathring{u}_n,\quad \mathring{v}_n=\bar\lambda_n u_n+\mathring{u}_n.$$

Define the operators L_1 and L_2 with the help of the equality

$$L\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} L_1\begin{pmatrix} u \\ \dot{u} \end{pmatrix} \\ L_2\begin{pmatrix} u \\ \dot{u} \end{pmatrix} \end{pmatrix}.$$

Lemma 3.1. *The operator* L_1 *is* R*-linear homeomorphism between* $\widetilde{W}_n^{n+2}$ *and* $W_{n,\mathrm{C}}$

Proof. Let us see first that L_1 is continuous.

$$\| L_1\begin{pmatrix} u \\ \dot{u} \end{pmatrix} \|_{W_{n,\mathrm{C}}} = \| v \|_{W_{n,\mathrm{C}}} = (\sum_{k \in \mathrm{Z}} (1+|k|)^{2n} \cdot |\lambda_k u_k + \dot{u}_k|^2)^{1/2}$$

$$\leq \sqrt{2}(\sum_{k \in \mathrm{Z}} (1+|k|)^{2n} \cdot (|\lambda_k u_k|^2 + |\dot{u}_k|^2))^{1/2} \leq c_1(\sum_{k \in \mathrm{Z}} ((1+|k|)^{2n+4} \cdot |u_k|^2 + (1+|k|^{2n} \cdot |\dot{u}_k|^2))^{1/2}$$

$$\leq c_1(\| u \|^2_{W_{n+2}} + \| \dot{u} \|^2_{W_n})^{1/2} \leq c_1 \| \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \|_{\widetilde{W}_n^{n+2}}.$$

The operator L_1 is inversible. Really $L_1\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = v$, $u_n = (v_n - \bar{v}_{-n}) / (\lambda_n - \bar{\lambda}_n)$, $\dot{u}_n = (\lambda_n \bar{v}_{-n} - \bar{\lambda}_n v_n) / (\lambda_n - \bar{\lambda}_n)$ and from v we can uniquely restore $\begin{pmatrix} u \\ \dot{u} \end{pmatrix}$. Since $\lim \lambda_n / n^2 = -\delta + i\delta$, then L_1 and L_1^{-1} are correctly defined, i. e. L_1 is surrection. With the help of the open mappings theorem (see for example [6], p. 88) we obtain that L_1 is open and hence L_1^{-1} is continuous.

Corollary: *Since* L_1 *is linear operator then* L_1 *is a diffeomorphism. By analogy* L_2 *is also a diffeomorphism.*

Consider again equation (2.3). Let $S = B = \widetilde{W}_1^3$. The function $\Phi : \widetilde{W}_s^k \to \widetilde{W}_{\min(k,s)}^k$ for each integer $k, s \geq 1$ is an analytic. The linear operator $A : \widetilde{W}_3^5 \to \widetilde{W}_1^3$ is continuous. According to the general definitions from 2., we already know what is the meaning of the terms "classical" or "generalized" solution. Let us do in (2.3) change of variables

$$\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = L(w) = \begin{pmatrix} L_1 \, w \\ L_2 \, w \end{pmatrix}.$$

We obtain: $L\frac{dw}{dt} = \frac{d}{dt} Lw = \frac{d}{dt}\begin{pmatrix} v \\ \dot{v} \end{pmatrix} = L \circ (A+\Phi)(w) = L \circ (A+\Phi) \circ L_1^{-1} v.$

As $L_1^{-1} v = w = L_2^{-1} \dot{v}$ the equation mentioned above is equivalent to the following two equations:

$$\frac{d}{dt} v = L_1 \circ (A+\Phi) \circ L_1^{-1} v, \ \frac{d}{dt} \dot{v} = L_2 \circ (A+\Phi) \circ L_2^{-1} \dot{v}. \tag{3.1}$$

If $w \in D_A$, then $v \in L_1(D_A) = W_{3,\mathrm{C}}$. The operators $L_1 A L_1^{-1}$ and $L_2 A L_2^{-1}$ are correctly defined in $W_{3,\mathrm{C}}$. Really

$$L_1 A L_1^{-1}(W_{3,\mathrm{C}}) = L_1 A(\widetilde{W}_3^5) \subset L_1(\widetilde{W}_1^3) = W_{1,\mathrm{C}}$$

$$L_2 A L_2^{-1}(W_{3,\mathrm{C}}) = L_2 A(\widetilde{W}_3^5) \subset L_1(\widetilde{W}_1^3) = W_{1,\mathrm{C}}.$$

The operators $L_1 \circ \Phi \circ L_1^{-1}$ and $L_2 \circ \Phi \circ L_2^{-1}$ are correctly defined in $W_{3,\mathrm{C}}$, too. Denote $\widetilde{\Phi}(v) = L_1 \circ \Phi \circ L_1^{-1} v$ and $\tilde{\tilde{\Phi}}(\dot{v}) = L_2 \circ \Phi \circ L_2^{-1} \dot{v}$. The operator L by construction diagonalize the linear operator A. Consequently in Fourier co-ordinates (3.1) looks like this:

$$\frac{d}{dt}\, \nu_n=\lambda_n\nu_n+\widetilde{\Phi}(\nu)_n,\quad \frac{d}{dt}\,\mathring{\nu}_n=\bar{\lambda}_n\mathring{\nu}_n+\mathring{\widetilde{\Phi}}(\mathring{\nu})_n. \tag{3.2}$$

If $u\in W_k$, then $\bar{u}_n=u_{-n}$ and $\bar{\nu}_n=\mathring{\nu}_{-n}$. From (3.1) we obtain that the both groups of equatons are conjugated and from the first group of equations in (3.2) we can restore completly the second. Further down, as is usually done, we work only with the first group (which is conjugated to the second), all the time keeping in mind the second. Just this we shall mean referring further to (3.2).

Consider in more details the nonlinear operator $\widetilde{\Phi}$, $\widetilde{\Phi}=L_1\circ\Phi\circ L_1^{-1}$. It is easy to see that

$$\widetilde{\Phi}(\nu)=g\Big(\sum_{n\in Z}\frac{\nu_n-\bar{\nu}_{-n}}{\lambda_n-\bar{\lambda}_n}e^{inx}\Big)+f\Big(\sum_{n\in Z}\frac{\lambda_n\bar{\nu}_{-n}+\bar{\lambda}_n\nu_n}{\lambda_n-\bar{\lambda}_n}\Big).$$

Here we shall introduce another two classes of operators, which will be important for us. They will be defined on the analogy of $\mathscr{H}$ and $\mathscr{M}$ in the complex Banach spaces $W_{k,C}$.

The class of analytic operators, defined in some neighbourhood of the origin in $W_{1,C}$, we shall denote by $\mathscr{H}_C$. The class of the homogeneous of order three operators $h\in\mathscr{H}_C$, which in Fourier co-ordinates, have the following form:

$$h(u)_n=\sum_{p+q+s=n}\big(h^n_{p,q,s}u_pu_qu_s+h^n_{\bar{p},q,s}\bar{u}_{-p}u_qu_s+h^n_{p,\bar{q},s}u_p\bar{u}_{-q}u_s$$
$$+h^n_{p,q,\bar{s}}u_pu_q\bar{u}_{-s}+h^n_{\bar{p},\bar{q},s}\bar{u}_{-p}\bar{u}_{-q}u_s+h^n_{\bar{p},q,\bar{s}}\bar{u}_{-p}u_q\bar{u}_{-s}+h^n_{p,\bar{q},\bar{s}}u_p\bar{u}_{-q}\bar{u}_{-s}+h^n_{\bar{p},\bar{q},\bar{s}}\bar{u}_{-p}\bar{u}_{-q}\bar{u}_{-s}\big)$$

where there is an operator $\theta\in\mathscr{M}$, defined in W_1, such that holds the underwritten Assumption 1., we shall denote by $\mathscr{M}_C$.

Assumption 1. For any four integers p, q, s, n such that $p+q+s=n$ holds:

$$\max\big(|h^n_{p,q,s}|,\ |h^n_{\bar{p},q,s}|,\ \dots,\ |h^n_{\bar{p},\bar{q},\bar{s}}|\big)\le\theta^n_{p,q,s}\ge 0.$$

If we have two operators $h\in\mathscr{M}_C$ and $\theta\in\mathscr{M}$ satisfying Assumption 1., then we shall say that θ majorize h. If is given the operator h, formally defined with the help of the set: $\{h^n_{p,q,s},\ h^n_{\bar{p},q,s},\ \dots,\ h^n_{\bar{p},\bar{q},\bar{s}}\}^{n\in Z}_{p+q+s=n}$ and exists an operator $\theta\in\mathscr{M}$ which majorize h, then it follows $h\in\mathscr{M}_C$. Indeed then the power series written for $h(u)_n$ (where $h(u)_n$ is the n's Fourier coefficient of $h(u)$) is majorized by the corresponding power series, written for $\theta'(u)_n$. Here $\theta'\in\mathscr{M}_C$ and is defined by the equality

$$\theta'(u)=\theta\Big(\sum_{k\in Z}(u_k+\bar{u}_{-k})\exp(ikx)\Big).$$

Remember now that $g\in\mathscr{H}$, $f\in\mathscr{M}$ and consequently $\widetilde{\Phi}\in\mathscr{H}_C$. In accordance with the general definitions from 1 we already have a definition for a classical and generalized solution of the problem (3.2). Here $D_{L_1AL_1^{-1}}=W_{3,C}$, $B=W_{1,C}$.

3.2. Let us do in the perturbated equation (2.1) (2.2), written in the equivalent form (3.2) correct change of variables $\nu=u+h(u)=H(u)$, where $h\in\mathscr{M}_C$. For suitable $r>0$ H is a diffeomorphism between $S_{W_{k,C}}(0,r)$ (a ball in $W_{k,C}$ with a centre in the origin and radius r) and its image $H(S_{W_{k,C}}(0,r))$. Indeed then $(I+h)^{-1}=\sum_{k=0}^{\infty}(-h)^k$. Now we can realize that

$$H(S_{W_{1,C}}(0,\ r)) \cap D_{L_1 A L_1^{-1}} = H(S_{W_{1,C}}(0,\ r) \cap D_{L_1 A L_1^{-1}}).$$

That is true because $H(D_{L_1 A L_1^{-1}}) = H(W_{3,C}) \subset W_{3,C} = D_{L_1 A L_1^{-1}}$. Hence the change $v = H(u)$ in (3.2) is actually correct. Let us reduce still further r so that the set $\{u \in W_{1,C} : \| \frac{dh}{du} \| < 1\}$ contain completely $S_{W_{1,C}}(0,\ r)$. It is possible because the sets are neighbourhoods of the origine. Then holds

$$\left(\frac{dH}{du}\right)^{-1} = \left(I + \frac{dh}{du}\right)^{-1} = \sum_{k=0}^{\infty} \left(-\frac{dh}{du}\right)^k$$

and consequently the equation (3.2) has in variables $u = H^{-1}(v)$ in a neighbourhood of the zero in $W_{1,C}$ the following form: $\frac{dv}{dt} = \frac{dH}{du}\ \frac{du}{dt} = L_1 \circ (A + \Phi) \circ L_1^{-1} \circ H(u)$ or

$$\text{(3.3)} \qquad \frac{du}{dt} = \sum_{k=0}^{\infty} \left(-\frac{dh}{du}\right)^k \circ L_1 \circ (A + \Phi) \circ L_1^{-1} \circ (I + h)(u).$$

Denote $\widetilde{A} = L_1 A L_1^{-1}$. Nullify the homogeneous terms of order three on the right-hand side of (3.3) (there are not such terms of order two) we obtain the equation: $-\frac{dh}{du}\widetilde{A}u + \widetilde{A}h(u) + \widetilde{\Phi}_3(u) = 0$ for determination of the operator h. (Remind that $\widetilde{\Phi}(u) = \Sigma_{k=3}^{\infty} \widetilde{\Phi}_k(u)$).

Here we shall give the following definition: Let $f\colon W_{i,C} \to W_{j,C},\ g\colon W_{j,C} \to W_{j,C}$ are differentiable operators, $j \leq i$, and $g(v) \in W_{i,C}$ for $v \in W_{i,C}$. We call the operator $\frac{df}{dv} g(v) - \frac{dg}{dv} f(v)$, $v \in W_{i,C}$ commutator of f and g and denote by $[f(v),\ g(v)]$. For the linear operator $\widetilde{A}$ we shall use the symbol $[\widetilde{A}v,\ g(v)] = \widetilde{A}g(v) - \frac{dg}{dv}\widetilde{A}v$. The equation for determination of h takes a form

$$\text{(3.4)} \qquad [\widetilde{A}u,\ h(u)] + \widetilde{\Phi}_3(u) = 0$$

and is called a homologous equation connected with the linear operator $\widetilde{A}$ ([8], p. 166). But $\frac{dh}{du} v = 3\widetilde{h}(u,\ u,\ v)$. Now (3.4) can be written down in the following form:

$$\text{(3.5)} \qquad \widetilde{A}\,\widetilde{h}(u,\ u,\ u) - \widetilde{h}(\widetilde{A}u,\ u,\ u) - \widetilde{h}(u,\ \widetilde{A}u,\ u) - \widetilde{h}(u,\ u,\ \widetilde{A}u) = -\widetilde{\Phi}_3(u).$$

Recalling that $h \in \mathcal{M}_C$ and taking into consideration that $\lambda_k = \lambda_{-k}$, we obtain:

$$\text{(3.6)} \qquad \Sigma h^n_{p,q,s}\, u_p u_q u_s (\lambda_n - \lambda_p - \lambda_q - \lambda_s) + \Sigma h^n_{\bar{p},q,s} (\lambda_n - \bar{\lambda}_p - \lambda_q - \lambda_s)\, \bar{u}_{-p} u_q u_s$$

$$+ \Sigma h^n_{p,\bar{q},s}\, v_p \bar{u}_{-q} u_s (\lambda_n - \lambda_p - \bar{\lambda}_q - \lambda_s) + \Sigma h^n_{p,q,\bar{s}} (\lambda_n - \lambda_p - \lambda_q - \bar{\lambda}_s)\, u_p u_q \bar{u}_{-s}$$

$$+ \Sigma h^n_{\bar{p},\bar{q},s}\, \bar{u}_{-p} \bar{u}_{-q} u_s (\lambda_n - \bar{\lambda}_p - \bar{\lambda}_q - \lambda_s) + \Sigma h^n_{\bar{p},q,\bar{s}} (\lambda_n - \bar{\lambda}_p - \lambda_q - \bar{\lambda}_s)\, \bar{u}_{-p} u_q \bar{u}_{-s}$$

$$+ \Sigma h^n_{p,\bar{q},\bar{s}}\, u_p \bar{u}_{-q} \bar{u}_{-s} (\lambda_n - \lambda_p - \bar{\lambda}_q - \bar{\lambda}_s) + \Sigma h^n_{\bar{p},\bar{q},\bar{s}} (\lambda_n - \bar{\lambda}_p - \bar{\lambda}_q - \bar{\lambda}_s)\, \bar{u}_{-p} \bar{u}_{-q} \bar{u}_{-s}$$

$$= -\widetilde{\Phi}_3(u)_n = -\frac{1}{2\pi} \int_0^{2\pi} g_3\Big(\sum_{k \in Z} \frac{u_k - \bar{u}_{-k}}{\lambda_k - \bar{\lambda}_k} e^{ikx}\Big) e^{-inx}\, dx$$

$$- \frac{1}{2\pi} \int_0^{2\pi} f\Big(\sum_{k \in Z} \frac{\lambda_k \bar{u}_{-k} - \bar{\lambda}_k u_k}{\lambda_k - \bar{\lambda}_k} e^{ikx}\Big) e^{-inx}\, dx,$$

where the symbol Σ means summation by all integers p, q, s, such that $p+q+s=n$. The eigenvalues of the linear operator $\tilde{A}$ (defined in (3.1) and (3.2)) are $\{\lambda_k, \bar{\lambda}_k : k \in \mathbf{Z}\}$. Since we want to solve the homologous equation (3.6) we have to calculate the resonances of order three (for a definition of resonance see [8], p. 164). Let at least two of the integers $|p|$, $|q|$, $|s|$ are less or equal to N, where N is a fixed natural number. Then the following two lemmas hold:

Lemma 3.2. *If $\delta=0$ and $n=p+q+s$, then any possible resonance correlation between the complex numbers $\lambda_p, \lambda_q, \lambda_s, \lambda_n, \bar{\lambda}_p, \bar{\lambda}_q, \bar{\lambda}_s, \bar{\lambda}_n$ has a form $\lambda_n=\lambda_p+\lambda_q+\bar{\lambda}_s$, where $n=p$, $q=-s$ or $n=q$, $p=-s$.*

Lemma 3.3. *For all nonresonance (in the sence of Lemma 3.2) correlation there exist real numbers $\varepsilon_N>0$, $\delta_N>0$, such, that if $n=p+q+s$ and $\delta \in [0, \delta_N]$, then $|\lambda_n \pm \overset{(-)}{\lambda_p} \pm \overset{(-)}{\lambda_q} \pm \overset{(-)}{\lambda_s}| > \varepsilon_N$.*

Consider the function $\varepsilon(x)=|\sqrt{1+x^2}|-|x|$. For $x \geq 0$ we have $\varepsilon'(x)=x/\sqrt{1+x^2}-1=(x-\sqrt{x^2+1})/\sqrt{1+x^2}<0$, $\varepsilon''(x)=(1+x^2)^{-3/2}>0$.
Consequently for $x \geq 0$ $\varepsilon(x)$ is a strictly decreasing function. It is easy to see from the last equality that $\varepsilon(x)$ is convex. Then for any $a \geq 0$ the function $f(x)=\varepsilon(x)-\varepsilon(x+a)$ is decreasing. Let us assume that for $\delta=0$ Lemma 3.3 is proved. Let $N_\infty > N$ is an integer, such, that if $n>N_\infty$ then $n^2-(n-1)^2>2(\alpha+N^2)+n$, i. e. $N_\infty>2(\alpha+N^2)+1$. Remind that $\lambda_n=\frac{\alpha\delta}{2}-\delta n^2+\sqrt{\frac{\alpha^2\delta^2}{4}-\alpha\delta^2 n^2-\delta^2 n^4-1-n^2}$ and $\alpha>0$, $\delta>0$. We shall assume that $\delta>0$ is such that $\operatorname{Re}\lambda_n=\alpha\delta/2-\delta n^2$. If $|p|$, $|q|$, $|s|$, $|n|<N_\infty$, then $|\lambda_n(\delta)-\lambda_n(0)| \leq \delta c_2$ for a suitable positive constant c_2. Then for $\delta \in [0, \varepsilon_N/8c_2]$ Lemma 3.3. is proved, if only instead of ε_N we replace $\varepsilon_N/2$. If at least one of the integers $|p|$, $|q|$, $|s|$, $|n|$ is greater than N_∞, then for any nonresonance correlation $\varphi(\delta)=\lambda_n \pm \overset{(-)}{\lambda_p} \pm \overset{(-)}{\lambda_q} \pm \overset{(-)}{\lambda_s}$ we have $|\operatorname{Re}\varphi(\delta)| \geq \delta m$, where $m=\max(|p|, |q|, |s|, |n|)$. On the other hand, $|\operatorname{Im}\varphi(\delta)-\operatorname{Im}\varphi(0)| \leq \delta^2 m^2 c_3$ for a suitable positive constant c_3, $i\operatorname{Im}\varphi(0)=\varphi(0)$ and we obtain

$$|\operatorname{Im}\varphi(\delta)| \geq |\operatorname{Im}\varphi(0)| - |\operatorname{Im}\varphi(\delta)-\operatorname{Im}\varphi(0)| \geq \varepsilon_N-\delta^2 m^2 c_3.$$

But $|\varphi(\delta)| \geq \max(|\operatorname{Re}\varphi(\delta)|, |\operatorname{Im}\varphi(\delta)|)=\max(\varepsilon_N-\delta^2 m^2 c_3, \delta m)$. Now, if $\delta m<\sqrt{\varepsilon_N/2c_3}$, then $\varepsilon_N-\delta^2 m^2 c_3>\varepsilon_N/2$ and consequently Lemma 3.3. holds. For the proof of Lemma 3.3. to be complete we have to prove it only in the case $\delta=0$.

Proof of Lemma 3.2. and Lemma 3.3. in the case $\delta=0$.

3.2.1. Obviously $|\lambda_n-\bar{\lambda}_p-\bar{\lambda}_q-\bar{\lambda}_s| \geq 4$.

3.2.2. Consider the complex number $\lambda_n-\bar{\lambda}_p-\bar{\lambda}_q-\lambda_s$.

i) $pqsn=0$. Let for example $p=0$ and $qsn \neq 0$. Then $n-q-s=0$ and

$$\lambda_n-\bar{\lambda}_p-\bar{\lambda}_q-\lambda_s=i(1+|n|+|q|-|s|+\varepsilon(n)+\varepsilon(q)-\varepsilon(s)).$$

As $n=q+s$, then $1+|n|+|q|-|s|$ is an odd integer. On the other hand, $|\varepsilon(n)+\varepsilon(q)-\varepsilon(s)|<2\varepsilon(1)$ and $\varepsilon(1)=0{,}414\ldots$. Consequently $|\lambda_n-\bar{\lambda}_p-\bar{\lambda}_q-\lambda_s|>0{,}1$. For $s=0$ $|\lambda_n-\bar{\lambda}_p-\bar{\lambda}_q-\lambda_s| \geq 2$. If $p=0$ and $qsn=0$, for example $q=0$, then

$$|\lambda_n-\bar{\lambda}_p-\bar{\lambda}_q-\lambda_s|=|2+|n|-|s|+\varepsilon(n)-\varepsilon(s)|=2,$$

because $n=s+0+0$. The other cases are considered by analogy.

ii) $pqsn \neq 0$. Then

$$|\lambda_n-\bar{\lambda}_p-\bar{\lambda}_q-\lambda_s|=|n|+|p|+|q|-|s|+\varepsilon(n)+\varepsilon(p)+\varepsilon(q)-\varepsilon(s).$$

If $|n|+|p|+|q|-|s|$ is even and is not zero, then its modulus is greater than two. But $|\varepsilon(n)+\varepsilon(p)+\varepsilon(q)-\varepsilon(s)|<3\varepsilon(1)$. If $|n|+|p|+|q|-|s|=0$, then $|n|, |p|, |q| < |s|$ and

$$|\varepsilon(n)+\varepsilon(p)+\varepsilon(q)-\varepsilon(s)| \geqq \max(\varepsilon(n), \varepsilon(p), \varepsilon(q)) \geqq \varepsilon(N).$$

Consequently in this case

$$|\lambda_n-\bar{\lambda}_p-\bar{\lambda}_q-\lambda_s| \geqq \min(\varepsilon(N), 0.5).$$

3.2.3. Consider the complex number $\lambda_n-\bar{\lambda}_p-\lambda_q-\lambda_s$.

i) $pqsn=0$. If at least two of the integers p, q, s, n are zeros, obvious solutions are $p=q=0, n=s$; $p=s=0, n=q$; $p=-s, n=q=0$; $n=s=0, p=-q$.

If just one of the integers p, q, s, n is zero, for example n, we obtain:

$$|\lambda_n-\bar{\lambda}_p-\lambda_q-\lambda_s|=|1+|p|-|q|-|s|+\varepsilon(p)-\varepsilon(q)-\varepsilon(s)|,$$

$|\varepsilon(p)-\varepsilon(q)-\varepsilon(s)|<2\varepsilon(1)<0.9$ and hence $|\lambda_n-\bar{\lambda}_p-\lambda_q-\lambda_s|>0{,}1$. The cases p, q or $s=0$ are considered by analogy.

ii) $pqsn\neq0$. Essentially this is the only nontrivial case. We have

$$|\lambda_n-\bar{\lambda}_p-\lambda_q-\lambda_s|=||n|+|p|-|q|-|s|+\varepsilon(n)+\varepsilon(p)-\varepsilon(q)-\varepsilon(s)|.$$

Because of $|\varepsilon(n)+\varepsilon(p)-\varepsilon(q)-\varepsilon(s)|<4\varepsilon(1)<1{,}8$ we conclude $|n|+|p|-|q|-|s|=0$. (Otherwise $|\lambda_n-\bar{\lambda}_p-\lambda_q-\lambda_s|>0{,}2$). Thus $|n|-|q|=|s|-|p|$. Without loss of generality we can assume $|n|\leq|q|$. If $|n|=|q|$, then $|s|=|p|$ and we obtain the solutions $n=q$, $s=-p$. If $n=-q$, then $-2q=p+s$, for $p=-s$ we have $n=q=0$, $s=-p$ and for $p=s$ we have $n=-q=p=s$. The more general solution $n=s$, $p=-q$ subsumes the last solution. The above solutions are so to say obvious. We shall show that in the considered case there are not others.

And so, let $|n|-|q|=|s|-|p|$ and for example $|n|>|q|$, $n\neq s$. Then $|\lambda_n-\bar{\lambda}_p-\lambda_q-\lambda_s|=|\varepsilon(|n|)-\varepsilon(|q|)-\varepsilon(|s|)-\varepsilon(|p|))|$. Let for a definiteness $|q|\leq N$. We have

$$\begin{aligned}|\lambda_n-\bar{\lambda}_p-\lambda_q-\lambda_s| &= |\varepsilon(|q|)-\varepsilon(|q|+|n|-|q|)-(\varepsilon(|p|)-\varepsilon(|p|+|s|-|p|))| \\ &\geqq \min(|\varepsilon(|q|)-\varepsilon(|n|)-(\varepsilon(|q|+1)-\varepsilon(|n|+1))|, \\ &\quad |\varepsilon(|q|)-\varepsilon(|n|)-(\varepsilon(|q|-1)-\varepsilon(|n|-1))|)=\alpha(|n|, |q|).\end{aligned}$$

The last inequality is true, because as was said earlier, the function $f(x)=\varepsilon(x)-\varepsilon(x+a)$ is strictly decreasing for $a\geqq0$, $x\geqq0$. Let $c_4>0$ is such, that for $|n|>c_4$, $|\varepsilon(n)-\varepsilon(|n|+1)<(\varepsilon(N)-\varepsilon(N+1))/2$. Recalling that $|q|\leq N$, we obtain

$$\alpha(|n|, |q|)>(\varepsilon(N)-\varepsilon(N+1))/2 \quad (\text{for } |n|>c_4).$$

On the other hand, there are only a finite number of integers q and n such that $|n|\leq c_4$ and $|q|\leq N$. Observing that $\alpha(|n|, |q|)\neq0$ for $|n|\neq|q|$, we obtain there exists $\varepsilon_N>0$ so that for any values of p, q, s, n such that $|n|-|q|=|s|-|p|$, $pqsn\neq0$, $|n|>|q|$, $n\neq s$ holds $|\lambda_n-\bar{\lambda}_p-\lambda_q-\lambda_s|>\varepsilon_N\neq0$ for a suitable constant ε_N. The condition $|n|>|q|$ is not restrictively. The cases $|s|\leq N$ or $|p|\leq N$ are considered by analogy.

3.2.4. The case $\lambda_n-\lambda_p-\lambda_q-\lambda_s=0$ is considered by analogy with 3.2.2. So Lemma 3.2 and Lemma 3.3. are proved.

Now in the homologous equation (3.6) we can define the coefficients of h. If at least two of the numbers $|p|, |q|, |s|$ are greater than N or that is not so, but the coefficients of h with such indices are multiplicated in (3.6) by a resonance (in the sense of Lemma 3.2.), then for such coefficient we replace

$$h^n_{\bar{p},q,s}=h^n_{p,\bar{q},s}=h^n_{p,q,\bar{s}}=h^n_{\bar{p},\bar{q},s}=h^n_{\bar{p},q,\bar{s}}=h^n_{p,\bar{q},\bar{s}}=h^n_{\bar{p},\bar{q},\bar{s}}=h^n_{p,q,s}=0.$$

In all remaining cases we calculate uniquely the coefficients of h by the homologous equation (3.6). Because of Lemma 3.3. the modulus of any "small denominator" will be greater than ε_N. In other words, the just defined operator is from the set $\mathcal{M}_C$. Let us write down the normal form of (2.3). From (3.3) we obtain:

$$\text{(3.7)}\quad \frac{du}{dt}=\sum_{n=0}^{\infty}\left(-\frac{dh}{du}\right)^n\circ(\tilde{A}+\tilde{\Phi})\circ H(u)=\tilde{A}u+\sum_{n=0}^{\infty}\left(-\frac{dh^n}{dn}\right)\circ(\tilde{A}u,\ h(u)]+\tilde{\Phi}\circ(I+h)(u)).$$

Taking into consideration the definition of h, we obtain in Fourier co-ordinates:

$$\text{(3.8)}\qquad \frac{d}{dt}u_n=\lambda_n u_n+u_n\Big(\sum_{k=-N}^{N}\Phi^n_k.|u_k|^2\Big)+\dots,$$

where the dots mean nonlinear terms of order three, five and more. The nonlinear terms of order three contain only addends of the tipe: $u_p u_q u_s$, $\bar{u}_{-p}u_q u_s$, $\bar{u}_{-p}\bar{u}_{-q}u_s$, $\bar{u}_{-p}\bar{u}_{-q}\bar{u}_{-s}$, where at least two of the integers $|p|$, $|q|$, $|s|$ are greater than N. We used the symbols:

$$\Phi^n_k=6f^n_{n,k,-k}\frac{(-\bar{\lambda}_n\lambda_k\bar{\lambda}_k)}{(\lambda_n-\bar{\lambda}_n)|\lambda_k-\bar{\lambda}_k|^2}-6g^n_{n,k,-k}/(\bar{\lambda}_n-\lambda_n).|\lambda_k-\bar{\lambda}_k|^2,\quad n\neq k$$

$$\Phi^n_n=3f^n_{n,n,-n}.\bar{\lambda}_n.|\lambda_n|^2/(\lambda_n-\bar{\lambda}_n)^3-3g^n_{n,n,-n}/(\lambda_n-\bar{\lambda}_n)^3.$$

Now we shall show that after the change of variables $u=H(v)$ the equation (3.2) has a form $\frac{du}{dt}=\tilde{A}u+\hat{\Phi}(u)$, where $\hat{\Phi}$ is a nonlinear operator of the class $\mathcal{H}_C$. Indeed according to (3.7):

$$\hat{\Phi}(u)=\sum_{n=0}^{\infty}\left(-\frac{dh}{du}\right)\circ([\tilde{A}u,\ h(u)]+\tilde{\Phi}\circ(I+h)(u)).$$

By the definition of h (see (3.4)) we conclude that the operator $[\tilde{A}u,\ h(u)]$ is of the class $\mathcal{M}_C$. On the other hand, $\tilde{\Phi}\in\mathcal{H}_C$, $h\in\mathcal{M}_C$ and then $\tilde{\Phi}\circ(I+h)\in\mathcal{H}_C$. But $(I+\frac{dh}{du})^{-1}$ is a linear operator and consequently $\hat{\Phi}\in\mathcal{H}_C$. Further for us will be important as well the equation

$$\text{(3.9)}\qquad \frac{d}{dt}u_n=\lambda_n u_n+u_n\Big(\sum_{k=-N}^{N}\Phi^n_k.|u_k|^2\Big).$$

In other words (3.9) is the "shortened" equation (3.8).

Example. If the system of linear equations

$$\operatorname{Re}\lambda_n+\sum_{k=-N}^{N}y_k.\operatorname{Re}\Phi^n_k=0,\quad n=0,\ \pm1,\dots,\pm N$$

has a solution $(y_0,\ y_{\pm1},\dots,y_{\pm N})$ and $y_i\geq 0$ for $i=0,\ \pm1,\dots,\pm N$, then the set $T=\{u=\sum_{n=-N}^{N}u_n e^{inx}:\ |u_n|=\sqrt{y_n},\ n=0,\ \pm1,\dots,\pm N\}$ is an invariant torus of equation (3.9). Its dimension is equal to the number of the nonzero co-ordinates $y_0, y_{\pm 1}, \dots, y_{\pm N}$. On the torus (3.9) looks like this:

$$\frac{d}{dt} u_n = iu_n(\operatorname{Im} \lambda_n + \sum_{k=-N}^{N} \operatorname{Im} \Phi_k^n . y_k), \quad n=0, \pm 1, \dots, \pm N$$

and consequently T is filled with a family of periodic trajectories with frequencies

$$\operatorname{Im} \lambda_n + \sum_{k=-N}^{N} \operatorname{Im} \Phi_k^n . y_k, \quad n=0, \pm 1, \dots, \pm N.$$

4. Conditions for stability of the invariant tori. If the invariant torus of equation (3.9) is stable, then is naturally to expect in some neighbourhood of it to exist an invariant stable torus of equation (3.8). This is proved in **6**. In that section we shall investigate the condition which has to satisfy the nonlinearity Φ in order to possess (3.9) an invariant asymptotic stable torus. The idea which we shall follow is the idea of the classical bifurcation of Andronov — Hopf. Denote

$$W_N^+ = \{u = \sum_{n \in \mathbf{Z}} u_n e^{inx} : u \in W_{1,\mathrm{C}}, \quad \operatorname{Im} u_n = 0, \quad \operatorname{Re} u_n \geq 0, \quad |n| \leq N\}.$$

Define the map $p: W_{1,\mathrm{C}} \to W_N^+$ thus: $p(u) = \Sigma_{|n| \leq N} |u_n| e^{inx} + \Sigma_{|n|>N} u_n e^{inx}$. The triple $(W_{1,\mathrm{C}}, W_N^+, p)$ we shall call a degenerated bundle with base W_N^+, fibred space $W_{1,\mathrm{C}}$ and projection p.

Proceed in (3.9) to new co-ordinates $u_n = \rho_n . \exp(i\theta_n)$, $|n| \leq N$. We obtain $\frac{d\rho_n}{dt} = -i\rho_n \frac{d\theta_n}{dt} + \rho_n(\lambda_n + \Sigma_{k=-N}^{N} \rho_k^2 . \Phi_k^n)$, $|n| \leq N$. It is clear that (3.9) is equivalent to the following three equations:

$$\frac{d\rho_n}{dt} = \rho_n(\operatorname{Re} \lambda_n + \sum_{k=-N}^{N} \rho_k^2 . \operatorname{Re} \Phi_k^n) \quad |n| \leq N, \tag{4.1}$$

$$\frac{d\theta_n}{dt} = \operatorname{Im} \lambda_n + \sum_{k=-N}^{N} \rho_k^2 . \operatorname{Im} \Phi_k^n \quad |n| \leq N, \tag{4.2}$$

$$\frac{du_n}{dt} = u_n(\lambda_n + \sum_{k=-N}^{N} \rho_k^2 . \Phi_k^n) \quad |n| > N. \tag{4.3}$$

If the system (4.1), (4.3) has in W_N^+ a stationary solution $\tau = \Sigma_{k=-N}^{N} \tau_k e^{ikx}$, $\tau_k = \mathrm{const}$, then (3.9) has in $W_{1,\mathrm{C}}$ an invariant torus $T_\tau = p^{-1}\tau$, coinciding with the fibre over the point τ of the base W_N^+. The dimension of T_τ is equal to the number of the nonzero co-ordinates τ_k, $k=0, \pm 1, \dots, \pm N$. The existence of such a torus is garanteed by the following:

Assumption 2. The system of equalities

$$(\operatorname{Re} \lambda_n + \sum_{k=-N}^{N} \tau_k^2 \operatorname{Re} \Phi_k^n)\tau_n = 0, \quad n=0, \pm 1, \dots, \pm N$$

has a positive solution $\tau = (\tau_0, \tau_{\pm 1}, \dots, \tau_{\pm N})$, i. e. such that $\tau_i \geq 0$ for $i=0, \pm 1, \dots, \pm N$ and at least one of the numbers τ_i, $i=0, \pm 1, \dots, \pm N$ is not zero. Moreover is fulfilled the condition: $\det (\operatorname{Re} \Phi_k^n)_{k,n=0}^{N,N} \neq 0$.

We shall note that the second condition in Assumption 2. is very important (although in [9] is omitted), because it ensure the uniqueness of the solution of the system in Assumption 2. The case

$$\det(\operatorname{Re}\Phi_k^n)_{k,n=0}^{N,N}=0$$

is also interesting but leads to manifolds, which are not homeomorphic to finite-dimensional tori (for example the manifold $\{|z_1|^2+|z_2|^2=1;\ z_1,\ z_2\in\mathbf{C}\}$ is not homeomorphic to the torus $\{|z_i|^2=1;\ i=1,\ 2,\ 3,\ z_i\in\mathbf{C}\}$) and that is why here will not be considered.

If Assumption 2. holds, then the system (4.1), (4.2), (4.3) (and also the equivalent equation (3.9)) has an invariant torus $T_\tau=p^{-1}\tau$. It is clear the torus T_τ is asymptotic stable if and only if the stationary solution τ of the system (4.1), (4.3) possesses that property. It was the reason to be introduced the last system.

i) Case of a periodic solution.

Let for some natural number m, $|m|\leq N$ holds $\operatorname{Re}\lambda_m>0$, $\operatorname{Re}\Phi_m^m<0$. Then (4.1), (4.3) has a stationary solution $\tau=(\tau_0,\ \tau_{\pm1},\ \dots,\ \tau_{\pm N})$, where $\tau_m=(-\operatorname{Re}\lambda_m/\operatorname{Re}\Phi_m^m)^{1/2}$ and $\tau_n=0$ for $n\neq m$. Replace now $x_n=\rho_n-\tau_n$, $|n|\leq N$. We obtain

$$\frac{d}{dt}x_m=(\operatorname{Re}\lambda_m+3\operatorname{Re}\Phi_m^m.\tau_m^2)x_m+\dots$$

$$\frac{d}{dt}x_n=(\operatorname{Re}\lambda_n+\operatorname{Re}\Phi_m^n\tau_m^2)x_n+\ \dots\ n\neq m,\ |n|\leq N \tag{4.4}$$

$$\frac{d}{dt}u_s=(\lambda_s+\Phi_m^s\tau_m^2)x_s=\ \dots \qquad |s|>N.$$

Taking into consideration that $\operatorname{Re}\lambda_m+3\operatorname{Re}\Phi_m^m=-2\operatorname{Re}\lambda_m<0$, we conclude the zero solution of (4.1), (4.3) and simultaneously a periodic solution of (3.9) is asymptotic stable if for some $c_5>0$ holds:

$$\operatorname{Re}\lambda_n-\frac{\operatorname{Re}\lambda_m}{\operatorname{Re}\Phi_m^m}\cdot\operatorname{Re}\Phi_m^n<-c_5.\delta$$

for any $n\neq m$. (For the proof see [3], chapter 2A.) We shall note that the spectrum of the unbounded operator $\tilde{A}$ is a discrete and this is an essentiall condition.

ii) Case of l-dimensional torus.

Let Assumption 2. is satisfied and the number of the nonzero co-ordinates is just l. Then the equation (3.9) possesses an invariant l-dimensional torus $T_\tau=d^{-1}\tau$, where τ is a stationary solution of (4.1), (4.3), $\tau=(\tau_0,\ \tau_{\pm1},\ \dots,\ \tau_{\pm N})$. In co-ordinates $x_n=\rho_n-\tau_n$, $|n|\leq N$ the system (4.1), (4.3) takes a form:

$$\frac{dx_n}{dt}=x_n(\operatorname{Re}\lambda_n+\sum_{k=-N}^{N}\tau_k^2.\operatorname{Re}\Phi_k^n)+2\sum_{k=-N}^{N}\tau_n\tau_k\operatorname{Re}\Phi_k^n.x_k+\ \dots$$

$$\frac{du_n}{dt}=u_n(\lambda_n+\sum_{k=-N}^{N}\tau_k^2\Phi_k^n)+\ \dots, \tag{4.5}$$

where the dots mean nonlinear terms. The zero solution of the system (4.5) and simultaneously a stationary solution of the system (4.1), (4.3) and invariant l-dimensional torus of equation (3.9) is asymptotic stable if the following natural assumption is satisfied. (See [3], chapter 2A.)

Assumption 3. Consider the numbers:

i) the eigenvalues of the $l\times l$ matrix (α_{ij})

$$\alpha_{ij}=2\tau_i\tau_j\operatorname{Re}\Phi_j^i,\ i\neq j,\ \alpha_{ii}=\operatorname{Re}\lambda_i+\sum_{k=-N}^{N}\tau_k^2\operatorname{Re}\Phi_k^i+2\tau_i^2.\operatorname{Re}\Phi_i^i$$

i, j take these integer values from the interval $[-N,\ N]$, for which $\tau_i\neq0$, $\tau_j\neq0$.

ii) for all integers n, such that $|n|>N$ or $|n|\leq N$ but $\tau_n=0$, we consider the numbers $\mathrm{Re}\,\lambda_n+\Sigma_{k=-N}^{N}\tau_k^2\,.\,\mathrm{Re}\,\Phi_k^n$.

Then there exists function $R(\delta)$, $\delta>0$, such that any of the above number lies in the left semi-plane (of the plane $\mathbf{C}$) and the distance between it and the imaginary axis is greater than $R(\delta)$, where $\lim_{\delta\to 0} R(\delta)/\delta=c_8$ and $c_8>0$.

Let us observe now that the topology of equation (3.9) is determined by the numbers $\mathrm{Re}\,\Phi_k^n$, which depend only upon the nonlinearities f and g_3. Introduce in $\mathcal{M}$ the following norm: If $\varphi, \psi\in\mathcal{M}$ then $\|\varphi-\psi\| = \sup_{k,n=0,\pm 1,\pm 2,\dots} |\varphi_k^n-\psi_k^n|$, where $\varphi_k^n=6\varphi_{n,k,-k}^n$, $k\neq n$, $\varphi_n^n=3\varphi_{n,n,-n}^n$. This norm in a natural way induces a norm in $\mathcal{M}^2=\mathcal{M}\times\mathcal{M}$. The following theorem is the central result of the present paper.

Theorem 4.1. *For any natural number l we can find open sets $D_l\subset\mathcal{M}^2$, $\Delta_l\subset\mathbf{R}^1$ and a constant δ_l such that if $(g_3, f)\in D_l$, $\alpha\in\Delta_l$ and $\delta\in(0, \delta_l)$, then equation* (2.3) *possesses an invariant asymptotic stable manifold (in $\tilde{W}_1^3$) homeomorphic to l-dimensional torus. Moreover, $\max_{u\in T_\delta}\|u\|_{\tilde{W}_1^3}=0(\sqrt{\delta})$, $\min_{u\in T_\delta}\|u\|_{\tilde{W}_1^3}=0(\sqrt{\delta})$, where T_δ is the above manifold.*

In order to prove Theorem 4.1. first we shall prove the following:

Theorem 4.2. *Let the nonlinearities $g\in\mathcal{H}$, $f\in\mathcal{M}$ are such that for some constant δ_N and any $\delta\in(0, \delta_N)$ Assumption 2. and Assumption 3. are satisfied. Then equation* (3.8) *possesses an invariant asymptotic stable manifold $\Gamma_\delta\subset W_{1,\mathbf{C}}$, homeomorphic to l-dimensional torus. The manifold can be represented in the form $\Gamma_\delta=\varphi(T_\tau)$, where φ is homeomorphism,*

$$T_\tau=\{u\in W_{1,\mathbf{C}}:\ |u_k|=\tau_k,\ |k|\leq N;\ u_k=0,\ |k|>N\}$$

and $\max_{u\in T_\tau}\|\varphi(u)-u\|_{W_{1,\mathbf{C}}}<c_7\,.\,\delta$, $c_7=\mathrm{const}$.

In more details proofs of Theorem 4.1. and Theorem 4.2. are given in **6.** Here we give a short scheme of the proof of Theorem 4.2. without going into technicalities.

Let B_δ is a suitable δ-neighbourhood of the invariant torus of (3.9). Assume the dimension of T_τ is $2N+1$. Define the projection

$$\pi:\ B_\delta\to T_\tau,\ \pi(u)=\sum_{k=-N}^{N}\tau_k\,.\,\exp(ikx+i\arg u_k).$$

For any sufficiently small δ the triple (B_δ, T_τ, π) is local trivial bundle with base T_τ, projection π and fibred space B_δ. Let the shift operator along the trajectories of equation (3.9) is N_t, where $t>0$. Define on the class ω, consisting of sections of the bundle (B_δ, T_τ, π) an operator $L:\omega\to\omega$, so that for any

$$\varphi\in\omega\ \{(L\varphi)(u):\ u\in T_\tau\}=\{N_t(\varphi(u)):\ u\in T_\tau\},\ t=\mathrm{const}.$$

With the metric $\rho(\varphi_1, \varphi_2)=\max_{u\in T_\tau}\|\varphi_1(u)-\varphi_2(u)\|_{W_{1,\mathbf{C}}}$ the set ω turns into a complete metric space, while the operator L is contraction mapping (which is naturally because of the asymptotic stability of the invariant torus of (3.9)). That is why the operator L has a unique stationary point $\eta=L\eta$ which possesses the feature asymptotic stability. The graphics of η $\Gamma_\delta=\{\eta(u):\ u\in T_\tau\}$ by construction is an invariant asymptotic stable manifold of the shift operator along the trajectories of (3.8) $-N_t$. The projection π is homeomorphism between the section η an the base T_τ. Hence Γ_δ is homeomorphic to $2N+1$-dimensional torus.

5. The shift operator along the trajectories of equation (3.8). The shift operator along the trajectories of equation (3.8) — N_t will be represented in Fourier co-ordinates like a sum consisting of the shift operator along the trajectories of equation (3.9) and additional "small" terms. Estimates of the "small" terms will be done but it will be used only in **6.** So the present section has solely subsidiary character.

As the operator $N_t u$ is differentiable with respect to u in $W_{1,C}$ (see the Appendix), then for any initial condition $u \in W_{1,C}$ holds

$$N_t u = \sum_{k=1}^{3} g_k(u, t)/k! + B(u, t).$$

(See the Taylor's formula in [4], 8.14.3) Here g_k are homogeneous operators of order k with respect to u and also for some constant $c_8 > 0$, $\| B(u) - B(v) \| \le c_8 \| u - v \| . (\| u \|^3 + \| v \|^3)$, $B(u) = B(u, t)$, $\| B(u) \| = 0(\| u \|^4)$. Replacing in (3.8) and equalize the homogeneous terms of one and the same order, we obtain

$$\frac{d}{dt} g_1(u, t) = \widetilde{A} g_1(u, t) \qquad , \quad g_1(u, 0) = 0$$

$$\frac{d}{dt} g_2(u, t) = \widetilde{A} g_2(u, t) + 2G_2(g_1(u, t)) \qquad , \quad g_2(u, 0) = 0$$

$$\frac{d}{dt} g_3(u, t) = \widetilde{A} g_3(u, t) + 6G_3(g_1(u, t)) + 3\frac{dG_2}{du}\Big|_{g_1(u,t)} . g_2(u, t), \quad g_3(u, 0) = 0,$$

where $\widetilde{A}$ is the linear part on the right hand-side of (3.8) while G_2 and G_3 are the nonlinear parts of order two and three, respectively. As $G_2 \equiv 0$ we obtain:

$$g_1(u, t) = (\exp(t\widetilde{A}))u, \quad g_2(u, t) = (\exp(t\widetilde{A}))g_2(u, 0) = 0,$$

$$g_3(u, t) = 6 \int_0^t \exp((t-\tau)\widetilde{A}) . G_3(\exp(t\widetilde{A})u) d\tau.$$

The above can be grounded by the Cauchy's theorem for existence and uniqueness of solution of equation (2.3) which is proved in the Appendix.

Further down we shall assume that Assumption 2. and Assumption 3. are satisfied. For the sake of brevity and without loss of generality we shall assume $\tau_k \neq 0$ for $k = 0, \pm 1, \ldots, \pm N$, so that the dimension of the torus T_τ will be $2N+1$. In Fourier co-ordinates the operator N_t has a form:

$$(N_t u)_n = g_1(u, t)_n + g_3(u, t)_n/6 + \beta(u, t)_n, \quad g_1(u, t) = \exp(t\lambda_n) . u_n$$

$$g_3(u, t)_n = 6 \int_0^t e^{(t-\tau)\lambda_n} . \sum_{k=-N}^{N} (u_n | u_k |^2 . \Phi_k^n . | e^{t\lambda_k} |^2 . e^{t\lambda_n}) d\tau + v(u, t)_n$$

$$= 6u_n e^{t\lambda_n} . \sum_{k=-N}^{N} \left(\frac{e^{2t\mathrm{Re}\lambda_k} - 1}{2\mathrm{Re}\,\lambda_k} \cdot \Phi_k^n . | u_k |^2\right) + v(u, t)_n,$$

where $v(u, t) = 6 \int_0^t e^{(t-\tau)\widetilde{A}} . \widetilde{v}(e^{\tau\widetilde{A}} u) d\tau, \quad L_1^{-1} u = \begin{pmatrix} \vartheta \\ \dot{\vartheta} \end{pmatrix}$

$$\text{and } \widetilde{v}(u)_n = \sum_{p,q,s} \left(\frac{(u_p - \bar{u}_{-p})(u_q - \bar{u}_{-q})(u_s - \bar{u}_{-s})}{(\lambda_p - \bar{\lambda}_p)(\lambda_q - \bar{\lambda}_q)(\lambda_s - \bar{\lambda}_s)} \cdot g_{p,q,s}^n + \dot{\vartheta}_p \dot{\vartheta}_q \dot{\vartheta}_s . f_{p,q,s}^n\right),$$

where the symbol $\Sigma_{p,q,s}$ means summation with respect to all integers p, q, s, such that $p+q+s=n$ and at least two of the integers $|p|$, $|q|$ or $|s|$ are greater than N.

Then

$$(4.1)\qquad (N_t u)_n = u_n e^{t\lambda_n}(1+\sum_{k=-N}^{N}(e^{2t\operatorname{Re}\lambda_k}-1)\,.\,\Phi_k^n\,.\,|u_k|^2/2\operatorname{Re}\lambda_k)\nu(u)_n + B(u)_n$$

$$= u_n e^{it\operatorname{Im}\lambda_n}\,.\,(1+e^{t\operatorname{Re}\lambda_n}-1+\sum_{k=-N}^{N} e^{t\operatorname{Re}\lambda_n}\,.\,\frac{e^{2t\operatorname{Re}\lambda_k}-1}{2\operatorname{Re}\lambda_k}\cdot\Phi_k^n\,.\,|u_k|^2)+\nu(u)_n+\mathrm{B}(u)_n$$

$$= u_n e^{it\operatorname{Im}\lambda_n}(1+t\operatorname{Re}\lambda_n\,.\,\mu_n(\delta)+t\sum_{k=-N}^{N}\tilde{\mu}_k^n(\delta)\,.\,\Phi_k^n\,.\,|u_k|^2)+\nu(u)_n+\mathrm{B}(u)_n.$$

Here $\mu_n(\delta)=(e^{t\operatorname{Re}\lambda_n}-1)/(t\operatorname{Re}\lambda_n)=1+O(\delta)$

$$\tilde{\mu}_k^n(\delta)=e^{t\operatorname{Re}\lambda_n}\,.\,(e^{2t\operatorname{Re}\lambda_k}-1)/(2t\operatorname{Re}\lambda_k)=(1+O(\delta))(1+O(\delta))=1+O(\delta).$$

Note that $\mu_k(\delta)$, $\tilde{\mu}_k^n(\delta)\in\mathrm{R}$.

Further we shall calculate $|(N_t u)_n|$ and $\arg(N_t u)_n$ for $|u|\leq N$. That will help us to use better Assumption 3. (which we assume to be fulfilled). The idea of the next calculation is this: if the complex number $a+bi$, where $b\ll a$ is given, then $|a|\sim|a+bi|$ and $b\sim\arg(a+bi)$. Of course the symbol "$\sim$" will be defined.
For $n=0,\ \pm1,\dots,\ \pm N$ we find

$$|(N_t u)_n| = \rho_n\,.\,|1+t\mu_n(\delta)\operatorname{Re}\lambda_n+t\sum_{k=-N}^{N}\tilde{\mu}_k^n(\delta)\Phi_k^n\,.\,|u_k|^2+\frac{\beta_n(u)+\nu_n(u)}{\rho_n}\,.\,e^{-i\theta_n-it\operatorname{Im}\lambda_n}|$$

$$=\rho_n((1+t\mu_n(\delta)\operatorname{Re}\lambda_n+t\sum_{k=-N}^{N}\tilde{\mu}_k^n(\delta)\operatorname{Re}\Phi_k^n\,.\,|u_k|^2+\operatorname{Re}(\frac{\beta_n(u)+\nu_n(u)}{\rho_n}e^{-i\theta_n-it\operatorname{Im}\lambda_n}))^2$$

$$+(t\sum_{k=-N}^{N}\tilde{\mu}_k^n(\delta)\,.\operatorname{Im}\Phi_k^n\,.\,|u_k|^2+\operatorname{Im}(\frac{\beta_n(u)+\nu_n(u)}{\rho_n}\,.\,e^{-i\theta_n-it\operatorname{Im}\lambda_n}))^2)^{1/2}$$

$$=\rho_n((1+\zeta_1(u))^2+(\zeta_2(u))^2)^{1/2}.$$

By the last equality we defined implicitly the functionals $\zeta_1(u)$ and $\zeta_2(u)$. Taking into consideration that when $u\in B_\delta$ $\zeta_1(u)$ and $\zeta_2(u)$ are "small" with respect to the number one, we obtain

$$(N_t u)_n| = \rho_n(1+\zeta_1(u)+((\zeta_1(u))^2+(\zeta_2(u))^2)/2)+\rho_n(\sum_{k=2}^{\infty}(2\zeta_1(u)+(\zeta_1(u))^2+(\zeta_2(u))^2)^k.a_k)\quad a_k\in\mathrm{R}.$$

Denote $\tilde{\gamma}_n(u)=\rho_n(((\zeta_1(u))^2+(\zeta_2(u))^2)/2+\sum_{k=2}^{\infty}(2\zeta_1(u)+(\zeta_1(u))^2+(\zeta_2(u))^2)^k a_k)$.

Then we obtain

$$(5.2)\qquad |(N_t u)_n| = \rho_n(1+\zeta_1(u))+\tilde{\gamma}_n(u)).$$

Consider the just defined functional $\tilde{\gamma}_n$. We find consequently:

$$|\sum_{k=-N}^{N}\tilde{\mu}_k^n(\delta)\operatorname{Re}\Phi_k^n\,.\,|u_k|^2-\sum_{k=-N}^{N}\tilde{\mu}_k^n(\delta)\operatorname{Re}\Phi_k^n\,.\,|\vartheta_k|^2|$$

$$\leq 2\sum_{k=-N}^{N}|\Phi_k^n|\,.\,(|u_k|-|\vartheta_k|)\,.\,(|u_k|+|\vartheta_k|)\leq c_8\|u-\vartheta\|(\|u\|+\|\vartheta\|),$$

$$\|\cdot\|=\|\cdot\|_{W_{1,C}},\quad u_n=\rho_n^n\exp(i\theta_n^u),\quad \vartheta_n=\rho_n^\vartheta\,.\exp(i\theta_n^\vartheta).$$

$$|\operatorname{Re}(\frac{\beta_n(u)}{\rho_n^u}e^{-i\theta_n^u-it\operatorname{Im}\lambda_n})-\operatorname{Re}(\frac{\beta_n(\vartheta)}{\rho_n^\vartheta}\,.\,e^{-i\theta_n^\vartheta-it\operatorname{Im}\lambda_n})|\leq c_9\delta\,.\,\|u-\vartheta\|.$$ Indeed $\beta_n(u)=O(\delta^2)$

$$\rho_n^u\rho_n^\vartheta = O(\sqrt{\delta}),\quad |\beta_n(u)-\beta_n(\vartheta)| = \|u-\vartheta\|\,.\,O(\delta^{3/2}),$$
$$|\rho_n^u-\rho_n^\vartheta| \le \|u-\vartheta\|,\quad |e^{i\theta_n^u}-e^{i\theta_n^\vartheta}| \le \|u-\vartheta\|\,.\,\tilde{c}_9\,.\,\delta^{-1/2}.$$

By analogy

$$\left|\operatorname{Re}\left(\frac{\nu_n(u)}{\rho_n^u}e^{-i\theta_n^u-it\operatorname{Im}\lambda_n}\right)-\operatorname{Re}\left(\frac{\nu_n(\vartheta)}{\rho_n^\vartheta}e^{-i\theta_n^\vartheta-it\operatorname{Im}\lambda_n}\right)\right| \le c_{10}\,.\,\delta^{1/2}\,.\,\|u-\vartheta\|.$$

Thus we obtain

$$|\zeta_1(u)-\zeta_1(\vartheta)| \le c_{11}\delta^{1/2}\,.\,\|u-\vartheta\|. \tag{5.3}$$

In a way analogical to that one proves

$$|\zeta_2(u)-\zeta_2(\vartheta)| \le c_{12}\,.\,\delta^{1/2}\,.\,\|u-\vartheta\|. \tag{5.4}$$

Now with the help of (5.3) and (5.4) we conclude

$$|\tilde{\gamma}_n(u)-\tilde{\gamma}_n(\vartheta)| \le c_{13}\,.\,\delta^2\,.\,\|u-\vartheta\|. \tag{5.5}$$

Further $i\arg(N_tu)_n = \log\dfrac{(N_tu)_n}{|(N_tu)_n|}$

$$i\arg(N_tu)_n - i\theta_n - it\operatorname{Im}\lambda_n = \log\frac{1+\zeta_1(u)+i\zeta_2(u)}{1+\zeta_1(u)+\tilde{\gamma}_n(u)/\rho_n^u}$$

$$=\log\left(1+(i\zeta_2(u)-\tilde{\gamma}_n(u)/\rho_n^u)/(1+\zeta_1(u)+\tilde{\gamma}_n(u)/\rho_n^u)\right)$$

$$=(i\zeta_2(u)-\tilde{\gamma}_n(u)/\rho_n^u)/(1+\zeta_1(u)+\tilde{\gamma}_n(u)/\rho_n^u)+((i\zeta_2(u)-\tilde{\gamma}_n(u)/\rho_n^u)/(1+\zeta_1(u)+\tilde{\gamma}_n(u)/\rho_n^u))^2+\ldots$$

For the Taylor's series expansion to be correct we naturally used

that $\left|\dfrac{i\zeta_2(u)-\tilde{\gamma}_n(u)/\rho_n^u}{1+\zeta_1(u)-\tilde{\gamma}_n(u)/\rho_n^u}\right|<1$ for sufficiently small δ.

Let $f_\delta(u)=(i\zeta_2(u)-\tilde{\gamma}_n(u)/\rho_n^u)/(1+\zeta_1(u)+\tilde{\gamma}_n(u)/\rho_n^u)$. Then $f_\delta(u)=0(\delta)$ for $u\in B_\delta$

$$|f_\delta(u)-f_\delta(\vartheta)| = |(i\zeta_2(u)+\tilde{\gamma}_n(u)/\rho_n^u+i\zeta_2(u)\zeta_1(\vartheta)+i\zeta_2(u)\,.\,\tilde{\gamma}_n(\vartheta)/\rho_n^\vartheta$$
$$-\tilde{\gamma}_n(u)\zeta_1(\vartheta)/\rho_n^u-i\zeta_2(\vartheta)\zeta_1(u)+\zeta_1(u)\,.\,\tilde{\gamma}_n(\vartheta)/\rho_n^\vartheta-i\zeta_2(\vartheta)\tilde{\gamma}_n(u)/\rho_n^u+\tilde{\gamma}_n(\vartheta)\tilde{\gamma}_n(u)/(\rho_n^u\,.\,\rho_n^\vartheta)/|(1+\zeta_1(u)$$
$$+\tilde{\gamma}_n(u)/\rho_n^u)(1+\zeta_1(\vartheta)+\gamma_n(\vartheta)/\rho_n^\vartheta)| \le c_{14}\,.\,(|\zeta_2(u)-\zeta_2(\vartheta)|+|\tilde{\gamma}_n(u)/\rho_n^u-\tilde{\gamma}_n(\vartheta)/\rho_n^\vartheta|+|\zeta_2(u)\,.\,\zeta_1(\vartheta)$$
$$-\zeta_1(u)\,.\,\zeta_2(\vartheta)|+|\zeta_2(u)\tilde{\gamma}_n(\vartheta)/\rho_n^\vartheta-\zeta_2(\vartheta)\tilde{\gamma}_n(u)/\rho_n^u|+|\tilde{\gamma}_n(u)\zeta_1(\vartheta)/\rho_n^u-\tilde{\gamma}_n(\vartheta)\,.\,\zeta_1(u)/\rho_n^\vartheta|).$$

Estimate the addends $|\zeta_2(u)\,.\,\zeta_1(\vartheta)-\zeta_1(u)\,.\,\zeta_2(\vartheta)| \le |\zeta_2(u)(\zeta_1(\vartheta)-\zeta_1(u))|+|\zeta_1(u)(\zeta_2(\vartheta)-\zeta_2(u))|$ $\le c_{15}\,.\,\delta^{3/2}\,.\,\|u-\vartheta\|$ (we used (5.4)) $|\tilde{\gamma}_n(u)\,.\,\zeta_1(\vartheta)/\rho_n^u-\tilde{\gamma}_n(\vartheta)\zeta_1(u)/\rho_n^\vartheta| \le |(\tilde{\gamma}_n(u)\,.\,\zeta_1(\vartheta)\,.\,\rho_n^\vartheta$ $-\tilde{\gamma}_n(\vartheta)\,.\,\zeta_1(u)\,.\,\rho_n^u)/\rho_n^u\,.\,\rho_n^\vartheta| \le |(\tilde{\gamma}_n(u)-\tilde{\gamma}_n(\vartheta))\zeta_1(\vartheta)\,.\,\rho_n^\vartheta+\tilde{\gamma}_n(\vartheta)(\zeta_1(\vartheta)-\zeta_1(u))\,.\,\rho_n^u+\tilde{\gamma}_n(\vartheta)\,.\,\zeta_1(u)(\rho_n^\vartheta$ $-\rho_n^u)|/|\rho_n^u\,.\,\rho_n^\vartheta| \le c_{16}\,.\,\delta^{5/2}\,.\,\|u-\vartheta\|$. An analogical estimate holds for the last addend. Ultimately we obtain $|f_\delta(u)-f_\delta(\vartheta)| \le c_{17}\,.\,\|u-\vartheta\|\,.\,\delta^{3/2}$.

Denote $\tilde{\zeta}_n(u)=f_\delta(u)-i\zeta_2(u)$. As in the above inequalities we obtain: $|\tilde{\zeta}_n(u)-\tilde{\zeta}_n(\vartheta)|$ $\le c_{18}\|u-\vartheta\|\,.\,\delta^{3/2}$.

Now we can write down: $-i\theta_n - it.\operatorname{Im}\lambda_n + i\arg(N_t u)_n = \Sigma_{k=1}^{\infty}(f_\delta(u))^k = i\zeta_2(u)+\zeta_n(u)$ $+\Sigma_{k=2}^{\infty}(f_\delta(u))^k = i\zeta_2(u)+i\tilde{\tilde{\zeta}}_n(u)$, where we have denoted $i.\tilde{\tilde{\zeta}}_n(u)=\tilde{\zeta}_n(u)+\Sigma_{k=2}^{\infty}(f_\delta(u))^k$. Then we obtain

$$|\tilde{\tilde{\zeta}}_n(u)-\tilde{\tilde{\zeta}}_n(\vartheta)|\leq c_{19}.\|u-\vartheta\|.\delta^{3/2}. \tag{5.6}$$

Denote $\mu_n^\rho(u)=\operatorname{Re}(\nu_n(u).\exp(-i\theta_n^u-it.\operatorname{Im}\lambda_n))$, $\mu_n^\theta(u)=\operatorname{Im}(\nu_n(u).(\exp(-i\theta_n^u-it .\operatorname{Im}\lambda_n))/\rho_n^u)$.

Generalize the obtained results: The shift operator along the trajectories of equation (3.8) — N_t maps a point $u=\Sigma_{k\in Z}u_k e^{ikx}\in B_\delta$ in this way:

For $|m|\leq N$, where $u_m=\rho_m.\exp(i\theta_m)$

$$\rho_m\to\rho_m(1+\zeta_1(u))+\tilde{\gamma}_m(u)=\rho_m(1+t\mu_m(\delta).\operatorname{Re}\lambda_m+t\sum_{k=-N}^{N}.\tilde{\mu}_k^m(\delta).\operatorname{Re}\Phi_k^m|u_k|^2)+\mu_n^\rho(u)+\gamma_m(u)$$

$$\theta_m\to\theta_m+t\operatorname{Im}\lambda_m+\zeta_2(u)+\tilde{\tilde{\zeta}}_m(u)=\theta_m+t\operatorname{Im}\lambda_m+t\sum_{k=-N}^{N}\tilde{\mu}_k^m(\delta).\operatorname{Im}\Phi_k^m.|u_k|^2+\mu_m^\theta(u)+\xi_m(u),$$

where $\gamma_m(u)=\tilde{\gamma}_m(u)+\operatorname{Re}(\beta_m(u).\exp(-i\theta_m-it\operatorname{Im}\lambda_m))$

$$\xi_m(u)=\tilde{\tilde{\zeta}}_m(u)+\operatorname{Im}(\beta_m(u)(\exp(-i\theta_m-it\operatorname{Im}\lambda_m))/\rho_m^u)$$

and as it is easy to calculate by the help of (5.5), (5.6)

$$|\gamma_m(u)-\gamma_m(\vartheta)|\leq c_{20}\cdot\|u-\vartheta\|.\delta^{3/2},\quad |\xi_m(u)-\xi_m(\vartheta)|\leq c_{21}\|u-\vartheta\|.\delta. \tag{5.7}$$

For $|n|>N$ we have

$$u_n\to u_n.\exp(t\lambda_n).(1+\sum_{k=-N}^{N}\frac{e^{2t\operatorname{Re}\lambda_k}-1}{2\operatorname{Re}\lambda_k}\cdot\Phi_k^n\cdot|u_k|^2)+\nu_n(u)+\beta_n(u).$$

Up to the end of **5.** we shall give some definitions and prove a lemma concerning the operator N_t. Define the spaces:

$$W_N=\{u\in W_{1,C}:u_n=0,\ |n|>N\},\quad W_\perp=\{u\in W_{1,C}:u_n=0,\ |n|\leq N\}$$

$$W^+=\{u\in W_{1,C}:\ \operatorname{Im}u_n=0,\ \operatorname{Re}u_n\geq 0,\ n\in\mathbf{Z}\}$$

$W_N^+=W^+\cap W_N$ and denote by P_N, $P_\perp$ the natural projectors, corresponding to the decomposition of $W_{1,C}$ in a direct sum $W_{1,C}=W_N\oplus W_\perp$.

$$P_N:\ W_{1,C}\to W_N,\quad P_\perp:\ W_{1,C}\to W_\perp.$$

Define also $P_N^+:\ u\to(|u_0|,\ |u_{\pm1}|,\dots,|u_{\pm N}|,\ 0,\ 0,\dots)$

$$P:\ u\to P_N^+u+P_\perp u=(|u_0|,\ |u_{\pm1}|,\dots,|u_{\pm N}|,\ u_{\pm(N+1)},\dots).$$

Consider the operator $\bar{N}:\ W_N^+\times W_\perp\to W_N^+\times W_\perp$ defined like this:

$$\rho_m\to\rho_m(1+t.\mu_m(\delta).\operatorname{Re}\lambda_m+t\sum_{k=-N}^{N}\tilde{\mu}_k^m(\delta).\operatorname{Re}\Phi_k^m.\rho_k^2)\quad |m|\leq N$$

$$\bar{N}:\ u_n\to u_n.(\exp(t\lambda_n)).(1+\sum_{k=-N}^{N}\frac{\exp(2t\operatorname{Re}\lambda_k)-1}{2\operatorname{Re}\lambda_k}\cdot\Phi_k^n\cdot\rho_k^2)\quad |n|>N.$$

If Assumption 2. is satisfied, then the system

$$(\mu_m(\delta)\,.\,\mathrm{Re}\,\lambda_m+\sum_{k=-N}^{N}\widetilde{\mu}_k^m(\delta)\,.\,\mathrm{Re}\,\Phi_k^m\,.\,\rho_k^2)\,.\,\rho_m=0.\quad m=0,\ \pm1,\ldots,\ \pm N$$

has a positive solution $\rho=(\rho_0,\ \rho_{\pm1},\ \rho_{\pm2},\ldots,\ \rho_{\pm N})$, $\rho_i\geqq 0$, $|i|\leqq N$. Really the distance between the stationary solution ρ of the operator $\bar{N}$ and the stationary solution τ of equation (3.9) is $O(\delta^{3/2})$, because $\mu_n(\delta)\cdot\mathrm{Re}\,\lambda_n=\mathrm{Re}\,\lambda_n+O(\delta^2)$, $\widetilde{\mu}_k^n(\delta)=1+O(\delta)$ and then $\rho_k^2-\tau_k^2=O(\delta^2)$ hence $\rho_k-\tau_k=O(\delta^{3/2})$. For us will be more conviniently to operate with the stationary point ρ of the operator $\bar{N}$ than with the stationary solution τ. As we noted $\rho_k-\tau_k=O(\delta^{3/2})$ for $k=0,\ \pm1,\ldots,\ \pm N$ and then ρ will satisfy Assumption 3. In the above reasonings we used that $\tau_k=O(\sqrt{\delta})$ for $k=0,\ \pm1,\ldots,\ \pm N$. It is easy to achieve that, replacing in Assumption 2. $\tau_k=\tau_{-k}$. Since $\mathrm{Re}\,\lambda_n=O(\delta)$ and $\det(\mathrm{Re}\,\Phi_k^n)_{k,n=0}^{N,N}\neq 0$ we obtain $\tau_k=O(\sqrt{\delta})$.

Denote $W^b=\{u\in W_{1,C}:\ \mathrm{Im}\,u_n=0,\ n\in\mathbf{Z}\}$, $W_N^b=W^b\cap W_N$.

Lemma 5.1. *Let hold Assumption 2. and Assumption 3. Then there exists a constant $c>0$, such that for the Frechet derivative of the operator $\bar{N}$ in the point ρ is fulfilled*

$$\|\frac{d\bar{N}}{d\rho}\|<1-c\delta$$

for any $\delta\in(0,\ \delta_c)$, where δ_c depends only on c.

Proof. We have $\frac{d\bar{N}}{d\rho}:\ W_N^b\oplus W_\perp\to W_N^b\oplus W_\perp$. Let $u=\Sigma_{n\in Z}u_n\exp(inx)\in W_N^b\oplus W_\perp$. For $|n|\leqq N$ holds

$$\bar{N}(\rho+u)_n=(\rho_n+u_n)(1+\mu_n(\delta)\,\mathrm{Re}\,\lambda_n+t\sum_{k=-N}^{N}\widetilde{\mu}_k^n(\delta)\,.\,\mathrm{Re}\,\Phi_k^n\,.\,(\rho_k+u_k)^2)$$

$$=u_n(1+t\mu_n(\delta)\,\mathrm{Re}\,\lambda_n+t\sum_{k=-N}^{N}\widetilde{\mu}_k^n(\delta)\,.\,\mathrm{Re}\,\Phi_k^n\,.\,\rho_k^2)+2\rho_n\sum_{k=-N}^{N}t\widetilde{\mu}_k^n(\delta)\,.\,\mathrm{Re}\,\Phi_k^n\,.\,\rho_k u_k+\ldots$$

For $|m|>N$ holds

$$\bar{N}(\rho+u)_m=u_m\,.\,e^{t\lambda_m}\,.\,(1+\sum_{k=-N}^{N}\frac{e^{2t\,\mathrm{Re}\,\lambda_k}-1}{2\,\mathrm{Re}\,\lambda_k}\,.\,\Phi_k^m\,.\,\rho_k^2)+\ldots,$$

where the dots mean nonlinear addends with respect to u. Taking into consideration that $\mu_n(\delta)=1+O(\delta)$, $\widetilde{\mu}_k^n(\delta)=1+O(\delta)$, $\rho_k^2=\tau_k^2+O(\delta^2)$, we find that the eigenvalues of the matrix $(\widetilde{\alpha}_{ij})$, $i,j=0,\ \pm1,\ldots,\ \pm N$

$$\widetilde{\alpha}_{ij}=2\rho_i\rho_j\,.\,\widetilde{\mu}_j^i(\delta)\,.\,\mathrm{Re}\,\Phi_j^i\,.\,t\quad i\neq j$$

$$\widetilde{\alpha}_{ii}=t\mu_i(\delta)\,.\,\mathrm{Re}\,\lambda_i+2t\rho_i^2\,.\,\widetilde{\mu}_i^i(\delta)\,.\,\mathrm{Re}\,\Phi_i^i+t\sum_{k=-N}^{N}\rho_k^2\,\mathrm{Re}\,\Phi_k^i\,.\,\widetilde{\mu}_k^i(\delta)$$

(which are eigenvalues of the operator $\frac{d\bar{N}}{d\rho}-I$) satisfy condition i) of Assumption 3. Moreover, as the limits $\lim\widetilde{\alpha}_{ij}/\delta$ exist, then the distance between the eigenvalues of (α_{ij}) and the origine of C is $O(\delta)$. Hence the first $2N+1$ eigenvalues of the operator $\frac{d\bar{N}}{d\rho}$ which are obtained when we consider the restriction of the above operator on the invariant space. W^b_N lie in a circle with a radius $1-c_{22}\delta$ for some constant $c_{22}>0$. It remains to calculate the rest eigenvalues:

$$e^{t\,\mathrm{Re}\,\lambda_n}\Big(1+\sum_{k=-N}^{N}\frac{e^{2t\,\mathrm{Re}\,\lambda_k}-1}{2\,\mathrm{Re}\,\lambda_k}\,.\,\mathrm{Re}\,\Phi^n_k\,.\,\rho_k^2\Big)=1+t\,\mathrm{Re}\,\lambda_n$$

$$+t\sum_{k=-N}^{N}(1+\chi^k_n(\delta))\,.\,\mathrm{Re}\,\Phi^n_k\,.\,\rho^2_k+O((\mathrm{Re}\,\lambda_n)^2).$$

But $\lambda_n=\alpha\delta/2-\delta n^2+\sqrt{\alpha^2\delta^2/4-\alpha\delta^2n^2-\delta^2n^4-1-n^2}$ and consequently $\exp(t\,\mathrm{Re}\,\lambda_n)\leq\exp(\alpha t\delta/2)$. Now we shall make sure that condition ii) of Assumption 3. remains valid for the numbers

$$\tilde{\tilde{\lambda}}_n=e^{t\lambda_n}\,.\,\Big(1+\sum_{k=-N}^{N}\frac{e^{2t\,\mathrm{Re}\,\lambda_k}-1}{2\,\mathrm{Re}\,\lambda_k}\cdot\Phi^n_k\,.\,\rho_k^2\Big),\ |n|>N.\ \text{Really}$$

$$1+\chi^k_n(\delta)\overset{\mathrm{def}}{=\!=}\exp(t\,\mathrm{Re}\,\lambda_n)\,.\,\exp(2t\,\mathrm{Re}\,\lambda_k)-1)/2\,\mathrm{Re}\,\lambda_k$$

$$=\exp(t\,\mathrm{Re}\,\lambda_n)(1+O(2t\,\mathrm{Re}\,\lambda_k))\leq\exp(\alpha t\delta/2)(1+O(2t\,\mathrm{Re}\,\lambda_k))=1+O_k(\delta),\ \lim\frac{O_k(\delta)}{\delta}=\tilde{c}_k\neq 0.$$

As the index k takes here a finite number of values then condition ii) of Assumption 3. is fulfilled and hence for all sufficiently small numbers $\delta>0$ and a suitable constant c_{23}, the real part of the number $\tilde{\lambda}_n=e^{t\,\mathrm{Re}\,\lambda_n}(1+\Sigma^N_{k=-N}(e^{2t\,\mathrm{Re}\,\lambda_k}-1)\,.\,\Phi^n_k\,.\,\rho^2_k/2\,\mathrm{Re}\,\lambda_k),|n|>N$ lies in the interval $[-1+c_{23}\delta,\ 1-c_{23}\delta]$. Now we shall prove that for a suitable constant $c_{24}>0$ holds $|\mathrm{Im}\,\tilde{\lambda}_n|\leq c_{24}\,.\,\delta$. Indeed

$$|\,\mathrm{Im}\,\tilde{\lambda}_n|=\Big|\sum_{k=-N}^{N}\frac{e^{2t\,\mathrm{Re}\,\lambda_k}-1}{2\,\mathrm{Re}\,\lambda_k}\,.\,\mathrm{Im}\,\Phi^n_k\,.\,\rho^2_k\,.\,e^{t\,\mathrm{Re}\,\lambda_n}\Big|=\Big|\sum_{k=-N}^{N}t(1+\chi^k_n(\delta))\,.\,\mathrm{Im}\,\Phi^n_k\,.\,\rho_k^2)\Big|\leq c_{24}\delta$$

(we used that the numbers $|\Phi^n_k|$ are limited). Possibly decreasing the constant $c>0$, we conclude that the numbers $\tilde{\lambda}_n$ lie in a circle with a radius $1-c\delta$ and centrum — the origine in C. To the $\tilde{\lambda}_n$ corresponds an eigenvalue $\tilde{\tilde{\lambda}}_n$ of the operator $\frac{d\bar{N}}{d\rho}$ and $|\tilde{\lambda}_n|=|\tilde{\tilde{\lambda}}_n|$. In this way we conclude that for all sufficiently small numbers $\delta>0$ the eigenvalues of the operator $\frac{d\bar{N}}{d\rho}$ lies in a circle with a radius $1-c\delta$ and hence $\|\frac{d\bar{N}}{d\rho}\|<1-c\delta$.

6. Asymptotic stable invariant tori. Let B_δ is a δ-neighbourhood of the torus $T_\rho=\{u\in W_{1,\mathrm{C}}:|u_n|=\rho_n,\ |n|\leq N;\ u_n=0,\ |n|>N\}$. The last set was invariant for the operator $\bar{N}$. Define the projection $\pi:B_\delta\to T_\rho$, $\pi u=\Sigma^N_{k=-N}\rho_k\,.\,\exp(ikx+i\arg u_k)$. The triple (B_δ,T_ρ,π) is local trivial bundle with base T_ρ projection π and fibred space B_δ (for suitable δ and neighbourhood B_δ).

Proof of Theorem 4.2.

Where it is necessary we shall identify the elements $(u_0,\ u_{\pm1},\dots,\ u_{\pm N},\ 0,\ 0,\dots)\in T_\rho$ and $(\arg u_0,\ \arg u_{\pm1},\dots,\arg u_{\pm N})\in\mathrm{R}_{2N+1}/2\pi\mathrm{Z}^{2N+1}$ of the $2N+1$-dimensional torus T_ρ and the standard $2N+1$-dimensional torus obtained by factorization of R_{2N+1} with respect to the lattice $2\pi\mathrm{Z}^{2N+1}$. In this way already is defined the sum $\theta+\varphi\in T_\rho$ for arbitrary elements of $T_\rho-\theta,\varphi$. Denote by ω the set of sections of the bundle (B_δ,T_ρ,π) satisfying the Lipschitz's condition with exponent $\delta^{2/3}$ in the following sence: for any $\varphi\in\omega$ and $\theta_1,\ \theta_2\in T_\rho$ holds the estimate

$$\|P\varphi(\theta_1)-P\varphi(\theta_2)\|_{W_{1,\mathrm{C}}}\leq\delta^{2/3}\,.\,\|\theta_1-\theta_2\|_{\mathrm{R}_{2N+1}}.$$

The operator P was defined in 5. In the set ω we shall define the metric

$$\rho(\varphi,\ \psi)=\sup_{\theta\in T_\rho}\|\varphi(\theta)-\psi(\theta)\|_{W_{1,C}}$$

which turn it into a complete metric space. The invariant torus of the operator N_t we shall search as a section of the set ω. Define the operator $L:\omega\rightarrow\omega$ mapping a section $\varphi\in\omega$ into a section $L\varphi$ which graphics coincides with the set $\{N_t\varphi(u):\ u\in T_\rho\}$.

It is clear if some $\varphi\in\omega$ is a stationary point of the operator L, then its graphics will be the invariant torus in question. Now our aim shall be to prove that the mapping L is contraction (which is naturally because of the asymptotic stability). That will be done by three stages.

6.1. Here will be obtained some auxiliary estimates. Define the three-linear symmetrical functionals $\varphi_n:\ W_{1,C}\times W_{1,C}\times W_{1,C}\rightarrow\mathbf{C}$ regarding that $\varphi_n(u,\ \vartheta,\ z)$ for $u=\vartheta=z$ is equal to

$$\varphi_n(u,u,u)=\begin{cases} t\sum\limits_{k=-N}^{N}\widetilde{\mu}_k^n(\delta)\,.\,\mathrm{Re}\,\Phi_k^n\,.\,|u_k|^2\,.\,u_n, & |n|\leq N\\ t\sum\limits_{k=-N}^{N}e^{t\lambda_n}\,.\,\dfrac{e^{t\,\mathrm{Re}\,\lambda_k}-1}{2t\,\mathrm{Re}\,\lambda_k}\,.\,\Phi_k^n\,.\,|u_k|^2\,.\,u_n, & |n|>N.\end{cases}$$

Denote $G_n(u,\ \vartheta)=t\sum\limits_{k=-N}^{N}\widetilde{\mu}_k^n(\delta)\,.\,\mathrm{Im}\,\Phi_k^n\,.\,|u_k|\,.\,|\vartheta_k|$

$$\varphi(u,\ \vartheta,\ z)=\sum_{n\in\mathbf{Z}}\varphi_n(u,\ \vartheta,\ z)\,.\,\exp(inx)$$

$$\eta(u)=(\gamma_0(u),\ \gamma_{\pm1}(u),\ \dots,\ \gamma_{\pm N}(u),\ \beta_{\pm(N+1)}(u),\ \dots)$$

$$\mu^\rho(u)=(\mu_0^\rho(u),\ \mu_{\pm1}^\rho(u),\ \dots,\ \mu_{\pm N}^\rho(u),\ \nu_{\pm(N+1)}(u),\ \dots).$$

As it was noted at the beginning of 5, $\|\beta(u)-\beta(\vartheta)\|\leq c_8\|u-\vartheta\|\,.\,(\|u\|^3+\|\vartheta\|^3)$ and from (5.7) we find $\|\eta(u)-\eta(\vartheta)\|\leq c_{25}\delta^{3/2}\,.\,\|u-\vartheta\|$, $\|\eta(u)\|\leq c_{26}\delta^2$; $u,\ \vartheta\in B_\delta$. Let $u\in B_\delta$, $y=Pu$, $\theta=\pi u$. The shift operator along the trajectories of equation (2.8) maps a point $u\in B_\delta$ thus:

$$y\rightarrow Pe^{t\widetilde{A}}y+\varphi_n(y,\ y,\ y)+\mu^\rho(u)+\eta(u)$$

$$\theta_n\rightarrow\theta_n+t\,\mathrm{Im}\,\lambda_n+G_n(u,\ u)+\mu_n^\theta(u)+\xi_n(u),\quad |n|\leq N \tag{6.1}$$

$$\|\eta(u)-\eta(\vartheta)\|\leq c_{25}\delta^{3/2}\,.\,\|u-\vartheta\|,\ \ |\xi_n(u)-\xi_n(\vartheta)|\leq c_{21}\delta\,.\,\|u-\vartheta\|.$$

Denote $\rho_\theta=(\rho_0\exp(i\theta_0),\ \rho_{\pm1}\exp(i\theta_{\pm1}),\ \dots,\ \rho_{\pm N}\exp(i\theta_{\pm N}),\ 0,\ 0,\ \dots)$.

Lemma 6.1. *For arbitrary points $u,\ \vartheta\in W_{1,C}$ holds* $\|u-\vartheta\|\leq\|u\|\,.\,\|\theta-\varphi\|+\|Pu-P\vartheta\|$.

Proof. Let $\theta=\pi u$, $\varphi=\pi\vartheta$ and

$$u_\varphi=(|u_0|\,.\,\exp(i\varphi_0),\ |u_{\pm1}|\exp(i\varphi_{\pm1}),\ \dots,\ |u_{\pm N}|\exp(i\varphi_{\pm N}),\ u_{\pm(N+1)},\ \dots).$$

Then $\|u-\vartheta\|\leq\|u-u_\varphi\|+\|u_\varphi-\vartheta\|\leq\|u\|\,.\,\|\exp(i\theta)-\exp(i\varphi)\|+\|Pu-P\vartheta\|$. But $|\exp(i\theta_k)-\exp(i\varphi_k)|=|\theta_k-\varphi_k|$ then $\|u-\vartheta\|\leq\|u\|\,.\,\|\theta-\varphi\|+\|Pu-P\vartheta\|$.

Corollary. For any two elements $\theta,\ \varphi\in\mathbf{R}_{2N+1}$ holds $\|\rho_\theta-\rho_\varphi\|\leq\|\rho\|\,.\,\|\theta-\varphi\|$.

Let u and ϑ are such that $\rho+u,\ \rho+\vartheta\in B_\delta$. Denote $\theta=\pi u$, $\varphi=\pi\vartheta$. Then $\rho_\theta+u\in\pi^{-1}\theta$, $\rho_\varphi+\vartheta\in\pi^{-1}\varphi$. In order to prove that L is contraction in the complete metric space ω, we are to estimate the term $PN_t(\rho_\theta+u)-PN_t(\rho_\varphi+\vartheta)$. From (6.1) we obtain

$$(6.2)\qquad PN_t(\rho_\theta+u)-PN_t(\rho_\varphi+\vartheta)=Pe^{t\widetilde{A}}(\rho+Pu-\rho-P\vartheta)+\varphi(\rho+Pu,\ \rho+Pu,\ \rho+Pu)-\varphi(\rho+P\vartheta,\ \rho+P\vartheta,\ \rho+P\vartheta)+\eta(\rho_\theta+u)-\eta(\rho_\varphi+\vartheta)+\mu^\theta(u+\rho_\theta)-\mu^\rho(\rho_\varphi+\vartheta).$$

Using the symbols of (6.1), we have $\bar{N}$: $y\to Pe^{t\widetilde{A}}y+\varphi(y,\ y,\ y)$. Then $\frac{d\bar{N}}{d\rho}y=Pe^{t\widetilde{A}}y+3\varphi(\rho,\ \rho,\ y)$. In this way from (6.2) after calculations we obtain the equivalent:

$$(6.3)\quad PN_t(\rho_\theta+u)-PN_t(\rho_\varphi+\vartheta)=\frac{d\bar{N}}{d\rho}(Pu-P\vartheta)+3\varphi(\rho,\ Pu+P\vartheta,\ Pu-P\vartheta)+\varphi(Pu,\ Pu,\ Pu-P\vartheta)+\varphi(Pu,\ P\vartheta,\ Pu-P\vartheta)+\varphi(P\vartheta,\ P\vartheta,\ Pu-P\vartheta)+\eta(\rho_\theta+u)-\eta(\rho_\varphi+\vartheta)+\mu^\rho(u+\rho_\theta)-\mu^\rho(\rho_\varphi+\vartheta).$$

Remains to estimate the terms $\eta(\rho_\theta+u)-\eta(\rho_\varphi+\vartheta)$ and $\mu^\rho(\rho_\theta+u)-\mu^\rho(\rho_\varphi+\vartheta)$. Using (6.1 and Lemma 6.1., we find

$$(6.4)\quad \|\eta(\rho_\theta+u)-\eta(\rho_\varphi+\vartheta)\|\leq c_{25}\delta^{3/2}.\|\rho_\theta+u-\rho_\varphi-\vartheta\|\leq c_{25}\delta^{3/2}(\|\rho_\theta-\rho_\varphi\|+\|u-\vartheta\|)\leq\|Pu-P\vartheta\|\,c_{26}\delta^{3/2}+c_{26}\delta^{3/2}.(\|\rho\|.\|\theta-\varphi\|+\|u\|.\|\theta-\varphi\|).$$

On the other hand, taking into consideration the definition of $\mu_n^\rho(u)$ and $\nu_n(u)$ (for $|n|\leq N$) we find

$$(6.5)\qquad \|\mu^\rho(u+\rho_\theta)-\mu^\rho(\vartheta+\rho_\varphi)\|\leq c_{27}.\|u+\rho_\theta-\vartheta-\rho_\varphi\|.\delta^{1/2}.(\|u\|+\|\vartheta\|)\leq c_{27}\delta^{3/2}.(\|\rho\|.\|\theta-\varphi\|+\|u\|.\|\theta-\varphi\|+\|Pu-P\vartheta\|).$$

By analogy with this one proves

$$(6.6)\qquad |\mu_n^\theta(u+\rho_\theta)-\mu_n^\theta(\vartheta+\rho_\varphi)|\leq c_{28}\delta.(\|\theta-\varphi\|.(\|\rho\|+\|u\|)+\|Pu-P\vartheta\|)$$

(which will be used further). In this way we obtain the estimate

$$(6.7)\qquad \|PN_t(\rho_\theta+u)-PN_t(\rho_\varphi+\vartheta)\|\leq\|\frac{d\bar{N}}{d\rho}\|.\|Pu-P\vartheta\|+3\|\varphi\|.\|\rho\|.\|Pu+P\vartheta\|.\|Pu-P\vartheta\|+\|\varphi\|.\|Pu\|^2.\|Pu-P\vartheta\|+\|\varphi\|.\|Pu\|.\|P\vartheta\|.\|Pu-P\vartheta\|+\|\varphi\|.\|P\vartheta\|^2.\|Pu-P\vartheta\|+c_{29}\delta^{3/2}.(\|\rho\|.\|\theta-\varphi\|+\|u\|.\|\theta-\varphi\|+\|Pu-P\vartheta\|)\leq(1-\frac{c}{2}\delta).\|Pu-P\vartheta\|+c_{29}\delta^2.\|\theta-\varphi\|_{\mathbf{R}^{2N+1}}.$$

Here we used again $\|\rho\|=O(\delta^{1/2})$; $\|u\|$, $\|\vartheta\|=O(\delta)$, $\|\varphi\|<\infty$.

Now we shall estimate from below the quantity $\|\pi N_t(\rho_\theta+u)-\pi N_t(\rho_\varphi+\vartheta)\|$. From (6.1) we obtain $\pi N_t(\rho_\theta+u)_n=\theta_n+t\,\mathrm{Im}\,\lambda_n+G_n(\rho_\theta+u,\ \rho_\theta+u)+\mu_n^\theta(\rho_\theta+u)+\xi_n(\rho_\theta+u)$. Using that $G_n(\rho_\theta+u,\ \rho_\theta+u)=G_n(\rho+Pu,\ \rho+Pu)$, we reach to the identity

$$\pi N_t(\rho_\theta+u)_n-\pi N_t(\rho_\varphi+\vartheta)_n=\theta_n-\varphi_n.$$

Now with the help of (6.1), (6.6) and Lemma 6.1. we can write down:

$$|\pi N_t(\rho_\theta+u)_n-\pi N_t(\rho_\varphi+\vartheta)_n|\geq|\theta_n-\varphi_n|-c_{30}\delta^{1/2}.\|Pu-P\vartheta\|-c_{31}\delta.\|\rho_\theta+u-\rho_\varphi-\vartheta\|\geq|\theta_n-\varphi_n|-c_{30}.\delta^{1/2}.\|Pu-P\vartheta\|-c_{21}\delta(\|\rho\|.\|\theta-\varphi\|+\|u\|.\|\theta-\varphi\|+\|Pu-P\vartheta\|)\geq|\theta_n-\varphi_n|-c_{31}\delta^{1/2}.\|Pu-P\vartheta\|-c_{32}\delta^{3/2}.\|\theta-\varphi\|.$$

Then

(6.8) $\| \pi N_t(\rho_\theta+u)-\pi N_t(\rho_\varphi+\vartheta)\|_{W_{1,C}} \geqq (1-c_{32}\delta^{3/2}).\|\theta-\varphi\|_{R^{2N+1}}-c_{31}\delta^{1/2}.\|Pu-P\vartheta\|_{W_{1,C}}.$

6.2. Now we shall check that the operator L is correctly defined upon the metric space ω, i. e. for any $\psi\in\omega$ holds $L\psi\in\omega$.

Denote $S=R_{2N+1}/2\pi Z^{2N+1}$. Let the section $\psi\in\omega$ is fixed. Consider the operator $\pi N_t\psi: S\to S$ defined in (6.1) thus

$$\pi N_t\psi: \theta\to\pi(N_t\psi(\theta)).$$

It is obvious the operator is continuous. From inequality (6.8) we see that it is inversible and the inverse operator is also continuous. In other words, $\pi N_t\psi$ is homeomorphism between S and S. But S is both open and closed topological space and consequently the set $\pi N_t\psi(S)\subset S$ is such a space. It proves that $\pi N_t\psi(S)=S$. According to the above, for arbitrary $\varphi_1, \theta_1\in S$ there exist points $\varphi, \theta\in S$, such that $\pi N_t(\psi(\varphi))=\varphi_1, \pi N_t(\psi(\theta))=\theta_1$, i. e.

$$(L\psi)(\varphi_1)=N_t(\psi(\varphi)), \quad (L\psi)(\theta_1)=N_t(\psi(\theta)).$$

Let us check if $L\psi$ satisfies the Lipschitz's condition.

$$\begin{aligned}&\| P((L\psi)(\varphi_1))-P((L\psi)(\theta_1))\|/\|\varphi_1-\theta_1\|\\ &=\| PN_t(\psi(\varphi))-PN_t(\psi(\theta))/\|\pi N_t(\psi(\varphi))-\pi N_t(\psi(\theta))\|\\ &=\| PN_t(\rho_\varphi+\vartheta)-PN_t(\rho_\theta+u)\|/\|\pi N_t(\rho_\varphi+\vartheta)-\pi N_t(\rho_\theta+u)\|,\end{aligned}$$

where $\pi\vartheta=\varphi$, $\pi u=\theta$, $\psi(\theta)=\rho_\theta+u$, $\psi(\varphi)=\rho_\varphi+\vartheta$.

As $\psi\in\omega$ using inequalities (6.7) and (6.8), we obtain

$$\begin{aligned}&\| P(L\psi)(\varphi_1)-P(L\psi)(\theta_1)\|/\|\varphi_1-\theta_1\|\\ &\leq((1-\tfrac{c}{2}\delta)\|Pu-P\vartheta\|+c_{29}\delta^2\|\theta-\varphi\|)/(1-c_{32}\delta^{3/2}.\|\theta-\varphi\|-c_{31}\delta^{1/2}.\|Pu-P\theta\|)\\ &=((1-\tfrac{c}{2}\delta).\|Pu-P\vartheta\|/\|\theta-\varphi\|+c_{29}\delta^2)/(1-c_{32}\delta^{3/2}-c_{31}\delta^{1/2}.\|Pu-P\vartheta\|/\|\theta-\varphi\|).\end{aligned}$$

As $\rho_\theta+u=\psi(\theta)$, $\rho_\varphi+\vartheta=\psi(\varphi)$, then

$$\frac{\|P\psi(\theta)-P\psi(\varphi)\|}{\|\theta-\varphi\|}=\frac{\|Pu-P\vartheta\|}{\|\theta-\varphi\|}\leq\delta^{2/3}$$

and ultimately

$$\begin{aligned}\| P(L\psi)(\varphi_1)-P(L\psi)(\theta_1)\|/\|\varphi_1-\theta_1\| &\leq((1-\tfrac{c}{2}\delta)\delta^{2/3}+c_{29}\delta^2)/(1-c_{32}\delta^{3/2}-c_{31}\delta^{1/2}.\delta^{2/3})\\ &\leq\delta^{2/3}(1-c\delta/2+c_{29}\delta^{4/3})/(1-c_{32}\delta^{3/2}-c_{31}\delta^{7/6})\leq\delta^{2/3}\end{aligned}$$

for sufficiently small δ. The only thing we have not proved is that $(L\psi)(\theta)\in B_\delta$ for any $\theta\in S$. In the next item 6.3. we shall see that L is contraction mapping. If the neighbourhood B_δ has a form $B_\delta=\{u\in W_{1,C}: \operatorname{dist}(u, T_\rho)<c.\delta\}$, it will be enough to assert that $(L\psi)(\theta)$ for any θ belongs to B_δ and consequently $L\psi\in\omega$.

6.3. Here we shall prove that L is contraction mapping. By analogy with (6.8) may be proved the inequality

(6.9) $\|\pi N_t(\rho_\theta+u)-\pi N_t(\rho_\varphi+\vartheta)\|\leq(1+c_{32}\delta^{3/2}).\|\theta-\varphi\|+c_{31}\delta^{1/2}.\|Pu-P\vartheta\|$, $\pi u=\theta$, $\pi\vartheta=\varphi$

Let $\psi_1, \psi_2 \in \omega$ and $\theta \in S$ is an arbitrary point. We denote $\theta_1 = \pi N_t \psi_1(\theta)$, $\theta_2 = \pi N_t \psi_2(\theta)$. Then from (6.9) we obtain

$$(6.10) \qquad \| \pi N_t(\psi_1(\theta)) - \pi N_t(\psi_2(\theta)) \| = \| \theta_1 - \theta_2 \| \leq c_{31} \delta^{1/2} . \| P\psi_1(\theta) - P\psi_2(\theta) \|.$$

Using Lemma 6.1, we have $\|(L\psi_1)(\theta_1) - (L\psi_2)(\theta_1)\| \leq \| P(L\psi_1)(\theta_1) - P(L\psi_2)(\theta_2) \| + \| P(L\psi_2)(\theta_2) - P(L\psi_2)(\theta_1) \| = \| PN_t\psi_1(\theta) - PN_t\psi_2(\theta) \| + \| P(L\psi_2)(\theta_2) - P(L\psi_2)(\theta_1) \|$ (we use (6.7))

$$\leq (1 - \frac{c}{2}\delta) \| P\psi_1(\theta) - P\psi_2(\theta) \| + \delta^{2/3} . \| \theta_1 - \theta_2 \|$$

(we use 6.10)

$$\leq (1 - \frac{c}{2}\delta) . \| \psi_1(\theta) - \psi_2(\theta) \| + c_{31}\delta^{7/6} . \| \psi_1(\theta) - \psi_2(\theta) \| \leq (1 - \frac{c\delta}{3}) . \rho(\psi_1, \psi_2).$$

Now we can observe that the operator $\pi N_t \psi$, as we saw at the beginning of item 6.2. is surjection and consequently the following inequality holds $\rho(L\psi_1, L\psi_2) \leq (1 - c\delta/3) . \rho(\psi_1, \psi_2)$, i. e. the operator L is contraction mapping.

We denote by ζ the zero section of the bundle (B_δ, T_ρ, π). Let us estimate the distance between ζ and $L\zeta$. Using (6.1) and (6.5), we obtain:

$$\rho(\zeta, L\zeta) = \sup_{\theta \in T_\rho} \| (L\zeta)(\pi N_t \rho_\theta) - \zeta(\pi N_t \rho_\theta) \|$$

$$= \sup \| Pe^{t\widetilde{A}} \rho + \varphi(\rho, \rho, \rho) + \eta(\rho_\theta) + \mu^\rho(\rho_\theta) - \rho \| = \sup_{\theta \in T_\rho} \| \eta(\rho_\theta) \| \leq c_{25}\delta^{3/2} . \rho_\theta \| \leq c_{33}\delta^2.$$

For an arbitrary section $\psi \in \omega$ we obtain:

$$\rho(\zeta, L\psi) \leq \rho(L\zeta, L\psi) + \rho(\zeta, L\zeta) \leq (1 - \frac{c\delta}{3}) . \rho(\zeta, \psi) + c_{33} . \delta^2$$

$$\leq (1 - \frac{c\delta}{3}) . c_0\delta + c_{33}\delta^2 = c_0\delta(1 - \frac{c\delta}{3} + \frac{c_{33}}{c_0} . \delta).$$

Now possibly increasing the radius of B_δ, i. e. the constant c_0, we obtain $\rho(\zeta, L\zeta) \leq c_0\delta$.

We shall note that it was possible, because the constant c_{33} does not depends upon the constant c_0. So we saw that L is correctly defined and is contraction mapping. Hence L has in ω unique stationary point η and $L\eta = \eta$. The graphics $\Gamma_\delta = \{\eta(\theta) : \theta \in T_\rho\}$ of the section η is an invariant manifold of the shift operator along the trajectories of equation (3.8) — N_t. As L is contraction mapping, then η possesses the feature asymptotic stability. The projection π is a homeomorphism between Γ_δ and T_ρ. Thereby Theorem 4.2. in the case, when the torus determined by Assumption 2. is an odd dimensional is proved. The even-dimensional case, however, is considered by analogy. Suffice it to assume for example $\tau_0 = 0$.

In order to prove Theorem 4.1. we have to show at least one pair of nonlinearities $g \in \mathscr{H}$, $f \in \mathscr{M}$ for which Assumption 2. and Assumption 3. are satisfied (see the formulation of Theorem 4.2.).

Example. Let us note that for $\delta = 0$

$$\operatorname{Re} \Phi_k^n = \frac{3}{4} \operatorname{Re} f^n_{n,k,-k} + \frac{3}{4} \operatorname{Im} g^n_{n,k,-k} / (\sqrt{1+n^2} . (1+k^2)), \quad k \neq n$$

$$\operatorname{Re} \Phi_n^n = \frac{3}{8} \operatorname{Re} f^n_{n,n,-n} + \frac{3}{8} \operatorname{Im} g^n_{n,n,-n} / (1+n^2)^{3/2}$$

(see the definition of Φ_k^n). Moreover, for $|k| \leq N, |\lambda_k(\delta) - \lambda_k(0)| \leq c_{34} . \delta$, for some constant c_{34}. A simple example of nonlinearities f and g satisfying the conditions of Theorem 4.2. are

$$f(u_t)=u_t^3-\frac{3u_t}{\pi}\int_0^{2\pi}u_t^2dx;\quad g\in\mathscr{H},\quad \operatorname{Im} g^n_{n,k,-k}=0.$$

For example $g(u)=\sin u-u$. Really $(\vartheta^3)_n=\Sigma_{p+q+s=n}\vartheta_p\vartheta_q\vartheta_s$,

$$(\frac{3\vartheta}{\pi}\int_0^{2\pi}\vartheta^2dx)_n=6u_n\,.\sum_{k=0}^{\infty}|u_k|^2$$

and consequently the coefficients f^n_k aue equal to zero for $k\neq n$ and $f^n_{n,k,-k}=1$ for $k=n$. The system of linear equations from Assumption 2. in this case is diagonal and has a form

$$(\alpha\delta/2-\delta n^2-3\tau_n^2/8)\,.\,\tau_n=0,\ n=0,\ \pm1,\ldots,\ \pm N.$$

If $\alpha/2-N^2>0$, then the above system has a positive solution $(\tau_0,\tau_{\pm1},\ldots,\tau_{\pm N})$, where $\tau_i=\sqrt{\frac{8}{3}(\alpha\delta/2-\delta\,.\,n^2)}$. The corresponding matrix (α_{ij}) from Assumption 3. is diagonal and has eigenvalues $\{-\alpha\delta+2\delta n^2:\ n=0,\ \pm1,\ldots,\ \pm N\}$. They are negative because $\alpha-2N^2>0$. The inequalities from condition ii) of Assumption 3. are satisfied if $\alpha-2(N+1)^2<0$. Thus we conclude that if $\alpha\in(2N^2,\ 2(N+1)^2)$, then the Fourier coefficients satisfy Assumption 2. and Assumption 3., i. e. according to Theorem 4.2. equation (3.8) has an invariant asymptotic stable $2N+1$-dimensional torus. An arbitrary even-dimensional torus can be obtained if we replace the above function f by the function

$$f(u_t)=u_t^3-\frac{3u_t}{\pi}\int_0^{2\pi}u_t^2\,dx-\frac{3\alpha}{(\alpha-2)^2}\int_0^{2\pi}u_tdx\,.\,((\int_0^{2\pi}u_t\cos xdx)^2+(\int_0^{2\pi}u_t\sin xdx)^2).$$

If $\alpha\in(2N^2,\ 2(N+1)^2)$, then equation (3.8) possess $2N$-dimensional invariant asymptotic stable torus (the system linear equations from Assumption 2. has a solution $\tau_0=0$, $\tau_i>0$, $i=\pm1,\ldots,\ \pm N$).

If equation (3.8) possesses an invariant asymptotic stable torus, then the same is true for equation (2.3). Really if Γ_δ is the torus of equation (3.8), then $\Gamma_\delta=L_1\circ H(\Gamma_\delta)\subset\tilde{W}_1^3$ is an invariant asymptotic stable torus of equation (2.3). As all inequalities in Assumption 3. are uniformly strict then there exists a neighbourhood $D_e\subset\mathscr{M}_2$ of the point (g_3,f) and an interval $\Delta_e\subset\mathrm{R}$, such that for arbitrary nonlinearities $(g_3,\ f)\in D_e$ and constant $\alpha\in\Delta_e$ equation (2.3) possesses an invariant asymptotic stable torus with the same dimension. Thus the proof of Theorem 4.1. is completed.

7. Appendix. Let us consider the ordinary differential equation

$$\frac{d}{dt}u(t)=Lu(t)+\Phi(u(t))\tag{7.1}$$

in the Banach space $W_{1,C}$, where $\Phi:\ W_{1,C}\to W_{1,C}$ is nonlinear continuously differentiable operator and L is linear unbounded operator with everywhere dense domain of definition $D_L=W_{3,C}\cap W_{1,C}$. L is defined with the help of the equalities $L\exp(inx)=\lambda_n\exp(inx)$, where $\lambda_n=\alpha\delta/2-\delta n^2+(\alpha^2\delta^2/4-\alpha\delta^2n^2-\delta^2n^4-1-n^2)^{1/2}$.

Proposition 7.1. There exist constants $r>0$ and $T>0$ so that if $u_0\in S_{W_{1,C}}(0,\ r)$, then there is continuously differentiable with respect to t function $u(t)$, which takes values in $S_{W_{1,C}}(0,\ 2r)$ and satisfies equation (7.1) on the interval $[0,\ T]\ni t$.

Proof. We denote $W_{0,C}=L_{2,C}[0,\ 2\pi]$. Let us define the operator $B:\ W_{1,C}\to W_{0,C}$ in this way $B(u)=u_0+\Sigma_{n\in Z}\,n\,.\,u_n\,.\,e^{inx}$, where $u=\Sigma_{n\in Z}u_ne^{inx}$. The operators $B:\ W_{1,C}\to W_{0,C}$ and $B^{-1}:\ W_{0,C}\to W_{1,C}$ are continuous and $\|B\|=1$, $\|B^{-1}\|=2$. The operator $\Phi\circ B^{-1}$:

$W_{0,C} \to W_{1,C}$ is continuously differentiable. In conjunction with (7.1) let us consider the equivalent integral equation

$$u(t) = e^{tL} . u(0) + \int_0^t e^{(t-\tau)L} \Phi(u(\tau)) d\tau \tag{7.2}$$

and also the equation

$$y(t) = Be^{tL} . B^{-1} . y(0) + \int_0^t Be^{(t-\tau)L} . \Phi . B^{-1} . y(\tau) \, d\tau. \tag{7.3}$$

The vector function $u(t) = B^{-1} y(t)$ is solution of (7.2) if and only if $y(t)$ is solution of (7.3).

Let us define the metric space Ω, consisting of vector functions $y(t)$ of the time $t \in [0, T]$, taking values in the space $W_{0,C}$ and satisfying the conditions:

i) $y(0) = y_0$, where $y_0 \in S_{W_{0,C}}(0, r)$ is a fixed initial condition.

ii) $y(t) \in S_{W_{0,C}}(0, 2r)$ for $t \in [0, T]$.

With the metric $\rho(y_1, y_2) = \sup_{t \in [0,T]} \| y_1(t) - y_2(t) \|_{W_{0,C}}$ the set Ω turns into a complete metric space.

We shall prove that the operator $G: \Omega \to \Omega$ defined thus

$$G: y(t) \to Be^{tL} B^{-1} y_0 + B \int_0^t e^{(t-\tau)L} . \Phi \circ B^{-1} y(\tau) d\tau$$

is contraction mapping.

On the half-axis $r > 0$ there exists a function $K(r) > 0$, $K(r) = O(r)$, $r \to 0$, such that for arbitrary $y_1, y_2 \in S_{W_{0,C}}(0, r)$ is fulfilled

$$\| \Phi \circ B^{-1} y_1 - \Phi . B^{-1} y_2 \|_{W_{1,C}} \le K(r) . \| y_1 - y_2 \|_{W_{0,C}}.$$

We obtain consequtively

$$|(Gy_1)(t)_n - (Gy_2)(t)_n| = | n e^{t\lambda_n} . \int_0^t e^{-\tau\lambda_n} . ((\Phi \circ B^{-1} y_1(\tau))_n - (\Phi \circ B^{-1} y_2(\tau))_n) d\tau |$$

$$\le | e^{t\lambda_n} | . \int_0^t e^{-\tau\lambda_n} | . K(r) . \| y_1(\tau) - y_2(\tau) \| \, d\tau \le \rho(y_1, y_2) . K(r) . | 1 - e^{t \operatorname{Re} \lambda_n} | / |\operatorname{Re} \lambda_n|.$$

We denote $\sigma_n = | 1 - \exp(t \operatorname{Re} \lambda_n) | / | \operatorname{Re} \lambda_n |$. In order to estimate the quantity $\| (Gy_1)(t) - (Gy_2)(t) \|_{W_{0,C}}$ we have to estimate first the power series $S = (\Sigma_{n \in Z} \sigma_n^2)^{1/2}$.

The obvious estimate $| \sigma_n | \le c_{35} / \delta(1 + n^2)$ and $S \le c_{35} (\Sigma_{n \in Z} (1 + n^2)^{-2})^{1/2} / \delta$ is not suitable for our purposes. In this way we prove the local solvability of equation (7.1) only in a domain with a radius $r = r(\delta) = O(\delta)$ while as we know the distance between the invariant manifold from Theorem 4.1. and the origine is $O(\sqrt{\delta})$. Such a minor gap is done in [9]. In our case the above difficulty can be got over in the following way:

Let us fix the parameter $\delta > 0$. If $| n | \ge \delta^{-1}$, then holds $| \sigma_n | \le 2 / | \operatorname{Re} \lambda_n | \le c_{36} / \delta(1 + n^2) \le c_{37} / (1 + | n |)$. If $| n | < \delta^{-1}$, then holds $| \sigma_n \le t \exp(t \operatorname{Re} \lambda_n(\delta_0))$, where $0 < \delta_0 < \delta$ and consequently $S \le (c_{38} t^2 / \delta + c_{39})^{1/2}$. If $t \le \delta^{1/2}$ we obtain ultimately $S \le c_{40}$, where c_{40} is a constant which does not depend upon the parameter δ. Possibly decreasing the constant $r > 0$, we obtain $K(r) . S \le 1/2$ and consequently $\rho(Gy_1, Gy_2) \le \rho(y_1, y_2)/2$, i. e. G is contraction mapping. Hence it follows that G has in Ω a unique stationary point $y(t)$, $t \in [0, T]$ which is simultaneously a solution of (7.3) with an initial condition $y(0) = y_0$. Taking into consideration that $\| u \|_{W_{1,C}} = \| B^{-1} y \| \le 2 \| y \|_{W_{0,C}} \le 2 \| u \|_{W_{1,C}}$, we conclude that $u = B^{-1} y$ is solution of (7.1), which satisfies the conditions of the Proposition. We shall note once again, that $T=O(\delta^{1/2})$ but r does not depend on δ.

After combining the above Proposition with Theorem 9.6. and Theorem 9.7 from [3], one can prove that the operator $u(t)=u(t, u(0))$ for fixed t is differentiable with respect to $u(0)$ (and it was used in the paper).

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