Physica D 34 (1989) 223–239
North-Holland, Amsterdam

BIFURCATIONS OF INVARIANT MANIFOLDS IN THE GENERALIZED HÉNON–HEILES SYSTEM

Ljubomir GAVRILOV*
Institute of Mathematics and Mechanics, Differential Equations, 1090 Sofia, Bulgaria

Received 14 March 1988
Revised manuscript received 18 July 1988
Communicated by H. Flaschka

The generalized Hénon–Heiles Hamiltonian $H = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + Aq_1^2 + Bq_2^2) - q_1^2q_2 - (\varepsilon/3)q_2^3$ in the integrable case $\varepsilon = 6$ is studied from topological point of view. The topology of the real level sets for all generic values of the constants of the motion, and all generic bifurcations of the Liouville tori and cylinders are described by making use of the reach algebraic structure of the problem.

1. Introduction

Consider the Hamiltonian system corresponding to the generalized Hénon–Heiles Hamiltonian

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2 + Aq_1^2 + Bq_2^2\right) - q_1^2q_2 - \frac{\varepsilon}{3}q_2^3. \tag{1.1}$$

It is known that this system possesses a second integral of motion (i.e. it is Liouville completely integrable) in the following three cases:
i) $A = B$, $\varepsilon = 1$ [14],
ii) $\varepsilon = 6$, and arbitrary A and B [15],
iii) $16A = B$, $\varepsilon = 16$ [16, 17]
which can be identified also by the Painlevé analysis [15].

The purpose of the present paper is to show that the study of the algebraic structure of the generalized Hénon–Heiles system in the integrable case $\varepsilon = 6$ leads to a detailed description of its *real phase space topology*.

According to the classical Liouville theorem, the generic invariant manifolds (i.e. the generic level sets) of a completely integrable Hamiltonian system consist of tori and cylinders on which the Hamiltonian flows are straight-line motions. In order to describe the topological nature of the Hamiltonian flows on the whole phase space we have to determine also the topological type of the non-generic invariant manifolds, and then to explain how the invariant manifolds topologically fit together, as the values of the constants of the motion vary, to make up all the phase picture. We note that the Smale's topological program [7] for studying Hamiltonian systems with symmetries is not applicable because the complete integrability of the most non-trivial integrable systems is due to ingeneous first integrals, i.e. integrals which do not come from obvious symmetries of the system.

Recently, Fomenko [13] classified all generic bifurcations of compact Liouville tori. He proved that any generic bifurcation is a composition of five basic types of bifurcations. This theorem, however, cannot be applied to the generalized Hénon–Heiles system as its invariant manifolds are never compact. The Fomenko's theorem is

*Research partially supported by the Committee for Science to the Council of Ministers of P.R. of Bulgaria by contract No. 911.

used in [20] where the topology of the free rotating 4-dimensional rigid body (Euler–Arnold equations on the Lie algebra so(4)) in the Manakov case is studied. We also note the papers of M.P. Kharlamov [11, 12] in which the topology of the celebrated Kowalevski's top and the Goryachev–Chaplygin top were studied in a quite different way.

In order to study the topology of the generalized Hénon–Heiles system in the integrable case $\varepsilon = 6$ we use, in contrast to [13, 11, 12], the rich algebraic structure of the problem. This structure is described in section 2. We prove that the generic complex invariant manifolds can be completed into Abelian varieties (i.e. the system is algebraically completely integrable) each of them being a two-sheeted covering of a Jacobi variety of a genus two hyperelliptic curve S (theorem 1). In section 3 we prove that the topological type of the generic real invariant manifolds depends only upon the number and the location of the ovals on S (lemma 4). This result is crucial for the paper. It implies a description of the topological type of the generic real invariant manifolds of the system. The non-generic invariant manifolds are studied in a similar way. This leads to a description of all generic bifurcations of the Liouville tori and cylinders. These are the main results of the paper and they are summarized in theorem 2 and theorem 3.

2. Algebraic structure

Consider the generalized Hénon–Heiles Hamiltonian (1.1) in the integrable case $\varepsilon = 6$. The corresponding Hamiltonian system is

$$\dot q_1 = p_1, \qquad \dot p_1 = 2q_1q_2 - Aq_1,$$
$$\dot q_2 = p_2, \qquad \dot p_2 = q_1^2 + 6q_2^2 - Bq_2, \tag{2.1}$$

and the second integral of motion reads [15]

$$F = q_1^4 + 4q_1^2q_2^2 + 4p_1(p_1q_2 - p_2q_1) - 4Aq_1^2q_2 + (4A - B)(p_1^2 + Aq_1^2). \tag{2.2}$$

Let $\mathbf{A}_{\mathbf{C}}$ denote the affine algebraic variety

$$\mathbf{A}_{\mathbf{C}} = \{H = h, F = f\} \subset \mathbf{C}^4. \tag{2.3}$$

$\mathbf{A}_{\mathbf{C}}$ will be called also complex invariant manifold, even though for some values of h, f, A, and B (the bifurcation set) it will not have a manifold structure.

As Wojciechowski [1] has shown, the Hamilton–Jacobi equation corresponding to the system (2.1) separates in u, v coordinates defined by the formula

$$q_1^2 = -4uv,$$
$$q_2 = u + v + (B - 4A)/4. \tag{2.4}$$

It is easy to check that the restrictions of p_1 and p_2 on $\mathbf{A}_{\mathbf{C}}$ can be expressed in terms of the u, v coordinates in the following way:

$$p_1 = \sqrt{-1}\,\frac{\sqrt{u}\sqrt{P(v)} - \sqrt{v}\sqrt{P(u)}}{2(u - v)},$$
$$p_2 = \frac{\sqrt{v}\sqrt{P(v)} - \sqrt{u}\sqrt{P(u)}}{2(u - v)}. \tag{2.5}$$

In the above expressions $P(z)$ denotes the polynomial

$$P(z) = 16z^4 - 8(6A - B)z^3 + (12A - B)(4A - B)z^2 + \left(8h - A(4A - B)^2\right)z - f. \tag{2.6}$$

Finally, as $\dot q_1 = \mathrm{d}\sqrt{-4uv}/\mathrm{d}t = p_1$ and $\dot q_2 = \mathrm{d}(u + v)/\mathrm{d}t = p_2$, we conclude that u and v satisfy the following system of differential equations:

$$\frac{\mathrm{d}u}{\sqrt{uP(u)}} + \frac{\mathrm{d}v}{\sqrt{vP(v)}} = 0,$$
$$\frac{u\,\mathrm{d}u}{\sqrt{uP(u)}} + \frac{v\,\mathrm{d}v}{\sqrt{vP(v)}} = -\tfrac{1}{2}\,\mathrm{d}t. \tag{2.7}$$

Suppose that the genus two hyperelliptic curve S: $\{w^2 = z \cdot P(z)\}$ is non-degenerated (i.e.

disc$(z \cdot P(z)) \neq 0$). Then, as it is well known, the equations (2.7) imply that the solutions of the system (2.1), lying on $\mathbf{A}_{\mathbf{C}}$, can be expressed in terms of genus two hyperelliptic theta functions associated with the curve S.

Let $\mathrm{d}\zeta_1, \mathrm{d}\zeta_2$ be a normal basis of the space of holomorphic differentials on S (with respect to some canonical homology basis on S) where

$$\mathrm{d}\zeta_1(z) = \frac{(a_1 z + b_1)\,\mathrm{d}z}{\sqrt{z \cdot P(z)}}, \qquad \mathrm{d}\zeta_2(z) = \frac{(a_2 z + b_2)\,\mathrm{d}z}{\sqrt{z \cdot P(z)}}. \tag{2.8}$$

Denote by S^2 the 2-fold symmetric product of S and let ζ be the Abel–Jacobi map [4]

$$\zeta: S^2 \to J(S): P_1 + P_2 \to \left(\sum_{i=1}^{2} \int_{P_0}^{P_i} \mathrm{d}\zeta_1, \sum_{i=1}^{2} \int_{P_0}^{P_i} \mathrm{d}\zeta_2 \right), \tag{2.9}$$

where P_0 is a fixed base point, and $J(S)$ is the Jacobi variety of the curve S.

After integrating the system (2.7) we obtain

$$\zeta(u+v) = z \in J(S), \tag{2.10}$$

where $z = -\frac{1}{2}at + t^0$, $a = (a_1, a_2)$, $t^0 = (t_1^0, t_2^0)$. Here t_1^0 and t_2^0 are arbitrary constants, and a_1, a_2 are defined in (2.8). Solving the Jacobi inversion problem (2.10) we can express the symmetric functions $u+v$ and $u \cdot v$ in terms of theta functions on $J(S)$. Thus, using the standard formulae [10, 6], and (2.4), we obtain

$$q_1(t) = c_1 \frac{\Theta_0(-\frac{1}{2}at + t^0)}{\Theta_\infty(-\frac{1}{2}at + t^0)}, \qquad q_2(t) = \frac{\partial^2}{\partial t^2} \ln \Theta_\infty(-\tfrac{1}{2}at + t^0) + c_2, \tag{2.11}$$

where $\Theta_0(z)$ and $\Theta_\infty(z)$ are suitable first order theta functions with characteristic, and c_1, c_2 depend only upon A, B, f, and h. Similar explicit formulae are derived by Ercolani (see [17]), under the additional assumption $A = B = 0$. Formulae (2.11) imply that the system (2.1) can be linearized on $J(S)$. The precise geometric characterization of $\mathbf{A}_{\mathbf{C}}$ (which will be used in section 3) is that it is an affine part of an Abelian variety $\tilde{\mathbf{A}}_{\mathbf{C}}$, the last being a two-sheeted unramified covering of $J(S)$. If $\mathbf{A}_{\mathbf{C}}$ is a generic invariant manifold then this result can be proved by the technique developed in [2] (see also [3] where the system (2.1) in the case $A = B = 0$ is studied). In fact, the above geometric characterization of $\mathbf{A}_{\mathbf{C}}$ remains valid for all values of A, B, f, and h, such that disc$(z \cdot P(z)) \neq 0$. This assertion is not very surprising, as the explicit formulae (2.11) hold if disc$(z \cdot P(z)) \neq 0$. It is, however, essential for section 3 where the non-generic real invariant manifolds are also studied.

To summarize, we state the following:

Theorem 1. If disc$(z \cdot P(z)) \neq 0$ then the affine algebraic variety $\mathbf{A}_{\mathbf{C}}$ is a smooth complex manifold, biholomorphically equivalent to the complex manifold $\tilde{\mathbf{A}}_{\mathbf{C}} \setminus D_\infty$, where $\tilde{\mathbf{A}}_{\mathbf{C}}$ is an Abelian variety and $D_\infty \subset \tilde{\mathbf{A}}_{\mathbf{C}}$ is a curve. $\tilde{\mathbf{A}}_{\mathbf{C}}$ is a two-sheeted unramified covering of the Jacobi variety $J(S)$ of the genus two hyperelliptic curve S: $\{w^2 = z \cdot P(z)\}$, such that D_∞ becomes a two-sheeted covering of the genus two hyperelliptic curve $\zeta(S) \cong S$. The Hamiltonian flows defined by H and F on $\mathbf{A}_{\mathbf{C}}$ extend holomorphically to flows on $\tilde{\mathbf{A}}_{\mathbf{C}}$ which are straight-line motions.

Later in this section we shall prove theorem 1. Suppose that disc$(z \cdot P(z)) \neq 0$. According to (2.11) the functions p_1, q_1, p_2, q_2 have no branch points and hence they can be considered as multivalued meromorphic functions on $J(S)$ (we recall that $p_1 = \dot{q}_1$, $p_2 = \dot{q}_2$). The expressions (2.4), (2.5) imply that $p_1^2, q_1^2, p_2, q_2, p_1 \cdot q_1$ are single-valued meromorphic functions on S^2 and hence they are single-valued on $J(S)$ too. Consider the Abelian variety $\tilde{\mathbf{A}}_{\mathbf{C}} = \mathbf{C}^2/\mathbf{Z}\{e_1, e_2, e_3, 2e_4\}$ where $J(S) = \mathbf{C}^2/\mathbf{Z}\{e_1, e_2, e_3, e_4\}$. If the basis e_1, e_2, e_3, e_4 of the period lattice is chosen in a proper way then the functions p_1, p_2, q_1, q_2 become single-valued on $\tilde{\mathbf{A}}_{\mathbf{C}}$. Let us fix such a basis. The natural projection

$$\pi: \tilde{\mathbf{A}}_{\mathbf{C}} \to J(S): \mathbf{C}^2/\mathbf{Z}\{e_1, e_2, e_3, 2e_4\} \to \mathbf{C}^2/\mathbf{Z}\{e_1, e_2, e_3, e_4\} \tag{2.12}$$

corresponds to the involution $(p_1, p_2, q_1, q_2) \to (-p_1, p_2, -q_1, q_2)$ on $\mathbf{A}_{\mathbf{C}}$. Consider the mapping

$$i: \mathbf{C}^4 \to \mathbf{CP}^7: (q_1, q_2, p_1, p_2) \to [f_0, f_1, \ldots, f_7],$$

where

$$\begin{aligned} f_0 &= 1, \\ f_1 &= q_1, \\ f_2 &= q_2, \\ f_3 &= p_1, \\ f_4 &= q_1^2, \\ f_5 &= 2q_2p_1 - q_1p_2, \\ f_6 &= 2p_1^2 - q_1(2q_1q_2 - Aq_1), \\ f_7 &= p_1p_2 - 2q_2(q_1q_2 - Aq_1). \end{aligned} \tag{2.13}$$

Lemma 1. The functions f_i, $i = 0, 1, \ldots, 7$, considered as single-valued meromorphic functions on $\tilde{\mathbf{A}}_{\mathbf{C}}$ provide a smooth embedding of $\tilde{\mathbf{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$.

Proof of theorem 1 *assuming the above lemma*. Let $D_\infty \subset \tilde{\mathbf{A}}_{\mathbf{C}}$ be the minimal divisor along which the functions p_1, p_2, q_1, q_2 blow up. According to (2.11) D_∞ coincides with the pre-image of the divisor $(\Theta_\infty(z)) \subset J(S)$ with respect to the projection π: $\tilde{\mathbf{A}}_{\mathbf{C}} \to J(S)$. As the pole divisor of the function z on S is $(z)_\infty = 2P_\infty$, where P_∞ is the "infinite" point on S, then the pole divisor of the function $u \cdot v$ on S^2, considered as meromorphic function on $J(S)$ is $2(\zeta(P_\infty) + \zeta(S))$. Further we shall suppose that the base point of the Abel–Jacobi map ζ is P_∞, i.e. $\zeta(P_\infty) = 0 \in J(S)$. It follows that $(\Theta_\infty(z)) = \zeta(S)$ (see (2.4), (2.11)) and $D_\infty = \pi_0^{-1}\zeta(S)$.

As the functions $f_0, f_1, \ldots, f_7$ provide an embedding of $\tilde{\mathbf{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$ (lemma 1) then $\overline{i(\mathbf{A}_{\mathbf{C}})}$ is biholomorphically equivalent to $\tilde{\mathbf{A}}_{\mathbf{C}}$. Identifying $\tilde{\mathbf{A}}_{\mathbf{C}}$ and $\overline{i(\mathbf{A}_{\mathbf{C}})}$ we denote the image of D_∞ in $\overline{i(\mathbf{A}_{\mathbf{C}})}$ by D_∞ as well. It is clear now that $\overline{i(\mathbf{A}_{\mathbf{C}})} = i(\mathbf{A}_{\mathbf{C}}) \cup D_\infty$.

Notice that i is a biholomorphic mapping between some neighbourhood $V_{\mathbf{A}_{\mathbf{C}}} \subset \mathbf{C}^4$ of $\mathbf{A}_{\mathbf{C}}$ and $i(V_{\mathbf{A}_{\mathbf{C}}}) \subset \mathbf{CP}^7$. Indeed, if $(q_1, q_2, p_1, p_2) \in \mathbf{A}_{\mathbf{C}}$ then

$$\det\left(\frac{\partial(f_1, f_2, f_3, f_5)}{\partial(q_1, q_2, p_1, p_2)}\right) = -q_1,$$

$$\det\left(\frac{\partial(f_1, f_2, f_3, f_7)}{\partial(q_1, q_2, p_1, p_2)}\right) = p_1$$

and hence $\operatorname{rank}(i) = 4$ (otherwise the equality $q_1 = p_1 = 0$ implies $f = 0$ and hence $\operatorname{disc}(z \cdot P(z)) = 0$). As $i(\mathbf{A}_{\mathbf{C}}) = \tilde{\mathbf{A}}_{\mathbf{C}} \setminus D_\infty$ is a smooth complex manifold, it is concluded that $\mathbf{A}_{\mathbf{C}}$ is also a smooth complex manifold.

It remains to prove that the Hamiltonian flows g_H^t, g_F^t corresponding to H and F linearize on $\overline{i(\mathbf{A}_{\mathbf{C}})} \cong \tilde{\mathbf{A}}_{\mathbf{C}}$. (2.11) implies that the flow g_H^t linearizes on $\tilde{\mathbf{A}}_{\mathbf{C}}$. The flow g_F^t can be extended holomorphically to a flow on $\tilde{\mathbf{A}}_{\mathbf{C}}$ (and hence it can be linearized on $\tilde{\mathbf{A}}_{\mathbf{C}}$) in the following standard way: If $z \in D_\infty$ we define $g_F^t(z) = g_H^{-t_1} g_F^t g_H^{t_1}(z)$ for suitable t_1. This definition does not depend upon the choice of t_1 as the above flows commute (see [2] for details). It completes the proof of theorem 1.

Proof of lemma 1. Denote for an arbitrary divisor $D \subset \tilde{\mathbf{A}}_{\mathbf{C}}$

$$\mathscr{L}(D) = \{ f \text{ meromorphic on } \tilde{\mathbf{A}}_{\mathbf{C}}, (f) \geq -D \}.$$

$\zeta(S) \subset J(S)$ is a translation of the Riemann theta divisor and hence it defines (1, 1) polarization on $J(S)$ [4]. As $\tilde{\mathbf{A}}_{\mathbf{C}}$ is a two-sheeted covering of $J(S)$ then $D_\infty = \pi^{-1} \circ \zeta(S)$ defines a (1, 2) polarization on $\tilde{\mathbf{A}}_{\mathbf{C}}$. It is concluded that $2D_\infty$ is a very ample divisor [5] and $\dim \mathscr{L}(2D_\infty) = 2 \cdot 4 = 8$. As the minimal divisor along which the functions q_1, q_2, p_1, p_2 (as functions on $\tilde{\mathbf{A}}_{\mathbf{C}}$) blow up is D_∞, then $f_i \in \mathscr{L}(k \cdot D_\infty)$ for some $k \in \mathbf{Z}$. We shall prove that $k=2$ and the functions f_i are linearly independent on $\tilde{\mathbb{A}}_{\mathbf{C}} \cong \overline{i(\mathbb{A}_{\mathbf{C}})}$ (and hence they form a basis of $\mathscr{L}(2D_\infty)$ and provide an embedding of $\tilde{\mathbb{A}}_{\mathbf{C}}$ into $\mathbf{CP}^7$). For that purpose we shall find the asymptotic expansions of the solutions of (2.1) as functions of the time t in a neighbourhood of a generic point $t^0 \in D_\infty$ (notice that the flow g_H^t and the curve D_∞ intersect transversely in the generic points of D_∞). This procedure is a simple linear algebra problem [2] and it is also a part of the well-known Painlevé property test [15]. We find that the system (2.1) admits only one family of Laurent solutions depending on three continuous parameters α, β, θ, and the constants A and B:

$$\begin{aligned} q_1(t) &= -\alpha t^{-1} + t\alpha(-\alpha^2 - 6A + B)/12 \\ &\quad + t^2\beta/2 + t^3\alpha\left[10\alpha^4 + 11\alpha^2(6A - B)\right. \\ &\quad \left.+ 90A^2 - 30AB + B^2\right]/720 \\ &\quad + t^4\beta(-5\alpha^2 - 6A + B)/120 + o(t^4), \\ q_2(t) &= t^{-2} + (B - \alpha^2)/12 \\ &\quad + t^2(-5\alpha^4 - 24\alpha^2 A + 4\alpha^2 B + B^2)/240 \\ &\quad + t^3\alpha\beta/6 + t^4\theta/4 + o(t^4). \end{aligned} \tag{2.14}$$

After substituting (2.14) into (2.13) ($p_1 = \dot{q}_1$, $p_2 = \dot{q}_2$) we obtain

$$\begin{aligned} f_0 &= 1, \\ f_1 &= -\frac{\alpha}{t} + \cdots, \\ f_2 &= \frac{1}{t^2} + \cdots, \\ f_3 &= \frac{\alpha}{t^2} + \cdots, \\ f_4 &= \frac{\alpha^2}{t^2} + \cdots, \\ f_5 &= \frac{1}{2t^2}\alpha(-\alpha^2 - 4A + B) + \cdots, \\ f_6 &= \frac{1}{2t^2}\alpha^2(-\alpha^2 - 6A + B) + \cdots, \\ f_7 &= -\frac{3\beta}{t^2} + \cdots. \end{aligned} \tag{2.15}$$

All complex constants α, β, and θ, such that $(p_1, p_2, q_1, q_2) \in \mathbb{A}_{\mathbf{C}}$, parametrize the pole divisor D_∞. Substituting (2.14) into the equations $H(p_1, p_2, q_1, q_2) = h$ and $F(p_1, p_2, q_1, q_2) = f$ and solving the simultaneous system for α, β, and θ we conclude that α and β satisfy the following equation:

$$\begin{aligned} &144\beta^2 + \alpha^8 + 2\alpha^6(6A - B) \\ &\quad + \alpha^4(12A - B)(4A - B) \\ &\quad + 4\alpha^2\left[A(4A - B)^2 - 8h\right] - 16f = 0. \end{aligned} \tag{2.16}$$

Eqs. (2.16) and (2.15) imply that the functions $f_0, f_1, \ldots, f_7$ are linearly independent of $\tilde{\mathbb{A}}_{\mathbf{C}}$, and hence lemma 1 is proved.

3. Topological analysis

In the present section we consider the system (2.1) as a system of real differential equations. The constants A, B, f, h will be real constants and the flows g_H^t, g_F^t will be real flows (i.e. $t \in \mathbb{R}$). Denote by $\mathbb{A}_{\mathbb{R}}$ the real invariant manifold of the system (2.1):

$$\mathbb{A}_{\mathbb{R}} = \{H = h, F = f\} \subset \mathbb{R}^4.$$

In this section we shall show that the information for the complex system (2.1), derived in section 2, suffices to describe the topological nature of the real flow g_H^t. It means (in the context of the present paper) that we have to describe

i) the topological type of $\mathbb{A}_{\mathbb{R}}$ for all constants A, B, f, h;
ii) how the sets $\mathbb{A}_{\mathbb{R}}$ topologically fit together, as A, B, f, h vary, to make up $\mathbb{R}^4\{q_1, q_2, p_1, p_2\}$.

As it will be seen from the next theorem, the topological type of $\mathbb{A}_{\mathbb{R}}$ may change only as the point (A, B, f, h) passes through the set

$$\mathbb{B} = \{(A, B, f, h) \in \mathbb{R}^4 : \operatorname{disc}(z \cdot P(z)) = 0\}.$$

For that reason the bifurcation set of the system (2.1) will be called the set $\mathbb{B}$.

Definition. The sets $\mathbb{B} \cap \{A = A_1, B = B_1\}$ and $\mathbb{B} \cap \{A = A_2, B = B_2\}$ are topologically equivalent provided that there exist continuous functions $A = A(s)$, $B = B(s)$, $s \in [0,1]$, such that $A(0) = A_1$, $B(0) = B_1$, $A(1) = A_2$, $B(1) = B_2$, and all sets $\mathbb{B} \cap \{A = A(s), B = B(s)\}$, $s \in [0,1]$ are homeomorphic each to other (with respect to the relative topology in $\mathbb{R}^4$).

Theorem 2. All topologically different intersections $\mathbb{B} \cap \{A = \text{const.}, B = \text{const.}\}$ are given in fig. 1. The set $\mathbb{R}^4 \backslash \mathbb{B}$ consists of 14 open, connected and non-intersecting with each other domains. The topological type of the invariant manifold $\mathbf{A}_{\mathbb{R}}$ may change only as the point (A, B, f, h) passes through the set $\mathbb{B}$ and, if $(A, B, f, h) \in \mathbb{R}^4 \setminus \mathbb{B}$, it is described in table I.

Remark. Each of the 20 sets pictured in fig. 1 corresponds to an arbitrary point (A, B), lying in one of the 20 sets shown in fig. 2, and having the same number as the intersection $\mathbb{B} \cap \{A = \text{const.}, B = \text{const.}\}$ pictured in fig. 1. The notation $m \cdot T + n \cdot C$ in table I means a disjoint union of m two-dimensional tori and n two-dimensional cylinders. The location of the 14 subdomains of $\mathbb{R}^4 \setminus \mathbb{B}$ is also shown in fig. 1.

Before proving theorem 2 we shall make some observations. Denote by τ: $S \to S$ the hyperelliptic involution $\tau(w, z) = (-w, z)$ and by $\bar{\tau}$: $S \to S$ the antiholomorphic involution $\bar{\tau}(w, z) = (\bar{w}, \bar{z})$, where S: $\{w^2 = z \cdot P(z)\}$. Introduce the following notation:

$$\mathring{R} = \{u + v \in S^2 \colon u = \tau(v)\},$$

$$R = \{u + v \in S^2 \colon u = P_\infty\} \cup \mathring{R}.$$

Consider the natural projection $S \setminus P_\infty \to \mathbb{C}$: $(w, z) \to z$. The context will distinguish a point $u \in S$ from the natural projection of this point on the z-plane. $\mathring{R}$ is the pre-image of the point $0 \in J(S)$, and R is the pre-image of the curve $\zeta(S) \subset J(S)$ in S^2 with respect to the Abel–Jacobi map ζ: $S^2 \to J(S)$. It is easy to check, using the Abel's theorem [4] that ζ: $S^2 \setminus \mathring{R} \to J(S) \setminus \{0\}$ is a biholomorphic mapping. As the functions q_1, q_2, p_1, p_2, considered as two-valued meromorphic functions on $J(S)$ (see (2.11)), blow up only along the curve $\zeta(S)$ then the formulae (2.4), (2.5) define a two-valued mapping μ: $S^2 \backslash R \to \mathbf{A}_{\mathbb{C}}$: $u + v \to (\pm q_1, q_2, \pm p_1, p_2)$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbf{A}_{\mathbb{R}} \subset \mathbf{A}_{\mathbb{C}} & \xrightarrow{\ i\ } & i(\mathbf{A}_{\mathbb{C}}) \cong \tilde{\mathbf{A}}_{\mathbb{C}} \setminus \mathrm{D}_\infty \\ \mu \uparrow & & \downarrow \pi \\ S^2 \backslash R & \underset{\zeta^{-1}}{\overset{\zeta}{\rightleftarrows}} & J(S) \setminus \zeta(S) \end{array} \tag{3.1}$$

(recall that i is biholomorphic mapping, and the projection π identifies the points (q_1, q_2, p_1, p_2) and $(-q_1, q_2, -p_1, p_2)$).

Definition. Each connected component of the set $\{u \in S \colon \bar{\tau}(u) = u\}$ is called an oval.

Using the definition of S we conclude that the natural projection of the ovals on the z-plane coincides with the set $\{z \in \mathbb{C} \colon z \in \mathbb{R}$ and $z \cdot P(z) \geq 0\}$. On the other hand, according to (3.1)

$$\zeta^{-1} \circ \pi \circ i(\mathbf{A}_{\mathbb{R}}) = \mu^{-1}(\mathbf{A}_{\mathbb{R}}) = \{u + v \in S^2 \backslash R \colon \mu(u + v) \in \mathbb{R}^4\},$$

and using (2.4), (2.5) we obtain

$$\begin{aligned} \zeta^{-1} \circ \pi \circ i(\mathbf{A}_{\mathbb{R}}) = \{ u + v \in S^2 \colon & u \in \mathbb{R}, \\ & v \in \mathbb{R},\ u^2 + v^2 \neq 0, \\ & uP(u) \geq 0,\ vP(v) \geq 0, \\ & vP(u) \leq 0,\ uP(v) \leq 0,\ uv \leq 0 \}. \end{aligned} \tag{3.2}$$

This implies immediately:

Lemma 2. Each connected component of the set $\overline{\zeta^{-1} \circ \pi \circ i(\mathbf{A}_{\mathbb{R}})}$ is a direct product of non-intersecting with each other ovals.

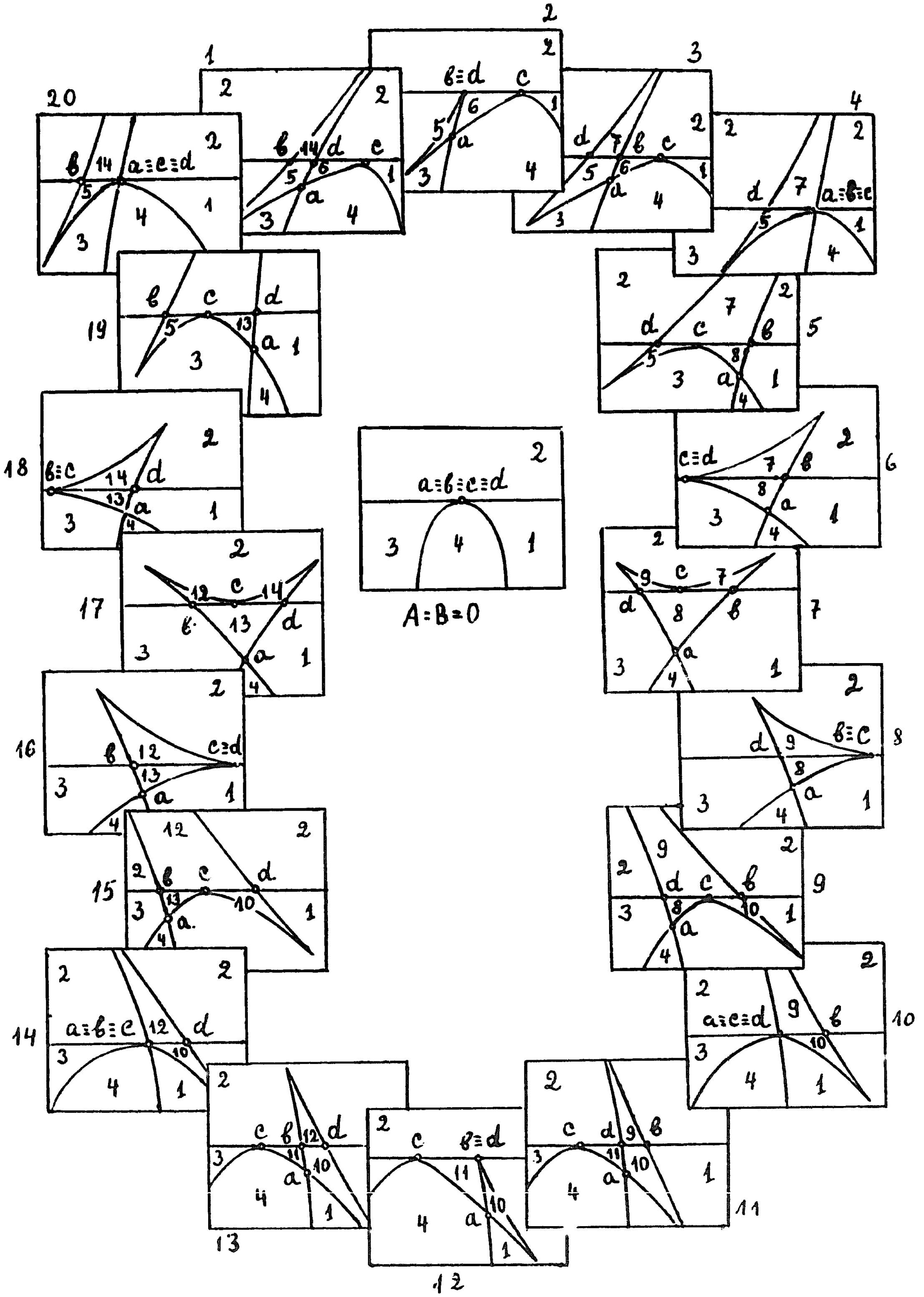

Fig. 1.

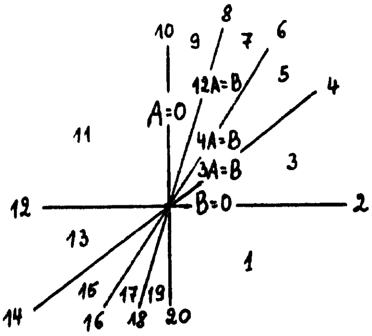

Fig. 2.

If the curve S possesses only one oval, then according to lemma 2, $\zeta^{-1}\circ\pi\circ i(\mathbb{A}_{\mathbb{R}})=\varnothing$. For that reason we shall suppose that S possesses at least two ovals which we denote by γ_1 and γ_2. As γ_1 and γ_2 do not intersect with each other, we can choose a canonical homology basis $\gamma_1, \gamma_2, \delta_1, \delta_2$. Let $\mathrm{d}\zeta_1, \mathrm{d}\zeta_2$ be the corresponding normal basis (2.8) of the space of holomorphic differentials on S, and

$$\lambda_i=\left(\int_{\gamma_i}\mathrm{d}\zeta_1, \int_{\gamma_i}\mathrm{d}\zeta_2\right), \qquad \lambda_{i+2}=\left(\int_{\delta_i}\mathrm{d}\zeta_1, \int_{\delta_i}\mathrm{d}\zeta_2\right), \qquad i=1,2,$$

be a basis of the period lattice on $J(S)$.

Lemma 3. Each connected component of the set $\overline{\pi\circ i(\mathbb{A}_{\mathbb{R}})}$ is a translation of the real two-dimensional torus $\mathbb{R}^2\{\lambda_1, \lambda_2\}/\mathbb{Z}\{\lambda_1, \lambda_2\}$ by a semiperiod.

Remark. Lemma 3 explains how to choose the "initial condition" $t^0=(t_1^0, t_2^0)$ in (2.11), in order to obtain real solutions. Namely, t^0 has to lie on one of the translates described in the above lemma (compare with the classical results of Kowalevski [9] and Kötter [18]).

Proof of lemma 3. First of all note that the relations

$$\int_{\gamma_i}\frac{\mathrm{d}z}{\sqrt{z\cdot P(z)}}\in\mathbb{R}, \quad \int_{\gamma_i}\frac{z\,\mathrm{d}z}{\sqrt{z\cdot P(z)}}\in\mathbb{R}, \quad i=1,2,$$

and $\int_{\gamma_i}\mathrm{d}\zeta_j=\delta_{ij}$ (δ_{ij} is the Kronecker delta) imply that $a_1, a_2, b_1, b_2\in\mathbb{R}$ (see (2.8)) and hence $\bar{\tau}^*\mathrm{d}\zeta_i=\overline{\mathrm{d}\zeta_i}$, $i=1,2$ (if in a local chart $\mathrm{d}\zeta_i=f(z)\mathrm{d}z$, then $\bar{\tau}^*\mathrm{d}\zeta_i=f(\bar{\tau}(z))\mathrm{d}\bar{\tau}(z)$). It follows that the antiholomorphic involution $\bar{\tau}$ induces, with the help of the Abel–Jacobi map ζ, an antiholomorphic involution on $J(S)$, which we denote by $\bar{\tau}$ as well. We have $\bar{\tau}$: $J(S)\to J(S)$: $(z_1, z_2)\to(\bar{z}_1, \bar{z}_2)$. The stationary set $\{z\in J(S)\colon \bar{\tau}(z)=z\}$ of $\bar{\tau}$ on $J(S)$ consists of the four tori described in lemma 3. As the real flows g_H^t and g_F^t are straight-line motions on $J(S)$, then each connected component of the set $\overline{\pi\circ i(\mathbb{A}_{\mathbb{R}})}$ is a two-dimensional real torus linearly embedded in $J(S)$, and hence it coincides with one of the above four tori. Thus lemma 3 is proved.

Table I

Domain:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Positive roots of $P(z)$	0	1	2	0	4	2	3	2	1	0	0	1	2	3
Negative roots of $P(z)$	2	1	0	0	0	0	1	2	3	4	2	3	2	1
Number of ovals on S	2	2	2	1	3	2	3	3	3	3	2	3	3	3
Connected components of $\overline{\pi\circ i(\mathbb{A}_{\mathbb{R}})}$	1	1	0	0	0	0	2	2	2	2	1	2	2	2
Connected components of $\overline{i(\mathbb{A}_{\mathbb{R}})}$	2	1	0	0	0	0	2	2	2	4	2	2	2	2
Topological type of $\mathbb{A}_{\mathbb{R}}$	2C	C	∅	∅	∅	∅	$T+C$	$T+2C$	3C	4C	2C	3C	$T+2C$	$T+C$

To prove theorem 2 we shall need also the following:

Lemma 4. Consider a connected component of the set $\pi \circ i(\mathbb{A}_{\mathbb{R}})$. According to (3.1) and lemma 2 the pre-image of this component in S^2 (with respect to the Abel–Jacobi map ζ) is a direct product of two ovals, which we denote by c_1 and c_2. The pre-image of the above connected component in $i(\mathbb{A}_{\mathbb{R}})$ (with respect to π) may consist of one torus, two tori, one cylinder, or two cylinders. More precisely, it consists of:

i) one torus, iff neither c_1 nor c_2 contains the point $P_\infty \in S$, but at least one of them contains the point $z = 0$ on S;
ii) two tori, iff the ovals c_1 and c_2 do not contain the points P_∞ and 0 on S;
iii) one cylinder, iff one of the ovals contains the point 0 and the second contains the point P_∞;
iv) two cylinders, iff one of the ovals c_1 and c_2 contains both the points P_∞ and 0 on S, or one of the ovals contains the point P_∞ and neither c_1 nor c_2 contains the point 0 on S.

Proof of lemma 4. Denote by N a connected component (real two-dimensional torus) of the set $\overline{\pi \circ i(\mathbb{A}_{\mathbb{R}})}$, corresponding to the ovals c_1 and c_2. As $\overline{i(\mathbb{A}_{\mathbb{C}})} \cong \tilde{\mathbb{A}}_{\mathbb{C}}$ is a two-sheeted covering of $J(S)$ (theorem 1), then the set $\pi^{-1}(N)$ may consist of a torus which doubly covers N, or it may consist of two tori each of them being a one-sheeted covering of N. If neither c_1 nor c_2 contains the point P_∞ on S then it is easy to check (using the Abel's theorem) that $N \cap \zeta(S) = \varnothing$, and hence $\pi^{-1}(N) \cap D_\infty = \varnothing$. This implies that the set $\pi^{-1}(N) = \pi^{-1}(N) \cap i(\mathbb{A}_{\mathbb{R}})$ consists of one or two tori (we recall that $\overline{i(\mathbb{A}_{\mathbb{C}})} = i(\mathbb{A}_{\mathbb{C}}) \cup D_\infty$). $\pi^{-1}(N)$ consists of two tori if and only if the restriction of the (two-valued) mapping π^{-1} on N has a trivial monodromy group. Equivalently, the set $\pi^{-1}(N)$ consists of two tori if and only if the restriction of the (two-valued) mapping μ on $c_1 \times c_2$ has a trivial monodromy group. The formulae (2.4), (2.5) imply that it is equivalent to the condition neither c_1 nor c_2 to contain the point 0 on S (as we suppose that c_1 and c_2 do not contain the point P_∞). Otherwise the set $\pi^{-1}(N)$ consists of one torus. Thus the assertions i) and ii) of lemma 4 are proved.

Suppose that at least one of the ovals c_1 and c_2 contains the point P_∞ on S. To fix the notation, let $P_\infty \in c_1$. It is easy to check (using again the Abel's theorem [4]) that $N \cap \zeta(S) = \zeta(c_2)$. As $\pi^{-1}(N)$ consists of one or two tori, and $\pi^{-1} \circ \zeta(c_2)$ consists of one or two circles then the set

$$\pi^{-1}(N) \cap i(\mathbb{A}_{\mathbb{R}}) = \pi^{-1}(N) \setminus \pi^{-1} \circ \zeta(c_2)$$

consists of
– two cylinders, if $\pi^{-1} \circ \zeta(c_2)$ consists of two circles,
– one cylinder, if $\pi^{-1} \circ \zeta(c_2)$ consists of one circle.

On the other hand, the set $\pi^{-1} \circ \zeta(c_2)$ consists of two circles if and only if the restriction of the (two-valued) mapping π^{-1} on the curve $\zeta(c_2)$ (or on an arbitrary translate of $\zeta(c_2)$ in $J(S)$) has a trivial monodromy group. Equivalently, the set $\pi^{-1} \circ \zeta(c_2)$ consists of two circles if and only if the restriction of the mapping μ on $c_2 \times \{s_0\}$, where s_0 is an arbitrary fixed point on S, has a trivial monodromy group. The last condition implies that the oval c_2 does not contain the point 0 on S (see (2.4), (2.5)). Otherwise the set $\pi^{-1} \circ \zeta(c_2)$ consists of one circle and hence the set $\pi^{-1}(N) \cap i(\mathbb{A}_{\mathbb{R}})$ consists of one cylinder. This completes the proof of lemma 4.

Proof of theorem 2. To determine the topological type of $i(\mathbb{A}_{\mathbb{R}})$ (and hence of $\mathbb{A}_{\mathbb{R}}$) it is enough, according to lemma 4, to determine the number and the location of the ovals on S. For that reason we shall study the bifurcation set $\mathbb{B}$. It is clear that $\tilde{\mathbb{A}}_{\mathbb{R}}$ may change its topological type only as (A, B, f, h) passes through the set $\mathbb{B}$ (if $\operatorname{disc}(z \cdot P(z)) \neq 0$ then $\mathbb{A}_{\mathbb{R}}$ is a smooth real manifold – see theorem 1). As

$$\mathbb{B} = \{\operatorname{disc}(z \cdot P(z) = 0\} = \{f = 0\} \cup \mathbb{B}_0,$$

where $\mathbb{B}_0 = \{\operatorname{disc}(P(z)) = 0\}$, then we shall study

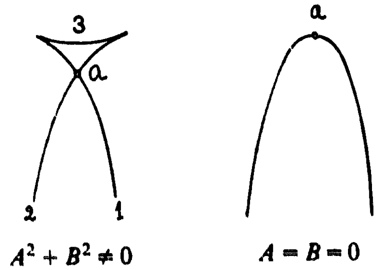

Fig. 3.

first the set $\mathbb{B}_0$. $\mathbb{B}_0$ is the set of all constants A, B, f, h, such that the polynomial $P(z)$ possesses a multiple root. Solving the simultaneous system

$$P(z)=0,$$
$$\frac{\mathrm{d}}{\mathrm{d}z}P(z)=0$$

for h and f we obtain the following parametrization of the intersection $\mathbb{B}_0\cap\{A=\text{const.}, B=\text{const.}\}$:

$$\begin{aligned} f&=-z^2(4z-4A+B)(12z-12A+B),\\ h&=(4z-4A+B)\left[-2z^2+(10A-B)z/4\right.\\ &\quad\left.+A(B-4A)/8\right]. \end{aligned} \tag{3.3}$$

If $A=B=0$ then the curve (3.3) is shown in fig. 3. If $A^2+B^2\neq 0$, the curve (3.3) has two cusp points, one normal crossing, and no points of inflexion (see fig. 3). Below we prove this assertion. After simple calculation we obtain the identity

$$\frac{\mathrm{d}f}{\mathrm{d}z}=8z\frac{\mathrm{d}h}{\mathrm{d}z}=-2z\big(96z^2+24(B-6A)z+(12A-B)(4A-B)\big).$$

The polynomial $\mathrm{d}h/\mathrm{d}z$ has two different real roots. If z_0 is such a root, as it is easy to check, the vectors

$$\left(\frac{\mathrm{d}^2h}{\mathrm{d}z^2}(z_0),\frac{\mathrm{d}^3h}{\mathrm{d}z^3}(z_0)\right)\quad\text{and}$$
$$\left(\frac{\mathrm{d}^2f}{\mathrm{d}z^2}(z_0),\frac{\mathrm{d}^3f}{\mathrm{d}z^3}(z_0)\right)$$

are linearly independent, i.e. z_0 is a cusp point. Using the identity $\mathrm{d}^2P(z)/\mathrm{d}z^2=-8\,\mathrm{d}h(z)/\mathrm{d}z$, we conclude that z_0 is a cusp point of the curve (3.3) if and only if z_0 is a root of $P(z)$ of multiplicity three.

If the curve (3.3) has a point of self-intersection, then the polynomail $P(z)$ possesses two pairs of double roots: $P(z)=16(z-z_1)^2(z-z_2)^2$, where $(f(z_1),h(z_1))=(f(z_2),h(z_2))$. Using the definition (2.6) of the polynomial $P(z)$ we conclude that z_1 and z_2 satisfy the identities

$$\begin{aligned} &-32(z_1+z_2)=8(B-6A),\\ &16\left(z_1^2+z_2^2+4z_1z_2\right)=(12A-B)(4A-B),\\ &-32z_1z_2(z_1+z_2)=8h-A(4A-B)^2,\\ &16z_1^2z_2^2=-f. \end{aligned}$$

The first two identities imply $z_1+z_2=(6A-B)/4$ and $z_1z_2=A(3A-B)/8$, and the second two

$$\begin{aligned} h&=-A^2(2A-B)/8,\\ f&=-A^2(3A-B)^2/4. \end{aligned} \tag{3.4}$$

We denote the point with coordinates (3.4) by a (see fig. 3). As $z_1\neq z_2$ then the vectors $(\mathrm{d}h(z_1)/\mathrm{d}z, \mathrm{d}f(z_1)/\mathrm{d}z)$ and $(\mathrm{d}h(z_2)/\mathrm{d}z, \mathrm{d}f(z_2)/\mathrm{d}z)$ are linearly independent and hence the point a is a norma¹ crossing.

If z is not a root of the polynomial $\mathrm{d}h(z)/\mathrm{d}z$ then the vector $(\mathrm{d}f(z)/\mathrm{d}z,\mathrm{d}h(z)/\mathrm{d}z)$ is collinear with the vector $(8z,1)$, and hence the curve (3.3) has no points of inflexion.

Consider a hyperplane $\{r_1A+r_2B=r_3\}$ in $\mathbb{R}^4\{A,B,f,h\}$. Using fig. 3 we can picture the intersection between $\mathbb{B}_0$ and the above hyperplane in the three-dimensional space $\mathbb{R}^3\{f,h,A\}$ or $\mathbb{R}^3\{f,h,B\}$ (see fig. 4). There is an obvious connection between this surface and the swallowtail surface known from the catastrophe theory [19]. It is not a quite unexpected fact. Indeed, the swallowtail surface can be characterized as the set of all points (a_1,a_2,a_3), such that the polynomial

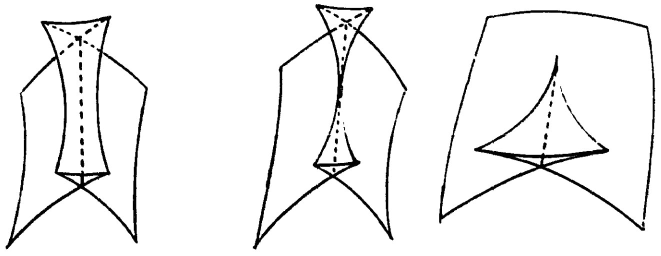

Fig. 4.

$z^4 + a_1 z^2 + a_2 z + a_3$ has a multiple root [19] (compare with (2.6)). The swallowtail surface can be found also in the bifurcation set of the celebrated Kowalevski top (see [12], formula (1.9)).

To study the set $\mathbb{B} = \mathbb{B}_0 \cup \{f = 0\}$ we consider again the intersection $\mathbb{B} \cap \{A = \text{const.}, B = \text{const.}\}$. The set $\mathbb{B}_0 \cap \{A = \text{const.}, B = \text{const.}\} \cap \{f = 0\}$ consists of three points b, c, and d with h-coordinates $B^3/216$, $A(4A - B)^2/8$, and 0 respectively (these coordinates are obtained after substituting $z = (12A - B)/12$, $z = 0$, and $z = (4A - B)/4$ in (3.3)). It is seen that $\{f = 0\}$ is the tangent plane of the (singular) manifold $\mathbb{B}_0$ at the point c. The topological type of the set $\mathbb{B} \cap \{A = \text{const.}, B = \text{const.}\}$ does not change, if moving the constants A and B, the points a, b, c, d remain all different. As $A(4A - B)^2/8 - B^3/216 = (3A - B)(12A - B)^2/216$ then the above condition amounts to

$$A \neq 0, \quad B \neq 0, \quad 3A - B \neq 0, \qquad (3.5)$$
$$4A - B \neq 0, \quad 12A - B \neq 0.$$

The inequalities (3.5) determine on the plane $\mathbb{R}^2\{A, B\}$ 10 non-intersecting with each other open domains and 10 half-lines (see fig. 2). The topological type of the set $\mathbb{B} \cap \{A = \text{const.}, B = \text{const.}\}$ does not change as the point (A, B) varies in one of the above open domains or half-lines. It is concluded that the sets $\mathbb{B} \cap \{A = A_1, B = B_1\}$ and $\mathbb{B} \cap \{A = A_2, B = B_2\}$ are topologically equivalent (see the definition before theorem 2) if and only if the points (A_1, B_1) and (A_2, B_2) lie in one and the same open domain or half-line shown in fig. 2.

The above arguments imply the following:

Lemma 5. All topologically different intersections $\mathbb{B} \cap \{A = \text{const.}, B = \text{const.}\}$ are shown in fig. 1. Each of these 20 intersections corresponds to a point (A, B) lying in one of the 20 sets shown in fig. 2, as it is explained in the remark after theorem 2.

Fig. 1 implies that the set $\mathbb{R}^4 \setminus \mathbb{B}$ consists of 14 open, connected, non-intersecting with each other domains. After simple calculations we find the number of positive and negative roots of $P(z)$, and hence the number of ovals on S in each of the above 14 domains (table I). Finally, using lemma 4 we determine the topological of $i(\mathbb{A}_{\mathbb{R}})$ (and hence of $\mathbb{A}_{\mathbb{R}}$) in these domains (table I). It completes the proof of theorem 2.

To this end we shall study all generic bifurcations of the set $\mathbb{A}_{\mathbb{R}}$.

Definition. A point $(A_0, B_0, f_0, h_0) \in \mathbb{B}$ is said to be generic provided that in a neighbourhood of this point the bifurcation set $\mathbb{B}$ is a smooth three-dimensional real manifold.

Definition. A bifurcation of the set $\mathbb{A}_{\mathbb{R}}$ is said to be generic provided that the point (A, B, f, h) passes, transversally to $\mathbb{B}$, through a generic point $(A_0, B_0, f_0, h_0) \in \mathbb{B}$.

If $M_1, M_2, \ldots, M_n$ are such sets, that $M_i \cap M_j$ consists of one point for $|i - j| = 1$, and $M_i \cap M_j = \varnothing$ for $|i - j| \neq 1, 0$, then we denote $\bigcup_{i=1}^{n} M_i$ by $M_1 \vee M_2 \vee \cdots \vee M_n$.

Consider the following bifurcations:

i) $T \to \varnothing$: the torus T collapses along its "axes" – the circle S^1 – and then vanishes, i.e. $T \to S^1 \to \varnothing$.

ii) $T \to T$: the torus T bifurcates into a torus with a doubled cycle, as it is shown in fig. 5. The singular manifold is homeomorphic to a non-trivial bundle with base S^1 and a fiber $S^1 \vee S^1$.

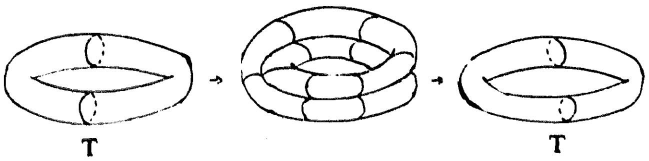

Fig. 5.

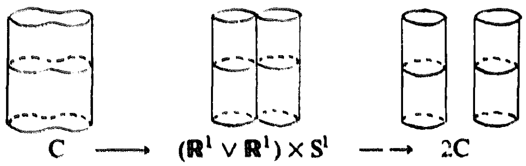

Fig. 6.

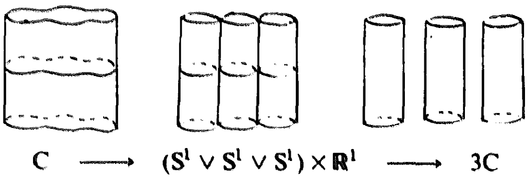

Fig. 7.

The bifurcations i) and ii) are also described in [11–13].

iii) $C \to \varnothing$: the cylinder C collapses along the line $\mathbb{R}^1$ and then vanishes, i.e. $C \to \mathbb{R}^1 \to \varnothing$.

iv) The bifurcations $C \to 2C$, $C \to 3C$, $2C \to 2C$, $T + C \to C$, and $T + 2C \to 2C$ are described in figs. 6–10, respectively.

v) The notation $C \sim C$, $2C \sim 2C$, etc. will mean that the corresponding set does not change its topological type.

If $M_1 \to M_2$ is an already defined bifurcation, then we denote by $M_2 \to M_1$ the "inverse" bifurcation.

Theorem 3. Any generic bifurcation of connected components of the invariant manifold $\mathbb{A}_{\mathbb{R}}$ can be found among the bifurcations i)–v). The precise description of all generic bifurcations of $\mathbb{A}_{\mathbb{R}}$ is given in table II.

Remark. The notation $i \xrightarrow{k} j$, used in table II, means a generic bifurcation between subdomains of $\mathbb{R}^4 \setminus \mathbb{B}$ with numbers i and j as in table I (see also fig. 1), and such that the point (A, B, f, h)

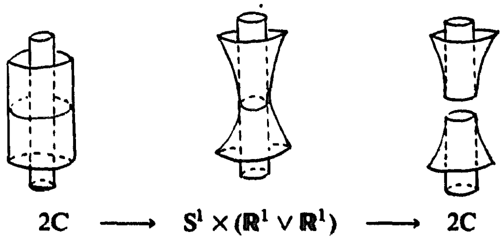

Fig. 8.

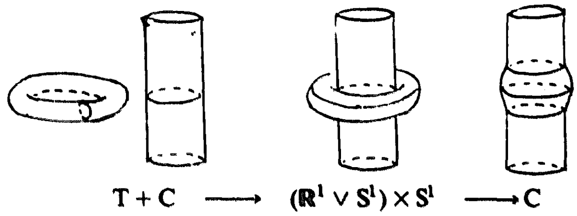

Fig. 9.

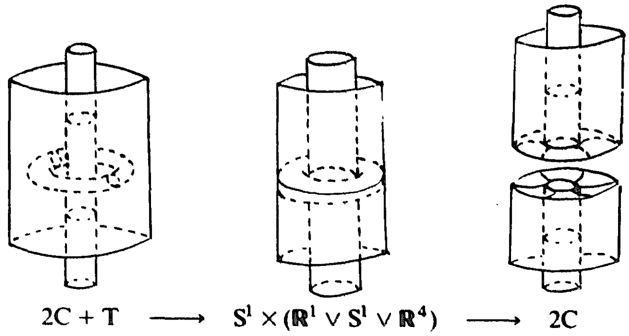

Fig. 10.

passes through a generic point $(A_0, B_0, f_0, h_0) \in \mathbb{B}$. The integer k is equal to the number of the branch of the curve $\mathbb{B}_0 \cap \{A = A_0, B = B_0\}$, shown in fig. 3, on which the point (f_0, h_0) lies. The notation $i \to j$, used in table II, means an arbitrary generic bifurcation between subdomains of $\mathbb{R}^4 \setminus \mathbb{B}$ with numbers i and j, respectively.

Proof of theorem 3. Consider a singular invariant manifold $\mathbb{A}^0_{\mathbb{R}}$, i.e. an invariant manifold corresponding to a point $(A_0, B_0, f_0, h_0) \in \mathbb{B}$. An arbitrary point lying on $\mathbb{A}^0_{\mathbb{R}}$ is a limit of a sequence of points lying on non-singular invariant manifolds. However, every point lying on a non-singular invariant manifold is parametrized by the formulas (2.4), (2.5) (see lemma 2 and (3.1)). After going to the limit in these formulas we conclude that the

Table II
List of all generic bifurcations.

$1 \longrightarrow 2$	$2 \longrightarrow 4$	$1 \longrightarrow 8$	$1 \xrightarrow{2} 10$	$1 \xrightarrow{3} 10$	$1 \longrightarrow 13$	$2 \longrightarrow 6$	$2 \xrightarrow{2} 7$	$2 \xrightarrow{3} 7$	$2 \xrightarrow{1} 9$	$2 \xrightarrow{3} 9$
$2C \longrightarrow C$	$C \longrightarrow \emptyset$ $C \longrightarrow \emptyset$	$2C \longrightarrow T+2C$	$\emptyset \longrightarrow C$ $\emptyset \longrightarrow C$ $2C \sim 2C$	$C \longrightarrow 2C$ $C \longrightarrow 2C$	$2C \longrightarrow T+2C$	$C \longrightarrow \emptyset$	$C \longrightarrow T+C$	$\emptyset \longrightarrow T$ $C \sim C$	$C \sim C$ $C \longrightarrow \emptyset$ $C \longrightarrow \emptyset$	$C \longrightarrow 3C$
$2 \longrightarrow 11$	$2 \xrightarrow{1} 12$	$2 \xrightarrow{3} 12$	$2 \xrightarrow{2} 14$	$2 \xrightarrow{3} 14$	$3 \longrightarrow 4$	$3 \longrightarrow 5$	$3 \longrightarrow 8$	$3 \longrightarrow 13$	$4 \longrightarrow 6$	$4 \longrightarrow 11$
$C \longrightarrow 2C$	$C \sim C$ $\emptyset \longrightarrow C$ $\emptyset \longrightarrow C$	$C \longrightarrow 3C$	$C \longrightarrow T+C$	$C \sim C$ $\emptyset \longrightarrow T$	$\emptyset \sim \emptyset$	$\emptyset \sim \emptyset$	$\emptyset \longrightarrow C$ $\emptyset \longrightarrow C$ $\emptyset \longrightarrow T$	$\emptyset \longrightarrow C$ $\emptyset \longrightarrow C$ $\emptyset \longrightarrow T$	$\emptyset \sim \emptyset$	$\emptyset \longrightarrow C$ $\emptyset \longrightarrow C$
$5 \longrightarrow 6$	$5 \longrightarrow 7$	$5 \longrightarrow 14$	$7 \longrightarrow 8$	$8 \longrightarrow 9$	$9 \longrightarrow 10$	$10 \longrightarrow 11$	$10 \longrightarrow 12$	$12 \longrightarrow 13$	$13 \longrightarrow 14$	$2 \longrightarrow 3$
$\emptyset \sim \emptyset$	$\emptyset \longrightarrow C$ $\emptyset \longrightarrow T$	$\emptyset \longrightarrow C$ $\emptyset \longrightarrow T$	$C \longrightarrow 2C$ $T \longrightarrow T$	$T \rightarrow \emptyset$ $\emptyset \rightarrow C$ $2C \sim 2C$	$C \longrightarrow 2C$ $2C \longrightarrow 2C$	$2C \sim 2C$ $C \rightarrow \emptyset$ $C \rightarrow \emptyset$	$2C \rightarrow 2C$ $2C \rightarrow C$	$2C \sim 2C$ $C \longrightarrow \emptyset$ $\emptyset \longrightarrow T$	$T \longrightarrow T$ $2C \longrightarrow C$	$C \longrightarrow \emptyset$

points lying on the singular invariant manifold $\mathbb{A}^0_{\mathbb{R}}$ are also parametrized by (2.4), (2.5), where u and v range according to (3.2). However, after going to the limit in (3.2) we could also obtain $u=v=0$ (as the curve S is degenerated for $(A_0, B_0, f_0, h_0) \in \mathbb{B}$). Using (2.4) we conclude that $q_1=0$ and $q_2=(B-4A)/4$. Substituting these equations in the identities $H=h$ and $F=f$ we obtain the curve

$$\{(q_1, q_2, p_1, p_2) \in \mathbb{R}^4\colon q_1=0,\ q_2=(B-4A)/4,\ p_1^2+p_2^2=2h-A(B-4A)^2/4\}. \tag{3.6}$$

Note that $q_1=0$, $q_2=(B-4A)/4$ and the identity $F=f$ imply $f=0$.

In view of the above arguments, our procedure of finding the topological type of the singular invariant manifold $\mathbb{A}^0_{\mathbb{R}}$ will consist of the following steps:

i) Using (3.2) we determine all possible pairs of intervals on $\mathbb{R}^1$ in which u and v range. Let Δ_1 and Δ_2 be such intervals. Without loss of generality we suppose that

$$u \in \Delta_1 \subset \{\tau \in \mathbb{R}^1\colon \tau \le 0\} \quad \text{and} \quad v \in \Delta_2 \subset \{\tau \in \mathbb{R}^1\colon \tau \ge 0\}.$$

ii) Using formulas (2.4), (2.5), as in the proof of lemma 4, we determine the topological types of the subsets of $\mathbb{A}^0_{\mathbb{R}}$ corresponding to $u \in \Delta_1, v \in \Delta_2$ but $u \neq v$, for all possible pairs (Δ_1, Δ_2).

iii) Determine the topological type of the set (3.6) (it is a circle, a point, or the empty set).

iv) The set $\mathbb{A}^0_{\mathbb{R}}$ is a union of the two non-intersecting sets described in ii) and iii). To determine the topological type of $\mathbb{A}^0_{\mathbb{R}}$ we take the closure of the set ii) and then identify the new points (if they exist) with the corresponding points of the set iii).

To determine the topological type of the set described in ii) we shall also use the following lemma:

Lemma 6. Suppose that the polynomial $z \cdot P(z)$ has a multiple real root α. Then $\mathbb{A}^0_{\mathbb{R}}$ is a smooth real manifold, except in the points parametrized by $u=\alpha \in \Delta_1, v \in \Delta_2$ or $v=\alpha \in \Delta_2, u \in \Delta_1$ (the intervals Δ_1 and Δ_2 are defined in step i) above).

Proof of lemma 6. Denote the real roots of $z \cdot P(z)$ by α_i, $1 \le i \le 5$. In a neighbourhood of a point on $\mathbb{A}^0_{\mathbb{R}}$ parametrized by a point (u_0, v_0), such that $u_0 \neq \alpha_i$, $v_0 \neq \alpha_j$, local coordinates are u and v as

$$\det\left(\frac{\partial(q_1, q_2)}{\partial(u, v)}\right)\bigg|_{\substack{u=u_0\\ v=v_0}} \neq 0 \quad (\text{see } (2.4)).$$

Suppose that $u_0=\alpha_i \in \Delta_1$, $v_0 \in \Delta_2$, $v_0 \neq \alpha_j$ and α_i is of multiplicity one. Then the parameters $\tilde{u}$ and v, where $u-\alpha_i=\tilde{u}^2$, provide local coordinates on $\mathbb{A}^0_{\mathbb{R}}$. Indeed

$$\frac{\partial q_2}{\partial \tilde{u}}\bigg|_{\substack{\tilde{u}=0\\ v=v_0}}=0, \qquad \frac{\partial q_2}{\partial v}\bigg|_{\substack{\tilde{u}=0\\ v=v_0}}=1,$$

and for fixed v we have $p_2=c_1+c_2\tilde{u}+o(\tilde{u})$, $c_2 \neq 0$ (see (2.5)). It follows that

$$\det\left(\frac{\partial(q_2, p_2)}{\partial(\tilde{u}, v)}\right)\bigg|_{\substack{\tilde{u}=0\\ v=v_0}} \neq 0.$$

Finally, if $u_0=\alpha_i \in \Delta_1$, $v_0=\alpha_j \in \Delta_2$, and $\mathrm{mult}(\alpha_i) = \mathrm{mult}(\alpha_j)=1$, we choose local coordinates $\tilde{u}$ and $\tilde{v}$, where $u-\alpha_i=\tilde{u}^2$, $v-\alpha_j=\tilde{v}^2$. Using (2.5) we obtain $p_2=c_3\tilde{u}+c_4\tilde{v}+o(|\tilde{u}|+|\tilde{v}|)$, $c_3 \neq 0$, $c_4 \neq 0$, and if $\alpha_i, \alpha_j \neq 0$, then

$$p_1=\sqrt{-1}\,\frac{\sqrt{\alpha_i}}{\sqrt{\alpha_j}}\left(\frac{\alpha_i}{\alpha_j}c_4\tilde{v}+c_3\tilde{u}+o(|\tilde{u}|+|\tilde{v}|)\right).$$

It is clear that

$$\det\left(\frac{\partial(p_1, p_2)}{\partial(\tilde{u}, \tilde{v})}\right)\bigg|_{\substack{\tilde{u}=0\\ \tilde{v}=0}} \neq 0.$$

If, for example, $\alpha_i=0$, then $q_1=2\sqrt{-1}\,\alpha_j\tilde{u}+$

$o(|\tilde{u}| + |\tilde{v}|)$, and hence

$$\det\left(\frac{\partial(q_1, p_2)}{\partial(\tilde{u}, \tilde{v})}\right)\Bigg|_{\substack{\tilde{u}=0\\ \tilde{v}=0}} \neq 0.$$

It is concluded that the subset of $\mathbf{A}^0_{\mathbb{R}}$, parametrized by u and v, where $u \in \Delta_1$, $v \in \Delta_2$, $u \neq \alpha$, $v \neq \alpha$, possesses a manifold structure. As it has no points of self-intersection (see (2.4),(2.5)), then lemma 6 is proved.

Now we shall present a few examples of determining the topological type of the singular invariant manifold $\mathbf{A}^0_{\mathbb{R}}$.

Example 1. Consider the generic bifurcation 2 → 3. In the domain 2 the polynomial $P(z)$ has two real roots α_1 and α_2, $\alpha_1 < 0 < \alpha_2$, and in the domain 3 it has also two different real roots α_1 and α_2, $0 < \alpha_1 < \alpha_2$ (see table I). For a generic bifurcation point $(A, b, f, h) \in \mathbb{B}$ we have $\alpha_1 = 0$ and hence $\Delta_1 = \{0\}$ and $\Delta_2 = [\alpha_2, \infty)$ (step i)). Formulas (2.4) and (2.5) imply $q_1 = p_1 = 0$ (as $f = 0$). This combined with $H = h$ determines the curve (step ii))

$$\tfrac{1}{2}p_2^2 = 2q_2^3 - \tfrac{1}{2}Bq_2^2 + h$$

(elliptic solution). (3.7)

Finally we note that the set (3.6) is the empty set. Using table I we obtain the bifurcation $C \to 0$ (the curve (3.7) is homeomorphic to $\mathbb{R}^1$).

Example 2. Consider the generic bifurcation $2 \xrightarrow{3} 9$. In the domain 2 the polynomial $P(z)$ has two real roots α_1 and α_2, $\alpha_1 < 0 < \alpha_2$, and in the domain 9 it has four real roots $\alpha_1 < \alpha_2 < \alpha_3 < 0 < \alpha_4$. For a generic bifurcation point $(A, B, f, h) \in \mathbb{B}$ lying on the branch 3 of $\mathbb{B}_0 \cap \{A = \text{const.}, B = \text{const.}\}$ (see fig. 3) we have $\alpha_1 < \alpha_2 = \alpha_3 < 0 < \alpha_4$ and $\Delta_1 = [\alpha_1, \alpha_2] \cup [\alpha_3, 0]$, $\Delta_2 = [\alpha_4, \infty)$. The set corresponding to $u \in [\alpha_1, \alpha_2)$, $v \in [\alpha_4, \infty)$ is diffeomorphic to a set of two non-intersecting infinite open smooth (see lemma 6) rectangles, shown in fig. 11. The set corresponding to $u = \alpha_2$,

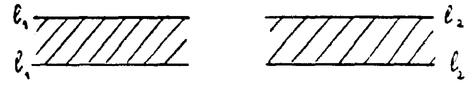

Fig. 11.

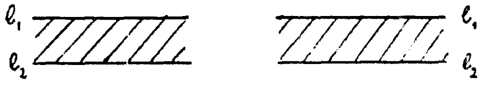

Fig. 12.

$v \in [\alpha_4, \infty)$ consists of two non-intersecting smooth curves (elliptic solutions) l_1 and l_2. They are identified with the sides of the rectangles, as it is shown in fig. 11. The set corresponding to $u \in (\alpha_3, 0]$, $v \in [\alpha_4, \infty)$ is also diffeomorphic to a set of two open non-intersecting rectangles, their sides being identified as it is shown in fig. 12. Identifying the rectangles shown in fig. 11 and fig. 12 along the lines l_1 and l_2 we conclude that the set described in step ii) above is homeomorphic to a set of three cylinders with two common lines – l_1 and l_2. As the set (3.6) is the empty set (because $f \neq 0$) then $\mathbf{A}^0_{\mathbb{R}}$ coincides with the set described above, and hence we obtain the bifurcation $C \to 3C$ (fig. 7).

Example 3. Consider the generic bifurcation 1 → 2. In the domain 1 the polynomial $P(z)$ has two different real roots α_1 and α_2, $\alpha_1 < \alpha_2 < 0$, and in the domain 2 it has also two roots α_1 and α_2, $\alpha_1 < 0 < \alpha_2$. For a generic bifurcation point $(A, B, f, h) \in \mathbb{B}$ we have $\alpha_2 = 0$ and hence $\Delta_1 = [\alpha_1, 0]$, $\Delta_2 = [0, \infty)$ (step i)). The subset of $\mathbf{A}^0_{\mathbb{R}}$, parametrized by $u \in [\alpha_1, 0)$ and $v \in (0, \infty)$, is diffeomorphic to four open, non-intersecting, smooth rectangles shown in fig. 13 (here we used lemma 6). The set corresponding to $u = 0, v \in (0, \infty)$ is diffeomorphic to two open half-lines l_1 and l_2. The set parametrized by $v = 0$ and $u \in [\alpha_1, 0)$ is

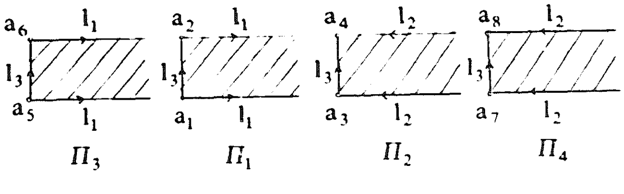

Fig. 13.

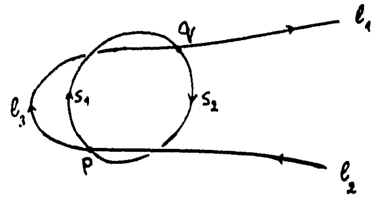

Fig. 14.

diffeomorphic to an open interval – l_3. The closure of the set $l_1 \cup l_2 \cup l_3$ is (according to (2.4), (2.5)) the elliptic solution (3.7) – see fig. 14. The different sides of the four open rectangles are identified (after going to the limits) with l_1, l_2, l_3, as it is shown in fig. 13 (step i)). The set (3.6) consists of a circle which intersects the elliptic solution (3.7) in two different points p and q with coordinates $q_1 = p_1 = 0$, $q_2 = (B - 4A)/4$, $p_2 = \pm(8h - A(4A - B)^2)^{1/2}/4$. These points divide the circle (3.6) into two open intervals S_1 and S_2 (see fig. 14). After going to the limits in fig. 13 and using formulas (2.4), (2.5) we conclude that the points (see fig. 13 and fig. 14) a_2 and a_6 have to be identified with q, the points a_3 and a_7 with p, the points a_1 and a_4 with the closure of s_1, and the points a_5 and a_8 with the closure of s_2. Of course the above identifications are not unique as they depend upon the choice of the signs in (2.4), (2.5). To determine the topological type of the set $\mathbb{A}^0_{\mathbb{R}} = \overline{\Pi} \cup \overline{\Pi}_2 \cup \overline{\Pi}_3 \cup \overline{\Pi}_4$ we consider first the sets $\overline{\Pi}_1 \cup \overline{\Pi}_2$ and $\overline{\Pi}_3 \cup \overline{\Pi}_4$ (fig. 15) and then identify these two cylinders along the elliptic solution $\overline{l_1 \cup l_2 \cup l_3}$ (fig. 15). The above standard topological manipulations imply that $\mathbb{A}^0_{\mathbb{R}}$ is homeomorphic to two cylinders with a common line, i.e. we obtain the bifurcation $2C \to C$ (see fig. 6).

Example 4. Consider the generic bifurcation $1 \to 8$. In the domain 1 the polynomial $P(z)$ has two real roots α_1 and α_2, $\alpha_1 < \alpha_2 < 0$, and in the domain 8 it has four real roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$. For a generic bifurcation point $(A, B, f, h) \in \mathbb{B}$ we have $\alpha_3 = \alpha_4 > 0$ and hence we find $\Delta_1 = [\alpha_1, \alpha_2]$ and $\Delta_2 = [0, \alpha_3] \cup [\alpha_4, \infty) = [0, \infty)$. The subset of $\mathbb{A}^0_{\mathbb{R}}$ parametrized by $u \in [\alpha_1, \alpha_2]$, $v \in [0, \alpha_3]$ consists of a torus. The

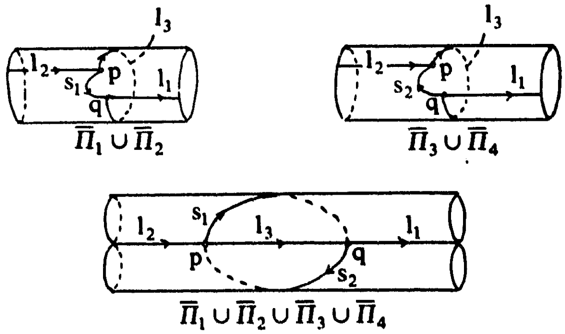

Fig. 15.

subset of $\mathbb{A}^0_{\mathbb{R}}$ parametrized by $u \in [\alpha_1, \alpha_2]$, $v \in [\alpha_4, \infty)$, consists of two infinite cylinders. The above subsets intersect each other into the set parametrized by $u \in [\alpha_1, \alpha_2]$, $v = \alpha_3 = \alpha_4$, the last being diffeomorphic to two non-intersecting circles. On the one hand these circles lie on different cylinders, and on the other hand they are homologous cycles on the torus. As (3.6) is the empty set, it is concluded that $\mathbb{A}^0_{\mathbb{R}}$ is homeomorphic to the singular invariant manifold shown in fig. 10, and hence (see table I) we obtain the bifurcation $2C \to T + 2C$.

Example 5. Consider the generic bifurcation $13 \to 14$. In the domain 13 the polynomial $P(z)$ has four real roots $\alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4$ and in the domain 14 it has also four real roots $\alpha_1 < 0 < \alpha_2 < \alpha_3 < \alpha_4$. For a generic bifurcation point $(A, B, f, h) \in \mathbb{B}$ we have $\alpha_2 = 0$ and hence $\Delta_1 = [\alpha_1, 0]$, $\Delta_2 = [0, \alpha_3] \cup [\alpha_4, \infty)$. The subset of $\mathbb{A}^0_{\mathbb{R}}$ parametrized by $u \in [\alpha_1, 0]$ and $v \in [\alpha_4, \infty)$ consists of two infinite cylinders, each of them being homeomorphic to $S^1 \times \mathbb{R}^1$. These cylinders are identified along the line parametrized by $u = \upsilon$, $v \in [\alpha_4, \infty)$. Using table I we obtain the bifurcation $2C \to C$ (fig 6). Consider now the subset of $\mathbb{A}^0_{\mathbb{R}}$ parametrized by $u \in [\alpha_1, 0]$ and $v \in [0, \alpha_3]$. One may prove along the same lines as in example 3 that the above subset of $\mathbb{A}^0_{\mathbb{R}}$ is homeomorphic to the set $[0,1] \times (S^1 \vee S^1)$, where the sets $\{0\} \times (S^1 \vee S^1)$ and $\{1\} \times (S^1 \vee S^1)$ are identified with the help of a mapping homotopic to a "central symmetry" of $S^1 \vee S^1$. It is clear that this set coincides with the singular invariant manifold, pictured in fig. 5. Using table I we obtain the bifurcation $T \to T$. The last assertion follows also from the Fomenko's theorem [13]. Indeed, according to this theorem, the only generic bifurcation of a type $T \to T$ is the one described in fig. 5.

The remaining generic bifurcations, described in table II, are studied in a similar way. Thus theorem 3 is proved.

Acknowledgements

Acknowledgements are due to Emil Horozov for the non-formal discussions.

References

[1] S. Wojciechowski, Phys. Lett. A100 (1984) 277.
[2] M. Adler and P. van Moerbeke, A systematic approach towards solving integrable systems, in: Perspectives in Mathematics (Academic Press, New York, 1988).
[3] M. Adler and P. van Moerbeke, Representing the Kowalewski and Hénon–Heiles motions as Manakov geodesic flows on SO(4) and a two-dimensional family of Lax pairs, Commun. Math. Phys. 113 (1988) 659.
[4] P. Griffiths and J. Harris, Principles of Algebraic Geometry (Wiley–Interscience, New York, 1978).
[5] S. Ramanan, Proc. London Math. Soc. 51 (1985) 231–245.
[6] B.A. Dubrovin, Russ. Math. Surveys 36 (1981) 11–92.
[7] S. Smale, Inventiones Mathematicae 10 (1970) 305–331.
[8] A.C. Newell, M. Tabor and Y.B. Zeng, Physica D29 (1987) 1–68.
[9] S. Kowalevski, Acta Math. 14 (1890) 81–93.
[10] Encyklopädie der Matematischen Wissenschaften, Band II, 2 Teil (Leipzig, 1901), pp. 766–768.
[11] M.P. Kharlamov, Sov. Math. Dokl. 28 (1983) 802–805.
[12] M.P. Kharlamov, PMM USSR 47 (1983) 737–743.
[13] V.V. Trofimov and A.T. Fomenko, Itogi Nauki Tech. 29 (1986) 3–108.
[14] Y. Aizawa and N. Saitô, J. Phys. Soc. Jpn. 32 (1972) 1636.
[15] T. Bountis, H. Segur and F. Vivaldi, Phys. Rev. A25 (1982) 1257–1264.
[16] B. Grammaticos, B. Dorizzi and R. Padjen, Phys. Lett. A89 (1982) 111.
[17] Y.F. Chang, M. Tabor and J. Weiss, J. Math. Phys. 23 (1982) 531–538.
[18] F. Kötter, Acta Math. 17 (1893) 209–263.
[19] V.I. Arnol'd, Catastrophe Theory (Moscow Univ., Moscow, 1983).
[20] A.A. Oshemkov, Russ. Math. Surveys 42 (6) (1987) 199–200.