COHOMOLOGICAL TOOL IN THE STUDY OF COMPLEX FOLIATIONS.

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Introduction.

In its whole generality, the study of the analytical classification of singularities of codimension one complex foliations may be reduced to the following question: given a germ of singular foliation $\mathfrak{F}$, describe as precisely as possible the quotient of its topological class up to the equivalence relation of analytical conjugacy, thus describe the following set

$$\mathcal{M}(\mathfrak{F}) := \left\{\mathfrak{F}' \sim_{\mathsf{top}} \mathfrak{F}\right\}\Big/\mathfrak{F}' \sim_{\mathsf{ana}} \mathfrak{F}.$$

Let us be more specific: two germs of foliations $\mathfrak{F}$ and $\mathfrak{F}'$ are *topologically equivalent* if there exists a germ of homeomorphism $\psi \in \text{Homeo}\,(\mathbb{C}^n, 0)$ that sends any leaf of $\mathfrak{F}$ to a leaf of $\mathfrak{F}'$, that is $\psi^*\mathfrak{F} = \mathfrak{F}'$. We will denote this situation by

$$\mathfrak{F}' \sim_{\mathsf{top}} \mathfrak{F}.$$

They are *analytically equivalent* if ψ can be chosen analytical. In that case, we will denote

$$\mathfrak{F}' \sim_{\mathsf{ana}} \mathfrak{F}.$$

In that case, the relation between $\mathfrak{F}$ and $\mathfrak{F}'$ can be written in a more explicit way. Let ω and ω' two holomorphic 1- forms defining respectively $\mathfrak{F}$ and $\mathfrak{F}'$. Then the map ψ is an analytical conjugacy if and only if there exists a germ of unity u such that

$$\begin{cases}\psi^*\omega = u\omega' \\ u(0) \neq 0\end{cases}$$

Indeed, if $\gamma : \left(\mathbb{C}^{n-1}, 0\right) \to \mathbb{C}^n$ is invariant hypersurface of $\mathfrak{F}$, that is $\gamma^*\omega = 0$, then $\psi^{(-1)} \circ \gamma$ is an invariant hypersurface for ω' because the following equalities hold:

$$\gamma^* u \circ \psi^{(-1)} \times \left(\psi^{(-1)} \circ \gamma\right)^* \omega' = \gamma^*\omega = 0$$

Describing $\mathcal{M}(\mathfrak{F})$ is an extremely tough problem. Therefore, in a first moment, our objective is to understand the *local* structure of this space around the point $[\mathfrak{F}]$ which is the analytical class of $\mathfrak{F}$. A standard approach to study such a local problem is to classify the small deformations of $\mathfrak{F}$ up to analytical equivalence for deformations. A first approach would be to deform $\mathfrak{F}$ by considering a germ of analytic family of holomorphic 1- forms $\omega_t,\ t \in (\mathbb{C}, 0)$ that coincides with ω when the parameter t is equal to 0. However, there is no reason for such a deformation to stay in the topological class of $\mathfrak{F}$! Indeed, we have to restrict ourselves to the deformations that preserve the topological type of $\mathfrak{F}$. There is the intervention of the unfoldings of foliations. This special kind of deformations will always be topologically trivial, and thus will fix the topological class. In the course of the lecture, we will give a complete classification of germs of unfoldings of foliations up to the analytical equivalence of unfoldings, and, in this way, we will be able to describe the local structure of $\mathcal{M}(\mathfrak{F})$.

Nevertheless, unfoldings are really tough to construct. Indeed, the situation is *a priori* desperate. Except when the foliation $\mathfrak{F}$ admits some kind of first integral - a situation which is highly not generic -, we are unable to give any explicit example. Hence, the question is, how a mathematical object so poor in example can be of any help ? The answer is that thanks the cohomological tool, we will be allowed to analyze deeply how unfoldings are build. One objective of this lecture could be to convince you that such an inflexible tool is actually useful and powerful.

Contents

1. Basic concepts.

A foliation of a manifold M can be defined as a special system of charts on M. Since we are going to deal only with codimension one foliations, we do not want introduce this general point of view and we will see a foliation as the data of a family of holomorphic $1-$forms $(\omega_i)_{i\in I}$, ω_i defined on an open set U_i, such that the union of the U_i's cover the whole manifold except maybe a closed sub-manifold. Moreover, the following properties are required:

- ω_i never vanishes on U_i and $\omega_i \wedge d\omega_i = 0$. The latter condition is *the integrability condition*.
- on $U_i \cap U_j$, there exists a non-vanishing function f_{ij} such that $\omega_i = f_{ij}\omega_j$.

We prefer the latter point of view because it highlights the fact that a foliation is given by a partial differential equation of order one. The integrability ensures that the above definition of foliation coincides with the standard one. This coincidence of definitions is the object of the next section.

1.1. **Integrability.**

1.1.1. *Flow of a vector field.* Let X be a germ of vector field in $(\mathbb{C}^n, 0)$

$$X = \sum a_i(x) \frac{\partial}{\partial x_i}.$$

The flow at time t of X is the map defined by

$$x \mapsto e^{(t)X} x$$

where $e^{(t)X} x$ is by definition the value at time t, $\gamma(t)$, of a solution $\gamma(s)$ of the system

$$\begin{cases} \gamma'(s) = X(\gamma(s)) \\ \gamma(0) = x \end{cases} .$$

In view of the theorem of Cauchy-Lipschitz, the flow is well defined for t small enough. It can be also defined by the following formula,

$$x \mapsto e^{(t)X} \cdot x = \sum_{k=0}^{\infty} t^k \frac{X^k \cdot x}{k!}.$$

Indeed, one has $\left(e^{(t)X} \cdot x\right)' = X\left(e^{(t)X} \cdot x\right)$ and $e^{(0)X} \cdot x = x$. The proof of the lemma given below is just a formal computation that relies on the fact that if $[X, Y] = X \cdot Y - Y \cdot X = 0$ then for any k and k' one has

$$X^k Y^{k'} = Y^{k'} X^k$$

Lemma 1. *Let X and Y be two commuting vector fields, that is $[X, Y] = 0$. Then their flow commute, i.e., for any t and t' we have*

$$e^{(t)X} \circ e^{\left(t'\right)Y} = e^{\left(t'\right)Y} \circ e^{(t)X}.$$

When $[X, Y] \neq 0$, one can express the composed flow $e^{(1)X} \circ e^{(1)Y}$ as the flow of a well defined vector field. This is *the Campbell-Hausdorff formula*[1]:

Theorem 2. (Campbell-Hausdorff Formula) *Let X and Y two vector fields. Then*

$$e^{(1)X} \circ e^{(1)Y} = e^{(1)\langle X, Y\rangle}$$

where

$$\langle X, Y\rangle = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X,[X, Y]] - \frac{1}{12}[Y,[X, Y]] + \cdots .$$

[1]See for instance [21]

In particular, if $[X,Y]=0$ then $\langle X,Y\rangle = X+Y$. This formula is very important and is a deep result although it may appeared simply as a technical lemma. There are explicit expressions of $\langle X,Y\rangle$, for instance the formula of Dynkin [22]:

$$\langle X,Y\rangle = \sum_{m\geq 1}\frac{(-1)^{m-1}}{m}\sum_{p_i+q_i>0}\frac{\left[\underbrace{X,\cdots,X}_{p_1 \text{ times}},\underbrace{Y,\cdots,Y}_{q_1 \text{ times}},\cdots,\underbrace{X,\cdots,X}_{p_m \text{ times}},\underbrace{Y,\cdots,Y}_{q_m \text{ times}}\right]_*}{p_1!q_1!\cdots p_m!q_m!}$$

where $[X_1,X_2,\cdots,X_p]_* = [X_1,[X_2,[\cdots,X_p]]\cdots]$.

1.1.2. *Integrability.* As already mentioned, a smooth codimension one foliation can be defined as a system of local charts $x=(x_1,x_2,\cdots,x_n)$ such that the changes of coordinates are written $x\to(h(x_1),\ldots)$. The leaves of the foliation are locally defined by $x_1=cst$. It can also be defined by an holomorphic germ of $1-$form Ω such that $\Omega(0)\neq 0$, with the condition $\Omega\wedge d\Omega=0$. Below, we explain why these two definitions are equivalent.

Theorem 3. (Lie [21]) *A germ of smooth foliation can be equivalently defined by*

(1) *A local chart* $x=(x_1,x_2,\cdots,x_n)$ *such that* $x_1=cst$ *are the leaves.*
(2) *A germ of* $1-$*form* $\Omega=u(x)\,dx_1$ *with* $u(0)\neq 0$.
(3) *A germ of* $1-$*form* Ω *such that* $\Omega(0)\neq 0$ *and* $\Omega\wedge d\Omega=0$.

Proof. The equivalence between (1) and (2) is obvious, the implication (2) $\Longrightarrow$ (3) also. It remains to proof that (3) $\Longrightarrow$ (2): let us write

$$\Omega=\sum_{i=1}^{n}a_i(x)\,dx_i.$$

We can suppose that $a_1(0)\neq 0$. Let us consider the family of vector fields defined by

$$X_i=\frac{\partial}{\partial x_i}-\frac{a_i(x)}{a_1(x)}\frac{\partial}{\partial x_1}.$$

These vector fields are tangent to the one form Ω. Let us consider the application

$$\Phi(x) = e^{(x_2)X_2} \circ \cdots \circ e^{(x_n)X_n}(x_1, 0, \ldots, 0).$$

The hypothesis $\Omega \wedge d\Omega = 0$ implies that $[X_i, X_j] = 0$ for any $i \neq j$. Hence, the map Φ can be also written

$$e^{\left(x_{\sigma(2)}\right)X_{\sigma(2)}} \circ \cdots \circ e^{\left(x_{\sigma(n)}\right)X_{\sigma(n)}}(x_1, 0, \ldots, 0)$$

for any permutation σ. The integral curve of $\frac{\partial}{\partial x_i}$ consists in the curve defined by $x_j = cst_j$ for $j \neq i$. If we fix x_j for $j \neq i$ and make vary x_i then $\Phi(x)$ moves along the integral curve of X_i: indeed, we can write

$$\Phi(x) = e^{(x_i)X_i} \circ \psi(x_1, 0, \ldots, 0).$$

and it is easily seen that $\psi(x_1, 0, \ldots, 0)$ does not depend on x_i. Thus the map satisfies $\Phi^* X_i = u_i \frac{\partial}{\partial x_i}$ for some function u_i. Hence, we have

$$\Phi^*\Omega\left(\frac{\partial}{\partial x_i}\right) = 0$$

for any $i \geq 2$. Therefore, $\Phi^*\Omega$ can be written $\Phi^*\Omega = u(x)\, dx_1$. Now, any change of chart $\Psi = (\Psi_1, \cdots, \Psi_n)$ satisfies

$$\Psi^* dx_1 \wedge dx_1 = \sum \frac{\partial \Psi_1}{\partial x_i} dx_i \wedge dx_1 = 0,$$

hence Ψ_1 depends only on x_1. □

1.2. **Object of interest in this lecture.** In this lecture, we consider a germ of foliation $\mathfrak{F}$ given by an holomorphic 1-form ω with an isolated singularity in $(\mathbb{C}^2, 0)$. The 1-form ω is the following expression

$$\omega = a(x, y)\, dx + b(x, y)\, dy$$

where a and b belongs to $\mathbb{C}\{x, y\}$ whose disc of convergence has a strictly positive radius. Since, the singularity is isolated, a and b have no common factor. The *multiplicity* of $\mathfrak{F}$ is defined as the integer

$$\nu_0(\mathfrak{F}) = \nu_0(\omega) = \min(\nu_0(a), \nu_0(b)). \tag{1.1}$$

A *leaf* is a curve $\gamma : U \subset \mathbb{C} \to \mathbb{C}^2$ that lies in the open set of definition of ω such that $\gamma^*\omega = 0$, i.e., if $\gamma(t) = (x(t), y(t))$ then

$$a(x(t), y(t))\, x'(t) + b(x(t), y(t))\, y'(t) = 0 \qquad \text{for all } t \in U.$$

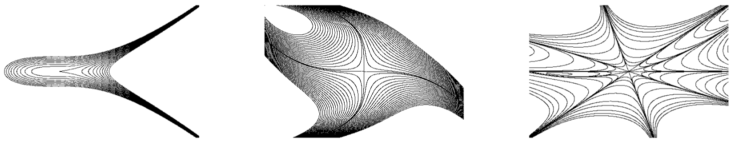


FIGURE 1.1. Example of real trace of singularities in $\mathbb{C}^2$.

Thus, the leaves of the foliation can be also seen as solutions of the first order differential equation

$$\frac{dy}{dx} = -\frac{a(x, y)}{b(x, y)}.$$

A *separatrix* of is an irreducible curve S that is analytic at the origin of $(\mathbb{C}^2, 0)$ such that $S \backslash 0$ is invariant by $\mathfrak{F}$.

Theorem 4. (C. Camacho, P. Sad [4]) *The foliation $\mathfrak{F}$ admits at least one separatrix.*

It may happen that there is an infinite number of separatricies, for instance when $\omega = xdy - ydx$. We refer to this case as the *dicritical* case. In this lecture,

All foliations are supposed to be non-dicritical.

Our main problem is to classify foliations, and so 1−forms up to the analytical equivalence, thus to describe the following set

$$\mathcal{M}(\mathfrak{F}) := \left\{ \omega' \sim_{\text{top}} \omega \right\} \Big/ \omega' \sim_{ana} \omega.$$

where $\omega' \sim_{ana} \omega$ means there exist ψ and u with $\psi^*\omega = u\omega'$. Here, $\psi^*\omega$ stands for the pull-back 1−form: if $\omega = a(x,y)\,dx + b(x,y)\,dy$ and $\psi = (\psi_1, \psi_2)$ then

$$\begin{aligned} \psi^*\omega &= a(\psi_1,\psi_2)\,d\psi_1 + b(\psi_1,\psi_2)\,d\psi_2 \\ &= \left(a(\psi_1,\psi_2)\frac{\partial\psi_1}{\partial x} + b(\psi_1,\psi_2)\frac{\partial\psi_2}{\partial x}\right)dx \\ &+ \left(a(\psi_1,\psi_2)\frac{\partial\psi_1}{\partial y} + b(\psi_1,\psi_2)\frac{\partial\psi_2}{\partial y}\right)dy. \end{aligned}$$

2. A first example « by hand ».

Let us consider a first example. Let $\mathfrak{F}$ be a foliation given by a $1-$form

$$\omega = \omega_0 + \omega_1 + \dots$$

where ω_i is the homogeneous part of degree $i+1$. The $1-$form ω_0 is the linear part and is written

$$\omega_0 = (ax+by)\,dy + (cx+dy)\,dx.$$

with $a,\ b,\ c$ and d complex numbers. Suppose that the matrix

$$\begin{pmatrix} a & b \\ -c & -d \end{pmatrix}$$

has two eigenvalues λ_1 and λ_2 whose quotient is a complex number that is not a real number. In this situation, we say that ω has *an hyperbolic singularity.* In particular, the matrix A is diagonalizable since $\lambda_1 \neq \lambda_2$: hence, there exists P such that $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = P^{-1}AP$. Consider the linear automorphism defined by $\Phi(x,y) = P\begin{pmatrix} x \\ y \end{pmatrix}$. The pullback of ω by Φ is written

$$\Phi^*\omega_0 = \lambda_1 x dy + \lambda_2 y dx.$$

Thus we have $\frac{\Phi^*\omega}{\lambda_2} = \frac{\lambda_1}{\lambda_2} x dy + y dx + \cdots$. Following Camacho-Sad, we called the so defined ratio $\frac{\lambda_1}{\lambda_2}$ the *residue* of ω. The initial foliation $\mathfrak{F}$ is given in some coordinates by a $1-$form

$$\omega = \lambda x dy + y dx + \text{ higher order terms.}$$

Lemma 5. *Let* $\omega = \lambda x dy + y dx + \cdots$ *and* $\omega' = \lambda' x dy + y dx + \cdots$ *be two hyperbolic singularities. If they are analytically equivalent, then* $\lambda = \lambda'$ *or* $\lambda = \frac{1}{\lambda'}$.

Proof. From the definition, there exist a conjugacy ψ and a unity such that

$$\psi^* \omega = u \times \omega'. \tag{2.1}$$

Thus if we look to the linear part of the relation 2.1, from the remark above, there exist a matrix $P = \begin{pmatrix} q & r \\ s & t \end{pmatrix}$ and a complex number $w = u(0,0) \neq 0$ such that

$$\begin{aligned} \begin{pmatrix} \lambda & 0 \\ 0 & -1 \end{pmatrix} P &= wP \begin{pmatrix} \lambda' & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} \lambda q & \lambda r \\ -s & -t \end{pmatrix} &= w \begin{pmatrix} q\lambda' & -r \\ s\lambda' & -t \end{pmatrix}. \end{aligned}$$

In particular, $-t = -wt$, hence $w = 1$ or $t = 0$. Suppose first, $w = 1$ then $q\lambda = q\lambda'$. If $q = 0$ then $\lambda r = -r$ implies $r = 0$ since $\lambda \neq -1$. But $q = r = 0$ is impossible. Thus $q \neq 0$ and $\lambda = \lambda'$. If now $t = 0$, then we have the system

$$\begin{aligned} \lambda q &= wq\lambda' \\ \lambda r &= -wr \\ -s &= ws\lambda'. \end{aligned}$$

Since $r \neq 0$, $w = -\lambda$ and thus $-s = -s\lambda\lambda'$. However, $s \neq 0$ since $t = 0$ therefore $\lambda\lambda' = 1$. □

The next result is due to Koenigs and has been extended in higher dimension by Poincar.

Theorem 6. (Koenigs-Poincar [13]) *The* $1-$*form* ω *is analytically equivalent to its linear part.*

Proof. We give below two different nearly complete proofs.

[1]. Let us suppose that in some coordinates $\omega = \omega_0 + \omega_N + \dots$ where ω_N is homogeneous of degree N. Consider Φ_N a biholomorphism that is written $\Phi_N = (x + A_N, y + B_N)$ where A and B are homogeneous of degree N. Now, a simple computation ensures that

$$\begin{aligned}\Phi_N^*\omega &= \omega_0 \\ &+ \left(\lambda x \frac{\partial B_N}{\partial x} + B_N + y\frac{\partial A_N}{\partial x}\right) dx \\ &+ \left(y\frac{\partial A_N}{\partial y} + A_N + y\frac{\partial B_N}{\partial y}\right) dx + \omega_N + \dots\end{aligned}$$

thus, we can make the term of order N disappear once A and B are solutions of

$$\begin{cases}(\lambda i + 1)\, b_i + (i+1)\, a_{i+1} & = -\alpha_i \quad i = 0 \dots N-1 \\ (\lambda(N-i) + 1)\, a_i + (N-i+1)\, b_{i+1} & = -\beta_i \quad i = 0 \dots N-1 \\ (\lambda N + 1)\, b_N & = -\alpha_N \\ \lambda a_N & = -\beta_N\end{cases}$$

where $A_N = \sum_{i=0}^N a_i x^i y^{N-i}$ and $B_N = \sum_{i=0}^N b_i x^i y^{N-i}$, which can clearly be solved for $\lambda \notin \mathbb{R}$. Thus there exist biholomorphisms $\Phi_1 = \mathrm{Id}$, $\Phi_2, \cdots$ and Φ_n such that

$$(\Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_n)^* \,\omega = \omega_0 + (\text{ terms of order } n)\,.$$

Thus there is a formal change of coordinates

$$\hat{\Phi} := \lim_{n\to\infty} \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_n$$

such that $\hat{\Phi}^*\omega = \omega_0$. This formal map can be proved to be convergent using standard techniques.[2]

[2]. From the theorem of Briot-Bouquet [17], there exist two smooth transverse curves $S_1 \cup S_2$ that are separatricies. If we choose these

[2] See for instance, *Classification analytique des feuilletages holomorphes singuliers*, J. Mozo Monografias del IMCA, 2010

two curves as axes of coordinates, the 1− form is written in these new coordinates where $S_1 \cup S_2 = \{xy = 0\}$

$$\omega = \lambda x \left(1 + a\left(x, y\right)\right) dy + y \left(1 + b\left(x, y\right)\right) dx.$$

Let us consider a transverse curve $T = \{x = \epsilon\}$. Let γ be the path defined by $\gamma\left(t\right) = \left(x\left(t\right) = \epsilon e^{2i\pi t}, 0\right)$. Using the theorem of Cauchy-Lipschitz, there exists a unique solution $y\left(t\right)$ of the differential equation

$$\lambda x\left(t\right)\left(1 + a\left(x\left(t\right), y\left(t\right)\right)\right) y'\left(t\right) + y\left(t\right)\left(1 + b\left(x\left(t\right), y\left(t\right)\right)\right) x'\left(t\right) = 0$$

with $y\left(0\right) = z$. This solution is defined on a open segment $[0, t_0[$. It can be shown that for z small enough, t_0 can b taken strictly greater than 1 : indeed, since the coefficient on dx vanishes when $y = 0$, this is a consequence of some *a priori* control. The point $y\left(1\right)$ is denoted by $h_{\mathfrak{F},T}\left(z\right)$: the map $z \mapsto h_{\mathfrak{F},T}\left(z\right)$ is called the *holonomy map* of $\mathfrak{F}$. Using the theorem of Cauchy-Lipschitz with parameters yields the analycity of $h_{\mathfrak{F},T}$.

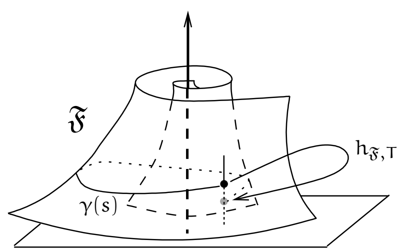


FIGURE 2.1. The holonomy map

Now, we have the relation

$$\frac{h_{\mathfrak{F},T}\left(z\right)}{z} = \exp\left(\int_0^1 \frac{y'\left(t\right)}{y\left(t\right)} dt\right) = \exp\left(-\frac{1}{\lambda}\int_0^1 \frac{x'\left(t\right)}{x\left(t\right)} \frac{1 + b\left(x\left(t\right), y\left(t\right)\right)}{\left(1 + a\left(x\left(t\right), y\left(t\right)\right)\right)} dt\right).$$

When z goes to 0, the solution $y\left(t\right)$ goes uniformly to the zero solution. Thus, using Lebesgue convergence theorem, we have

$$\lim_{z \to 0} \frac{h_{\mathfrak{F},T}\left(z\right)}{z} = \exp\left(-\frac{1}{\lambda}\int_0^1 \frac{x'\left(t\right)}{x\left(t\right)} \frac{1 + b\left(x\left(t\right), 0\right)}{\left(1 + a\left(x\left(t\right), 0\right)\right)} dt\right) = e^{-\frac{2i\pi}{\lambda}} = \eta$$

in view of the residues formula. Thus, we are led to the next relation

$$\left|h'_{\mathfrak{F},T}(0)\right| = |\eta| \neq 1.$$

since $\lambda \notin \mathbb{R}$. We are going to prove that there exists a germ of biholomorphism $\phi(z) = z + \cdots$ such that

$$\phi \circ h_{\mathfrak{F},T} = \eta\phi(z).$$

We can suppose $|\eta| < 1$ considering $h^{-1}_{\mathfrak{F},T}$ if necessary. Let us write $h_{\mathfrak{F},T}(z) = \eta z + \sum_{i\geq 2} h_i z^i$ and $\phi(z) = \sum_{i\geq 1} l_i z^i$. Thus

$$\begin{aligned}\sum_{i\geq 1} l_i \left(\eta z + \sum_{i\geq 2} h_i z^i\right)^k &= \eta z + \sum_{i\geq 2} l_i \eta z^i \\ \sum_{i\geq 2} l_i \left(\eta - \eta^i\right) z^i &= \sum_{j\geq 2}\left(\sum_{k=1}^{j-1} l_k \underbrace{\sum_{i_1+i_2+\ldots+i_k=j} h_{i_1}\cdots h_{i_k}}_{\left[h^k_{\mathfrak{F},T}\right]_j}\right) z^j\end{aligned}$$

Therefore, the coefficients l_i's of ϕ have to satisfy the following linear induction:

$$\forall j > 0, \quad l_j = \frac{1}{\eta - \eta^j}\sum_{k=1}^{j-1} l_k \left[h^k_{\mathfrak{F},T}\right]_j. \tag{2.2}$$

Now, for any $1 > \eta' > \eta$, there exists a small neighborhood $|z| < \epsilon < 1$ such that $|h_{\mathfrak{F},T}(z)| < \eta'|z|$. Hence, for any $k \geq 1$, $\left|h^k_{\mathfrak{F},T}(z)\right| < \left(\eta'\right)^k \epsilon^k$. Using the Cauchy's formula yields the following inequality

$$\left|\left[h^k_{\mathfrak{F},T}\right]_j\right| < \left(\eta'\right)^k \epsilon^{k-j}.$$

Now, suppose that for any $l < j$, we have $|l_l| < A\alpha^l$ for some constants A and α . Then, introducing these inequalities in 2.2 gives us

$$|l_j| < \frac{1}{\eta - \eta^j} \sum_{k=1}^{j-1} A\alpha^k \left(\eta'\right)^k \epsilon^{k-j} < \frac{2}{\eta} \frac{A}{\epsilon^j} \alpha\eta' \epsilon \frac{1 - \left(\alpha\eta'\epsilon\right)^j}{1 - \alpha\eta'}$$

It is possible to choose α so that the inequality

$$\frac{2}{\eta} \frac{A}{\epsilon^j} \alpha\eta' \epsilon \frac{1 - \left(\alpha\eta'\epsilon\right)^j}{1 - \alpha\eta'} < A\alpha^j$$

holds: indeed, the quotient of the two members of the above inequality is equivalent to $cst \times \left(\eta'\right)^j$ while j goes to $+\infty$: therefore, it tends to zero. To finish the proof, consider the foliation $\mathfrak{L}_\lambda$ given by,

$$\omega = \lambda x dy + y dx.$$

Its holonomy computed on the transverse T is exactly $h_{\mathfrak{L}_\lambda, T} : z \to \eta z$. The two holonomies $h_{\mathfrak{L}_\lambda, T}$ and $h_{\mathfrak{F}, T}$ are conjugated by ϕ. We are going to prove that this ϕ can be extended on a neighborhood of 0 in a conjugacy of the foliations. Consider the fibration $\Pi : (x, y) \to x$. On a small neighborhood both foliations are transverse to this fibration. We make the following construction: let (x, y) be a point in the neighborhood of 0. We consider a path γ in $y = 0$ that links x and ϵ. We can choose

$$\gamma(t) = (t|x| + (1-t)\epsilon) e^{it\arg(x)}.$$

Let $\tilde{x}$ be the extremity of the lifting path of γ in the leaf of $\mathfrak{F}$. We denote by $\Phi(x, y)$ the extremity of the lifting path of γ^{-1} in the leaf of $\mathfrak{L}_\lambda$. It can be shown that Φ is bounded near $x = 0$ and thus can be holomorphically extended in view of the Riemann extension result. The biholomorphism Φ is the checked conjugacy. □

Lemma 7. *The two forms* $\lambda x dy + y dx$ *and* $\lambda' x dy + y dx$ *where* λ *and* λ' *are complex number not real are topologically equivalent.*

Proof. Let us consider the universal covering $\mathbb{C}^2 \to (\mathbb{C}^*)^2$ defined by $(x, y) \mapsto (e^x, e^y)$. The pullback of the two foliations as defined in the lemma are linear foliations given by the level of the function $\lambda y + x$ and

$\lambda' y + x$. Let us consider $\mathbb{C}^2$ as $\mathbb{R}^4$. The leaves are 2 -planes given by the cartesian equations:

$$\begin{cases} ax_1 - bx_2 + y_1 &= c_1 \\ bx_1 - ax_2 + y_2 &= c_2 \end{cases} \text{ and } \begin{cases} a'x_1 - b'x_2 + y_1 &= c_1 \\ b'x_1 - a'x_2 + y_2 &= c_2 \end{cases}$$

where $x = x_1 + ix_2$, $y = y_1 + iy_2$,$\lambda = a + ib$ and $\lambda' = a' + ib'$. Since these two foliations are transverse to the fibration $\pi : (x_1, x_2, y_1, y_2) \mapsto (x_1, y_1)$, one can construct ϕ the following way: to any point p in a leaf L, we associate the point p' at the intersection of the fiber $\pi^{-1}(\pi(p))$ and the leaf L' passing trough the intersection of L and $\pi^{-1}(0)$. Since ϕ preserves the fibration, it can be pushed down through the universal covering in a conjugacy of the foliations that can be extended on a neighborhood of 0. □

The proof of the next result is delicate and will not be given. It is a corollary of the topological invariance of the multiplicity: this is not a easy result. This had been proved by Camacho and Sad in [2].

Theorem 8. [2] *Let $\mathfrak{F}$ be a foliation topologically equivalent to the foliation given by $\lambda x dy + y dx$. Then $\mathfrak{F}$ is given by a $1-$form with a linear part whose residue is a complex number not real.*

In this section, we have proved that any hyperbolic singularity is analytically equivalent to the singularity given by some linear form $\lambda x dy + y dx$ with $\lambda \notin \mathbb{R}$. We have also admit that any singularity topologically equivalent to an hyperbolic one, is also hyperbolic. Finally, we have noticed that if two $1-$forms $\lambda x dy + y dx$ and $\lambda' x dy + y dx$ led two analytically equivalent foliation then $\lambda = \lambda'$ or $\lambda = \frac{1}{\lambda'}$. The conclusion of this study is that the moduli space of any hyperbolic singularity is the quotient $\mathbb{C}\backslash\mathbb{R} \big/ \lambda \sim \frac{1}{\lambda}$. Therefore, we have identified the moduli space of a hyperbolic singularity:

$$\mathcal{M}(\lambda x dy + y dx = 0) \sim \mathbb{H} = \{\lambda \in \mathbb{C} | \mathrm{Im}(\lambda) > 0\}.$$

3. Unfolding of germs of singular foliations and equisingularity.

3.1. **Basic definitions.** Hereafter, we give the central definition of this lecture. We follow the definition introduced in [15]

Definition 9. An *unfolding* of $\mathfrak{F}$ with parameters in $(\mathbb{C}^p, 0)$ is a germ of foliation $\mathbf{F}$ of $\left(\mathbb{C}^{2+p}, 0\right)$ of codimension one such that

(1) the leaves of $\mathbf{F} \backslash \mathrm{Sing}\,(\mathbf{F})$ are transverse to the leaves of the *vertical* foliation given by the fibers of the projection on the space of parameters $\pi : \left(\mathbb{C}^{2+p}, 0\right) \to (\mathbb{C}^p, 0)$.
(2) if ν_0 stands for the embedding $\nu_0 : \left(\mathbb{C}^2, 0\right) \to \left(\mathbb{C}^{2+p}, 0\right)$ $\nu_0(x) = (x, 0)$ then $\nu_0^* \mathbf{F} = \mathfrak{F}$.

To be more specific, we can also give this definition in terms of holomorphic 1−forms, which is absolutely equivalent to the one above.

Definition 10. An *unfolding* of ω with parameters in $(\mathbb{C}^p, 0)$ is a germ of holomorphic 1-form Ω of $\left(\mathbb{C}^{2+p}, 0\right)$ with

$$\Omega = \mathbf{a}(x, y, t)\, dx + \mathbf{b}(x, y, t)\, dy + \sum_{i=1}^{p} \mathbf{c}_i(x, y, t)\, dt_i$$

where $t = (t_1, \ldots, t_p)$ and

(1) $\Omega \wedge d\Omega = 0$ (*the integrability condition*).

(2) $\mathbf{a}(x,y,0) = a(x,y)$ and $\mathbf{b}(x,y,0) = b(x,y)$ *(the initial condition).*
(3) the ideal $(\mathbf{c}_1, \cdots, \mathbf{c}_p)$ is a sub-ideal of the radical ideal $\sqrt{(\mathbf{a},\mathbf{b})}$ *(the transversality condition).*

In this alternative definition, the first condition ensures that Ω induces a foliation in $\mathbb{C}^{2+p}$, while the second condition, that this foliation restricted to the fiber of $\pi^{(-1)}(0)$ coincides with the one defined by ω. Finally, the third is equivalent to the transversality assumption of the first definition: indeed the locus of tangency between the unfolding and the fibration is the set of zeros of (a,b). The hypothesis of transversality ensures that $\mathcal{Z}(a,b) \subset \mathcal{Z}(a,b,c_1,\ldots,c_n) = \text{Sing}(\mathbf{F})$, which implies that $c_i \in \sqrt{(a,b)}$. The first condition is obviously the strongest one. Indeed, suppose ω given. If one wants to unfold ω with one parameter, one has to set $\Omega = \omega + t\eta + c dt$ where η is a 1- form and c a function and to solve the partial differential equation $\Omega \wedge d\Omega = 0$. Actually, we are unable to give any solution to the latter equation except when ω has a very simple form from the foliated point of view.

An unfolding induces in a very natural way a deformation in the standard sense: indeed, one can set $\mathfrak{F}_t = \nu_t^* \mathbf{F}$ where $\nu_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^{2+p}, 0)$, $\nu_t(x) = (x,t)$. The transversality condition ensures that $\mathfrak{F}_t$ is actually a foliation. The family $\mathfrak{F}_t$ is an analytical deformation of $\mathfrak{F}_0 = \mathfrak{F}$. One can also defined this deformation as the analytical family of 1—forms

$$\omega_t = \nu_t^* \Omega = \mathbf{a}(x,y,t)\, dx + \mathbf{b}(x,y,t)\, dy.$$

Example 11. The first family of examples comes from the classical theory of singularities. Indeed, suppose that $\mathfrak{F}$ is given by the level of an holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with an isolated singularity. For any family of p functions $g_1, \ldots,\ g_p$, we can set

$$\begin{aligned} F(x,y,t) &= f(x,y) + \sum_{i=1}^{p} t_i g_i(x,y). \\ \Omega = dF &= df + \sum_{i=1}^{p} t_i dg_i + \sum_{i=1}^{p} g_i dt_i \end{aligned}$$

Hence, the $1-$form Ω is an unfolding of $\omega = df$. For instance, on can consider the deformation of the function $(x, y) \to xy$ defined by $F(x, y, t) = xy + tx^2 = x(y + tx)$.This kind of explicit examples are more or less the only known examples.

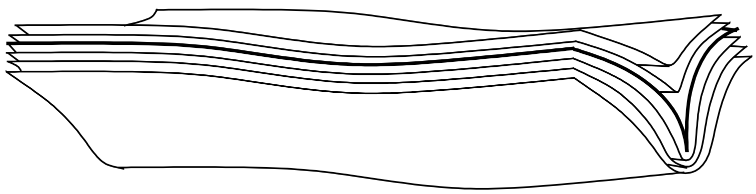


FIGURE 3.1. A picture of what looks like an unfolding.

Definition 12. Two unfoldings $\mathbf{F}$ and $\mathbf{F}^{'}$ of $\mathfrak{F}$ with parameter in $(\mathbb{C}^p, 0)$ are said to be analytically (topologically, formally, ...) equivalent if there exists a germ of biholomorphism $\Phi : \left(\mathbb{C}^{2+p}, 0\right) \to \left(\mathbb{C}^{2+p}, 0\right)$such that

(1) Φ commutes with the projection π.
(2) $\Phi|_{\pi^{(-1)}(0)} = \text{Id}$.
(3) $\Phi^* \mathbf{F}^{'} = \mathbf{F}$.

We will denote

$$\mathbf{F} \sim \mathbf{F}^{'}$$

In particular, an unfolding $\mathbf{F}$ of $\mathfrak{F}$ with p parameters is said to be *analytically trivial* if it is analytically equivalent to the trivial unfolding $\mathfrak{F} \times (\mathbb{C}^p, 0)$.

We can also make change in the space of parameter. Indeed, if $\mathbf{F}$ is an unfolding with $(\mathbb{C}^p, 0)$ as space of parameters and $\Lambda : (\mathbb{C}^q, 0) \to (\mathbb{C}^p, 0)$ an analytical map, we can consider the unfolding $\mathbf{F}_\Lambda$ defined by

$$\mathbf{F}_\Lambda = (\text{Id}, \Lambda)^* \mathbf{F}.$$

The unfolding $\mathbf{F}_\Lambda$ is an unfolding with $(\mathbb{C}^q, 0)$ as space of parameters.

3.2. **Equisingularity.** According to a theorem due at first to Seidenberg [20, 17], any germ of singularity of foliation admits a *process of reduction of its singularities*: this is a map $E : (A, D) \to (\mathbb{C}^2, 0)^3$ composed of a finite number of standard blowing up of points such that the pull-back foliation $E^*\mathfrak{F}$ has only *reduced singularities*, i.e., singularities of the following kind:

(1) $E^*\mathfrak{F}$ is locally smooth.
(2) $E^*\mathfrak{F}$ is locally given by a 1−form $\widetilde{\omega}$ such that is some coordinates the first jet of $\widetilde{\omega}$ is
   (a) $J_1\widetilde{\omega} = \lambda x dy + y dx$ where $\lambda \in \mathbb{C}\backslash\mathbb{Q}^-$
   (b) $J_1\widetilde{\omega} = x dy$

The *exceptional divisor* D is the pre-image of 0, $D = E^{(-1)}(0)$. For example, the singularity called *cusp* $\omega = d\left(x^2 - y^3\right) = 2x dx - 3y^2 dy$ can be reduced by three successive blowing-ups: to be more precise, the reduction of the foliation is the same as the reduction of the curve $x^2 - y^3 = 0$.

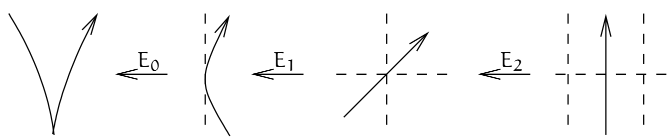


FIGURE 3.2. Reduction of singularities of the cusp.

For a large class of foliations, this property holds:

Definition 13. A foliation is said to be of *general kind*[4] when all the singularities that appear in the reduction process belong to the class 1 or 2.(a) .

A foliation of general kind and its separatricies share the same process of reduction [2, 16].

[3]The notation A refers to the fact that in french the manifold obtained after reduction is often called *Arbre de rduction*.

[4]Also, called *generalized curve.*

Now, given a unfolding $\mathbf{F}$ and the underling deformation $\mathfrak{F}_t$, one can consider the family E_t of associated process of reduction. The property of equisingularity requires that this family of process *depends continuously on the parameter* t. Hereafter, we are going to give a precise definition but for the moment, let us give an example of *what is not* an equisingular unfolding. We hope that the following simple example will be enough to understand the sense of the equisingularity property since the precise definition is a bit awkward.

Example 14. Consider the family of functions $f_t = x\left(ty + x + y^2\right)$ for $t \in (\mathbb{C}, 0)$ and the family of foliations given by $\omega_t = df_t$. Then, for $t \neq 0$, one blowing-up is enough to reduce the singularity of ω_t whereas ω_0 needs two successive blowing-ups to be reduced. Here, the non-equireducibility comes from the bifurcation of the singular locus when t is near 0: actually, for $t \neq 0$ there are three singular points $(0,0)$, $(0,-t)$ and $\left(\frac{t^2}{8}, -\frac{t}{2}\right)$ that collapse while t goes to 0 . We could nearly adopt the non-bifurcation property as the definition of equisingularity but, for technical reason we choose the following definition:

Definition 15. An unfolding $\mathbf{F}$ of $\mathfrak{F}$ with parameters in $(\mathbb{C}^p, 0)$ is said equisingular if there exists a manifold $\mathbf{A}$ of dimension $2+p$ which is a neighborhood of a compact divisor $\mathbf{D}$ such that

(1) there is an holomorphic map $\Pi : (\mathbf{A}, \mathbf{D}) \to (\mathbb{C}^p, 0)$ which is a surjective submersion over $(\mathbb{C}^p, 0)$ whose fibers are transverse to $\mathbf{D}$ and that is a surjective submersion on any irreducible component of $\mathbf{D}$.
(2) there is an holomorphic map $\mathbf{E} : (\mathbf{A}, \mathbf{D}) \to \left(\mathbb{C}^{2+p}, 0\right)$ such that
  (a) $\pi \circ \mathbf{E} = \Pi$ where $\pi : \left(\mathbb{C}^{2+p}, 0\right) \to (\mathbb{C}^p, 0)$ defined by $\pi(x, t) = t$.
  (b) the leaves of $\mathbf{E}^*\mathbf{F}$ are transverse to the fiber of Π.
  (c) that for any $t \in \mathbb{C}^p, \mathbf{E}|_{\pi^{(-1)}(t) \subset \mathbf{A}}$ is the process of reduction of singularities of $\mathbf{F}_t$.

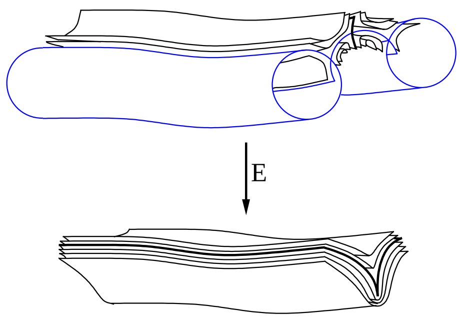


FIGURE 3.3. An unfolding and its process of reduction of singularities.

If $\mathbf{F}$ is equisingular, then the singular locus Sing $(\mathbf{F})$ of $\mathbf{F}$ is an irreducible germ of smooth submanifold of dimension p. Up to some change of coordinates, we can suppose that, in the coordinates $(x,t) \in \mathbb{C}^{2+p}$, Sing $(\mathbf{F})$ is equal to $\{x=0\}$. Indeed, if the singular locus has a bifurcation, this bifurcation still appeared after the reduction process, which is impossible according to the local triviality property of unfolding, a property that we will be proved below.

3.3. **Fundamental properties.** The following lemma is a fundamental property

Lemma 16. [5] *Let $\mathbf{F}$ be an unfolding of $\mathfrak{F}$, given by ω, with parameters in $(\mathbb{C},0)$. Let us write the $1-$ form defining $\mathbf{F}$*

$$\Omega = \mathbf{a}(x,y,t)\,dx + \mathbf{b}(x,y,t)\,dy + \mathbf{c}(x,y,t)\,dt.$$

Then $\mathbf{F}$ is analytically trivial if one of the following equivalent properties is satisfied

(1) $\mathbf{c} \in (\mathbf{a},\mathbf{b})$
(2) There exists a germ of vector field X such that $\Omega(X) = 0$ and $D\pi(X) = \frac{\partial}{\partial t}$.

Proof. The equivalence between the two properties is trivial. Suppose now that $\mathbf{F}$ is analytically trivial. In view of the definition, there exists $\Phi = (\Phi_1, \Phi_2, t)$ such that $\Phi^*\Omega = u\omega$ where u is a unity. In particular,

$$\mathbf{a}(\Phi_1, \Phi_2, t)\frac{\partial \Phi_1}{\partial t} + \mathbf{b}(\Phi_1, \Phi_2, t)\frac{\partial \Phi_2}{\partial t} + \mathbf{c}(\Phi_1, \Phi_2, t) = 0,$$

which ensures that $\mathbf{c} \in (\mathbf{a}, \mathbf{b})$. Conversely, if $\mathbf{c} \in (\mathbf{a}, \mathbf{b})$ we can set $X = \mathbf{u}\frac{\partial}{\partial x} + \mathbf{v}\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$ where $\mathbf{ua} + \mathbf{vb} + \mathbf{c} = 0$. For t small enough, one can consider the flow at time t of the vector field X. Let us consider now $\Phi(x, y, t) = e^{(t)X}(x, y, 0)$: it is easily seen that Φ commutes with π, is equal to Id when restricted to the fiber $\pi^{-1}(0)$ and satisfies $\Phi^* X = u\frac{\partial}{\partial t}$ for some function u. Since $\Omega(X) = 0$, one has $\Phi^*\Omega\left(\frac{\partial}{\partial t}\right) = 0$. Hence, $\Phi^*\Omega$ can be written

$$\Phi^*\Omega = \mathbf{a}'(x, y, t)\, dx + \mathbf{b}'(x, y, t)\, dy.$$

The integrability of $\Phi^*\Omega$ yields the relation

$$\mathbf{b}'\frac{\partial \mathbf{a}'}{\partial t} = \mathbf{a}'\frac{\partial \mathbf{b}'}{\partial t}.$$

Therefore, there exists a unity u such that $\mathbf{a}' = u(x, y, t)\,\mathbf{a}(x, y, 0)$ and $b' = u(x, y, t)\,\mathbf{b}(x, y, 0)$. Hence, we are led to the relation

$$\frac{1}{u}\phi^*\Omega = \omega$$

which is the lemma. □

Corollary 17. *Let $\mathbf{F}$ be an unfolding of $\mathfrak{F}$, given by ω, with parameter in $(\mathbb{C}^p, 0)$. Let us write the $1-$ form defining $\mathbf{F}$*

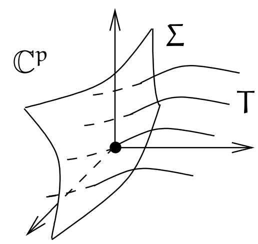
$$\Omega = \mathbf{a}(x, y, t)\, dx + \mathbf{b}(x, y, t)\, dy + \sum_{i=1}^{p} \mathbf{c}_i(x, y, t)\, dt_i.$$

Then $\mathbf{F}$ is analytically trivial if one of the following equivalent properties is satisfied

(1) $(\mathbf{c}_1, \cdots, \mathbf{c}_p) \subset (\mathbf{a}, \mathbf{b})$
(2) There exists a family of germs of vector fields X_i such that $\Omega(X_i) = 0$ and $D\pi(X_i) = \frac{\partial}{\partial t_i}$.

We can be even more precise:

Proposition 18. *Let $\mathbf{F}$ be an unfolding of $\mathfrak{F}$ with p parameter. Let $T = \sum a_i(t)\frac{\partial}{\partial t_i}$ be a germ of non-vanishing vector field in $(\mathbb{C}^p, 0)$. Let Σ be a germ of smooth hypersurface in $(\mathbb{C}^p, 0)$ transverse to T. Let Λ be the projection of $\mathbb{C}^p$ to Σ along the leaves and i_Σ the embedding of $\Sigma \subset \mathbb{C}^p$. Suppose that there exists a germ of vector field X tangent to $\mathbf{F}$ such that $D\pi(X) = T$. Then we have the following isomorphism of unfoldings*



$$\mathbf{F} \sim \mathbf{F}_{i_\Sigma \circ \Lambda}$$

This proposition highlights the fact that T is a direction along which the unfolding is trivial. The proof consists simply on making a change of coordinates in the space of parameters so that T becomes $\frac{\partial}{\partial t_1}$ and applying the previous results. Notice that since $D\pi$ commutes with the Lie bracket, if T and T' satisfy the hypothesis then $\left[T, T'\right]$ also. Thus, the $\mathcal{O}_{(\mathbb{C}^p,0)}$-modules generated by the set of regular vector fields along which the unfolding is trivial is involutive[5] and so defines a germ of smooth foliation in $(\mathbb{C}^p, 0)$.

The fundamental properties that will allow us to use cohomological tools is the local triviality of unfolding

Proposition 19. *Let $\mathbf{F}$ be an equireducible unfolding of $\mathfrak{F}$ with parameters in $(\mathbb{C}^p, 0)$. Suppose that $\mathfrak{F}$ has a reduced singularity. Then $\mathbf{F}$ is analytically trivial.*

Proof. We consider each possible case for $\mathfrak{F}$

(1) If $\mathfrak{F}$ is smooth then, in view of the definition, $\mathbf{a}(0,0,0) \neq 0$ or $\mathbf{b}(0,0,0) \neq 0$. In any case, the ideal $(\mathbf{a}, \mathbf{b})$ is the whole space $\mathbb{C}\{x, y, t\}$. Thus, the ideal inclusion in corollary 17 is obvious.

[5] Involutive means invariant by Lie bracket.

(2) Suppose now that $\mathfrak{F}$ is singular with $J_0\omega = \lambda x dy + y dx$. Then it is known that ω admits two smooth and transverse separatricies. From the equisingularity property, for any t small enough $\mathfrak{F}_t$ is also reduced and of same type. Hence, we get a analytical family of curves that induces two smooth transverse invariant hypersurfaces for Ω which are also transverse to the fibration π. We can straightened these two hypersurfaces on $xy = 0$ with a conjugacy that commutes with π. Hence, we can suppose that Ω is written

$$\Omega = x\mathbf{a}dy + y\mathbf{b}dx + \sum_{i=1}^{p} \mathbf{c}_i dt_i$$

where $\mathbf{a}$ and $\mathbf{b}$ are unity. But now, the transversality condition ensures that $(\mathbf{c}_1, \cdots, \mathbf{c}_p) \subset \sqrt{(x\mathbf{a}, y\mathbf{b})} = \sqrt{(x,y)} = (x,y) = (x\mathbf{a}, y\mathbf{b})$. In view of corollary 17, the unfolding is trivial.

(3) Suppose finally that $\mathfrak{F}$ is singular and that $J_0\omega = x dy$. Since, $dJ_0\omega = dx \wedge dy$, we have $d\Omega = u(x,y,t)\, dx \wedge dy + \cdots$ with $u(0,0,0) \neq 0$. The integrability condition yields

$$u\mathbf{c_i} \in (\mathbf{a}, \mathbf{b}).$$

This property is more general and known as the Kupka-Reeb phenomena [12].

□

Thus we are able now to establishe the property announced in the introduction: the topological triviality of unfolding.

Proposition 20. *Let $\mathbf{F}$ be an equireducible unfolding with parameters in $\mathbb{C}^p$. Then $\mathbf{F}$ is topologically trivial.*

Proof. Consider the reduction $\mathbf{E}$ of the singularities of the unfolding $\mathbf{F}$. Any singularity of $\mathbf{E}|^*_{\pi^{-1}(0)}\mathfrak{F}$ is reduced. Thus, we can cover the exceptional divisor $\mathbf{D}$ with a covering $\{\mathbf{U}_i\}_{i\in I}$ such that the unfolding is trivial along $\mathbf{U}_i$. Therefore, there exists a family of tangent holomorphic vector fields $\left\{X_i^j\right\}_{i\in I}$ defined on $\mathbf{U}_i$ such that $D\pi\left(X_i^j\right) = \frac{\partial}{\partial t_j}$. Using a $\mathcal{C}^\infty$ partition of the unity adapted to the covering $\{\mathbf{U}_i\}_{i\in I}$, it is possible to glue together the vector fields X_i^j to get a global tangent $\mathcal{C}^\infty$ vector field X^j. Therefore, there exists a $\mathcal{C}^\infty$ conjugacy defined on $(\mathbf{A}, \mathbf{D})$ that conjugates the unfolding $\mathbf{F}$ and the trivial unfolding $\mathbf{E}|^*_{\pi^{-1}(0)}\mathfrak{F} \times (\mathbb{C}^p, 0)$. This conjugacy can be push down as a conjugacy on $\mathbb{C}^2 \setminus \{0\} \times (\mathbb{C}^p, 0)$ that can be extended continuously on $\left(\mathbb{C}^{2+p}, 0\right)$. □

4. Cech cohomology.

In this chapter, we would like to introduce the Cech cohomology for it is going to be a central tool to analyze unfolding. The objective is not to explore all the properties of this cohomology but only to mention its main properties in order to make this lecture almost self contained. For any precision, we refer to [8].

4.1. **Sheaf of groups.** Hereafter, we give the definition of sheaf of group.

Definition 21. Let M be a topological space. A sheaf $\mathcal{S}$ is a the data of a group $\mathcal{S}(U)$ for any open set U of M called *the section of* $\mathcal{S}$ *over* U and of *restriction functions* $\rho_{UV} : \mathcal{S}(U) \to \mathcal{S}(V)$ for $V \subset U$ such that

(1) For any $W \subset V \subset U$, we have $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$
(2) If $\sigma_U \in \mathcal{S}(U)$ and $\sigma_V \in \mathcal{S}(V)$ such that $\sigma_U = \sigma_V$ on $U \cap V$ then there exists $\sigma_{U\cup V} \in \mathcal{S}(U \cup V)$ with $\sigma_{U\cup V}|_U = \sigma_U$ and $\sigma_{U\cup V}|_V = \sigma_V$.
(3) If $\sigma \in \mathcal{S}(U \cup V)$ satisfies $\sigma|_U = 0$ and $\sigma|_V = 0$ then $\sigma = 0$.

Example 22. Let M be any complex manifold. We consider the sheaf $\mathcal{O}_M$ of holomorphic function on M. In this case, $\mathcal{O}_M(U)$ is the set of holomorphic functions on U. The restriction map are simply the standard restrictions. One can also consider the sheaf TM of holomorphic tangent vector field. Here, the section over U has a structure of $\mathcal{O}_M(U)$ module: such a sheaf is called a sheaf of $\mathcal{O}_M-$modules. Finally, the last important sheaf is the sheaf of $\mathcal{O}_M$-module of holomorphic $p-$forms $\Omega^p(M)$. Let us also mention the sheaf of locally constant functions with values in $\mathbb{Z}$ or $\mathbb{C}$.

4.2. **Cech Cohomology for abelian group.** Let M be a manifold and $\mathcal{S}$ a sheaf on M. Let $M = \bigcup_{i\in I} U_i$ a covering. For any $k \geq 0$ we define the set of k-cochains as the following set

$$C^k\left(M, \{U_i\}_{i\in I}, \mathcal{S}\right) = \prod_{i_0\cdots i_k} \mathcal{S}\left(U_{i_0} \cap \cdots \cap U_{i_k}\right)$$

and a border operator by

(4.1)

$$\delta^k(\sigma)|_{U_{i_0}\cap\cdots\cap U_{i_{k+1}}} = \sum_{j=0}^{k+1} (-1)^j \, \sigma\left(U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_{k+1}}\right)\Big|_{U_{i_0}\cap\cdots\cap U_{i_{k+1}}}.$$

This operator satisifies the relation of complex

$$\delta^k \circ \delta^{k-1} = 0.$$

The Cech cohomology is then defined as follows

Definition 23. For any $k \geq 0$

$$H^k\left(M, \{U_i\}_{i\in I}, \mathcal{S}\right) = \frac{\ker \delta^k}{\mathrm{im}\delta^{k-1}} = \frac{\{\text{ the set of } k-\text{cocycles}\}}{\{\text{the set of } k-\text{coboundary}\}}.$$

Moreover, taking the inductive limit on the system of coverings of M, we define

$$H^k(M, \mathcal{S}) = \lim_{\{U_i\}_{i\in I}\rightarrow} H^k\left(M, \{U_i\}_{i\in I}, \mathcal{S}\right)$$

Let us describe the very first groups H^0 and H^1. By definition

$$H^0\left(M, \{U_i\}_{i\in I}, \mathcal{S}\right) = \frac{\ker \delta^0}{\mathrm{im}\delta^{-1}} = \ker \delta^0$$

and thus is equal to

$$\left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathcal{S}(U_i) \,\Big|\, \delta^0\left((s_i)_{i\in I}\right) = (s_i - s_j) = 0 \right\} = \mathcal{S}(M).$$

Hence, the first cohomological group in the sense of Cech is the set of global sections of the sheaf defined over M. Now, we have also

$$H^1\left(M, \{U_i\}_{i\in I}, \mathcal{S}\right) = \frac{\ker \delta^1}{\mathrm{im}\delta^0}$$

which can be explicitely described as

$$\frac{\left\{(s_{i_1 i_2}) \in \prod_{i_1 i_2 \in I^2} \mathcal{S}\left(U_{i_1} \cap U_{i_2}\right) | s_{i_1 i_2} + s_{i_2 i_3} + s_{i_3 i_1} = 0\right\}}{\left\{s_{i_1} - s_{i_2} \middle| (s_{i_1}) \in \prod_{i_1 \in I} \mathcal{S}\left(U_{i_1}\right)\right\}}.$$

Example 24. Let us consider the following problem: suppose that g is an holomorphic function on a manifold M that never vanishes. We would like to know if this function is the exponential of a function f. If M is compact then g is a constant and the problem is trivial. Else, we can cover $M = \bigcup_{i\in I} U_i$ where U_i and all the intersections $U_i \cap U_j$ are connected. For any U_i, there exists a function $f_i \in \mathcal{O}(U_i)$ satisfying $g = e^{f_i}$. Thus on $U_i \cap U_j$ we have $e^{f_i} = e^{f_j}$ and then $f_i = f_j + 2i\pi k_{ij}$ where $k_{ij} \in \mathbb{Z}$. Clearly

$$k_{ij} + k_{jl} + k_{li} = 0.$$

Thus $\{k_{ij}\}$ is a $1-$cocycle with values in the sheaf of locally constant integer functions. Suppose that k_{ij} is a coboundary: therefore, it can be written

$$k_{ij} = k_i - k_j.$$

The family of functions $f_i - 2i\pi k_i$ can be glued in an holomorphic function such that $e^f = g$. Therefore, we get the following result: if $H^1(M, \mathbb{Z}) = 0$ or if M is compact then $\mathcal{O}^*_M = e^{\mathcal{O}_M}$.

4.2.1. *Cech cohomology for non-abelian groups.* For non abelian groups, we can in a simple way define the first two groups of cohomology.

Definition 25. Let S be a sheaf of non-abelian group. We define

$$H^0(M, \mathcal{S}, \{U_i\}) = \mathcal{S}(M)$$

and

$$H^1(M, \mathcal{S}, \{U_i\}) = \frac{\{(s_{i_1 i_2}) | s_{i_1 i_2} \circ s_{i_2 i_3} \circ s_{i_3 i_1} = \mathrm{Id}\}}{(s_{ij}) \sim \left(s_i \circ s_{ij} \circ s_j^{(-1)}\right) \Big| (s_i) \in C^0}$$

With this naive approach, it is impossible to define in a coherent way the cohomological groups for bigger orders.

4.3. Mayer-Vietoris Sequence.

Theorem 26. *Let M be a manifold covered by a covering $M = \bigcup_{i \in I} U_i$. Then there exists an exact sequence*

$$\begin{aligned} 0 \quad \rightarrow \quad & H^0(M) \rightarrow \bigoplus_{i \in I} H^0(U_i) \rightarrow \bigoplus_{i,j \in I} H^0(U_i \cap U_j) \\ \rightarrow \quad & H^1(M) \rightarrow \bigoplus_{i \in I} H^1(U_i) \rightarrow \bigoplus_{i,j \in I} H^1(U_i \cap U_j) \rightarrow \dots \end{aligned}$$

In the case of a covering $M = U_1 \cup U_2$, the exact sequence is given by

$$0 \rightarrow H^0(M) \rightarrow H^0(U_1) \oplus H^0(U_2) \xrightarrow{\delta} H^0(U_1 \cap U_2) \xrightarrow{i} H^1(M) \rightarrow \dots$$

where $\delta(s_1, s_2) = s_1 - s_2$ and $i(s_{12}) = [s_{12}]$.

The proof of this result will not be given here because it is a bit technical but not very deep: it requires to introduce the standard cohomology of sheaves.

4.4. Long exact sequence associated to short one.

Let $\mathcal{S}$ and $\mathcal{S}'$ two sheaves of abelian groups : an application f from $\mathcal{S}$ two $\mathcal{S}'$is a collection of morphims of groups $f(U) : \mathcal{S}(U) \rightarrow \mathcal{S}'(U)$ that commute with the maps of restriction. The kernel of such an application is defined as the sheaves $\ker f$ whose sets of sections $\ker f(U)$ is the kernel of $f(U)$. This is a sheaves. Suppose that $\mathcal{S} \subset \mathcal{S}'$, the quotient $\mathcal{S}'/\mathcal{S}$ is defined as the sheaf whose set of sections $\mathcal{S}'/\mathcal{S}(U)$ is the set of family $(s_{V_i})_{i \in I}$ where $\{V_i\}$ is a covering of U such that

- s_{V_i} is a section of $\mathcal{S}'$.
- the difference $s_{V_i} - s_{V_j}$ is a section of $\mathcal{S}$ on $V_i \cap V_j$.

F or example, a global section of the quotient sheaf is a family s_i of local sections of $\mathcal{S}'$ defined on U_i, $M = \cup_{i\in I} U_i$, such that s_i and s_j are equal modulo a section of $\mathcal{S}$. In particular, the application

$$\mathcal{S}'(M) \longrightarrow \mathcal{S}'/\mathcal{S}(M)$$

does not have to be onto: it is the case precisely when $H^1(M, \mathcal{S}) = 0$. More generally, we have

Theorem 27. *Let $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}''$ three sheaves such that there exists an exact sequence*

$$0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{S}'' \to 0.$$

Then there exists a long exact sequence in cohomology

$$0 \to H^0(\mathcal{S}) \to H^0\left(\mathcal{S}'\right) \to H^0\left(\mathcal{S}''\right) \xrightarrow{\delta} H^1(\mathcal{S}) \to H^1\left(\mathcal{S}'\right) \to \cdots.$$

This theorem is just an application of the *Snake's lemma.* The map δ is defined the following way: consider an element of $s \in H^0\left(\mathcal{S}''\right)$. It is a family $s = (s_i)_{i\in I}$ of sections of S' for some covering $\{U_i\}_{i\in I}$ such that $s_i - s_j = s_{ij}$ is a section of S. The class of $[s_{ij}]$ in $H^1(\mathcal{S})$ is exactly the image $\delta(s)$.

Example 28. Let us consider the following exact sequence of sheaves on any complex manifold:

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{\cdot}} \mathcal{O}^* \to 0.$$

The morphism of sheaves of groups $\cdot \mapsto e^{\cdot}$ is actually onto since it is locally onto. The long exact sequence in cohomology associated to this short one is

$$0 \to \mathbb{Z} \to \mathcal{O}(M) \to \mathcal{O}^*(M) \to H^1(M, \mathbb{Z}) \to \cdots.$$

Thus we recover the following fact: if $H^1(M, \mathbb{Z}) = 0$ then the map $\mathcal{O}(M) \xrightarrow{e^{\cdot}} \mathcal{O}^*(M)$ is onto.

4.5. **Acyclic covering and holomorphy domain.** To compute the cohomology of Cech is *a priori* a though problem since it is defined as an inductive limit. However, if the covering has the property of *acyclicity* then one can compute the Cech cohomology with respect to this covering.

Theorem 29. [8] *If* $M = \bigcup_{i\in I} U_i$ *is an acyclic covering, i.e., for any intersection* $U_{i_1 i_2 \cdots i_n} = U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_n}$ *and any* $k \geq 1$*, we have*

$$H^k\left(U_{i_1 i_2 \cdots i_n}, \mathcal{S}\right) = 0$$

Then, there is a canonical isomorphism

$$H^k\left(M, \mathcal{S}\right) \simeq H^k\left(M, \{U_i\}_{i\in I}, \mathcal{S}\right).$$

It remains to find acyclic covering of a given manifold. It can be done introducing the notion of holomorphy domain.

Definition 30. A domain U of $\mathbb{C}^n$ is a domain of holomorphy if and only if for any point z_0 in the boundary ∂U there exists an holomorphic function on U that cannot be extended at z_0.

Any open set of $\mathbb{C}$ is a domain of holomorphy since if $z_0 \in \partial U$ one can consider the function $f(z) = \frac{1}{z-z_0}$. This result is false in higher dimension: for example, any open set $U \backslash m \subset \mathbb{C}^p,\ p \geq 2$ where $m \in U$ is not an holomorphy domain: any holomorphic function defined in a neighborhood of p except maybe at p can be actually extended at p. This is the Hartogs lemma. We are going to use it a few times, thus we mention it:

Theorem 31. *Let* f *be an holomorphic function defined in a neighborhood of* $(0,0)$ *in* $\mathbb{C}^2$ *except maybe at* $(0,0)$*. Then* f *can be extended holomorphically at* $(0,0)$*.*

Proof. This is a consequence of the Cauchy formula: for any $(x,y) \in U \subset \mathbb{C}^2$ we can set

$$\tilde{f}(x,y) = \frac{1}{2i\pi}\int_{|\zeta|=\rho} \frac{f(\zeta,y)}{\zeta - x} d\zeta.$$

The funciton $\tilde{f}$ is an holomorphic function in the whole U and coincides with f whenever $y \neq 0$. Thus $f = \tilde{f}$ on $U \setminus \{(0,0)\}$. □

To find acyclic covering, we can use the following result:

Theorem 32. [10, 6] *Let M be a domain of holomorphy (or more generally a Stein space*[6] *) and S a coherent*[7] *sheaf of $\mathcal{O}_M-$modules. Then for any $i \geq 1$*

$$H^i(M, S) = 0.$$

This result is accepted without proof: it is very deep and tough one.

[6]see *Complex algebraic and analytic geometry*, J.P. Demailly

[7]The definition of coherent sheaf will not be given: this means more or less that the sheaf of A-module has a finite rank that does not change with the fiber of the sheaf.

5. Moduli of unfolding.

Let $\mathbf{F}$ be an unfolding with parameters in $(\mathbb{C}^p, 0)$ of $\mathfrak{F}$. Let $\mathbf{E} : (\mathbf{A}, \mathbf{D}) \to \left(\mathbb{C}^{2+p}, 0\right)$ be the process of reduction of $\mathbf{F}$. We will denote $E : (A, D) \to \left(\mathbb{C}^2, 0\right)$ the process of reduction of $\mathfrak{F}$. The tree (A, D) is embedded in $(\mathbf{A}, \mathbf{D})$ as the fiber of π over the parameter 0. We consider the sheaf $\mathbf{G}_p(\mathfrak{F})$ of groups defined as follows: its base is the analytical space $D \subset A$. For any $U \subset D$, a section of $\mathbf{G}_p(\mathfrak{F})$ over U is a germ of automorphism Φ defined in a neighborhood of U in $A \times (\mathbb{C}^p, 0)$ such that

(1) Φ commutes with the projection $A \times (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$.
(2) Φ is equal to Id when $t = 0$.
(3) Φ lets invariant the trivial unfolding of $\mathfrak{F}$, i.e.,

$$\Phi^* (\mathfrak{F} \times (\mathbb{C}^p, 0)) = \mathfrak{F} \times (\mathbb{C}^p, 0).$$

Notice that the conditions (2) and (3) imply that Φ lets globally invariant each leaf of $\mathfrak{F} \times (\mathbb{C}^p, 0)$.

5.1. Cohomological interpretation for unfoldings.

The cohomological interpretation of unfoldings is contained in the following statement:

Theorem 33. [15] *There is a one to one correspondence between the set of analytical classes of equireducible unfoldings of $\mathfrak{F}$ with parameters in $(\mathbb{C}^p, 0)$ and $H^1(D, \mathbf{G}_p(\mathfrak{F}))$.*

Proof. Let us consider an unfolding $\mathbf{F}$ with parameters in $(\mathbb{C}^p, 0)$. In view of the proposition 19, one can cover $D = \cup_{i\in I} U_i$ such that the unfolding is analytically trivial along U_i. Therefore, there exists a local conjugacy Φ_i defined in a neighborhood of U_i such that $\Phi_i^* \mathbf{F} = \mathfrak{F} \times (\mathbb{C}^p, 0)$. In particular, the application $\Phi_i \circ \Phi_j^{(-1)}$, defined whenever $U_i \cap U_j$ is not empty, is a section of $\mathbf{G}_p(\mathfrak{F})$ over $U_i \cap U_j$. Hence, the collection $\left\{\Phi_i \circ \Phi_j^{-1}\right\}$ defines a cocycle in $Z^1\left(D, \{U_i\}_{i\in I}, \mathbf{G}_p(\mathfrak{F})\right)$ and thus an element of $H^1\left(D, \{U_i\}_{i\in I}, \mathbf{G}_p(\mathfrak{F})\right) = H^1(D, \mathbf{G}_p(\mathfrak{F}))$. Let us prove that this construction is well defined and induces a one to one correspondence with the set of analytical classes of equireducible unfoldings. Suppose, first, that we choose some others trivializations $\Phi_i^{'}$. Then $\Phi_i \circ \left(\Phi_i^{'}\right)^{-1}$ is an element of $\mathbf{G}_p(\mathfrak{F})(U_i)$. Since holds the following equality

$$\Phi_i \circ \Phi_j^{-1} = \Phi_i \circ \left(\Phi_i^{'}\right)^{-1} \circ \Phi_i^{'} \circ \left(\Phi_j^{'}\right)^{-1} \circ \Phi_j^{'} \circ (\Phi_j)^{-1},$$

the two collections $\left\{\Phi_i \circ \Phi_j^{(-1)}\right\}$ and $\left\{\Phi_i^{'} \circ \Phi_j^{'(-1)}\right\}$ define the same cohomological class. Therefore, the construction above defines an application. Let us prove that the image of $\mathbf{F}$ depends only on its analytical class. Suppose that $\mathbf{F}^{'}$ and $\mathbf{F}$ are analytically equivalent: the associated conjugacy ϕ can be lifted-up in a global conjugacy Φ defined from $\mathbf{A}$ to $\mathbf{A}^{'}$, the respective trees associated to $\mathbf{F}$ and $\mathbf{F}^{'}$. Therefore, if $\{\Phi_i\}_{i\in I}$ is a family of trivializations for $\mathbf{F}$ then $\{\Phi_i \circ \Phi\}_{i\in I}$ is a family of trivializations for $\mathbf{F}^{'}$. Hence, the two cocycles associated to $\mathbf{F}$ and $\mathbf{F}^{'}$ can be chosen equal.

In this way, we define a map from the set of analytical classes of equireducible unfoldings of $\mathfrak{F}$ with parameters in $(\mathbb{C}^p, 0)$ to the cohomological group $H^1(D, \mathbf{G}_p(\mathfrak{F}))$. It remains to prove that this correspondence is one to one. It is clearly an embedding: indeed, suppose that $\mathbf{F}$ and $\mathbf{F}^{'}$ induce two cocycles that are equivalent in $H^1(D, \mathbf{G}_p(\mathfrak{F}))$. Then, there exist a covering $\{U_i\}_{i\in I}$ and a $0-$cocycle $\{\psi_i\}_{i\in I}$ such that

$$\psi_i \circ \Phi_i \circ \Phi_j^{-1} \circ \psi_j^{(-1)} = \Phi_i^{'} \circ \Phi_j^{'(-1)}.$$

Thus, the family $\Phi_i^{'(-1)} \circ \psi_i \circ \Phi_i$ defines a global conjugacy between $\mathbf{F}$ and $\mathbf{F}^{'}$ in their reduction trees. This conjugacy can be pushed down using a standard argument of Hartogs: the pushed down conjugacy is holomorphic on $\mathbb{C}^2 \backslash \{0\} \times \mathbb{C}^p$ since the blowing-up process realize the following isomorphisms

$$\mathbf{A}^{'} \backslash \mathbf{D}^{'} \simeq \mathbf{A} \backslash \mathbf{D} \simeq \mathbb{C}^2 \backslash \{0\} \times \mathbb{C}^p.$$

Hence, following Hartogs lemma, it can be extended on $(\mathbb{C}^{2+p}, 0)$.

To prove that the correspondence is onto, let us consider a 1-cocycle

$$\{\Phi_{ij}\}_{i,j} \in Z^1 \left(D, \{U_i\}_{i \in I}, \mathbf{G}_p(\mathfrak{F})\right).$$

We construct a manifold $\widetilde{\mathbf{A}}$ by gluing the family of products $U_i \times (\mathbb{C}^p, 0)$ with the identification defined by $\{\Phi_{ij}\}_{i,j}$

$$\widetilde{\mathbf{A}} = \coprod_{i \in I} U_i \times (\mathbb{C}^p, 0) /_{x \sim \Phi_{ij}(x)}.$$

Since, Φ_{ij} lets globally invariant each leaf of the product foliation $\mathfrak{F} \times (\mathbb{C}^p, 0)$, the manifold $\widetilde{\mathbf{A}}$ admits a foliation of codimension 1 also constructed by a gluing trick

$$\widetilde{\mathbf{F}} = \coprod_{i \in I} \mathfrak{F}|_{U_i} \times (\mathbb{C}^p, 0) /_{x \sim \Phi_{ij}(x)}.$$

This manifold is equipped with a fibration $\widetilde{\mathbf{A}} \to (\mathbb{C}^p, 0)$ whose fibers are transverse to the foliation $\widetilde{\mathbf{F}}$. The foliated manifold $\widetilde{\mathbf{A}}$ is the neighborhood of a compact divisor $\widetilde{\mathbf{D}}$. A theorem of Grauert [9] ensures that the manifold $\widetilde{\mathbf{A}}$ is biholomorphic to a deformation of A. Let us denote this deformation $\mathbf{A}$. The manifold $\mathbf{A}$ comes with a codimension one foliation that can be pushed down as a codimension one foliation $\mathbf{F}$ in $\left(\mathbb{C}^{2+p}, 0\right)$. It is easily seen that $\mathbf{F}$ is an unfolding of $\mathfrak{F}$ whose associated cocycle is equivalent to the original cocycle $\{\Phi_{ij}\}_{i,j}$. □

The interest of the previous result is limited for the space $H^1(D, \mathbf{G}_p(\mathfrak{F}))$ has a very poor structure. At best, it is a set ! It does even not have a natural topology. However, this result - and more precisely its proof -

does produce a recipe to construct every unfoldings: glue together pieces of trivial unfoldings with sections of $\mathbf{G}_p(\mathfrak{F})$.

5.2. **Foliated Kodaira-Spencer map.** If $H^1(D, \mathbf{G}_p(\mathfrak{F}))$ were a manifold in the classical sense, the first question one would ask is: what is its dimension ? This dimension would be also the dimension of the tangent space $\mathcal{T}_{\mathfrak{F}} H^1(D, \mathbf{G}_p(\mathfrak{F}))$. To analyze what $\mathcal{T}_{\mathfrak{F}} H^1(D, \mathbf{G}_p(\mathfrak{F}))$ could be we introduce *the foliated Kodaira-Spence map*: this name refers to the construction of Kodaira and Spencer to analyze the deformations of the complex structure of a compact manifold [11]. Let us consider a cocycle $\{\Phi_{ij}\}_{i,j}$ in $H^1(D, \mathbf{G}_p(\mathfrak{F}))$ and write

$$\Phi_{ij}(x, y, t) = \left(\Phi^1_{ij}(x, y, t), \Phi^2_{ij}(x, y, t), t\right).$$

Since Φ_{ij} lets invariant the trivial unfolding $\mathfrak{F} \times (\mathbb{C}^p, 0)$, one can see that the vector field

$$\frac{\partial}{\partial t_k}\Phi_{ij} = X_{ij} = \left.\frac{\partial \Phi^1_{ij}(x, y, t)}{\partial t_k}\right|_{t=0} \frac{\partial}{\partial x} + \left.\frac{\partial \Phi^2_{ij}(x, y, t)}{\partial t_k}\right|_{t=0} \frac{\partial}{\partial y}$$

is tangent to $\mathfrak{F}$. Thus, the collection $\left\{\frac{\partial}{\partial t_k}\Phi_{ij}\right\}_{i,j}$ defines a 1- cocycle with values in the sheaf of tangent vector fields to $\mathfrak{F}$. We will denote this sheaf $\mathcal{T}\mathfrak{F}$. We let the reader verify that the class of $\left\{\frac{\partial}{\partial t_k}\Phi_{ij}\right\}_{i,j}$ in $H^1(D, \mathcal{T}\mathfrak{F})$ does depend only on the class of $\{\Phi_{ij}\}_{i,j} \in H^1(D, \mathbf{G}_p(\mathfrak{F}))$. Therefore, for any unfolding $\mathbf{F}$ of $\mathfrak{F}$ with p parameters, we obtain an application

$$\mathcal{T}_0\mathbb{C}^p \xrightarrow{\partial \mathbf{F}} H^1(D, \mathcal{T}\mathfrak{F}).$$

This map is called the *Kodaira-Spencer map* of the unfolding $\mathbf{F}$.

Remark 34. The map $\partial\mathbf{F}$ can be also computed the following way: since $\mathbf{F}$ is locally trivial after the process of reduction, there exist a covering $\{U_i\}_{i\in I}$ of D and a family of vector fields X^j_i defined near U_i such that X^j_i is tangent to the unfolding and satisfies $D\pi\left(X^j_i\right) = \frac{\partial}{\partial t_j}$. The class of the 1-cocycle $\left.X^j_i - X^j_l\right|_{t=0}$ in $H^1(D, \mathcal{T}\mathfrak{F})$ is the image of $\frac{\partial}{\partial t_j}$ by $\partial\mathbf{F}$.

Proposition 35. *For any $\mathbb{C}$-linear map from $\mathcal{T}_0\mathbb{C}^p$ to $H^1(D, \mathcal{T}\mathfrak{F})$, there exists an unfolding $\mathbf{F}$ of $\mathfrak{F}$ with p parameters whose it is the Kodaira-Spencer map.*

Proof. Let us denote by X_{ij}^k the image of $\frac{\partial}{\partial t_k}$ by the given $\mathbb{C}$−linear application. The unfolding obtained thanks to the cocycle

$$\Phi_{ij}(x, y, t) = \left(e^{(t_1)X_{ij}^1} \circ \cdots \circ e^{(t_p)X_{ij}^p}(x, y), t\right)$$

has the checked Kodaira-Spencer map. □

The geometrical meaning of the previous proposition is that one can interpret the $\mathbb{C}$-space $H^1(D, \mathcal{T}\mathfrak{F})$ as the tangent space to $H^1(D, \mathbf{G}_p(\mathfrak{F}))$ at $\mathfrak{F}$.

5.3. **Moduli for infinitesimal unfolding of foliation.** We want to give a proof of the following result which is due to J.-F. Mattei: let $\mathfrak{F}$ be a general[8] non dicritical foliation.

Theorem 36. [15] *The $\mathbb{C}$-space $H^1(D, \mathcal{T}\mathfrak{F})$ has a finite dimension equal to*

$$\delta(\mathfrak{F}) = \sum_{c \in \Sigma} \frac{(\nu_c - 1)(\nu_c - 2)}{2}$$

where Σ is the set of singular points appearing in the process of reduction and $\nu_c = \nu_{\mathfrak{F}}(c)$ is the multiplicity of the singular point c when it appeared in this process. Moreover, an unfolding $\mathbf{F}$ is universal if and only its the Kodaira-Spencer map is one to one.

In the theorem *universal* means the following: for any other unfolding $\mathbf{G}$ with n parameters, there exists a germ of function $\Lambda : (\mathbb{C}^n, 0) \to \left(\mathbb{C}^{\delta(\mathfrak{F})}, 0\right)$ such that

$$\mathbf{F}_\Lambda \sim \mathbf{G}.$$

In other words, any other unfoldings can be factorized through $\mathbf{F}$.

[8]See the definition 13.

Example 37. Let us compute the dimension $\delta(\mathfrak{F})$ for the foliation given by

$$\omega = \mathrm{d}\left(xy\left(y^2 - x^2\right)\right).$$

This foliation is reduced after on blowing-up: indeed, in the local chart, the foliation is given by

$$\frac{E^*\omega}{x^3} = \frac{1}{x^3}\mathrm{d}\left(x^4 t\left(t^2 - 1\right)\right) = 4t\left(t^2 - 1\right)\mathrm{d}x + x\left(3t^2 - 1\right)\mathrm{d}t.$$

Thus, the singularity at $(0,0)$ is written $-4t\mathrm{d}x - x\mathrm{d}t + \ldots$, so it is reduced. This is the same for the other four singularities. Therefore, for any singularity c obtained after the blowing-up, the multiplicity $\nu_{\mathfrak{F}}(c)$ is equal to 1. Moreover, at the origin of $\mathbb{C}^2$, $\nu_{\mathfrak{F}}(0)$ is equal to 3. Therefore, the checked dimension is equal to

$$\dim_{\mathbb{C}} H^1(D, \mathcal{T}\mathfrak{F}) = \frac{(3-1)(3-2)}{2} = 1.$$

Actually, the foliation given by ω admits four separatricies that are transverse: the only analytical invariant highlighted by the above computation is the cross-ratio of the four points in the tangent cone of ω.

Example 38. One can also consider the *double-cusp* example

$$\omega = \mathrm{d}\left(\left(x^2 - y^3\right)\left(y^2 - x^3\right)\right).$$

The dimension is 1. But in that case, the geometric interpretation of this modulus is not as clear as in the previous example.

Proof. (Theorem 36) The proof is an induction on the length of the process of reduction.

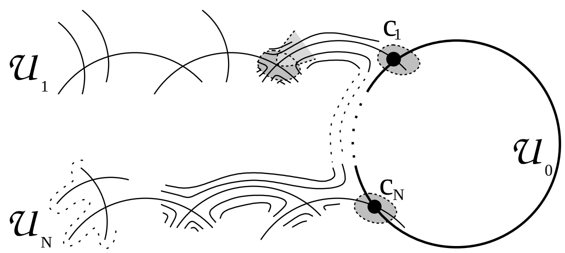


FIGURE 5.1. Covering of the divisor for Mayer-Vietoris theorem.

The Mayer-Vietoris sequence associated to the above covering reduces to the following exact sequence

$$0 \to H^1(D_0, \mathcal{T}\mathfrak{F}) \to H^1(D, \mathcal{T}\mathfrak{F}) \to \bigoplus_{i=1}^{n} H^1(D_i, \mathcal{T}\mathfrak{F}) \to 0$$

which gives the dimension statement since

$$\dim H^1(D, \mathcal{T}\mathfrak{F}) = \dim H^1(D_0, \mathcal{T}\mathfrak{F}) + \dim \bigoplus_{i=1}^{n} H^1(D_i, \mathcal{T}\mathfrak{F}).$$

Hereafter, we make the first step of the induction. Let us consider X a vector field with an isolated singularity that is tangent to the foliation $\mathfrak{F}$. Let E be the standard blowing-up of the origin of $\mathbb{C}^2$. We suppose that $E^*\mathfrak{F}$ is reduced. We consider the Stein[9] covering $D = U_1 \cup U_2$ where in each U_i, E is given by

$$E(x_1, y_1) = (x_1, y_1 x_1) \quad \text{and} \quad E(x_2, y_2) = (y_2 x_2, y_2)$$

with $y_2 = y_1 x_1$ and $x_2 = \frac{1}{y_1}$

[9]A Stein covering is a covering where the open sets are Stein

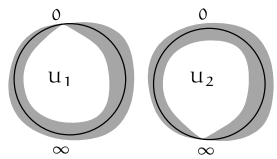


FIGURE 5.2. Covering of D.

We have the following equalities

$$\begin{aligned} \mathcal{T}\mathfrak{F}(U_1) &= \left(\sum_{(i,j)\in\mathbb{N}^2} a_{ij}x_1^i y_1^j\right)\frac{E^*X}{x_1^{\nu_0-1}} \\ \mathcal{T}\mathfrak{F}(U_2) &= \left(\sum_{(i,j)\in\mathbb{N}^2} b_{ij}x_2^i y_2^j\right)\frac{E^*X}{y_2^{\nu_0-1}} \\ \mathcal{T}\mathfrak{F}(U_1\cap U_2) &= \left(\sum_{(i,j)\in\mathbb{N}\times\mathbb{Z}} c_{ij}x_1^i y_1^j\right)\frac{E^*X}{x_1^{\nu_0-1}} \end{aligned}$$

Moreover, there is no three intersection, thus we have the following indentification

$$H^1(D_0,\mathcal{T}\mathfrak{F}) = \frac{\mathcal{T}\mathfrak{F}(U_2\cap U_1)}{\mathcal{T}\mathfrak{F}(U_1)\oplus\mathcal{T}\mathfrak{F}(U_2)}.$$

Hence, the dimension of this space is the number of obstructions in the resolution of the cohomological equation

$$\sum_{(i,j)\in\mathbb{N}\times\mathbb{Z}} c_{ij}x_1^i y_1^j = \sum_{(i,j)\in\mathbb{N}^2} b_{ij}x_1^j y_1^{j-i-\nu_0+1} + \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{ij}x_1^i y_1^j$$

This number is exactly $\frac{(\nu_0-1)(\nu_0-2)}{2}$. Futhermore, the cohomological space is generated as a finite dimensional $\mathbb{C}$-space by the cocycles

$$\frac{x_1^i}{y_1^j}\left(\frac{E^*X}{x_1^{\nu_0-1}}\right) \text{ with } 0\le i\le\nu_0-3,\ 1\le j\le\nu_0-2-i$$

Let us prove now that if the Kodaira-Spencer map is one to one then the unfolding is universal. Let $\mathbf{G}$ be an other unfolding of $\mathfrak{F}$ with m parameters. We consider the cocycles defining $\mathbf{G}$ and $\mathbf{F}$ denoted by $g = \{g_{ij}\}_{ij}$ and $f = \{f_{ij}\}_{ij}$ with

$$\begin{aligned} g_{ij}(x,y,s) &= \left(g^1_{ij}(x,s),,s\right) \\ f_{ij}(x,y,t) &= \left(f^1_{ij}(x,t),t\right) \end{aligned}$$

where $s \in \mathbb{C}^m$ and $t \in \mathbb{C}^{\delta(\mathfrak{F})}$. We define a new a cocycle with $m+\delta(\mathfrak{F})$ parameters by setting

$$f\Box g = \left\{(x,s,t) \to \left(f^1_{ij}\left(g^1_{ij}(x,t),s\right),s,t\right)\right\}.$$

The unfolding $\mathbf{F}\Box\mathbf{G}$ associated this cocycle has the following property

$$\begin{aligned} \mathbf{F}\Box\mathbf{G}|_{s=0} &= \mathbf{G} \\ \mathbf{F}\Box\mathbf{G}|_{t=0} &= \mathbf{F}. \end{aligned}$$

Let us denote by T the embedding $T(t) = (0,t)$. For any point (s_0,t_0) in the space of parameters of $\mathbf{F}\Box\mathbf{G}$ the Kodaira-Spencer map

$$\partial\mathbf{F}\Box\mathbf{G}_{(s_0,t_0)} : \mathcal{T}_{(s_0,t_0)}\mathbb{C}^{m+\delta(\mathfrak{F})} \to H^1\left(D_{(s_0,t_0)}, \mathcal{T}\mathbf{F}\Box\mathbf{G}|_{(s_0,t_0)}\right)$$

admits a kernel: the distribution so defined in $\left(\mathbb{C}^{m+\delta(\mathfrak{F})},0\right)$ is involutive and thus integrable[10]. Hence, it induces a germ of smooth foliation $\mathcal{N}$ that has the following property: for any leaf L of $\mathcal{N}$, the unfolding $\mathbf{F}\Box\mathbf{G}|_L$ is analytically trivial. Denoting by $\mathcal{N}(0)$ the kernel of $\partial\mathbf{F}\Box\mathbf{G}_{(0,0)}$, we have the following exact sequence:

$$0 \to \mathcal{N}(0) \to \mathcal{T}_0\mathbb{C}^{m+\delta(\mathfrak{F})} \to H^1(D,\mathcal{T}\mathfrak{F}) \to 0.$$

Since, the application $\partial\mathbf{F} : \mathcal{T}_0\mathbb{C}^{\delta(\mathfrak{F})} \to H^1(D,\mathcal{T}\mathfrak{F})$ is bijective, the embedding $T : \mathcal{T}_0\mathbb{C}^{\delta(\mathfrak{F})} \to \mathcal{T}_0\mathbb{C}^{m+\delta(\mathfrak{F})}$ is transverse to the direction $\mathcal{N}(0)$ of the leaf of $\mathcal{N}$ passing through 0. Thus, the picture is the following:

[10] See the proposition 18

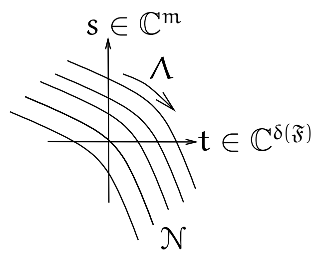


FIGURE 5.3. Proof of the universality property.

Therefore, one can choose $\lambda(s) = \Lambda(s, 0)$ where Λ is the projection along the leaves from $\mathbb{C}^m$ to $\mathbb{C}^{\delta(\mathfrak{F})}$. □

5.4. A quick application.

Theorem 39. *Let $\mathfrak{F}$ a foliation given by the level of a function f. Suppose that Sep$(\mathfrak{F})$ is a union of four smooth transverse curves. Then there exists a system of coordinates (x, y) such that $\mathfrak{F}$ is given by the level of a function*

$$f_\alpha = xy(y + x)(y + \alpha x)$$

where $\alpha \in \mathbb{CP}^1 \backslash \{0, 1, \infty\}$.

Proof. Let us consider the one parameter unfolding $\mathbf{F}$ given by

$$\Omega = d(xy(y + x)(y + tx))$$

where $t \in \mathbb{CP}^1 \backslash \{0, 1, \infty\}$. We are going to prove that, for any point t_0 the unfolding $\mathbf{F}$ with parameter $t \in (\mathbb{C}, t_0)$ is universal. The multiplicity of $\mathfrak{F}_{t_0} = \{df_{t_0} = 0\}$ is equal to three and the foliation is reduced after one blowing-up. Hence, the dimension of $H^1(D_0, \mathcal{T}\mathfrak{F}_{t_0})$ is one and this space is generated by $\frac{1}{y_1}\left(\frac{E^*X}{x_1^2}\right)$. Therefore, it is enough to prove that the Kodaira-Spencer map of $\mathbf{F}$ is not the null map. We are going to compute the image of $\frac{\partial}{\partial t}$ by $\partial\mathbf{F}$ using the remark 34. After one blowing-up, Ω is written

$$\begin{aligned}\Omega &= d\left(x_1^4 y_1 (y_1+1)(y_1+t)\right) = d\left(x_1^4 P(y_1)\right) \\ &= x_1^4 P'(y_1)\, dy_1 + 4x_1^3 P(y_1)\, dx_1 + x_1^4 y_1 (y_1+1)\, dt\end{aligned}$$

Let us find $X_1 = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial y_1} + \frac{\partial}{\partial t}$ such that $\Omega(X_1) = 0$. It corresponds to the equation

$$x_1 P'(y_1)\, b + 4P(y_1)\, a + x_1 y_1 (y_1+1) = 0.$$

Consider a Bezout relation between P' and P, $UP + VP' + 1 = 0$ where the degree of U is one and the degree of V is 2. Then a solution of the equation $\Omega(X_1) = 0$ is written

$$X_1 = \frac{1}{4} x_1 y_1 (y_1+1)\, U \frac{\partial}{\partial x_1} + y_1 (y_1+1)\, V \frac{\partial}{\partial y_1} + \frac{\partial}{\partial t}.$$

In the other chart one can choose

$$X_2 = x_2^2 (x_2+1)\, \tilde{V} \frac{\partial}{\partial x_2} + \frac{1}{4} \tilde{U} x_2^2 (x_2+1)\, y_2 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial t}$$

where $\tilde{U}$and $\tilde{V}$ are polynomial functions, solutions of the Bezout relation $\tilde{U}Q + \tilde{V}Q' + 1 = 0$ where $Q = x_2(x_2+1)(1+tx_2)$ and $d^{\circ}\tilde{U} = 1$. The vector field $X_1 - X_2$ is tangent to the foliation $\mathfrak{F}$ and is the image of $\frac{\partial}{\partial t}$ by the Kodaira-Spencer map. There exists a function ϕ holomorphic on $U_1 \cap U_2$ such that

$$X_1 - X_2 = \phi \frac{E^* X}{x_1^2}.$$

To evaluate the function ϕ, we look at the coefficient on $\frac{\partial}{\partial y_1}$ of $X_1 - X_2$ and divide it by $P(y_1)$:

$$\phi = \frac{V(y_1)}{y_1 + t} + \frac{1}{y_1^2 (y_1+t)} \tilde{V}\left(\frac{1}{y_1}\right).$$

The non triviality of the Kodaira-Spencer map can be red in the coefficient of $\frac{1}{y_1}$ in the Laurent development of ϕ. This coefficient is $t\tilde{V}\left(\frac{-1}{t}\right)$ and is not zero since $\tilde{V}$ cannot vanish at $\frac{-1}{t}$.

To finish the proof we make the following construction. Suppose that the foliation $\mathbf{F}$ is given by a function $f = f_1 f_2 f_3 f_4$. Since, $f_1 = 0$ and

$f_2 = 0$ are smooth and transverse is it possible to choose them as axes. Hence, in this new system of coordinates, the foliation $\mathfrak{F}$ is given by

$$xy\left(y+x+\sum_{i+j\geq 2}^{n}a_{ij}x^iy^j\right)\left(y+\alpha x+\sum_{i+j\geq 2}^{n}b_{ij}x^iy^j\right).$$

We can suppose that the perturbing part are polynomial functions thanks to a result due to Mather [14]. Now let us consider the unfolding defined by

$$\mathbf{G}_{a_{ij},b_{ij}}(x,y)=\mathrm{d}\left(xy\left(y+x+\sum_{i+j\geq 2}^{n}a_{ij}x^iy^j\right)\left(y+\alpha x+\sum_{i+j\geq 2}^{n}b_{ij}x^iy^j\right)\right).$$

It satisfies the following relation:

$$\mathbf{G}_{0,0}(x,y)=\mathrm{d}\left(xy\left(y+x\right)\left(y+\alpha x\right)\right).$$

According to the universality result, for a_{ij} and b_{ij} small enough, the following isomorphism holds

$$\mathbf{G}_{a_{ij},b_{ij}}(x,y)\sim xy\left(y+x\right)\left(y+\alpha x\right).$$

Now, the function $\mathbf{G}_{a_{ij},b_{ij}}(x,y)$ verifies the next functional relation: for any $\lambda > 0$

$$\mathbf{G}_{a_{ij},b_{ij}}(\lambda x,\lambda y)=\lambda^N\mathbf{G}_{\lambda\cdot a_{ij},\lambda\cdot b_{ij}}(x,y)$$

with $\lambda\cdot a_{ij}$ and $\lambda\cdot b_{ij}$ goes to zero while λ tends to zero. Hence for λ small enough, we conclude

$$\begin{aligned}\mathbf{G}_{a_{ij},b_{ij}}(x,y) &\sim \mathbf{G}_{a_{ij},b_{ij}}(\lambda x,\lambda y)=\lambda^N\mathbf{G}_{\lambda\cdot a_{ij},\lambda\cdot b_{ij}}(x,y)\\ &\sim xy\left(y+x\right)\left(y+\alpha x\right).\end{aligned}$$

□

The function f_α and its parameter are not unique: for example, if one permutes the variables x and y, one can see that $f_\alpha \sim f_{\frac{1}{\alpha}}$. Actually, for any automorphism σ of $S = \mathbb{P}^1\backslash\{0,1,\infty\}$ we have $f_\alpha \sim f_{\sigma(\alpha)}$. Thus, α is unique in ${}^{S}/_{\mathrm{Aut}(S)}$, which is a ramified covering of S.

6. Moduli space of curve.

Let S be a germ of curve in $(\mathbb{C}^2, 0)$. We denote by $\mathcal{M}(S)$ the moduli space of the curve S which is the set of curves topologically equivalent to S up to analytical equivalence. I want to present an application of the cohomological interpretation of unfoldings.

6.1. **Finite determinacy for germs of curve.** Let us consider a curve S. For any blowing-up process E, we denote by $\mathrm{Att}(E, S)$ the set of points $E^{-1}(0) \cap \overline{E^{-1}(S \backslash 0)}$. If E_S is the reduction process of S, we construct by induction the following sequence of blowing-up process:

- $E_0^S = E_S$ and
- E_{n+1}^S is $E_n^S \circ E$ where E is the standard blowing-up with centers at $\mathrm{Att}(E_n^S, S)$.

We have the following result which is consequence of a standard result due to Mather [14].

Theorem 40. [7] *There exists an integer n depending only on the topological class of S such that for any curve S' topologically equivalent to S, if*

$$\mathit{Att}\left(E_n^S, S\right) = \mathit{Att}\left(E_n^S, S'\right)$$

then S and S' are analytically equivalent.

The proof consists in building two equations of S and S', say, $f = 0$ and $f' = 0$ in $\mathbb{C}\{x, y\}$ such that $f - f' \in (x, y)^N$ for N big enough:

it ensures, following [14], that f and f' are analytically conjugated as functions. Thus, the curves $S = \{f = 0\}$ and $S' = \{f' = 0\}$ are also analytically conjugated.

6.2. Curves V.S. Foliations.

The proof of the following result relies deeply on the use of the unfoldings of foliations.

Theorem 41. [7] *Let $\mathfrak{F}$ be a general non dicritical foliation and $S = \mathrm{Sep}(\mathfrak{F})$. Then, the natural map between moduli spaces*

$$\mathcal{M}(\mathfrak{F}) \longrightarrow \mathcal{M}(S)$$

is onto.

In this theorem the goal is to construct a foliation with a prescribed set of separatricies. Once a curve S' is fixed in the topological class of S, the difficult part of the statement is to find a foliation $\mathfrak{F}'$ in the topological class of $\mathfrak{F}$. The idea is to unfold $\mathfrak{F}$ toward a foliation $\mathfrak{F}'$ with S' as separatricies. The proof given below is not complete because it is extremely technical, but my goal is only to convince you that the unfoldings are useful tools.

Let $E : (A, D) \to (\mathbb{C}^2, 0)$ be the reduction of singularities of $\mathfrak{F}$. We denote by $\mathcal{O}$ the sheaf of holomorphic functions on A with D as base space: for any point $p \in D$, the stack $\mathcal{O}_p$ is the set of germs of analytical functions defined in a neighborhood of p in A. Let $\mathfrak{M}$ be the subsheaf of $\mathcal{O}$ defined by the pre-image of the maximum ideal at the origin of $\mathbb{C}^2$. Finally, we denote by $\mathcal{T}S$ the sheaf of vector fields tangent to the transform of S by E.

Lemma 42. *For any integer n, the natural map induced by the inclusion of sheaves $\mathcal{T}\mathfrak{F} \subset \mathcal{T}S$*

$$H^1(D, \mathfrak{M}^n\mathcal{T}\mathfrak{F}) \longrightarrow H^1(D, \mathfrak{M}^n\mathcal{T}S)$$

is onto.

In a more concrete way, this lemma can be stated as follows: for any $1-$cocycle X_{ij} with values in $\mathfrak{M}^n\mathcal{T}S$ there exist a cocycle T_{ij} in $\mathfrak{M}^n\mathcal{T}\mathfrak{F}$

and a 0-cocycle X_i in $\mathfrak{M}^n\mathcal{T}S$ such that the following relation holds

$$X_{ij} = X_i + T_{ij} - X_j$$

Proof. Let ω be a 1-form with an isolated singularity that defines $\mathfrak{F}$ at the origin of $\mathbb{C}^2$. We consider the morphism $E^*\omega(\cdot)$ as a sheaf morphism from $\mathcal{T}S$ to $\mathcal{O}$ defined by

$$X \mapsto E^*\omega(X).$$

Thus, we get an exact sequence of sheaves

$$0 \to \mathcal{T}S \to \mathcal{T}\mathfrak{F} \to \mathcal{O}.$$

Let f be a reduced equation of the separatricies of S. We are going to prove that the image of $\mathcal{T}\mathfrak{F} \to \mathcal{O}$ is equal to $(E^*f)\,\mathcal{O}$. The proof can be done locally. Let p be a point of D where $E^*\mathfrak{F}$ is smooth. Let $x = 0$ a local reduced equation of D at p and y a transverse coordinate. There exists an integer m such that

$$E^*\omega = u(x,y)\,x^{m-1}dx \qquad E^*f = v(x,y)\,x^m.$$

Let X be a section $\mathcal{T}S$ at p. If the vector field X is written

$$X = a(x,y)\,x\frac{\partial}{\partial x} + b(x,y)\,\frac{\partial}{\partial y}$$

Then, one computes

$$E^*\omega(X) = u(x,y)\,x^m a(x,y) = \frac{u(x,y)}{v(x,y)}a(x,y)\,E^*f.$$

Thus, to reach any element $\alpha(x,y)\,E^*f$ of $(f \circ E)\,\mathcal{O}$, one can choose

$$X = \alpha(x,y)\,\frac{v(x,y)}{u(x,y)}x\frac{\partial}{\partial x}.$$

The proof is exactly the same at any type of point of D as singular points for $\mathfrak{F}$ or singular points of D. The basic properties used here is the correspondence between the multiplicities of $E^*\omega$ and E^*f which is ensured by the general kind hypothesis - see for instance [2].

Hence, we have the following exact sequence of sheaves

$$0 \to \mathcal{T}S \to \mathcal{T}\mathfrak{F} \to (f \circ E)\,\mathcal{O} \to 0.$$

Therefore, by multiplying it by $\mathfrak{M}^n$, we get

$$0 \to \mathfrak{M}^n \mathcal{T}S \to \mathfrak{M}^n \mathcal{T}\mathfrak{F} \to \mathfrak{M}^n (f \circ E)\, \mathcal{O} \to 0.$$

The long exact sequence in cohomology associated to short previous one is the sequence

$$\cdots \to H^1(D, \mathfrak{M}^n \mathcal{T}S) \to H^1(D, \mathfrak{M}^n \mathcal{T}\mathfrak{F}) \to H^1(D, \mathfrak{M}^n (f \circ E)\, \mathcal{O}) \to 0.$$

The last zero of this sequence comes from the fact that one can cover D with Stein open sest with no 3 by 3 intersections. Now we are going to prove that

$$H^1(D, \mathfrak{M}^n (f \circ E)\, \mathcal{O}) = 0.$$

First, we start with the proof that $H^1(D, \mathcal{O})$ is equal to 0. This is an induction on the length n of the reduction process. At step $n = 1$, it is a computation of Laurent series. The induction uses the Mayer-Vietoris sequence.

Case 1. $n = 1$. In this case, $D = \mathbb{CP}^1$ and D in cover by two Stein open sets U_1 and U_2 with Stein neighborhood in A denoted by $U_1 \subset \mathcal{U}_1$ and $U_2 \subset \mathcal{U}_2$. These two neighborhoods some admit systems of coordinates (x_1, y_1) and (x_2, y_2) such that $x_1 = y_2 = 0$ are local equations of the trace of D. Moreover, on $U_1 \cap U_2$, the change of coordinates is written

$$\begin{aligned} x_1 &= y_2 x_2 \\ y_1 &= \frac{1}{x_2} \end{aligned}$$

We can compute the cohomology group with this covering since the open sets U_1 and U_2 are Stein. Hence, we have the following identification

$$H^1(D, \mathcal{O}) \simeq \mathcal{O}(U_1 \cap U_2) \big/ \mathcal{O}(U_1) + \mathcal{O}(U_2)$$

Now, the module $\mathcal{O}(U_1 \cap U_2)$ is the set of series $\sum_{(i,j) \in \mathbb{N} \times \mathbb{Z}} a_{ij} x_1^i y_1^j$ while $\mathcal{O}(U_1)$ and $\mathcal{O}(U_2)$ correspond respectively to the series

$$\sum_{(i,j) \in \mathbb{N} \times \mathbb{N}} a_{ij} x_1^i y_1^j \quad \text{and} \quad \sum_{(i,j) \in \mathbb{N} \times \mathbb{N}} a_{ij} x_2^i y_2^j.$$

Therefore the cohomological equation to solve is

$$\begin{aligned}\sum_{(i,j)\in\mathbb{N}\times\mathbb{Z}} a_{ij}^{12}x_1^iy_1^j &= \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{ij}^{1}x_1^iy_1^j + \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{ij}^{2}x_2^iy_2^j \\ &= \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{ij}^{1}x_1^iy_1^j + \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} a_{ij}^{2}x_1^iy_1^{i-j}\end{aligned}$$

which can always be solved for any data of a family $\left\{a_{ij}^{12}\right\}_{ij}$. Therefore, we have the desired vanishing property

$$H^1\left(\mathbb{P}^1,\mathcal{O}\right)=0$$

Case 2. $n \implies n+1$. Let us consider the covering of D used in the proof of the theorem 36. The Mayer-Vietoris sequence associated to this covering yields the next exact sequence

$$\begin{aligned} 0 \to\ & H^0(D,\mathcal{O}) \to \\ & \underbrace{\bigoplus_{i\in I} H^0(U_i,\mathcal{O}) \bigoplus H^0(U_0,\mathcal{O}) \to \bigoplus_{i\in I} H^0(U_0\cap U_i,\mathcal{O}) \to}_{=H^1(D_0,\mathcal{O})=0} \\ & H^1(D,\mathcal{O}) \to \underbrace{\bigoplus_{i\in I} H^1(U_i,\mathcal{O}) \bigoplus H^1(U_0,\mathcal{O})}_{=0\text{ (hypothesis of induction)}}. \end{aligned}$$

Hence $H^1(D,\mathcal{O})=0$.

Now, the sheaf $\mathfrak{M}^n(f\circ E)$ is generated by its global sections. Therefore, we are finally led to

$$H^1(D,\mathfrak{M}^n(f\circ E)\,\mathcal{O})=0.$$

□

Let us denote by $\mathbf{G}^n(\mathfrak{F})$ the sheaf of automorphisms of $\mathfrak{F}$ that are equal to Id on D, that leave invariant each leaf of $\mathfrak{F}$ and that are tangent to Id with orders define by the ideal sheaf $\mathfrak{M}^n$: if Φ is a section at p of $G^n(\mathfrak{F})$ then the coordinates of $\Phi-\mathrm{Id}$ belong to $\mathfrak{M}^n$. Let us denote also by $\mathbf{G}^n(S)$ the sheaf of automorphisms of S that are equal to Id on D and that are tangent to Id with orders defined also by $\mathfrak{M}^n$.

Theorem 43. *Let $\mathfrak{F}$ be a general foliation and $S = \mathrm{Sep}(\mathfrak{F})$. Then, the natural map*

$$H^1\left(D, \mathbf{G}^1(\mathfrak{F})\right) \longrightarrow H^1\left(D, \mathbf{G}^1(S)\right)$$

is onto.

The proof of this result is based upon the following remark: let us consider a 1- cocycle $\{\Phi_{ij}\}_{i,j\in I^2}$ with values in $\mathbf{G}^1(S)$. Suppose that Φ_{ij} is the flow $e^{(1)X_{ij}}$[11] where $X_{ij} \in \mathfrak{M}^1\mathcal{T}S$. In view of the lemma 42, X_{ij} can be written

$$X_{ij} = X_i + T_{ij} - X_j.$$

Suppose that all these vector fields commute then

$$\Phi_{ij} = e^{(1)X_{ij}} = e^{(1)X_i} e^{(1)T_{ij}} \left(e^{(1)X_j}\right)^{(-1)}.$$

Thus Φ_{ij} is in the image of the map of the theorem. However, the cocycle Φ_{ij} may not be in a flow. Moreover, there is no reason for the vector fields appearing in the lemma 42 to commute. Actually, the proof is a way two overcome these two difficulties using the Campbell-Hausdorff formula[12]

Proof. Let us consider a 1-cocycle $\{\Phi_{ij}\}_{i,j\in I^2}$ with values in $\mathbf{G}^1(S)$. Let us consider a very big integer n. There exists a $1-$cocycle X_{ij} such that $\Phi_{ij} = e^{(1)X_{ij}} \circ \tilde{\Phi}_{ij}$ where $\tilde{\Phi}_{ij} \in \mathbf{G}^n(S)$. Actually, one can write

$$\Phi_{ij} = \mathrm{Id} + \sum_{\nu\geq 1} \left(\Phi_{ij}^{1\nu}, \Phi_{ij}^{2\nu}\right)$$

and take $X_{ij} = \sum_{\nu\geq 1}^{n} \Phi_{ij}^{1\nu} \frac{\partial}{\partial x} + \Phi_{ij}^{2\nu} \frac{\partial}{\partial y}$. Now let us consider, a cohomological relation given by the lemma 42 associated to X_{ij}

$$X_{ij} = X_i + T_{ij} - X_j.$$

[11] This is really an assumption, for there are diffeomorphisms that are not the flow of some vector fields

[12] See the first part.

Then, composing the flow of each above vector fields yields the next expression

$$e^{(1)X_{ij}} \circ e^{-(1)X_i} e^{-(1)T_{ij}} e^{(1)X_j} = e^{(1)\langle\langle\langle X_{ij}, -X_i\rangle, -T_{ij}\rangle, X_j\rangle}.$$

Now in view of the Campbell-Hausdorff formula, one has

$$\begin{aligned}\langle\langle\langle X_{ij}, -X_i\rangle, -T_{ij}\rangle, X_j\rangle &= \underbrace{X_{ij} - X_i + T_{ij} + X_j}_{=0} \\ &- \frac{1}{2}[X_{ij}, X_j] - \frac{1}{2}[X_{ij}, T_{ij}] + \frac{1}{2}[X_{ij}, X_j] + \cdots\end{aligned}$$

Moreover, this is an exercise to show the following inclusion:

$$[\mathfrak{M}^n \mathcal{T}S, \mathfrak{M}^m \mathcal{T}S] \subset \mathfrak{M}^{m+n} \mathcal{T}S.$$

Therefore, $\langle\langle\langle X_{ij}, -X_i\rangle, -T_{ij}\rangle, X_j\rangle$ belongs to $\mathfrak{M}^2 \mathcal{T}S$. Thus

$$e^{(1)X_{ij}} = e^{(1)X_i} e^{(1)T_{ij}} e^{-(1)X_j} \circ \phi_{ij}^2$$

where ϕ_{ij}^2 belongs to $\mathbf{G}^2(S)$. Then we do the same construction as before:

$$\begin{aligned}\phi_{ij}^2 &= e^{(1)X_{ij}^2} \circ \tilde{\phi}_{ij}^2 \\ X_{ij}^2 &= X_i^2 + T_{ij}^2 - X_j^2 \\ e^{(1)X_{ij}^2} &= e^{-(1)X_i^2} e^{-(1)T_{ij}^2} e^{(1)X_j^2} \circ \phi_{ij}^3.\end{aligned}$$

Thus, we are led to

$$\begin{aligned}e^{(1)X_{ij}} &= e^{(1)X_i} e^{(1)T_{ij}} e^{-(1)X_j} \circ \phi_{ij}^2 \\ &= e^{(1)X_i} e^{(1)T_{ij}} e^{-(1)X_j} e^{(1)X_i^2} e^{(1)T_{ij}^2} e^{-(1)X_j^2} \circ \phi_{ij}^3 \circ \tilde{\phi}_{ij}^2 \\ &= e^{(1)X_i + X_i^2} e^{(1)T_{ij} + T_{ij}^2} e^{(1) - X_j - X_j^2} \circ \underbrace{\Delta_{ij} \circ \phi_{ij}^3 \circ \tilde{\phi}_{ij}^2}_{\in \mathbf{G}^3(S)}\end{aligned}$$

and so on. With a trivial induction on n, one gets the following statement: for any n there exists a relation

$$\Phi_{ij} = e^{X_i} \circ e^{T_{ij}} \circ e^{X_j} \circ \Phi_{ij}^n$$

where $\Phi_{ij}^n \in \mathbf{G}^n(S)$. To finish the proof, we use a stability argument which is the lemma 44 given just below: it ensures that for n great enough, the cocycles $e^{X_i} \circ e^{T_{ij}} \circ e^{X_j} \circ \Phi^n_{ij}$ and $e^{X_i} \circ e^{T_{ij}} \circ e^{X_j}$ are equivalent as cocycles. Therefore, Φ_{ij} is equivalent to $e^{T_{ij}}$ as cocycle, which is the theorem. □

Lemma 44. ***(Stability)*** [18] *For n big enough, the natural map*

$$H^1(D, \mathbf{G}^n(\mathfrak{F})) \to H^1\left(D, \mathbf{G}^1(\mathfrak{F})\right)$$

is the trivial map.

This lemma means more ore less that any unfolding tangent enough to the trivial unfolding is actually analytically trivial. Its infinitesimal counterpart is the following result: for n great enough the map $H^1(D, \mathfrak{M}^n \mathcal{T}\mathfrak{F}) \to H^1(D, \mathcal{T}\mathfrak{F})$ is the zero map. This result is a corollary of a more general statement that one can find in [3].

The cohomological part of the proof is established. Now, let us present the geometric part of the proof: let S' be a curve topologically equivalent to $S = \text{Sep}(\mathfrak{F})$. It is known that in this situation, the process of reduction of S' and S have the same topological type. In particular, they have the same dual graph [23]. Let us consider a bijection σ between there dual graphs. Let us denote by D the exceptional divisor of the reduction process of $\mathfrak{F}$. For any $d \subset D$ irreducible, we will denote by

$$\Sigma_d(\mathfrak{F}) = \Sigma_d(S)$$

the set of singular point of $E^*S \cup D$. The proof of the result consists in unfolding the foliation $\mathfrak{F}$ toward a foliation with S' as separatricies: we make this construction with two consecutives unfoldings.

(1) First, we move the of $\Sigma_d(S)$ to put them in the same position as $\Sigma_{\sigma(d)}\left(S'\right)$.
(2) Then, we deform the separatricies obtained after the first step to reach exactly S'. The cohomological properties established before are going to be used here.

All these deformations have to be unfoldings. The first result we use is due to M. Seguy [19]:

Theorem 45. *There exists an unfolding* $\mathbf{F}$ *of* $\mathfrak{F}$ *with parameter in* $t \in U$ *that is open neighborhood of* $\overline{\mathbb{D}}$ *such that*

(1) $\mathbf{F}_0 = \mathfrak{F}$
(2) *for any irreducible component* d *of the exceptional divisor of the reduction* $\mathbf{F}_1$ *there exists a biholomorphism* $h_d : d \to \sigma(d)$ *such that* $h_d(\Sigma_d(\mathbf{F}_1)) = \Sigma_{\sigma(d)}\left(S'\right)$.

To prove this result, let us first mention the following lemma.

Lemma 46. *Let* $a^1(s), \cdots, a^n(s)$ *be a family of functions with values in* $\mathbb{CP}^1$ *such that* $a^i(s) \neq a^j(s)$ *for any* s. *There exists a* $\mathcal{C}^\infty$ *diffeomorphism from* $M = \widetilde{\mathbb{C}^2} \times \overline{\mathbb{D}}$ *to itself that sends the graph of* $a^i(s) \in \mathbb{CP}^1 \subset \widetilde{\mathbb{C}^2}$ *onto the horizontal line* $a^i(0) \times \overline{\mathbb{D}}$ *and that is holomorphic near each graph of* a^i.

Proof. We can cover M with a finite number of open set U_i $i = 1 \ldots n+1$ such that for $i = 1 \ldots n$ U_i is a small neighborhood of S_i with $U_i \cap U_j = \emptyset$. The open set U_{n+1} is a neighborhood of the complementary of $\cup_{i=1}^n U_i$ that does not meet any graph S_i. On each U_i it is possible to find an holomorphic vector field X_i tangent to S_i such that $\pi^* X_i = \frac{\partial}{\partial t}$ where $t \in \overline{\mathbb{D}}$. Using a unity partition adapted to the chosen covering, one can glue the X_i's in a $\mathcal{C}^\infty$ vector field X that is tangent to the curves S_i and projected on $\frac{\partial}{\partial t}$. The $\mathcal{C}^\infty$ diffeomorphism $(z,t) \to \left(e^{(t)X}(z,0), t\right)$ the desired properties. □

Using the lemma, we are going to perform the proof of the theorem 45 is the case of one blowing-up. The general case is a simple induction on the length of the process.

Proof. (Theorem 45). Suppose that $\mathfrak{F}$ is reduced after one blowing-up and let consider a set B of points distinct from $\Sigma_d(\mathfrak{F})$. Let us consider the manifold M and the diffeomorphism Φ build in the previous lemma applied to graphs S_i that links the set of points $\Sigma_d(\mathfrak{F})$ and B. The manifold M is foliated by $\mathfrak{F} \times \overline{\mathbb{D}}$. Let us consider a system of foliated charts on the complementary of the singular locus Sing $\left(\mathfrak{F} \times \overline{\mathbb{D}}\right)$:

$$\left(\zeta_1^i, \zeta_2^i, t\right)$$

such that the foliation is given locally by $\zeta_2 = \text{cst}$. We complete this system of charts with any chart $(\zeta_1^c, \zeta_2^c, t)$ in the neighborhood of the singular points c of Sing $\left(\mathfrak{F} \times \overline{\mathbb{D}}\right)$. Finally, we fix a transverse fibration $\rho : M \to \mathbb{CP}^1 \times \overline{\mathbb{D}}$. Let us consider the following system of charts on M

$$\tilde{M} : \begin{cases} \left(\zeta_1^i \circ \Phi \circ \rho, \zeta_2^i, t\right) & \text{in } M \text{ minus the singular locus} \\ (\zeta_1^c, \zeta_2^c, t) \circ \Phi & \text{else} \end{cases}.$$

One can verify that this system induces a complex structure on M, equips this manifold with a foliation still given by the equation $\zeta_2^i = \text{cst}$ and that the application $\Phi\left|_{\mathbb{CP}^1 \times \overline{\mathbb{D}}}\right.$ becomes a biholomorphism. Using the Grauert theorem [11, 9] on the uniqueness of the complex structure of a neighborhood of $\mathbb{CP}^1$ of self-intersection -1, we obtain that the manifold $\tilde{M}$ is biholomorphic to $\widetilde{\mathbb{C}^2} \times \overline{\mathbb{D}}$ equipped with the standard complex structure. We denote by G this biholomorphism. If we contract $\widetilde{\mathbb{C}^2} \times \overline{\mathbb{D}}$, we obtain a foliation $\mathbf{F}$ and an application $h_d = G \circ \Phi\Big|_{\mathbb{CP}^1 \times \{1\}}$ that satisfy the desired properties. The proof of the general case is an induction on the length of the reduction process. □

To finish the proof of the theorem 41, we make the following construction. According to the above theorem of Seguy, we can suppose that for any component d of the exceptional divisor of the reduction of $\mathfrak{F}$ - here we take a process of blowing-up that is actually a tree of finite determinacy for S using the theorem 40- there exists a biholomorphism $h_d : d \to \sigma(d)$ that conjugates the singular locus. Each map h_d can be extended in a small neighborhood of d and sends the trace of $D \cup S$ on the trace of $D' \cup S'$.

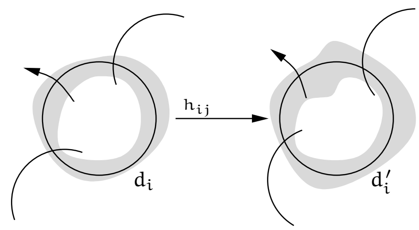


FIGURE 6.1. Semi-local conjugacy between the trees.

We still denote this extended map h_d. Thus, the family of applications defined by

$$h_{ij} = h_{d_i} \circ h_{d_j}^{(-1)}$$

is a cocycle that satisfies the following construction: if (A, D) is the tree of reduction of $\mathfrak{F}$ and $\left(A', D'\right)$ the one of S' then

$$\left(A', D'\right) \simeq \coprod_i U_i \Big/ x \sim h_{d_i} \circ h_{d_j}^{(-1)}(x)$$

We can suppose that the cocycle $\left\{h_{d_i} \circ h_{d_j}^{(-1)}\right\}_{i,j}$ belongs to $\mathbf{G}^1(S)$: we will omit the proof of this fact for it is technical and not very relevant. Therefore, the theorem is a consequence of the proposition 43: indeed, in view of this proposition, there exist a 1-cocycle $\{\phi_{ij}\}_{i,j} \in \mathbf{G}^1(\mathfrak{F})$ and a 0- cocycle $\{\psi_i\}_i \in \mathbf{G}^1(S)$ such that

$$h_{d_i} \circ h_{d_j}^{(-1)} = \psi_i \circ \phi_{ij} \circ \psi_j^{(-1)}.$$

Hence, we have the following isomorphism

$$\coprod_i U_i \Big/ x \sim h_{d_i} \circ h_{d_j}^{(-1)}(x) \overset{\{\psi_i\}_i}{\simeq} \coprod_i U_i \Big/ x \sim \phi_{ij}(x).$$

Since the cocycle $\{\phi_{ij}\}_{i,j}$ lets invariant each leaf of the foliation $\mathfrak{F}$, the manifold $\left(A', D'\right)$ admits a foliation defined by

$$\coprod_i \mathfrak{F}|_{U_i} \Big/ x \sim \phi_{ij}(x).$$

According to [1],the map Φ_{ij} is a flow: actually, it can be written

$$\Phi_{ij}(x) = e^{(\tau(x))X_{ij}} \cdot x$$

where τ is a function. The Grauert theorem and the finite determinacy for curve ensure that this foliation can be pushed down in an holomorphic foliation in a neighborhood of $\left(\mathbb{C}^2, 0\right)$ with a set of separatricies analytically equivalent to $S^{'}$. Moreover, this foliation is topologically equivalent to $\mathfrak{F}$ as it is an unfolding of $\mathfrak{F}$ defined by

$$\epsilon \in [0,1] \longrightarrow \coprod_i \mathfrak{F}|_{U_i} \Big/ x \sim e^{(\epsilon\tau(x))X_{ij}} \cdot x.$$

7. Exercises.

Exercise 1. Prove the lemma 1.

Exercise 2. Let X be a germ of vector field in $(\mathbb{C}^n, 0)$ with $X(0) = 0$. Show that for any $r > 0$ there exists an neighborhood of 0 such that the flow $(t, x) \to e^{(t)X}$ is defined for $(t, x) \in \mathbb{D}_r \times U$.

Exercise 3. We consider the foliation given by the one form $\omega = x^2 dy - (y - x)\, dx$. Show that $\{x = 0\}$ is the only separatrix.

Exercise 4. Show that the multiplicity as defined in 1.1 is an analytical invariant: for any $\phi \in \text{Diff}\left(\mathbb{C}^2, 0\right)$ we have $\nu_0(\phi^* \mathfrak{F}) = \nu_0(\mathfrak{F})$.

Exercise 5. $(*)$ Proof the formula of Dynkin.

Exercise 6. Give an explicit expression of the topological conjugacy constructed in the lemma 7.

Exercise 7. In this exercise, we propose an alternative proof of the linearization of the holonomy map $f = h_{\mathfrak{F},T}$.

(1) Show that we can suppose, $\left|f''(0)\right| < 2\epsilon\, |\eta|\, (1 - |\eta|)$ for some ϵ. Deduce that on a small disc around 0 we have $|f(z)| < \frac{|\eta||z|}{1-\epsilon(1-|\eta|)|z|}$.

(2) Compute the n^{th} iterative composition of $x \to \frac{|\eta| x}{1-\epsilon(1-|\eta|)x}$. Deduce that the family of holomorphic functions $\frac{f^{(n)}(z)}{\eta^n}$ is normal.

(3) Using the theorem of Montel[13] on the Cesaro sum $\frac{1}{n}\sum_{n=1}^{N}\frac{f^{(n)}(z)}{\eta^n}$, prove the linearization result.

Exercise 8. Let consider the deformation $\omega_t = xdy + tydx$. Show that this deformation cannot be induced by an unfolding.

Exercise 9. Let $\mathfrak{F}$ be the foliation defined by $\omega = x^3 d\left(\frac{y^2-x^3}{x^2}\right)$.

(1) Compute the reduction of singularities of ω.
(2) Let $\mathbf{F}$ be defined by $\Omega = \omega + x^3 dt$. Show that $\mathbf{F}$ is an unfolding in the sense of the definition 9. Show that $\mathbf{F}$ is not equisingular.[14]

Exercise 10. Compute the reduction of the singularity of the double cusp $\omega = d\left(\left(x^3-y^2\right)\left(x^2-y^3\right)\right)$. More generally, describe the reduction of the singularities of $\omega = d\left(x^p - y^q\right)$ where $p \wedge q = 1$[15]

Exercise 11. Write down explicitly the proof of the proposition 18.

Exercise 12. Show that the operators defines in 4.1 satisfies $\delta^k \circ \delta^{k-1} = 0$.

Exercise 13. Prove the theorem 27 with the Snake lemma applied to the following commutative diagram

$$\begin{array}{ccccc}
C^0(M,S) & \to & C^0\left(M,S'\right) & \to & C^0\left(M,S''\right) \\
\downarrow & & \downarrow & & \downarrow \\
C^1(M,S) & \to & C^1\left(M,S'\right) & \to & C^1\left(M,S''\right) \\
\downarrow & & \downarrow & & \downarrow \\
C^2(M,S) & \to & C^2\left(M,S'\right) & \to & C^2\left(M,S''\right) \\
\vdots & & \vdots & & \vdots
\end{array}.$$

[13] See for instance *Complex and real analysis* of W. Rudin.

[14] Look at the transversality property of $E^*\mathbf{F}$ where E is the common reduction of all the $\mathbf{F}_t$'s.

[15] Begin with $p = 2$ and $q = 3$. For the general case, introduce the Euclid algorithm between p and q.

Exercise 14. Show that the class of the cocycle defined by the formula 5.2 depends only on the class of $\{\Phi_{ij}\}$ and not on the choice of the representative element.

Exercise 15. Prove the remark 34.

Exercise 16. $(*)$ For the double cusp in the example 38, prove that the dimension of the moduli space is one using the formula 36. Show that the unfolding defines by $\Omega = d\left((1+t)\left(x^3-y^2\right)\left(x^2-y^3\right)\right)$ for $t \in (\mathbb{C}, 0)$ is universal.

Exercise 17. $(**)$ Suppose that ω admits an non analytically trivial unfolding Ω with one parameter such that the induced deformation Ω_t is analytically trivial as a deformation. Show that ω admits an integral factor, i.e., there exists a function f such that $d\left(\frac{\omega}{f}\right) = 0$.

Exercise 18. If $S = \mathbb{CP}^1 \backslash \{0, 1, \infty\}$, describe $\text{Aut}(S)$ and $S/\text{Aut}(S)$[16].

Exercise 19. $(*)$ Prove the following result of Mather: if f and g are two germs of analytical function in $\left(\mathbb{C}^2, 0\right)$ with an isolated singularity at 0 such that $f - g \in (x, y)^N$ for N great enough, then they analytically equivalent, i.e., there exists a germ of conjugacy Φ such that $f = g \circ \Phi$.

(1) Show that for N great enough $(x, y)^N \subset \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.
(2) Let us consider $F = tf + (1-t)g$. Prove that $\frac{\partial F}{\partial t} \in \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$.[17] Deduce that there exists a germ of vector field $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$ such that $dF(X) = 0$.
(3) Conclude.

[16] $\text{Aut}(S)$ is finite.
[17] Use the Nakayama lemma, see for instance *Algebra* of E. Artin.

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