Rigidity for dicritical germ of foliation in $\mathbb{C}^2$.

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Abstract

Considering the problem of moduli for a germ of singular holomorphic foliation in $\mathbb{C}^2$ leads naturally to point out some of its analytical invariants: the reduction of its singularities and the collection of its projective holonomy representations. It is of interest to know whether any *coherent* data of these invariants can be realized in a concrete foliation. The aim of this paper is to provide the infinitesimal obstructions to this problem for dicritical foliations, in sharp contrast with the non-dicritical case previously studied in [2].

Contents

Introduction and main statements.

A *germ of holomorphic foliation* $\mathcal{F}$ in $\mathbb{C}^2$ is the data up to multiplication by a unity $u \in \mathcal{O}^*$ of a germ of holomorphic 1-form at $0 \in \mathbb{C}^2$

$$\omega = a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y \tag{1}$$

where $a, b \in \mathbb{C}\{x,y\}$. A *separatrix* is a germ of analytical irreducible curve S such that $S \backslash \{0\}$ is a union of leaves of $\mathcal{F}$ and a foliation is *non-dicritical* when it has finitely many separatrix. From now on, $E : (\mathcal{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ denotes the process of reduction of singularities of $\mathcal{F}$ (see [6] for precise definition and existence) where $\mathcal{D}$ refers to the exceptional divisor $E^{-1}(0)$. By definition, the singularities along $\mathcal{D}$ of the pull-back $E^*\mathcal{F}$ are *reduced* so that, under some local change of coordinates, they belong to the following list

1. $\lambda x \mathrm{d}y + y\mathrm{d}x + \cdots$ terms of higher order, $\lambda \notin \mathbb{Q}^-$,
2. $x\mathrm{d}y + \ldots$ terms of higher order.

In the second case, the singularity is said to be a *saddle-node*. The formal normal form for such a singularity is given in [7]: up to a formal change of coodinates, the singularity is given by the following 1-form

$$(\zeta x^p - p)y\mathrm{d}x + x^{p+1}\mathrm{d}y, \quad p \in \mathbb{N}^*, \zeta \in \mathbb{C}.$$

The invariant curve $\{x = 0\}$ is the *strong invariant curve* and $\{y = 0\}$ the *weak* one. If the germ of the divisor $\mathcal{D}$ is the weak invariant curve, then the singularity is said to be a *tangent saddle-node*. Following [7], we recall that $\mathcal{F}$ is *in the second class* when none of the singularities of $E^*\mathcal{F}$ are tangent saddle-nodes.

Theorem (Realization theorem, [2]). *Let $\mathcal{F}_0$ be a germ of* **non-dicritical** *foliation in the second class and $E_0 : (\mathcal{M}_0, \mathcal{D}_0) \to (\mathbb{C}^2, 0)$ its process of reduction. Let E_t be any topologically trivial deformation of E_0. Then, there exists an isoholonomic deformation $\mathcal{F}_t$ of $\mathcal{F}_0$ such that E_t is precisely the process of reduction of $\mathcal{F}_t$.*

Roughly speaking, an isoholonomic deformation is a topologically trivial deformation, which lets invariant the projective holonomy representations (see [5][2] for more details). Hence, this theorem ensures the existence of a germ of foliation with process of reduction prescribed provided that the data of its projective holonomy representations comes from a non-dicritical foliation. This work is intended as an attempt to show how the above statement strongly depends on the non-dicritical assumption.

We follow [4] in calling $\mathcal{M}$-*simple* a germ of foliation, regular after blowing-up the origin whose closure of any invariant curve is an analytic curve passing through the origin. In this situtation, there is a finite number of invariant curves, called the *exceptional leaves*, which are tangent to exceptional projective line arising from the blowing-up. Any other invariant curve is transversal to the divisor. Thus, a $\mathcal{M}$-simpe foliation is dicritical.
We can now formulate our main result

Theorem. *Let $\mathcal{F}$ be a $\mathcal{M}$-simple foliation with at least four exceptional leaves. Then, there exists a topologically trivial deformation of the process of reduction of $\mathcal{F}$, which cannot be realized as an isoholonomical deformation of $\mathcal{F}$.*

The proof is based on the concept of *balanced equation of the separatrix* developed in the first section. Second section establishes the existence of infinitesimal obstructions in the realization problem for dicritical foliations in the second class: the basic ingredient is the balanced equation of separatrix. Last section contains the proof of the above result.

1 Dicritical foliations in the second class.

1.1 Multiplicity of a dicritical foliation.

In this section, we establish a link between the multiplicity

$$\nu_0(\mathcal{F}) = \min(\text{ord}(a), \text{ord}(b)),$$

$\mathcal{F}$ given by (1), and some numerical invariants produced by the process of reduction of $\mathcal{F}$.

The valence $v(D)$ of an irreducible component D of $\mathcal{D}$ is the number of irreducible components of $\mathcal{D}$, which intersect D. The integer $v_{\bar{d}}(D)$ refers to *the non-dicritical valence*, which is the number of non-dicritical components intersecting D. In what follows, $\mathfrak{M}$ stands for the sheaf $\mathcal{O}$-modules generated by the global sections E^*h with $h \in \mathcal{O}_2$ and $h(0) = 0$. It is a simple matter to get the following decomposition

$$\mathfrak{M} = \mathcal{O}\left(-\sum_{D\in\text{Comp}(\mathcal{D})} \nu(D)D\right),$$

where $\text{Comp}(\mathcal{D})$ refers to the set of irreducible components of $\mathcal{D}$. In this way, we obtain an integer $\nu(D)$ that is known as *the multiplicity of* D. This is also the multiplicity of a curve whose strict transform by E is smooth and attached to a regular point of D.

The following definition is introduced in [3]:

Definition 1.1. *Let $\mathcal{F}$ be a germ of foliation given by a 1-form*

$$\omega = a(x, y)dx + b(x, y)dy$$

1. *Let (S, p) be a germ of smooth invariant curve. If, in some coordinates, S is the curve $\{y = 0\}$ and p the point $(0, 0)$, then the integer* $\text{ord}_0 b(x, 0)$ *is called the indice of $\mathcal{F}$ at p with respect to S and is denoted by* $\text{Ind}(\mathcal{F}, S, p)$.
2. *Let (S, p) be a germ of smooth non-invariant curve. If, in some coordinates, S is the curve $\{y = 0\}$ and p the point $(0, 0)$, then the integer* $\text{ord}_0 a(x, 0)$ *is called the tangency order of $\mathcal{F}$ with respect to S and is denoted* $\text{Tan}(\mathcal{F}, S, p)$.

The following equality is proved in [3] and specializes to a result of [1] if $\mathcal{F}$ is non-dicritical.

Proposition 1.1 ([3]). *The multiplicity of $\mathcal{F}$ satisfies the equality*

$$\nu_0(\mathcal{F}) + 1 = \sum_{D\in\text{Comp}(d)} \nu(D)\rho(D)$$

where

1. *if D is non-dicritical,* $\rho(D) = -v_{\bar{d}}(D) + \sum_{q\in D} \text{Ind}(E^*\mathcal{F}, D, q)$.
2. *if D is dicritical,* $\rho(D) = 2 - v_{\bar{d}}(D) + \sum_{q\in D} \text{Tan}(E^*\mathcal{F}, D, q)$.

Note that this formula still holds if E is any morphism composed of blowing-up.
We will consider a notion of *balanced equation* defined as follows. A separatrix of $\mathcal{F}$ is said to be *isolated* if its strict transform by E is attached to a non-dicritical component. When D is dicritical, we call *pencil of* D the set of germs of invariant curves of $\mathcal{F}$ in $(\mathbb{C}^2, 0)$, whose strict transform is attached to D.

Definition 1.2. *A complete system of separatrix is the union of two germs of curves $Z \cup P$ where*

1. *Z is the union of isolated separatrix and, for each dicritical component D with valence smaller than 2, $2 - v(D)$ curves of the pencil of D.*
2. *P is the union of $v(D) - 2$ curves of the pencil of each dicritical component D with valence bigger than 3.*

A balanced equation of the separatrix is a germ of meromorphic function whose zeros and poles are respectively Z and P.

Note that the notion of balanced equation specializes to the standard equation of separatrix if $\mathcal{F}$ is non-dicritical.

Let us denote by $\mathcal{SNT}(\mathcal{F})$ the set of tangent saddle-node singular points of $E^*\mathcal{F}$.

Proposition 1.2. *Given any balanced equation of the separatrix F, we have*

$$\nu_0(F) = \nu_0(\mathcal{F}) + 1 + \sum_{s \in \mathcal{SNT}(\mathcal{F})} \sum_{D \in V(s)} \nu(D) \left(\mathrm{Ind}(E^*\mathcal{F}, D, s) - 1\right)$$

where $V(s)$ refers to the set of irreducible components containing the point s. In particular, the multiplicity of F does not depend on the choice of Z and P.

Proof: Let us write $F = \dfrac{N}{P}$ where N and P are holomorphic. The foliations dN and dP are reduced by the morphism E. Applying (1.1) to $\mathcal{F}$, dN and dP yields

$$\nu_0(\mathcal{F}) + 1 = \sum_{D \in \mathrm{Comp}(d)} \nu(D)\rho(D), \tag{2}$$

$$\nu_0(dN) + 1 = \sum_{D \in \mathrm{Comp}(d)} \nu_N(D)\rho_N(D), \tag{3}$$

$$\nu_0(dP) + 1 = \sum_{D \in \mathrm{Comp}(d)} \nu_P(D)\rho_P(D). \tag{4}$$

Since the multiplicities of the components only depend on E, for any component D of $\mathcal{D}$

$$\nu(D) = \nu_N(D) = \nu_P(D).$$

Now, when D is non-dicritical for $\mathcal{F}$, proposition (**??**) leads to

$$\rho(D) = \mathrm{Iso}(D) + \sum_{q \in D \cap \mathcal{SNT}(\mathcal{F})} (\mathrm{Ind}(E^*\mathcal{F}, D, q) - 1),$$

where $\mathrm{Iso}(D)$ is the number of isolated separatrix attached to D. If D is dicritical, $\nu_{\overline{d}}(D) = v(D)$ and $E^*\mathcal{F}$ is transversal to D. It follows that $\rho(D) = 2 - v(D)$. By definition, if D is non-dicritical, $\rho_N(D)$ is equal to $\mathrm{Iso}(D)$. Moreover, if D is dicritical with $v(D) \leq 2$, $\rho_N(D)$ is equal to $2 - v(D) = \rho(D)$. On any other component, the foliation $\{\mathrm{d}N = 0\}$ does not have any isolated separatrix and $\rho_N(D)$ vanishes. Substituting these equalities into (3) gives

$$\begin{aligned}\nu_0(N) = \nu_0(\mathrm{d}N) + 1 = &\sum_{\substack{D \in \mathrm{Comp}(d) \\ D \text{ non-dicritical or} \\ D \text{ dicritical and } v(D) \leq 2}} \nu(D)\rho(D) \quad + \\ &\sum_{\substack{D \in \mathrm{Comp}(d) \\ D \text{ non-dicritique}}} \nu(D) \sum_{q \in D \cap \mathcal{SNT}(\mathcal{F})} (\mathrm{Ind}(E^*\mathcal{F}, D, q) - 1). \quad (5)\end{aligned}$$

Since $E^*\mathcal{F}$ is reduced, for any point q in a dicritical component

$$\mathrm{Tan}(E^*\mathcal{F}, D, q) = 0.$$

Hence, for any dicritical components with valence greater than 3, the equality $\rho_P(D) = -\rho(D)$ holds. If not, $\rho_P(D)$ vanishes. Proceeding analogously to (5), we obtain

$$\nu_0(P) = \nu_0(dP) + 1 = -\sum_{\substack{D \in \mathrm{Comp}(d) \\ D \text{ dicritical and } v(D) \geq 3}} \nu(D)\rho(D). \quad (6)$$

The proposition is the combination of relations (5) and (6).

□

1.2 Dicritical foliation in the second class.

Definition 1.3. *$\mathcal{F}$ is in the second class when none of the singularities of $E^*\mathcal{F}$ are tangent saddle-nodes.*

The proposition to come follows easily of (1.2):

Proposition 1.3. *Let F be a balanced equation of $\mathcal{F}$. Then $\mathcal{F}$ is in the second class if and only if $\nu_0(F) = \nu_0(\mathcal{F}) + 1$.*

Our criterion agrees with the classical one for non-dicritical foliation [7]. The balanced equation of the separatrix of a dicritical foliation seems to be of independent interest. In what follows, we use it to study the realization problem in the dicritical class.

2 Analysis of dicritical obstructions.

2.1 Space of moduli of infinitesimal deformations.

The basic geometric ingredient of the proof of the realization theorem (see the introduction) is an infinitesimal level result, which is an easy consequence of the existence of the following exact sequence of sheaves on $\mathcal{M}$

$$0 \to \mathcal{X}_{\mathcal{F}} \to \mathcal{X} \to \mathcal{O}_{\mathcal{M}} \to 0. \tag{7}$$

Here, the fibre of $\mathcal{X}$ is the space of vector fields tangent to the total transform by E of the separatrix of $\mathcal{F}$ at the origin of $\mathbb{C}^2$. The sheaf $\mathcal{X}_{\mathcal{F}}$ is the sub-sheaf of $\mathcal{X}$ whose fibre is the space of vector field tangent to the foliation $E^*\mathcal{F}$. These two sheaves respectively correspond to the space of moduli $H^1(\mathcal{X})$ of infinitesimal deformations of $\mathcal{M}$ and to the space of moduli $H^1(\mathcal{X}_{\mathcal{F}})$ of infinitesimal isoholonomic deformations [5]. Since the first cohomology group of $\mathcal{O}_{\mathcal{M}}$ is trivial, one has the following sequence

$$H^1(\mathcal{X}_{\mathcal{F}}) \to H^1(\mathcal{X}) \to 0$$

which is the starting point of the proof of the realization theorem.

In this section, we establish an analogue of the sequence (7) for dicritical foliations.

Proposition 2.1. *The following conditions are equivalent:*

1. *$\mathcal{F}$ is in the second class.*
2. *Let F be a balanced equation for $\mathcal{F}$. Let us denote by Z_0 and Z_∞ the respective strict transforms by E of the curves $\{F = 0\}$ and $\{F = \infty\}$. Let $\mathcal{X}_{Z_0}$ be the sheaf of vector fields tangent to the divisor $\mathcal{D}$ and to Z_0. The sequence of sheaves*

$$0 \longrightarrow \mathcal{X}_{\mathcal{F}} \longrightarrow \mathcal{X}_{Z_0} \xrightarrow{E^*\frac{\omega}{F}(\cdot)} \mathcal{O}\left(-Z_\infty\right) \longrightarrow 0$$

is exact.

Let us first examine the multiplicity of the blown-up balanced equation along any irreducible component of the divisor

Lemme 2.1. *For any component D, we have the following alternative:*

1. *if D is non-dicritical, $\nu_D(F) = \nu_D(\mathcal{F}) + 1$,*
2. *if D is dicritical, $\nu_D(F) = \nu_D(\mathcal{F})$,*

Here and subsequently, $\nu_D(\mathcal{F})$ refers to the multiplicity of the blown-up $E^*\omega$ at a generic point of D where ω defines $\mathcal{F}$ in $(\mathbb{C}^2, 0)$.

Proof: The proof is an induction on the height of the component D in the blowing-up process. If D_0 is the exceptional projective line arising from the blowing-up of the origin, then $\nu_{D_0}(F)$ and $\nu_0(F)$ are equal; if D_0 is non-dicritical, $\nu_{D_0}(\mathcal{F}) = \nu_0(\mathcal{F})$ else $\nu_{D_0}(\mathcal{F}) = \nu_0(\mathcal{F}) + 1$. Therefore, the lemma at height 1 is the proposition (1.3). Assume the formula holds for height i and consider a component D of $\mathcal{D}^{i+1}$

obtained by the blowing-up E^i of the point $c \in \mathcal{D}^i$. Let F^i be the divided blown-up[1] of F by E^{*i}. Writing the expression of the blowing-up at c yields

$$\nu_D(\mathcal{F}) = \nu_c(E^{i*}\mathcal{F}) + \sum_{D_c \in \mathrm{V}(c)} \nu_{D_c}(\mathcal{F}) + \epsilon(D), \tag{8}$$

$$\nu_D(F) = \nu_c(F^i) + \sum_{D_c \in \mathrm{V}(c)} \nu_{D_c}(F). \tag{9}$$

Here and from now on, $\epsilon(D)$ is 0 if D is non-dicritical and 1 else. We recall that $V(s)$ refers to the set of irreducible components of $\mathcal{D}$ that contain s.

1. $V(c)$ **consists of one component** D_0**.**

   (a) D_0 **is non-dicritical**: let F_c be the germ of meromorphic function near the point c product of F^i and of a germ of equation of D_0. By definition, F_c is a balanced equation for $E^{i*}\mathcal{F}$. Proposition (1.3) gives

   $$\nu_c(F_c) = \nu_c(E^{i*}\mathcal{F}) + 1. \tag{10}$$

   Now, the construction ensures the equality

   $$\nu_c(F_c) = 1 + \nu_c(F_c^i). \tag{11}$$

   Moreover, the induction hypothesis shows the relation

   $$\nu_{D_0}(F) = \nu_{D_0}(\mathcal{F}) + 1. \tag{12}$$

   The relations (8), (9), (10), (11) and (12) give the result for the component D:

   $$\nu_D(F) = \nu_D(\mathcal{F}) + 1 - \epsilon(D).$$

   (b) D_0 **is dicritical**: in that case, one has to choose for F_c the germ F_c^i. Once again, F_c is a balanced equation for $E^{i*}\mathcal{F}$. Hence, $\nu_c(F_c)$ is equal to $\nu_c(E^{i*}\mathcal{F}) + 1$. Therefore, one gets the relation $\nu_c(F_c) = \nu_c(F_c^i)$. Under the induction hypothesis, $\nu_{D_0}(F)$ and $\nu_{D_0}(\mathcal{F})$ are equal. Combining the previous relations yields

   $$\nu_D(F) = \nu_D(\mathcal{F}) + 1 - \epsilon(D),$$

   D_0 being dicritical, $\epsilon(D)$ vanishes.

2. $V(c)$ **consists of two components** D_0 **and** D_1**.**

   (a) D_0 **and** D_1 **are non-dicritical**: let F_c be the product of F^i and of a germ of equation for $D_0 \cup D_1$. Since the function F_c is a balanced equation for $E^{i*}\mathcal{F}$, $\nu_c(F_c)$ is equal to

[1] $F^i = \dfrac{E^{*i}F}{h^m}$ where m is the bigger power of an equation h of the local divisor, which divides $E^{*i}F$.

$\nu_c(E^{i*}\mathcal{F})+1$. Now, in view of our construction, we have the equality $\nu_c(F_c) = 2 + \nu_c(F_c^i)$. Moreover, for any $D_c \in V(c)$, $\nu_{D_c}(F) = \nu_{D_c}(\mathcal{F}) + 1$. These equalities ensure that

$$\nu_D(F) = \nu_D(\mathcal{F}) + 1 - \epsilon(D).$$

(b) D_0 **is dicritical and** D_1 **is non-dicritical**: this case may be handled in much the same way.

□

Proof: (2.1) Let $\mathcal{F}$ be in the second class. It is clear that the kernel of $E^*\frac{\omega}{F}(\cdot)$ is $\mathcal{X}_{\mathcal{F}}$. It remains to prove that $E^*\frac{\omega}{F}(\cdot)$ is onto $\mathcal{O}(-Z_\infty)$. In what follows, giving a local expression of $E^*\frac{\omega}{F}$ thanks to the lemma (2.1), we solve the equation $E^*\frac{\omega}{F}(X) = g$ for g local section of $\mathcal{O}(-Z_\infty)$. Throughout the proof, u stands for a germ of unity in $\mathcal{O}^*$.

1. At a regular point of a non-dicritical component, which is neither a zero nor a pole of F:

$$E^*\frac{\omega}{F} = \frac{u}{x}\mathrm{d}x, \quad X = \frac{g}{u}x\frac{\partial}{\partial x}, \quad \mathcal{O}(-Z_\infty) = \mathcal{O}.$$

2. At a point of a dicritical component, which is neither a zero nor a pole of F:

$$E^*\frac{\omega}{F} = u\mathrm{d}y, \quad X = \frac{g}{u}\frac{\partial}{\partial y}, \quad \mathcal{O}(-Z_\infty) = \mathcal{O}.$$

3. At a singular point of a non-dicritical component or at singular point of $\mathcal{F}$, zero of F:

$$E^*\frac{\omega}{F} = u\frac{\mathrm{d}x}{x} + v\frac{\mathrm{d}y}{y}, \quad X = \frac{gx}{u}\frac{\partial}{\partial x}, \quad \mathcal{O}(-Z_\infty) = \mathcal{O}.$$

4. At a regular point of $\mathcal{F}$, zero of F:

$$E^*\frac{\omega}{F} = \frac{u}{y}\mathrm{d}y, \quad X = \frac{g}{u}y\frac{\partial}{\partial y}, \quad \mathcal{O}(-Z_\infty) = \mathcal{O}.$$

5. At a pole of F:

$$E^*\frac{\omega}{F} = uy\mathrm{d}y, \quad X = \frac{g}{uy}\frac{\partial}{\partial y}, \quad \mathcal{O}(-Z_\infty) = (y)\mathcal{O}.$$

Hence, in any case, the morphism $E^*\frac{\omega}{F}(\cdot)$ is onto the sheaf $\mathcal{O}(-Z_\infty)$. This ensures the exactness of the sequence.

□

2.2 Infinitesimal obstructions.

The long exact sequence in cohomology associated to the short exact sequence of sheaves in 2.1 is written

$$\cdots \to H^1(\mathcal{D}, \mathcal{X}_{\mathcal{F}}) \to H^1(\mathcal{D}, \mathcal{X}_{Z_0}) \to H^1(\mathcal{D}, \mathcal{O}(-Z_\infty)) \to 0.$$

This thus proved the first statement of the following proposition.

Proposition 2.2. *The space of infinitesimal obstructions to the realization problem is the $\mathbb{C}$-space of finite dimension $H^1(\mathcal{D}, \mathcal{O}(-Z_\infty))$. Its dimension is a topological invariant of the foliation.*

Let us first consider a germ of curve S in $(\mathbb{C}^2, 0)$ and $\tilde{S}$ the strict transform of S by E. The morphism E is a composition $E = E_0 \circ \cdots \circ E_N$ where E_i is the elementary standard blowing-up of a point. For any point c in a divisor $(E_0 \circ \cdots \circ E_j)^{-1}(0)$, $0 \geq j \geq N$, $\nu_c(S)$ refers to the multiplicity at c of the strict transform of S with respect to $E_0 \circ \cdots \circ E_j$. The following lemma may be establish in much the same way as theorem (2.1.3) in [5]. For the convenience of the reader we give a complete proof:

Lemme 2.2.

$$\dim_{\mathbb{C}} H^1(\mathcal{D}, \mathcal{O}(-\tilde{S})) = \sum_c \frac{v_c(S)(v_c(S)-1)}{2}$$

Proof: The proof is an induction on the length of the blowing-up process. Let E_0 be the blowing-up of the origin. Let us consider the canonical system of coordinates (x_1, y_1) and (x_2, y_2) in adapted neighborhood of $E_0^{-1}(0)$ such that the change of coordinates is written

$$y_2 = y_1 x_1, \; x_2 = \frac{1}{y_1}.$$

Let p be a reduced equation of P and p_1 and p_2 defined by

$$E^* p = x_1^{\nu_0(p)} p_1 \quad E^* p = y_2^{\nu_0(p)} p_2.$$

Hence, we can describe the space of global sections

$$\begin{aligned} H^0(V_1, \mathcal{O}(-\tilde{S})) &\simeq p_1 \mathbb{C}[[x_1, y_1]] \\ H^0(V_1 \cap V_2, \mathcal{O}(-\tilde{S})) &\simeq p_1 \mathbb{C}[[x_1]]((y_1)) \\ H^0(V_2, \mathcal{O}(-\tilde{S})) &\simeq \left\{ y_1^{-\nu_0(p)} \sum_{i,j \in \mathbb{N}^2} a_{ij} x_1^j y_1^{j-i} \,\middle|\, a_{ij} \in \mathbb{C} \right\} \end{aligned}$$

A simple computation shows the following isomorphisms

$$H^0(V_1 \cap V_2, \mathcal{O}(-\tilde{S})) \Big/ \bigoplus_{i=1,2} H^0(V_i, \mathcal{O}(-\tilde{S})) \simeq \mathbb{C}^{\frac{\nu_0(p)(\nu_0(p)-1)}{2}}.$$

We decompose the desingularization morphism $E = E_0 \circ E_1$ where E_0 is the first blowing-up of the origin. Let $\{s_1, \ldots, s_n\}$ refers to the intersection of D_0 and the strict transform of P. We denote the components of the exceptional divisor $\mathcal{D}_i = E_1^{-1}(s_i)$. For $i = 1, \dots, n$, consider $U_i(\epsilon) = B(s_i, \epsilon),\ \epsilon > 0$ be a disc for any smooth metric on D_0 such that U_i does not meet U_j for $j \neq i$. Let U_0 be the complementary of $\bigcup_{i=1,\dots,4} \overline{B}(s_i, \epsilon/2)$. Finally, let us denote by $\mathcal{U}_i$ the open set $E_1^{-1}(U_i)$. The system $\{\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_i\}$ provides a covering of the divisor $\mathcal{D}$ and the Mayer-Vietoris sequence for the sheaf $\mathcal{O}(-\tilde{S})$ is written

$$0 \to N \to H^1(\mathcal{D}, \mathcal{O}(-\tilde{S})) \to \bigoplus_i H^1(\mathcal{U}_i, \mathcal{O}(-\tilde{S})) \to \bigoplus_{ij} H^1(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}(-\tilde{S})) \to 0,$$

where N is given by

$$\bigoplus_i H^0(\mathcal{U}_i, \mathcal{O}(-\tilde{S})) \to \bigoplus_{ij} H^0(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}(-\tilde{S})) \to N \to 0.$$

Since $\mathcal{U}_i \cap \mathcal{U}_j$ is Stein and $\mathcal{O}(-\tilde{S})$ a coherent sheaf, $H^1(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}(-\tilde{S}))$ is trivial. Moreover, the morphism E_1 and the Hartogs argument induce following isomorphisms

$$\begin{aligned} H^0(\mathcal{U}_i, \mathcal{O}(-\tilde{S})) &\simeq H^0(U_i, \mathcal{O}(-\tilde{S})), \\ H^0(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}(-\tilde{S})) &\simeq H^0(U_i \cap U_j, \mathcal{O}(-\tilde{S})). \end{aligned}$$

Hence, N is identified with $H^1(D_0, \mathcal{O}(-\tilde{S}))$. All these remarks and an inductive limit on the neighborhood of $\mathcal{D}_i$ provide the next isomorphisms

$$H^1(\mathcal{D}, \mathcal{O}(-\tilde{S})) \simeq H^1(D_0, \mathcal{O}(-\tilde{S})) \oplus \bigoplus_i H^1(\mathcal{D}_i, \mathcal{O}(-\tilde{S})).$$

Therefore, the lemma is a straightforward computation from the hypothesis of induction and the formula above.

□

Proof: (2.2) Applying the previous lemma with $S = Z_\infty$ yields the finite dimension statement. Furthermore, looking at the formula, one can see that this dimension depends only on the topology of the process of reduction which is a topological invariant of the foliation [1].

□

3 The $\mathcal{M}$-simple examples.

Let r and $n \geq 2$ be positive integers and $\mathcal{F}_{n,r_1,\dots,r_n}$ the foliation given by the one form

$$x^{r+2} \mathrm{d} \left(\frac{x^{r+2} - \sum_{j=1}^{r+1} q_j x^{r+1-j} y^j}{x^{r+1}} \right).$$

Let us suppose that $Q(t)=\sum_{j=1}^{r+1}q_jt^j$ satisfies $Q'(t)=\prod_{i=1}^{n}(t-t_j)^{r_j}$. In [4], Klughertz proved that these foliations are topological normal forms for $\mathcal{M}$-foliation. To be more specific, any $\mathcal{M}$-foliation with n exceptional leaves tangent to the divisor with respective orders $r_1,\ldots,r_n$ is topologically equivalent to a foliation $\mathcal{F}_{n,r_1,\ldots,r_n}$. The position of the tangency points $t_1,\ldots,t_n$ may be chosen arbitrarily. Since the dimension of $H^1(\mathcal{D},\mathcal{O}(-Z_\infty))$ is a topological invariant, one can compute this dimension for the normal form in order to extend the result to any $\mathcal{M}$-foliation.

By a direct computation, one has $\nu_0(\mathcal{F}_{n,r_1,\ldots,r_n})=r+1$. The foliation is regular after one blowing and the exceptional divisor D_0 is not invariant. There are n integral curves S_i, $i=1..n$, which are tangent to the divisor. In the canonical coordinates $y=tx, x=x$, the points of tangency are given by $t=t_j$ and the r_j are the respective order of tangency. Hence, the foliation is completely reduced once one has reduced the germ of curves $S_i\cup(D_0)_{t_i}$. Therefore, the foliation has n isolated separatrix, which are the curves S_i if viewed after one blowing. Let us denote by h_i a reduced equation of S_i at the origin. One can see that $\nu_0(h_i)=r_i+1$. Futhermore, the component D_0 is dicritical with valence n. Let $\{\alpha_i\}_{i=1..n-2}$ be $n-2$ equations of curves of the pencil of D_0. By definition, the meromorphic function

$$F=\frac{h_1(x,y)h_2(x,y)\cdots h_n(x,y)}{\alpha_1(x,y)\alpha_2(x,y)\cdots\alpha_{n-2}(x,y)}$$

is a balanced equation for $\mathcal{F}$. In particular, since $\nu_0(\alpha_i)=1$,

$$\nu_0(F)=\sum_{i=1}^{n}\nu_0(h_i)-(n-2)=\sum_{i=1}^{n}(r_i+1)-n+2=r+2$$

Hence,

$$\nu_0(F)=\nu_0(\mathcal{F}_{n,r_1,\ldots,r_n})+1$$

which was predicted by (1.3). Moreover, in view of (2.2), the space of obstructions is of dimension $\frac{(n-2)(n-3)}{2}$. Hence, for any $\mathcal{M}$-simple foliation $\mathcal{F}$ with at least 4 exceptional leaves, there exists a deformation of the process of reduction, which cannot be followed by an isoholonomic deformation of $\mathcal{F}$. This completes the proof of the main statement of this paper.

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