CLASSIFICATION OF ABSOLUTELY DICRITICAL FOLIATIONS OF CUSP TYPE.

Y. GENZMER

02/03/2012

Abstract. We give a classification of absolutely dicritical foliation of cusp type, that is, the germ of singularities of complex foliation in the complex plane topologically equivalent to the singularity given by the level of the meromorphic function $\frac{y^2+x^3}{xy}$.

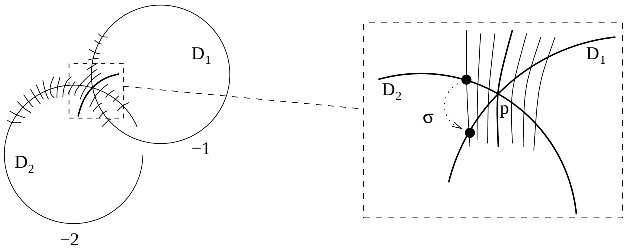
An important problem of the theory of singularities of holomorphic foliations in the complex plane is the construction of a geometric interpretation of the so-called *moduli of Mattei* of these foliations [10]. These moduli appear when one considers a very special kind of deformations called *the unfoldings*. Basically, the moduli of Mattei are precisely the moduli of germs of unfoldings of a given singular foliation. One of the major difficulty one meets looking at the mentionned geometric description is the lack of basic examples in the litterature. Actually, except when the foliation is given by the level of an holomorphic function, there exist none exemple. The purpose of the following article is not to solve the problem of Mattei even for the class of singularities we consider here but to describe this one as accuratly as possible in order to prepare the attack of the problem of moduli of Mattei.

The absolutely dicritical foliations of cusp type are good candidates to begin this study for the following reasons:

(1) their *transversal structure*, which usually is a very rich dynamic invariant [9], is very poor and can be completely understood.
(2) their number of Mattei moduli is 1.
(3) the topology of their leaves is more or less trivial.

Some results in the article might be quite easily extended to a larger class of absolutely dicritical foliations up to some technical and confusing additions. The risk would have been to miss the very first objective of this paper, that is, *to give an example.*

A germ of singularity of foliation $\mathcal{F}$ in $\left(\mathbb{C}^2,0\right)$ is said to be *absolutely dicritical* if there exists a sequence of blowing-up E such that $E^*\mathcal{F}$ is regular and transverse to each irreducible component of the exceptionnal divisor $E^{-1}(0)$. It is of *cusp type* if two successive blowing-up are sufficient. In that case the exceptionnal divisor $E^{-1}(0)$ is the union of two irreducible components $\mathbb{P}_1(\mathbb{C})$ of respective self-intersection -2 and -1. We denote them respectively D_2 and D_1.



The expression *cusp type* insists on the fact that the special leaf that passes trough the singular point of the divisor is analytically equivalent to the cuspidal singularity $y^2 + x^3 = 0$. The simplest example of an absolutely dicritical foliation is given by the levels of the rationnal function near $(0, 0)$

$$f = \frac{y^2 + x^3}{xy}.$$

We associate to $\mathcal{F}$ a germ $\sigma \in \mathrm{Diff}\left((D_2, p), (D_1, p)\right)$ as in the picture above. It is defined by the property that $x \in D_2$ and $\sigma(x) \in D_1$ belongs to the same local leaf. This germ is called *the transversal structure* of $\mathcal{F}$. This is the very first invariant of such a foliation. For the rationnal function above, the transversal structure σ reduces to the identity map in the standard coordinates associated to E.

The main result of this article is the following one: for any foliation $\mathcal{F}$ that is absolutely dicritical of cusp type we consider it topological class $\mathrm{Top}(\mathcal{F})$, that is the set of all foliations topologically equivalent to $\mathcal{F}$. The moduli space ${}^{\mathrm{Top}(\mathcal{F})}/_{\sim}$ of $\mathcal{F}$ is defined as the quotient of $\mathrm{Top}(\mathcal{F})$ by the analytical equivalence relation. Now we have,

Theorem 1. *The class $Top(\mathcal{F})$ is equal to the set of all absolutely dicritical foliations and its moduli space ${}^{Top(\mathcal{F})}/_{\sim}$ can be identified with the functionnal space $\mathbb{C}\{z\}$ up to the action of $\mathbb{C}^*$ defined by*

$$\epsilon \cdot (z \mapsto f(z)) = \epsilon^2 f(\epsilon z).$$

In this theorem, the germ of convergent series f is the image of the transversal structure σ by the Schwarzian derivative $S(\sigma) = \frac{3}{2}\left(\frac{\sigma'''}{\sigma'}\right) - \left(\frac{\sigma''}{\sigma'}\right)^2$. A quick lecture of the theorem would suggest that the transversal structure σ is the sole invariant of the foliation, which is not exactly true as it is highlighted in theorem (8).

We have to mention that it does exists *a lot of absolutely dicritical foliations*. Following a result due to F. Cano and N. Corral [3], the process E does not contain any obstruction to the existence of absolutely dicritical foliations. In other words, for any sequence of blowing-up E, there exists an absolutely dicritical foliation whose associated process of blowing-ups is exactly E.

1. Topological classification.

The topological classification is *trivial* as stated in a proposition to come in the sense that two absolutely dicritical foliations of cusp type are topologically equivalent. To prove this fact, we describe below the *model foliations* from which the absolutely dicritical foliations are build.

1.1. **Model foliations.** Let us consider the following model foliations

- $\mathcal{F}_2$ is given by the gluing of two copies of $\mathbb{C}^2$
$$\mathbb{C}^2 = (x_1, y_1) \qquad \mathbb{C}^2 = (x_2, y_2)$$
glued by $x_2 = \frac{1}{y_1}$ and $y_2 = y_1^2 x_1$ whose the neighborhood of $x_1 = y_2 = 0$ is transversaly foliated by $y_1 = cst$ and $x_2 = cst$. Topologically, this is a foliated neighborhood of a Riemann surface of genus 0 whose self-intersection is -2.
- $\mathcal{F}_1$ is given by the gluing of two copies of $\mathbb{C}^2$
$$\mathbb{C}^2 = (x_3, y_3) \qquad \mathbb{C}^2 = (x_4, y_4)$$
glued by $x_4 = \frac{1}{y_3}$ and $y_4 = y_3 x_3$ whose the neighborhood of $x_3 = y_4 = 0$ is transversaly foliated by $y_3 = cst$ and $x_4 = cst$. Topologically, this is a foliated neighborhood of a Riemann surface of genus 0 whose self-intersection is 1.

Following [2], any neighborhood of a Riemann surface A of genus 0 embedded in a manifold of dimension two with $A \cdot A = -2$ (resp. -1) and foliated by a transverse codimension 1 foliation is equivalent ot $\mathcal{F}_2$ (resp. $\mathcal{F}_1$). From this, it is easy to show that any $\left(\mathcal{C}^0, \mathcal{C}^\infty, \mathcal{C}^\omega\right)$ $-$isomorphism between two Riemann surfaces A_1 and A_2 as before can be extended in a neigborhood of A_1 and A_2 as a $\left(\mathcal{C}^0, \mathcal{C}^\infty, \mathcal{C}^\omega\right)$ $-$conjugacy of the foliations.

1.2. **Topological classification.** Let us first recall the following lemma:

Lemma 2. *Let σ be a germ in Diff $\left(\mathbb{P}^1, a\right)$, i.e., a germ of automorphism of a neighborhood of a in $\mathbb{P}^1$. Then there exists h a global homeomorphism of $\mathbb{P}^1$ such that h and σ coincide in a neighborhood of a.*

Proof. Let S_1 be a small circle around a in a domain where σ is defined. Its image $\sigma(S_1)$ is a topological circle. Consider S_2 a second circle such that the disc bounded by S_2 contains S_1 and $\sigma(S_1)$. The two coronas bounded respectively by S_1 and S_2 and $\sigma(S_1)$ and S_2 are homeomorphic. Actually, there exists an homeomorphism $\tilde{h}$ of the two coronas such that
$$\begin{aligned} \tilde{h}\Big|_{S_2} &= \mathrm{Id} \\ \tilde{h}\Big|_{S_1} &= \sigma. \end{aligned}$$
Therefore, we can define the homeomorphism h the following way: in the disc bounded by S_1, we set $h = \sigma$; in the corona bounded by S_1 and S_2, $h = \tilde{h}$; everywhere else we set $h = \mathrm{Id}$. Clearly, h satifies the properties in the lemma. $\square$

Proposition 3. *Two absolutely dicritical foliations of cusp type are topologically equivalent. The class Top $(\mathcal{F})$ is equal to the set of all absolutely dicritical foliations.*

Proof. Let us consider $\mathcal{F}_0$ and $\mathcal{F}_1$ two absolutely dicritical foliations of cusp type. Applying if necessary a linear change of coordinates to $\mathcal{F}_0$ for instance, we can suppose that both foliations are reduced by exactly the same sequence of two blowing-ups E. Let us write $E^{-1}(0) = D_2 \cup D_1$ and $D_2 \cap D_1 = \{p\}$. Let us consider σ_0 and σ_1 in Diff $((D_2, p), (D_1, p))$ the transversal structures of $\mathcal{F}_0$ and $\mathcal{F}_1$. According to the previous lemma, there exist h an homeomorphism of D_2 such that $h = \sigma_0^{-1} \circ \sigma_1$

in a neighborhood of p in D_2. Since, along D_2 or D_1 the foliations are transverse, there exist two homeomorphisms H_0 and H_1 defined respectively in a neighborhood of D_2 and D_1 such that

$$H_0^*\left(E^*\mathcal{F}_0\right) = E^*\mathcal{F}_1 \quad H_1^*\left(E^*\mathcal{F}_0\right) = E^*\mathcal{F}_1$$

and $H_0|_{D_2} = \mathrm{Id}$ and $H_1|_{D_1} = h$. Since $h = \sigma_0^{-1} \circ \sigma_1$, the automorphism $H_1 \circ H_0^{-1}$ of $E^*\mathcal{F}_0$ let invariant each leaf of $E^*\mathcal{F}_0$. Now, adapting the argument of the previous lemma yields the existence of H a global homeomorphism of $E^*\mathcal{F}_0$ defined in a neighborhood of D_1 letting invariant each leaf such that H and $H_1 \circ H_0^{-1}$ coincide in a neighborhood of p. Thus $\left(H^{-1} \circ H_1\right) \circ H_0^{-1}$ is equal to Id in a neighborhood of p. Therefore the collection $H^{-1} \circ H_1$ and H_0 glue in a global homeomorphism between $E^*\mathcal{F}_0$ and $E^*\mathcal{F}_1$. This homeomorphism can be blown down in a neighborhood of $\mathbb{C}^2$ and is a $\mathcal{C}^0$- conjugacy of the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$.

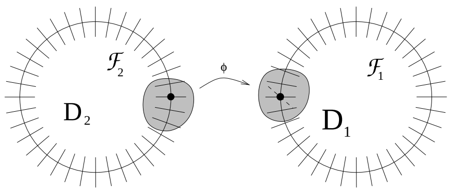
Now, if $\mathcal{F}_0$ is topologically equivalent to an absolutely dicritical foliation of cusp type, a theorem of C. Camacho and A. Lins Neto and P. Sad [1] ensures that the process of reduction of $\mathcal{F}_0$ is the one of an absolutely dicritical foliation. Since, they also shared the same dicritical components, $\mathcal{F}_0$ is absolutely dicritical of cusp type. □

2. Moduli space.

Consider a germ of biholomorphism ϕ written in the coordinates of the model foliations

$$(x_3, y_3) = \phi\,(x_1, y_1)\,, \qquad \phi\,(0,0) = (0,0)\,.$$

Suppose that it send the foliation defined by $y_1 = cst$ to the one defined by $y_3 = cst$ and that the curve $x_1 = 0$ is send to a curve transverse to $x_3 = 0$. With such a biholomorphism we can consider the foliation obtained by gluing the two models foliations $\mathcal{F}_2$ and $\mathcal{F}_1$ with the application ϕ



Following a classical result due to Grauert, this gluing is analytically equivalent to the neighborhood of the exceptionnal divisor obtained by a standard process of two successive blowing-ups [8]. The obtained foliation can be blown down in an absolutely dicritical foliation of cusp type at the origin of $\mathbb{C}^2$.

Remark 4. *Key remark.* Two foliations obtained by such an above gluing with the respective biholomorphisms ϕ and ψ are analytically equivalent if and only if there exists an automorphism Φ_2 of the foliation $\mathcal{F}_2$ and Φ_1 of the foliation $\mathcal{F}_1$ such that

$$\phi = \Phi_1 \circ \psi \circ \Phi_2.$$

Let us fix $\sigma \in \mathrm{Diff}\,(\mathbb{C}, 0)$. We consider the following biholomorphisms

$$g_\sigma\,(x_1, y_1) = (x_1 + \sigma\,(y_1)\,, \sigma\,(y_1)) \text{ and } \Phi_\alpha\,(x_1, y_1) = (x_1\,(1 + \alpha y_1)\,, y_1)$$

The composition $g_\sigma \circ \phi_\alpha$ send the foliation $y_1 = cst$ on it-self. Thus we can denote by $\mathcal{F}_{\sigma,\alpha}$ the foliation obtained by the above gluing

$$\mathcal{F}_{\sigma,\alpha} := \mathcal{F}_1 \coprod \mathcal{F}_2 / p \sim g_\sigma \circ \Phi_\alpha(p).$$

Now, moving the parameter α, we obtain an analytical family of absolutely dicritical foliations. Actually, the following property holds.

Theorem 5. *The germ of deformation $(\mathcal{F}_{\sigma,\alpha})_{\alpha\in(\alpha_0,\mathbb{C})}$ for α in a neighborhood of α_0 in $\mathbb{C}$ is a germ of equisingular semi-universal unfolding of $\mathcal{F}_{\sigma,\alpha_0}$ in the sense of Mattei* [10]*. In particular, for any germ of equisingular unfolding $(\mathcal{F}_t)_{t\in(\mathbb{C}^p,0)}$ with p parameters such that $\mathcal{F}_t|_{t=0} \sim \mathcal{F}_{\sigma,\alpha_0}$ there exists a map $\alpha : (\mathbb{C},\alpha_0) \to (\mathbb{C}^p,0)$ such that for all t $\mathcal{F}_t \sim \mathcal{F}_{\sigma,\alpha(t)}$.*

Before proving the above result, let us recall that an *unfolding* of a given foliation $\mathcal{F}$ is a germ $\mathbb{F}$ of codimension 1 foliation in $(\mathbb{C}^{2+p},0)$ transversal to the fiber of the projection $(\mathbb{C}^{2+p},0) \to (\mathbb{C}^p,0)$, $\pi : (x,t) \to t$ such that $\mathbb{F}|_{\pi^{-1}(0)} \sim \mathcal{F}$. The *equisingularity* property is a quite technical property to state. However, it means basically that the topology of the process of desingularization of the family of foliation $\mathbb{F}|_{t=\alpha}$ does not depend on α. For the details, we refer to [10].

Proof. **Step 1 -** Let us prove that the deformation $(\mathcal{F}_{\sigma,\alpha})_{\alpha\in(\mathbb{C},\alpha_0)}$ of $\mathcal{F}_{\sigma,\alpha_0}$ is induced by an unfolding. We can make the following *thick* gluing

$$\mathbb{F} := \mathcal{F}_1 \times (\mathbb{C},\alpha_0) \coprod \mathcal{F}_2 \times (\mathbb{C},\alpha_0) / (x_1,y_1,\alpha) \to \left((g_\sigma \circ \Phi_{\alpha_0}) \circ \left(\Phi_{\alpha_0}^{-1} \circ \Phi_\alpha\right)(x_1,y_1),\alpha\right).$$

where $\mathcal{F}_i \times (\mathbb{C},\alpha_0)$ stands for the product foliation: its leaves are the product of a leaf of $\mathcal{F}_i$ and of an open neighborhood of α_0 in $\mathbb{C}$. The codimension $1-$foliation $\mathbb{F}$ comes clearly with a fibration defined by the quotient of the map $\pi : (p,\alpha) \to \alpha$ whose fibers are transverse to the foliation. Thus, the above gluing is an unfolding. Now, the restriction $\mathbb{F}|_{\pi^{-1}(\alpha_0)} = \mathcal{F}_1 \coprod \mathcal{F}_2 / (g_\sigma \circ \Phi_{\alpha_0})$ is equal to $\mathcal{F}_{\sigma,\alpha_0}$. Finally, it is equisingular by construction. Thus, it satisfies all the properties of an equisingular unfolding in the sense of Mattei.

Step 2 - Let us consider the sheaf Θ whose base is the exceptionnal divisor $E^{-1}(0) = D = D_2 \cup D_1$ of tangent vector fields to the foliation $E^*\mathcal{F}_{\sigma,\alpha_0}$ and to the divisor $E^{-1}(0)$. The cohomological group $H^1(D,\Theta)$ represents the finite dimensionnal $\mathbb{C}$-space of infinitesimal unfoldings. Following [10], there exists a *Kodaira-Spencer map like* that associate to any unfolding with parameter in $(\mathbb{C}^p,0)$, its Kodaira Spencer derivative which is a linear map from $\mathbb{C}^P$ to $H^1(D,\Theta)$. The unfolding is semi-universal as in the theorem above if and only if its Kodaira Spencer derivative is a linear isomorphism.

We consider the covering of the exceptionnal divisor $E^{-1}(0)$ by two open sets U_1 and U_2 where U_1 and U_2 are respectively tubular neighborhood of D_1 and D_2. It is known that this covering is acyclic with respect to the sheaf Θ, i.e, $H^1(D_i,\Theta) = 0$. Therefore, following [7] to compute the cohomological group $H^1(D,\Theta)$ we can use this covering, that is to say, the following isomorphism

$$H^1(D,\Theta) \simeq \frac{H^0(U_1 \cap U_2,\Theta)}{H^0(U_1,\Theta) \oplus H^0(U_2,\Theta)}. \tag{2.1}$$

In view of the glued construction of $\mathcal{F}_{\sigma,\alpha_0}$, a $0-$cocycle X_{12} in $H^0(U_1 \cap U_2,\Theta)$ is trivial in $H^1(D,\Theta)$ if and only if the cohomological equation

$$X_{12} = X_1 - (g_\sigma \circ \Phi_{\alpha_0})^* X_2 \tag{2.2}$$

admits a solution where $X_1 \in H^0(U_1,\Theta)$ and $X_2 \in H^0(U_2,\Theta)$. Now, it is known [10][3] that the dimension of the $\mathbb{C}$ space$H^1(D,\Theta)$ is 1. Thus, to prove the result, it is enough to show that the image of the deformation $(\mathcal{F}_{\sigma,\alpha})_{\alpha\in(\alpha_0,\mathbb{C})}$ by the foliated Kodaira-Spencer map is not trivial in $H^1(D,\Theta)$. The foliation $\mathcal{F}_{\sigma,\alpha}$ is obtained from $\mathcal{F}_{\sigma,\alpha_0}$ by gluing with the automorphism

$$\Phi_{\alpha_0}^{-1}\circ\Phi_\alpha(x_1,y_1)=\left(x_1\frac{1+\alpha y_1}{1+\alpha_0 y_1},y_1\right).$$

Thus, its image by the Kodaira-Spencer map is the cocycle

$$\frac{\partial}{\partial\alpha}\Phi_{\alpha_0}^{-1}\circ\Phi_\alpha\bigg|_{\alpha=\alpha_0}=\frac{x_1y_1}{1+\alpha_0y_1}\frac{\partial}{\partial x_1}$$

Hence, the unfolding is semi-universal if and only if the equation

$$x_1y_1\frac{\partial}{\partial x_1}+\cdots=X_2-(g_\sigma\circ\Phi_{\alpha_0})^*X_1 \tag{2.3}$$

has no solution. This equation can be more precisely written in the following way

$$x_1y_1\frac{\partial}{\partial y_1}+\cdots=A_2(x_1,y_1)\,x_1\frac{\partial}{\partial x_1}-(g_\sigma\circ\Phi_{\alpha_0})^*\left(A_1(x_3,y_3)\,x_3\frac{\partial}{\partial x_3}\right)$$

where A_1 and A_2 are functions defined respectively in U_1 and U_2. Let us write the Taylor expansion of $A_2=\sum_{ij}a_{ij}^2x_1^iy_1^j$. In the coordinates (x_2,y_2) the function A_2 is written $A_2=\sum_{ij}a_{ij}^2x_2^{2i-j}y_2^i$. Therefore, if $a_{ij}^2\neq 0$ then $2i-j\geq 0$ and the monomial term y_1 cannot appear in the Taylor expansion of A_2. In the same way, the Taylor expansion of $A_1=\sum_{ij}a_{ij}^1x_3^iy_3^j$, satisfies $a_{ij}^1\neq 0\Rightarrow i\geq j$. Since X_1 vanishes along the exceptionnal divisor whose trace in U_1 is the diagonal $x_3=y_3$, we have $A_1=(x_3-y_3)\tilde{A}_1$. Thus, in the coordinates (x_1,y_1), X_1 is written

$$X_1=\tilde{A}_1(x_1(1+\alpha_0y_1)+\sigma(y_1),\sigma(y_1))\,(x_1(1+\alpha_0y_1)+\sigma(y_1))\,x_1\frac{\partial}{\partial x_1}.$$

If $\tilde{A}_1(0,0)=0$ then the term $y_1x_1\frac{\partial}{\partial y_1}$ of the Taylor expansion of the cocycle (2.3) cannot come from X_1. However, if $\tilde{A}_1(0,0)\neq 0$ then A_1 cannot be global. Therefore, the equation (2.3) cannot be solved, which proves the result. □

We observe that $\mathcal{F}_{\sigma,\alpha}$ is an unfolding over the whole $\mathbb{C}$. Actually in the course of the above proof, we obtain a more precise result

Corollary 6. *More generally, for any germ of function $A(x,y)$ with $A(0,0)\neq 0$, the $\mathbb{C}$-space $H^1(D,\Theta)$ for the foliation*

$$\mathcal{F}_1\coprod\mathcal{F}_2/(x_1,y_1)\to(x_1A(x_1,y_1)+\sigma(y_1),\sigma(y_1)).$$

is generated by the cocycle the image of $x_1y_1\frac{\partial}{\partial y_1}$ through the isomorphism (2.1). In particular, any deformation of the form

$$\epsilon\to(\mathcal{F}_1\coprod\mathcal{F}_2)_\epsilon/(x_1,y_1)\to(x_1A_\epsilon(x_1,y_1)+\sigma(y_1),\sigma(y_1))$$

where $\frac{\partial A_\epsilon}{\partial y_1}(0,0)$ does not depend on ϵ is locally analytically trivial.

As an easy consequence of the corollary, we obtain a theorem of normalization of the construction of absolutely dicritical foliations of cusp type.

Theorem 7. *Any absolutely dicritical foliation of cusp type is equivalent to some* $\mathcal{F}_{\sigma,\alpha}$.

Proof. Let us consider $\mathcal{F}$ an absolutely dicritical foliation of cusp type and let E be its associated reduction. Since along each component of the exceptionnal divisor the foliation is purely radial, there exists two automorphisms Φ_1 and Φ_2 that conjugates $\mathcal{F}$ respectively to the models $\mathcal{F}_1$ and $\mathcal{F}_2$ in the neighborhood of respectively D_1 and D_2. The cocycle of gluing is thus written $\Phi_1 \circ \Phi_2^{-1}$. Applying if necessary a global automorphism of Φ_1 that let invariant each leaf, we can suppose that $\Phi_1 \circ \Phi_2^{-1}$ send the exceptionnal divisor $x_1 = 0$ on the line $x_3 = y_3$. Since the cocycle conjugates the foliations $\mathcal{F}_2$ and $\mathcal{F}_1$, it can be written

$$(x_1, y_1) \mapsto (x_1 A(x_1, y_1) + \sigma(y_1), \sigma(y_1)).$$

for some $\sigma \in \mathrm{Diff}(\mathbb{C}, 0)$ and some $A \in \mathbb{C}\{x_1, y_1\}$ with $A(0,0) \neq 0$. Applying if necessary an automorphism of $\mathcal{F}_2$ defined by $(\epsilon x_3, \epsilon y_3)$ for some $\epsilon \neq 0$, we can suppose that $A(0,0) = 1$. Now we can write the cocycle

$$(x_1, y_1) \mapsto \left(x_1 \left(1 + \alpha y_1 + \tilde{A}(x_1, y_1)\right) + \sigma(y_1), \sigma(y_1)\right).$$

where no term of the form ay_1 appears in $\tilde{A}$. According to the corollary, the deformation parametrized by ϵ and defined by the gluing cocycle

$$(x_1, y_1) \mapsto \left(x_1 \left(1 + \alpha y_1 + \epsilon \tilde{A}(x_1, y_1)\right) + \sigma(y_1), \sigma(y_1)\right)$$

is locally analytically trivial. Thus the foliation obtained setting $\epsilon = 1$ and $\epsilon = 0$ are analytically equivalent and setting $\epsilon = 0$ yields a cocycle of the desired form. $\square$

The couple (α, σ) is unique up to conjugacies fixing any point of the exceptionnal divisor. However, once we authorize any kind of conjugacies, this couple is not unique anymore. But the ambiguity can be described.

Proposition 8. *Two normal forms* $\mathcal{F}_{\sigma,\alpha}$ *and* $\mathcal{F}_{\gamma,\alpha'}$ *are conjugated if and only if there are two homographies* h_0 *and* h_1 *such that*

$$\begin{cases} \sigma = h_1 \circ \gamma \circ h_0 \\ \frac{2}{5}\left(\alpha - \frac{3}{2}\frac{\sigma''(0)}{\sigma'(0)}\right) = \frac{2}{5}\left(\alpha' - \frac{3}{2}\frac{\gamma''(0)}{\gamma'(0)}\right) h_0'(0) - \frac{h_0''(0)}{h_0'(0)} \end{cases} \tag{2.4}$$

Proof. **Step 1 -** In view of our gluing construction and following the key remark (4), the existence of a conjugacy implies that there exist two automorphisms of respectively $\mathcal{F}_2$ and $\mathcal{F}_1$ written $\Phi_2 = (x_1 A_2(x_1, y_1), h_0(y_1))$ and $\Phi_1 = (x_3 A_1(x_3, y_3), h_1(y_3))$ such that

$$(x_1(1 + \alpha y_1) + \sigma(y_1), \sigma(y_1)) = \Phi_1 \circ \left(x_1 \left(1 + \alpha' y_1\right) + \gamma(y_1), \gamma(y_1)\right) \circ \Phi_2.$$

First, we obviously get the following relation $\sigma = h_1 \circ \gamma \circ h_0$. Moreover, if we look at the first component of the above relation we get

$$\begin{aligned} x_1(1 + \alpha y_1) + \sigma(y_1) = & \left(x_1 A_2(x_1, y_1)\left(1 + \alpha' h_0\right) + \gamma \circ h_0\right) \times \\ & A_1\left(x_1 A_2(x_1, y_1)\left(1 + \alpha' h_0\right) + \gamma \circ h_0, \gamma \circ h_0\right) \end{aligned}$$

If we compute the derivative $\frac{\partial}{\partial x_1}$ of the above relation and then set $x_1 = 0$, we get

$$\begin{aligned} 1+\alpha y_1 &= A_2\left(0,y_1\right)\left(1+\alpha^{'}h_0\right)\times \\ &\left(\gamma\circ h_0\frac{\partial A_1}{\partial x_1}\left(\gamma\circ h_0,\gamma\circ h_0\right)+A_1\left(\gamma\circ h_0,\gamma\circ h_0\right)\right) \end{aligned} \tag{2.5}$$

(1) Now, since Φ_1 preserve the curve $y = x$, we obtain

$$A_1\left(x,x\right)=\frac{h_1\left(x\right)}{x}$$

Thus, $A_1\left(0,0\right) = h_1^{'}\left(0\right)$. Setting $y_1 = 0$ in the relation above, we get $1 = A_2\left(0,0\right)A_1\left(0,0\right)$. Therefore, $A_2\left(0,0\right) = \frac{1}{h_1^{'}(0)}$. Now, let us write the Taylor expansion of A_1

$$A_1\left(x_3,y_3\right)=h_1^{'}\left(0\right)+rx_3+sy_3+\cdots.$$

Since, $A_1\left(x,x\right) = \frac{h_1(x)}{x}$, we have $r + s = \frac{h_1^{''}(0)}{2}$. Now, the biholomorphism $\left(x_3A_1\left(x_3,y_3\right),h_1\left(y_3\right)\right)$ is global: therefore, it can be push down and extended at the origin of $\mathbb{C}^2$ as a local automorphism written

$$\left(x,y\right)\mapsto\left(xA_1\left(x,\frac{y}{x}\right),h_1\left(\frac{y}{x}\right)xA_1\left(x,\frac{y}{x}\right)\right).$$

The second component of this expression is written

$$\frac{y}{\alpha x+\beta y}\left(h_1^{'}\left(0\right)x+rx^2+sy+\cdots\right)$$

where $\alpha = \frac{1}{h_1^{'}(0)}$ and $\beta = -\frac{h_1^{''}(0)}{2h_1^{'}(0)^2}$. It is extendable at $(0,0)$ if and only if the expression in parenthesis can be holomorphically divided by $\alpha x + \beta y$. Looking at the first jet of these expressions leads to

$$\begin{vmatrix}\beta & \alpha \\ s & h_1^{'}\left(0\right)\end{vmatrix}=0\Longrightarrow s=\frac{\beta h_1^{'}\left(0\right)}{\alpha}=-\frac{h_1^{''}\left(0\right)}{2}$$

Finally, we have $r = h_1^{''}\left(0\right)$.

(2) In the same way, let us write the Taylor expansion of $A_2\left(x_1,y_1\right) = \frac{1}{h_1^{'}(0)} + uy_1 + vy_1^2 + \cdots$. The second component of the expression of Φ_2 in the coordinates (x_2, y_2) is $y_2x_2^2h_0^2\left(\frac{1}{x_2}\right)A_2\left(y_2x_2^2,\frac{1}{x_2}\right)$ which is equal to

$$\frac{y_2}{\left(\alpha^{'}x_2+\beta^{'}\right)^2}\left(\alpha x_2^2+ux_2+v+y_2\left(\cdots\right)\right)$$

where $\alpha^{'} = \frac{1}{h_0^{'}(0)}$ and $\beta^{'} = -\frac{h_0^{''}(0)}{2h_0^{'}(0)^2}$. Since it is extendable at $x_1 = -\frac{\beta^{'}}{\alpha^{'}}$, there exists a constant Γ such that $\left(\alpha^{'}x_2+\beta^{'}\right)^2 = \Gamma\left(\alpha x_2^2+ux_2+v\right)$ Hence, we have the equality $u = 2\frac{\alpha\beta^{'}}{\alpha^{'}} = -\frac{h_0^{''}(0)}{h_0^{'}(0)h_1^{'}(0)}$.

Now, we can identified the coefficient of the equation (2.5)

It is

$$\begin{aligned}\alpha &= A_2(0,0)\left(\gamma'(0)\,h_0'(0)\,\frac{\partial A_1}{\partial x_1}(0,0)+\frac{h_1''(0)}{2}\gamma'(0)\,h_0'(0)+\alpha' h_0'(0)\,h_1'(0)\right)\\ &\quad +uh_1'(0)\\ &= \frac{3}{2}\gamma'(0)\,h_0'(0)\,\frac{h_1''(0)}{h_1'(0)}-\frac{h_0''(0)}{h_0'(0)}+\alpha' h_0'(0)\,.\end{aligned}$$

Using the relation $\sigma = h_1 \circ \gamma \circ h_0$ the above equality can be formulated as in the theorem.

Step 2 - We suppose that the conclusion of the statement is satisfied. Let us suppose that

$$h_1(z)=\frac{z}{\alpha+\beta z} \qquad h_0(z)=\frac{z}{a+bz}$$

Then we set

$$\begin{aligned}A_2(x_1,y_1) &= \alpha+2\frac{\alpha b}{a}y_1+\frac{\alpha b}{a^2}y_1^2\\ A_1(x_3,y_3) &= \frac{\alpha+\beta y_3}{(\alpha+\beta x_3)^2}.\end{aligned}$$

In view of the computations done in the first step, the two automorphisms Φ_1 and Φ_2 associated to A_1 and A_2 can be extended on U_1 and U_2, tubular beighborhood of D_1 and D_2. Moreover, we obtain the following relation

$$\begin{aligned}&(x_1(1+\alpha y_1+\Delta(x_1,y_1))+\sigma(y_1),\sigma(y_1))\\ &\quad=\Phi_1\circ\left(x_1\left(1+\alpha' y_1\right)+\gamma(y_1),\gamma(y_1)\right)\circ\Phi_2\end{aligned}$$

where Δ does not contain any monomial term in y_1. Now, using the proposition (6), we see that the deformation defined by

$$\epsilon\to(x_1(1+\alpha y_1+\epsilon\Delta(x_1,y_1))+\sigma(y_1),\sigma(y_1))$$

is analytically trivial, which ensures the theorem. □

Theorem 9. *The moduli space of absolutely dicritical foliations of cusp type can be identified with the functionnal space $\mathbb{C}\{z\}$ up to the action of $\mathbb{C}^*$ defined by*

$$\epsilon\cdot(z\mapsto\sigma(z))=\epsilon^2\sigma(\epsilon z)\,.$$

Proof. We can consider the following family parametrized by $\mathrm{Diff}(\mathbb{C},0)$

$$\sigma\in\mathrm{Diff}(\mathbb{C},0)\to\mathcal{F}_{\frac{3}{2}\frac{\sigma''(0)}{\sigma'(0)},\sigma}.$$

It is a complete family for absolutely dicritical foliations of cusp type: in any class of absolutely dicritical foliation of cusp type there is one that is analytically equivalent to one of the form $\mathcal{F}_{\frac{3}{2}\frac{\sigma''(0)}{\sigma'(0)},\sigma}$. Indeed, considering the foliation $\mathcal{F}_{\alpha',\gamma}$, we can choose h_0 such that $\frac{2}{5}\left(\alpha'-\frac{3}{2}\frac{\gamma''(0)}{\gamma'(0)}\right)h_0'(0)-\frac{h_0''(0)}{h_0'(0)}=0$. Therefore, setting $\sigma=\gamma\circ h_0$ ensures that $\mathcal{F}_{\alpha',\gamma}$ and $\mathcal{F}_{\frac{3}{2}\frac{\sigma''(0)}{\sigma'(0)},\sigma}$ are analytically equivalent. Moreover, if $\mathcal{F}_{\frac{3}{2}\frac{\sigma_0''(0)}{\sigma_0'(0)},\sigma_0}$ and $\mathcal{F}_{\frac{3}{2}\frac{\sigma_1''(0)}{\sigma_1'(0)},\sigma_1}$ are analytically equivalent then there exists $\epsilon \in \mathbb{C}^*$ and an homographie h_1 such that

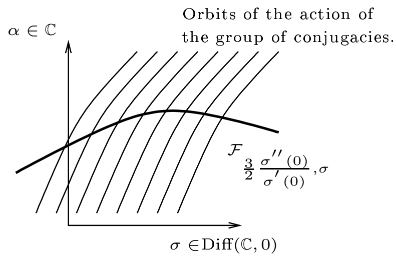
$$\sigma_0(z) = h_1 \circ \sigma_1 \circ (\epsilon z). \tag{2.6}$$

Indeed, the second homographie h_0 that appears in the proposition (8) has to be linear for the relations (2.4) ensures that $h_0''(0) = 0$. Thus, h_0 is written $z \mapsto \epsilon z$ for some ϵ. To simplify the relation (2.6), we use the Schwartzian derivative which is a surjective operator defined by

$$\mathcal{S} : \begin{cases} \mathrm{Diff}(\mathbb{C}, 0) \to \mathbb{C}\{z\} \\ y \mapsto \frac{3}{2}\left(\frac{y'''}{y'}\right) - \left(\frac{y''}{y'}\right)^2 \end{cases}$$

and satisfying the following property: the relation (2.6) is equivalent to $\mathcal{S}(\sigma_0)(z) = \epsilon^2 \mathcal{S}(\sigma_1)(\epsilon z)$. Therefore, the moduli space of absolutely dicritical foliation of cusp type is identified via the Schwartzian derivative to the quotient of $\mathbb{C}\{z\}$ up to the action of $\mathbb{C}^* \epsilon \cdot (z \mapsto \sigma(z)) = \epsilon^2 \sigma(\epsilon z)$. □

As mentionned in the introduction, this theorem does not state that the transversal structure σ is the sole analytical invariant of an absolutely dicritical foliation of cusp type. Indeed, the action of the group of conjugacies act transversaly to the transverse structures σ and to the moduli of Mattei α. The family $\mathcal{F}_{\frac{3}{2}\frac{\sigma''(0)}{\sigma'(0)},\sigma}$ is a complete tranversal set for this action



As a consequence of the above description of the moduli space of absolutely dicritical foliations, we should be able to prove the existence of a non algebrizable absolutely dicritical foliation using technics developped in [6].

3. Formal normal forms for 1-Forms.

It is known [3] that the valuation of a 1-form ω with an isolated singularity defining an absolutely dicritical foliation of cusp type is 3. Up to some linear change of coordinates, we can suppose that the singular point of the foliation after one blowing-up has $(0, 0)$ for coordinates in the standard coordinates associated to the blowing-up. Moreover, since the foliation is generically transversal to the exceptionnal divisor of the blowing-up of $0 \in (\mathbb{C}^2, 0)$, the homogeneous part of degree 3 of ω is tangent to the radial form $\omega_R = xdy - ydx$. Thus there exists an homogeneous polynomial function of degree 2 P_2 such that

$$\omega = P_2 \omega_R + \sum_{i \geq 4} (A_i(x, y)\, dx + B_i(x, y)\, dy).$$

After one blowing-up, the singular locus is given by the solutions of $P_2(1,y)=0$ and $P_2(x,1)=0$ in each chart. Thus P_2 is simply written ay^2 for some constant $a\neq 0$. After on blowing-up $(x,t)\mapsto(x,tx)$, the linear part near $(0,0)$ of the pull-back form is written

$$\left(A_4(1,0)+t\frac{\partial A_4}{\partial t}(1,0)+tB_4(1,0)\right)dx+xB_4(1,0)\,dt+xA_5(1,0)\,dx.$$

The absolutely dicritical property ensures that this linear part is non trivial and tangent to the radial vector field $tdx+xdt$. Hence, the following relations hold

$$A_4(1,0)=A_5(1,0)=0 \text{ and } \frac{\partial A_4}{\partial t}(1,0)+2B_4(1,0)=0$$

Finally, the form ω is written

$$\begin{aligned}\omega &= y^2\omega_R+\left(-2\alpha x^3+yQ_2(x,y)\right)ydx+\left(\alpha x^4+yQ_3(x,y)\right)dy\\ &\quad+\left(A_5(x,y)\,dx+B_5(x,y)\,dy\right)+\cdots\end{aligned}$$

where $\alpha\neq 0$.

Proposition 10. *The $1-$form ω is formally equivalent to a $1-$form written*

$$\begin{aligned}&y^2\omega_R+\alpha x^3(xdy-2ydx)+ax^3ydy\\ &+\sum_{n\geq 5}x^{n-1}\left((a_nx+b_ny)\,dx+(c_nx+d_ny)\,dy\right)\end{aligned}$$

where $a_5=0$. Moreover, this formal normal form is unique up to change of coordinates tangent to Id.

Proof. The action of a change of coordinates $\phi_n:(x,y)\to(x,y)+(P_n,Q_n)$ where P_n and Q_n are homogeneous polynomial functions of degree n does not modify the jet of order $n+1$ of ω. Moreover, the action on the homogeneous part of degree $n+2$ is written

$$\begin{aligned}J^{n+2}\left(\phi_n^*\omega\right) &= J^{n+2}\omega\\ &\quad+y^2\left(\left(x\frac{\partial Q_n}{\partial x}-y\frac{\partial P_n}{\partial x}+Q_n\right)dx+\left(x\frac{\partial Q_n}{\partial y}-x\frac{\partial P_n}{\partial x}+P_n\right)dy\right)\end{aligned}$$

We are going to verify that the linear morphism defined by

$$L:(P_n,Q_n)\mapsto\left(x\frac{\partial Q_n}{\partial x}-y\frac{\partial P_n}{\partial x}+Q_n,\,x\frac{\partial Q_n}{\partial y}-x\frac{\partial P_n}{\partial x}+P_n\right)$$

from the set of couples of homogeneous polynomial functions of degree n to itself is a one to one correspondance. To do so, let us compute the kernel of this morphism and let us write $P_n=\sum_{i=0}^n p_ix^iy^{n-i}$ and $Q_n=\sum_{i=0}^n q_ix^iy^{n-i}$. The coefficients of the components of $L(P_n,Q_n)$ on the monomial term x^iy^{n-i} are

$$\begin{aligned}q_i(i-1)-p_{i+1}(i+1) \quad & i = 0..n-1\\ q_n(n-1) \quad & i = n \qquad \text{and}\\ -p_i(n-i-1)+q_{i-1}(n-i+1) \quad & i = 1..n\\ p_0(n-1) \quad & i = 0.\end{aligned}$$

If (P_n, Q_n) is in the kernel then $q_n = 0$ and $p_0 = 0$. Moreover, applying the above relation with $i = 1$ and $i = n-1$ yields $p_2 = 0$ and $q_{n-2} = 0$. Now for $i = 1..n-1$ but $i \neq n-2$, a combination of the relations above ensures that

$$0 = q_i\,(i-1) - q_i\,(i+1)\,\frac{n-i}{n-i-2} \quad = \quad \frac{q_i}{n-i-2}\,(2-2n)$$

Thus $q_i = 0$ for $i = 0..n-1$. Therefore $(P_n, Q_n) = 0$ and L is an isomorphism. Thus, we can choose ϕ_n such that

$$J^{n+2}\left(\phi_n^* \omega\right) = x^{n-1}\left(\left(a_n x + b_n y\right) dx + \left(c_n x + d_n y\right) dy\right).$$

Clearly the composition $\phi_2 \circ \phi_3 \circ \cdots$ is formally convergent, which proves the proposition. □

4. Absolutely dicritical foliation admitting a first integral.

In this section, we study absolutely dicritical foliations that admit a meromorphic first integral. Such an existence can be completely red on the transverse structure.

Theorem 11. *Let $\mathcal{F}$ be an absolutely dicritical foliation of cusp type with σ as transverse structure. Then $\mathcal{F}$ admits a first integral if and only if there exists two non constant rationnal functions R_1 and R_2 such that*

$$R_1 \circ \sigma = R_2.$$

Notice that the existence of R_1 and R_2 does not depend on the equivalence class of σ modulo homographies.

Proof. Suppose that $\mathcal{F}$ admits a meromorphic first integral f. After blowing-up, the function f is a non constant rationnal function in restriction to each component of the divisor. Since for any point p, p and $\sigma(p)$ belongs to the same leaf, we have

$$f|_{D_1}(p) = f|_{D_2}(\sigma(p)).$$

Now, suppose there exist two rationnal function as in the lemma. According to some previous result, there exists α and γ such that the foliation $\mathcal{F}$ is analytically equivalent to $\mathcal{F}_{\alpha,\gamma}$. The application σ and γ are linked by a relation of the form

$$h_0 \circ \sigma \circ h_1 = \gamma$$

where h_0 and h_1 are homographies. Thus, setting $\tilde{R}_1 = R_1 \circ h_0^{-1}$ and $\tilde{R}_2 = R_2 \circ h_1$ yields $\tilde{R}_1 \circ \gamma = \tilde{R}_2$ where $\tilde{R}_1$ and $\tilde{R}_2$ are still rationnal. Now, let us go back to the construction of $\mathcal{F}_{\alpha,\gamma}$. We glue the models $\mathcal{F}_1$ and $\mathcal{F}_2$ around $(x_1, y_1) = 0$ and $(x_3, y_3) = 0$ by

$$(x_1, y_1) \mapsto (x_3 = x_1(1 + \alpha y_1) + \gamma(y_1), y_3 = \gamma(y_1))$$

Consider for $\mathcal{F}_1$ the first integral $F_1(x_1, y_1) = \tilde{R}_2(y_1)$ and for $\mathcal{F}_2$ the first integral $F_2(x_3, y_3) = \tilde{R}_1(y_3)$. Then these two meromorphic first integrals can be glued in a global meromorphic first integral since

$$F_2(x_3, y_3) = F_2(x_1(1 + \alpha y_1) + \gamma(y_1), \gamma(y_1)) = \tilde{R}_1(\gamma(y_1)) = \tilde{R}_2(y_1) = F_1(x_1, y_1).$$

Thus the absolutely dicritical foliation admits a meromorphic first integral. □

In view of this result, it is easy to produce a lot of examples of absolutely dicritical foliation admitting no meromorphic first integral setting for instance

$$\sigma(z) = e^z - 1.$$

Notice that the existence of the first integral depends only on the transversal structure σ and not on the value of the moduli of Mattei α. This is consistent with the fact that along an equireducible unfolding the existence of a meromorphic first integral for one foliation in the deformation ensures the existence of such a first integral for any foliation in the deformation.

Finally, since the topologically classification of absolutely dicritical foliations is *trivial,* the above result produce a lot of examples of couples of conjugated foliations such that only one of them admits a meromorphic first integral.

Hereafter we treated a special case, that is when the transversal structure σ is an homography.

Proposition 12. *Let $\mathcal{F}$ be an absolutely dicritical foliation of cusp type with an homographic transversal structure. Then, up to some analytical change of coordinates, $\mathcal{F}$ admits one of the following rationnal first integrals:*

(1) $f = \frac{y^2+x^3}{xy}$.
(2) $f = \frac{y^2+x^3}{xy} + x$

Proof. Let us consider the following germ of family of meromorphic functions with $(x, y, z) \in (\mathbb{C}^3, (0,0,0))$ defined by

$$f_z = \frac{y^2 + x^3 + zx^2y}{xy} = \frac{a}{b}.$$

For any z, the foliation associated to f_z is absolutely dicritical of cusp type. Let us prove that this family is an equireducible unfolding. We consider the integrable $1-$form $\Omega = adb - bda$. It is written

$$\left(2x^3y + zx^2y^2 - y^3\right) dx + \left(xy^2 - x^4\right) dy + x^3y^2 dz.$$

It defines an unfolding of the foliation given by f_0 with one parameter. Its singular locus is the $z-$axes and it is transversal to the fibers of the fibration $(x, y, z) \mapsto z$. Once we blow-up the $z-$axe, in the chart $E : (x, t, z) = (x, tx, z)$, the $1-$form Ω is written

$$\tilde{\Omega} = t(1 - zt) dx + \left(t^2 - x\right) dt + t^2 x dz.$$

Therefore, the singular locus of the pull-back foliation is still the $z-$axe in the coordinates (x, t, z) and in a neighborhood of $x = 0$ the foliation $\tilde{\Omega}$ is transverse to the fibration $z = cst$. If we blow-up again the z-axe we find

$$(1 - zx) dt + (1 - zt) dx + txdz$$

which is smooth. Since the curve $x = t = 0$ is invariant and since the foliation is still transverse to the fibration $z = cts$, the unfolding is equisingular. Now, this unfolding is analytically trivial if and only if the monomial term x^3y^2 belongs to the ideal generated by $2x^3y + zx^2y^2 - y^3$ and $xy^2 - x^4$ [5]. Setting $z = 0$ this would imply that $x^3y^2 \in \left(2x^3y - y^3, xy^2 - x^4\right)$ which is impossible. Thus, this unfolding is not analytically trivial and since the moduli space of unfolding of absolutely dicritical foliations is of dimension 1, it is also semi-universal.

Now, let us consider a foliation $\mathcal{F}$ as in the proposition. Up to some linear change of coordinate, we can suppose that after the reduction process its singular point and its transversal structure are the same as the function $\frac{x^2+y^3}{xy}$ that is to say $(0,0)$ and Id in the standard coordinates associated to the reduction process. Let us denote by $\mathcal{F}_0$ the foliation given by $\frac{x^2+y^3}{xy}$. We are going to construct an unfolding from $\mathcal{F}_0$ to $\mathcal{F}$. As always since the beginning of this article, we denote by D_1 and D_2 the two exceptionnal component of the divisor. In the neighborhood of each of them, both foliation are purely radial. Thus there exists two conjugacy Φ_1 and Φ_2 defined in the neighborhood of respectively D_1 and D_2 such that

$$\begin{array}{cc} \Phi_1^*\mathcal{F}_0 = \mathcal{F} & \Phi_2^*\mathcal{F}_0 = \mathcal{F} \\ \Phi_1|_{D_1\cup D_2} = \mathrm{Id} & \Phi_2|_{D_1\cup D_2} = \mathrm{Id} \end{array}.$$

Since, $\mathcal{F}_0$ and $\mathcal{F}$ have the same transversal structures, the cocycle $\Phi_1 \circ \Phi_2^{-1}$ is a germ automorphism of $\mathcal{F}_0$ near the singular point of the divisor that lets fix the points of the divisor and that let globally fix each leaf. It is easy to see that one can construct an isotopy from $\Phi_1 \circ \Phi_2^{-1}$ to Id in the group of germs of automorphisms of $\mathcal{F}_0$ near the singular point of the divisor that let fix each point of the divisor and that let globally fix each leaf. Let us denote by Φ_t this isotopy satisfying $\Phi_0 = \mathrm{Id}$ and $\Phi_1 = \Phi_1 \circ \Phi_2^{-1}$. The unfolding defined by the following glued construction

$$((\mathcal{F}_0, D_1) \times U) \coprod ((\mathcal{F}_0, D_2) \times U) \big/ (x,t) \sim (\Phi_t(x), t).$$

where U is an open neighborhood of $\{|t| \leq 1\}$ links $\mathcal{F}_0$ and $\mathcal{F}$. The meromophic first integral f_0 of $\mathcal{F}_0$ can be extended in a meromorphic first integral F of the whole unfolding [5]. Thus $F|_{t=1}$ is a meromorphic first integral of $\mathcal{F}$. By equisingularity $F|_{t=0}$ and $F|_{t=1}$ must have exactly the same number of irreducible components in their zeros and in their poles, which is the same number of irreducible components in the zeros and in the poles of F. They also must have the same topology since an unfolding is topologically trivial. Thus the foliation $\mathcal{F}$ admits a meromorphic first integral whose zero is exactly the leaf passing through the singular point of the exceptionnal divisor and whose poles are the union of two smooth curves attaching respectively to D_1 and D_2. Thererfore up to some change of coordinates, we can suppose that $\mathcal{F}$ has a meromorphic first integral of the form

$$f = \frac{\left(y^2 + x^3 + \Delta(x,y)\right)^a}{x^b y^c}$$

where the Taylor expansion of $\Delta(x,y)$ admits monomial term $x^i y^j$ with $2i + 3j > 6$. The absolutely dicritical property ensures that $a = b = c$. Therefore, we can suppose that $a = b = c = 1$. Let us denote by $\Lambda_\lambda(x,y)$ the homothetie $\Lambda_\lambda(x,y) = \left(\lambda^2 x, \lambda^3 y\right)$. Composing by Λ_λ at the right of f yields

$$\frac{f \circ \Lambda_\lambda}{\lambda} = \frac{y^2 + x^3 + \Delta_\lambda(x,y)}{xy}$$

For any $\lambda \neq 0$, the foliation given by f and by $\frac{f\circ\Lambda_\lambda}{\lambda}$ are analytically conjugated. But the deformation given by $\lambda \to \frac{f\circ\Lambda_\lambda}{\lambda}$ is an equisingular unfolding of f_0 since Δ_λ goes to 0 when $\lambda \to 0$. Using the semi-universality of the family introduced at the beginning of the proof, for λ small enough, there exists some α such that the following conjugacies holds

$$f \sim \frac{f \circ \Lambda_\lambda}{\lambda} \sim f_\alpha.$$

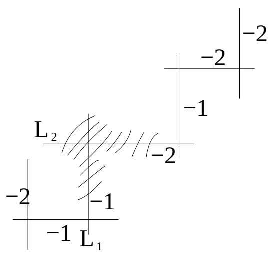
Now if $\alpha = 0$ then f is of type (1). If $\alpha \neq 0$, applying some well-chosen homothetie, we can suppose $\alpha = 1$. And f is of type (2). $\square$

Remark 13. In the last part of this article, we will prove that actually the two functions (1) and (2) of the previous result define two foliations analytically equivalent.

It is possible to construct some others examples of absolutely dicritical foliations of cusp type with a rationnal first integral: to do so, consider a foliation of degree 1 on $\mathbb{P}^2$. These are well-known [4]: they have three singular points counted with multiplicities and admit an integrating factor. For instance, the foliation given in homogeneous coordinates by the multivalued functions

$$[x:y:z] \to \frac{x^\alpha y^\beta}{z^{\alpha+\beta}} \quad \text{or} \quad [x:y:z] \mapsto \frac{Q}{z^2}$$

where Q is a non-degenerate quadratic form is of degree 1. When α and β are rationnal numbers, the foliation admits a rationnal first integral. Now consider two generic lines L_1 and L_2. Each of them is tangent to one leaf of the foliation. We can suppose that the tangency point is different from the intersection point of L_1 and L_2. Now, blow-up twice the tangency point on L_1 and thrice the tangency point on L_2. The final configuration is the following



Thus, the divisor $L_1 \cup L_2$ can be contracted toward a smooth algebraic manifold. The obtained singularity is naturally absolutely dicritical of cusp type and admits a rationnal first integral. For instance, if we consider the foliation given in affine coordinates by $xy = cst$ and $L_1 : x + y = 1$ and $L_2 : x - y = 1$, the transverse structure is equivalent to $\sigma(t) = t + 1$ and thus the foliation is equivalent to the functions of proposition (12). However, considering the foliation given by $x + y^2 = cst$ yields the transverse structure $t \mapsto \frac{1-\sqrt{1+12t+4t^2}}{2}$ which is not an homography.

5. Moduli of Mattei.

5.1. **The parameter space of the unfoldings.** As already explain, the deformation $\alpha \to \mathcal{F}_{\alpha,\sigma}$ is an unfolding with a set of paramater equal to $\mathbb{C}$. It is a natural problem to ask if two parameters define two foliations analytically equivalent. In order to do so, we introduced the following definition:

Definition 14. Let σ be an element of $\mathrm{Diff}(\mathbb{C}, 0)$. An homography h with $h(0) = 0$ is called an homographic symetry of f if and only if there exists an homography h_1 such that

$$h_1 \circ \sigma \circ h = \sigma. \tag{5.1}$$

We denote by $\mathcal{H}(f)$ the group of homographic symetries of f.

The following result is probably known but we do not find any reference in the litterature.

Lemma 15. *If $\mathcal{H}(f)$ is infinite then f is an homography and $\mathcal{H}(f)$ is the whole set of homographies fixing the origin.*

Proof. The relation (5.1) is equivalent to the functionnal equation

$$f \circ h(z) = \frac{1}{(h')^2} f(z)$$

where $f = S(\sigma)$ is the Schwartzian derivative of σ. Let us write $h(z) = \frac{z}{a+bz}$ and $f(z) = \sum_{n\geq 1} f_n z^n$.

(1) Suppose that $h^{'}(0)$ is not a root of unity. Then applying the above relation at $z = 0$ leads to $f(0) = 0$. Now, we have

$$a^2 \sum_{n\geq 1} f_n \frac{z^n}{(a+bz)^n} \quad = \quad (a+bz)^4 \sum_{n\geq 1} f_n z^n.$$

An easy induction on n show that for any n $f_n = 0$, thus $f = 0$ and σ is an homography.

(2) Suppose now $h^{'}(0) = 1$ then

$$\sum_{n\geq 0} f_n \frac{z^n}{(1+bz)^n} \quad = \quad (1+bz)^4 \sum_{n\geq 0} f_n z^n$$

Suppose that $b \neq 0$. If for any $n \leq N-1$ we have $f_n = 0$, let us have a look at the terms in x^{N+1} in the above equality. It is

$$-Nbf_N + f_{N+1} = 4bf_N + f_{N+1}$$

Thus $f_N = 0$. Which, proves by induction that f still is equal to zero.

(3) If $\mathcal{H}(f)$ is infinite, suppose it admits two elements h and g that did not commute, then $[h, g]$ is tangent to Id but is not the Id . Thus using the above computation, $f = 0$.

(4) Finally, if $h^{'}(0)$ is a root of unity, it is easly seen that $h^{\circ(n)} = \mathrm{Id}$ where n is the smallest integer such that $h^{'}(0)^n = 1$. Thus, suppose that the group $\mathcal{H}(f)$ is abelian and any element of finite order. We have an embedding

$$\mathcal{H}(f) \longrightarrow \mathrm{Aff}(\mathbb{C})$$

since, the sole element tangent to Id is the identity itself. Therefore, $\mathcal{H}(f)$ can be seen as abelian subgroup of $\mathrm{Aff}(\mathbb{C})$. Hence, the group has a fix point and can be seen as a subgroup of the linear transformations of $\mathbb{C}$. Now let us write the relation on the Schwartzian seen at ∞

$$f(1/(1/h(1/z))) = \frac{1}{h'\left(\frac{1}{z}\right)^2} f\left(\frac{1}{z}\right).$$

Setting, $u(z) = \frac{1}{z^4} f\left(\frac{1}{z}\right)$ yields $u(az+b) = \frac{1}{a^2} u(z)$. Since, $u = \frac{\alpha}{z^4} + \cdots$ we can consider the double primitive function $U = \iint u$ with $U(\infty) = 0$. This is a univalued holomorphic function defined near ∞. Finally, the function U satisfies the following functionnal relation

$$U(az+b) = U(z).$$

But in view of the dynamics of $\mathrm{Lin}(\mathbb{C})$, it is clear that if $\mathcal{H}(f)$ is infinite then $U = \mathrm{cst}$ and thus $u = 0$.

□

In the course of the proof of the above result, we obtain the following result

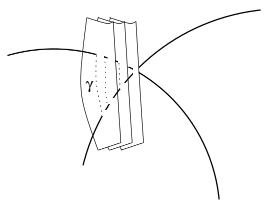
Corollary 16. *Let $\mathcal{M}$ be the quotient of $\mathbb{C}$ by the relation $\alpha \sim \alpha^{'}$ if and only if $\mathcal{F}_{\alpha,\sigma} \sim \mathcal{F}_{\alpha',\sigma}$ then there is only two possibilities*

(1) $\mathcal{M} = \{0\}$ *when σ is an homography - $\mathcal{F}_{\alpha,\sigma}$ is then analytically to $\frac{y^2+x^3}{xy}$.*
(2) $\mathcal{M} = {}^{\mathbb{C}}/_{H}$ *where H is a finite subgroup of $\mathit{Aff}(\mathbb{C})$.*

Genrerically, H is reduced to $\{Id\}$.

As an obvious consequence, the functions obtained in proposition (12) define two foliations analytically equivalent.

5.2. **Toward a geometric description of the moduli of Mattei.** It remains to give a geometric interpretation of the parameter α. A promising approach is the following. Near the singular point of the divisor, the leaf is conformally equivalent to a disc minus two points which are the intersections between the leaf and the exceptionnal divisor. If we consider in the leaf a path linking this two points, we obtain after taking the image of this path by E, an *asymptotic cycle* γ as defined in [11]which is not topologically trivial.



Therefore, considering the family of these cycles parametrized by a transversal parameter to the foliation yields a vanishing asymptotic cycle. We claim that the moduli of Mattei should be associated to the *length* of this vanishing asymptotic cycle: more precisely, it should be computed by the integral of some form along this vanishing cycle. Actually, it easy to prove the following: let ω be a $1-$form defining an absolutely dicritical foliation of cusp type and let η be any germ of 1 form. Then η is relatively exact with respect to ω, i.e., there exist two germs of holomorphic functions f and g such that

$$\eta = df + g\omega$$

if and only if the integral of η along any asymptotic cycle γ vanish. Thus, we think that in a sense that has to be worked out, the moduli of Mattei should be computed by the integral of some generator of the relative cohomology group of ω along the asymptotic vanishing cycle.

References

[1] C. Camacho, A. Lins Neto, and P. Sad. Topological invariants and equidesingularization for holomorphic vector fields. *J. Differential Geom.*, 20(1):143–174, 1984.

[2] C. Camacho, H. Movasati, and P. Sad. Fibered neighborhoods of curves in surfaces. *J. Geom. Anal.*, 13(1):55–66, 2003.

[3] Felipe Cano and Nuria Corral. Absolutely dicritical foliations. *Int. Math. Res. Not. IMRN*, (8):1926–1934, 2011.

[4] D. Cerveau, J. Déserti, D. Garba Belko, and R. Meziani. Géométrie classique de certains feuilletages de degré deux. *Bull. Braz. Math. Soc. (N.S.)*, 41(2):161–198, 2010.

[5] D. Cerveau and J.-F. Mattei. *Formes intégrables holomorphes singulières*. Astérisque, 1982.

[6] Y. Genzmer and L. Teyssier. Existence of non-algebraic singularities of differential equation. *J. Differential Equations*, 248(5):1256–1267, 2010.

[7] R. Godement. Théorie des faisceaux. *Hermann, Paris*, 1973.

[8] H. Grauert. Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen. *Inst. Hautes Études Sci. Publ. Math.*, (5):64, 1960.

[9] F. Loray. *Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux*. Preprint edition, 2006.

[10] J.-F. Mattei. Modules de feuilletages holomorphes singuliers. I. Équisingularité. *Invent. Math.*, 103(2):297–325, 1991.

[11] L. Teyssier. Analytical classification of singular saddle-node vector fields. *J. Dynam. Control Systems*, 10(4):577–605, 2004.