Moduli spaces for topologically quasi-homogeneous functions.

Yohann Genzmer*, Emmanuel Paul

September 29, 2015

Abstract

We consider the topological class of a germ of 2-variables quasi-homogeneous complex analytic function. Each element f in this class induces a germ of foliation ($\mathrm{d}f = 0$) and a germ of curve ($f = 0$). We first describe the moduli space of the foliations in this class and we give analytic normal forms. The classification of curves induces a distribution on this moduli space. By studying the infinitesimal generators of this distribution, we can compute the generic dimension of the moduli space for the curves, and we obtain the corresponding generic normal forms. [1]

Introduction

From any convergent series f in $\mathbb{C}\{x, y\}$, we can consider three different associated mathematical objects: a germ of holomorphic function defined by the sum of this series, a germ of foliation whose leaves are the connected components of the level curves $f =$ constants, and an embedded curve $f = 0$. Composing f on the left side by a diffeomorphism of $(\mathbb{C}, 0)$ may change the function but nor the foliation or the curve. Multiplying f by an invertible function u may change the function and the foliation but not the related curve. Therefore, there are three different analytic equivalence relations:

- The classification of functions (or right equivalence):

$$f_0 \sim_r f_1 \Leftrightarrow \exists \phi \in \text{Diff } (\mathbb{C}^2, 0),\ f_1 = f_0 \circ \phi.$$

- The classification of foliations (or left-right equivalence):

$$f_0 \sim f_1 \Leftrightarrow \exists \phi \in \text{Diff } (\mathbb{C}^2, 0),\ \psi \in \text{Diff } (\mathbb{C}, 0),\ \psi \circ f_1 = f_0 \circ \phi.$$

- The classification of curves:

$$f_0 \sim_c f_1 \Leftrightarrow \exists \phi \in \text{Diff } (\mathbb{C}^2, 0),\ \exists u \in \mathcal{O}_2, u(0) \neq 0, u f_1 = f_0 \circ \phi.$$

In the same way, one can define topological classifications requiring only topological changes of coordinates. In what follows, we are going to consider mostly the two last equivalence relations for foliations and curves, since the comparison between the two first analytic classifications has been studied in [1].

Finally, we emphasize that in our work, we will always require that the conjugacies that appear above will respect a fixed numbering of the branches of $f = 0$.

*The first author supported by the ANR under the project ANR-13-JS01-0002-01.
[1]**Keywords**: holomorphic foliation, moduli of curve, singularities.
A.M.S. class.: 34M35, 32S65, 32G13

A germ of holomorphic function $f_{\text{qh}} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is *quasi-homogeneous* if and only if f_{qh} belongs to its jacobian ideal $J(f_{\text{qh}}) = (\frac{\partial f_{\text{qh}}}{\partial x}, \frac{\partial f_{\text{qh}}}{\partial y})$. If f_{qh} is quasi-homogeneous, there exist coordinates (x, y) and positive coprime integers k and l such that the quasi-radial vector field $R = kx\frac{\partial}{\partial x} + ly\frac{\partial}{\partial y}$ satisfies

$$R(f_{\text{qh}}) = d \cdot f_{\text{qh}},$$

where the integer d is *the quasi-homogeneous* (k, l)*-degree* of f_{qh} [15]. In these coordinates, f_{qh} has some cuspidal branches and maybe axial branches, that is to say, f_{qh} is written

$$f_{\text{qh}} = cx^{n_\infty} y^{n_0} \prod_{b=1}^{p} (y^k + a_b x^l)^{n_b} \tag{1}$$

where c is a non vanishing complex number and the multiplicities satisfy $n_0 \geq 0$, $n_\infty \geq 0$ and $n_b > 0$. The complex numbers a_b are non vanishing numbers such that $a_b \neq a_{b'}$. Using a convenient analytic change of coordinates, we may suppose that $a_1 = 1$.
A germ of holomorphic function f is *topologically quasi-homogeneous* if the function f is topologically conjugated to a quasi-homogeneous function f_{qh}, that is to say there is a continuous right-equivalence between f and f_{qh}.

For any couple of coprime positive integers (k, l) with $k < l$ and $(p+2)$-uple (n) of integers in $\mathbb{N}^2 \times (\mathbb{N}^*)^p$, $(n) = (n_\infty, n_0, n_1, n_2, \cdots, n_p)$ we consider the topological class $\mathcal{T}_{(k,l),(n)}$ of f_{qh} defined in (1), that is the set of all functions topologically conjugated to f_{qh}. The first aim of this paper is to describe the moduli space defined by the quotient

$$\mathcal{M}_{(k,l),(n)} = \mathcal{T}_{(k,l),(n)}/_\sim$$

where $\sim$ refers to the left-right analytical equivalence. We give the infinitesimal description of this moduli space by making use of the cohomological tools considered by J.F. Mattei in [13]: the tangent space to the moduli space is given by the first Cěch cohomology group $H^1(D, \Theta_{\mathcal{F}})$, where D is the exceptional divisor of the desingularization of f_{qh}, and $\Theta_{\mathcal{F}}$ is the sheaf of germs of vector fields tangent to the desingularized foliation $\widetilde{\mathcal{F}}$ induced by $\mathrm{d}f_{\text{qh}} = 0$. Using a particular covering of D, we give a triangular presentation of the $\mathbb{C}$-space $H^1(D, \Theta_{\mathcal{F}})$ in Theorem (1.3). This description leads us to consider *triangular analytic normal forms*

$$N_a = x^{n_\infty} y^{n_0} \prod_{b=1}^{p} (y^k + \sum_{\{(b,d), \Phi(b,d) \in \mathbb{T}\} \cup \{(1,kl)\}} a_{b,d} m^d)^{n_b} \tag{2}$$

by perturbing the topological normal form (1) with some monomials m^d following an algorithm described in the subsection (1.2), in which the precise meaning of the indexation $\Phi(b, d)$ is defined. This family of analytic normal forms turns out to be semi-universal as established in Theorem (1.10). In this way, we obtain a local description of $\mathcal{M}_{(k,l),(n)}$. We finally give a global description of this moduli space in Theorem (1.15) and Theorem (1.16) by proving that any function in $\mathcal{T}_{(k,l),(n)}$ is actually conjugated to some normal form N_a , and that the parameter a is unique up to some weighted projective action of $\mathbb{C}^*$. All the results of this first part can be extended to the generic Darboux function:

$$f^{(\lambda)} = f_1^{\lambda_1} \cdots f_p^{\lambda_p}$$

with complex multiplicities λ_i. Nevertheless, we do not insert this extension here, since we have previously explain in [8] how to perform it in the topologically homogeneous case.

The second part of our work is dedicated to the study of the moduli space of curves in the quasi-homogeneous topological class. This problem is a particular case of an open problem known as

the *Zariski problem.* It has only a very few answers: Zariski [17] for the very first treatment of some particular cases, Hefez and Hernandes [5, 6] for the irreducible curves, Granger [9] in the homogeneous topological class and [2] for some results which are particular cases of our present result. Our strategy that we already introduced in a previous work [8], differs from all this works: we consider the integrable distribution $\mathcal{C}$ on the moduli space of foliations $\mathcal{M}_{(k,l),(n)}$ induced by the equivalence relation $\sim_c$: two foliations represented by two points in $\mathcal{M}_{(k,l),(n)}$ are in a same orbit of this distribution if and only if they induce the same curve up to analytic conjugacy. Studying the family of vector fields that induce the distribution $\mathcal{C}$ on $\mathcal{M}_{(k,l),(n)}$, we compute the dimension of the generic strata of the moduli space of curves $\mathcal{M}_{(k,l),(n)}/_{\mathcal{C}}$ in Theorem (2.7). We also give an algorithm in order to construct the corresponding generic normal forms in Theorem (2.8).
Since the cohomological description of the moduli space of foliations is known for a general one-form, we may expect that this strategy can be develop in a general topological class.

In order to keep a sufficiently readable text, we have postponed a lot of technical computations in appendix A.

We thank Jean-François Mattei for helpful discussions, and for suggesting improvements on a first version of this work.

Contents

1 The moduli space of foliations

In this section, we will consider a function f in the class $\mathcal{T}_{(k,l),(n)}$. The (k,l)-degree of a monomial $x^m y^n$ is $km + ln$. It induces a valuation on $\mathbb{C}\{x,y\}$ denoted by $\nu_{k,l}$.

Let f be a function in the topological class $\mathcal{T}_{(k,l),(n)}$. We know, from a theorem of Lejeune-Jalabert [10] that the desingularization process of f is identical to that of f_{qh}, that is to say: after a sequence of blowing-ups E, the exceptional divisor D is a chain of components isomorphic to $P^1(\mathbb{C})$, the strict transform of the cuspidal branches intersect the same component D_c, *the* *principal component*, and the strict transform of the axes, if they appear, intersect the end components of this chain : see Appendix A, and figure (2).

Lemma 1.1 (Prenormalization)**.** *There exists some coordinates (x, y) such that f is written*

$$f(x,y) = cx^{n_\infty} y^{n_0} \left(y^k + x^l + \cdots\right)^{n_1} \left(y^k + a_2 x^l + \cdots\right)^{n_2} \cdots \left(y^k + a_p x^l + \cdots\right)^{n_p}$$

where c is a non-vanishing complex number, $a_b,\ b = 2, \ldots, p$ are non-vanishing complex numbers with $a_b \neq a_{b'} \neq 1$, and the dots are terms of (k,l)-degree greater than kl.

Proof. Let f be a function topologically conjugated to f_{qh}. The number of branches, and their multiplicities are topological invariants. Therefore, we consider the following irreducible decomposition of f:

$$f = f_\infty^{n_\infty} f_0^{n_0} f_1^{n_1} \cdots f_p^{n_p}.$$

Since f has the same desingularization process as f_{qh}, if $n_0 > 0$ or $n_\infty > 0$, the strict transform of the corresponding branches appear on the end components. Therefore, their blowing-down are smooth transverse branches at 0, and we can choose coordinates (x, y) such that

$$f = x^{n_\infty} y^{n_0} f_1^{n_1} \cdots f_p^{n_p}.$$

Now, the strict transform of the other branches meet the principal component D_c. Using the blown-down formulas of proposition (3.2) in Appendix A, we obtain that:

$$f_i = \alpha_i y^k + \beta_i x^l + \cdots$$

with $\alpha_i \neq 0$ and $\beta_i \neq 0$. By factorizing α_i in each component f_i we obtain the existence of a non-vanishing constant c and a family of p non-vanishing complex number a_b, $b = 1, \ldots, p$ such that

$$f = cx^{n_\infty} y^{n_0} \left(y^k + a_1 x^l + \cdots\right)^{n_1} \cdots \left(y^k + a_p x^l + \cdots\right)^{n_p}$$

where the dots are terms of (k,l) −degree greater than kl. Finally, by applying a final change of coordinates of the form $(x, y) \to (\alpha x, y)$, we can suppose that $a_1 = 1$. □

Unless any precision is given, from now on, we will only consider system of coordinates (x, y) such that the function $f \in \mathcal{T}_{(k,l),(n)}$ has an expression as in the above lemma.

1.1 The infinitesimal description

Since the transverse structure of a foliation defined by a function is rigid, i.e. completely given by the discrete data of the multiplicities, any topologically trivial deformation is an unfolding as defined in [13]. We know from the same reference that the tangent space to the moduli space of unfoldings of a germ of analytic foliation $\mathcal{F}$ is the vector space: $H^1(D, \Theta_\mathcal{F})$, where $\Theta_\mathcal{F}$ is the sheaf on D of germs of holomorphic vector fields tangent to the desingularized foliation $\widetilde{\mathcal{F}}$. Furthermore, this vector space is a finite dimensional one, whose dimension δ is obtained by a formula involving the multiplicities of the foliation at the singular points appearing at each step of the blowing up process. In the present topological class, we will give an alternative description of this tangent space which will allow us to construct normal forms.

Let f be in $\mathcal{T}_{(k,l),(n)}$. We consider the saturated foliations $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ induced by df and E^*df, where E is the desingularization morphism of f.

Notation 1.2.

1. *We define two integers ε_0 and ε_∞ in $\{0, 1\}$ as follows: if $n_0 > 0$ then we set $\varepsilon_0 = 1$, else we set $\varepsilon_0 = 0$. We define ε_∞ the same way but relative to n_∞.*

2. *Let (u,v) be the unique couple of integers defined by the Bézout identity*

$$uk - vl = 1 \text{ with } 0 \le u < l,\ 0 \le v < k.$$

3. *We denote by ν_c the multiplicity of the desingularized foliation on the principal component D_c of the exceptional divisor. According to Proposition (3.4) in Appendix A, we have*

$$\nu_c = klp - k - l + k\varepsilon_\infty + l\varepsilon_0.$$

4. *Let $\mathbb{T}$ be the triangle in the real half plane (X,Y), $Y \ge 0$, delimited by*

$$\begin{array}{rcl} kX - (k-v)(Y-\nu_c) & > & 0 \\ lX - (l-u)(Y-\nu_c) & < & 0 \end{array}$$

The summit of this triangle is $(0,\nu_c)$. The directions of the non horizontal edges are given by the vectors

$$\vec{x} = (k-v,k) \quad \text{and } \vec{y} = (l-u,l).$$

Theorem 1.3. *There is an explicit linear isomorphism Ψ between $H^1(D,\Theta_{\mathcal{F}})$ and the $\mathbb{C}$-vector space freely generated by the set of integer points $e_{i,j} = (i,j)$ in the triangle $\mathbb{T}$.*

The expression of Ψ is given in the proof below. We give a presentation of the tangent space to the moduli space of a function in the topological class: $(k,l) = (3,5)$, $p = 4$, $n_0 = n_\infty = 0$, $n_1,\ldots,n_4$ arbitrary, obtained by this theorem in Appendix B, Figure (3).

Proof. Let us consider θ_f the vector field with an isolated singularity defined by

$$\theta_f = \frac{1}{\text{g.c.d.}\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)} \cdot \left(\frac{-\partial f}{\partial x}\frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right). \tag{3}$$

Let $\{U_0,U_\infty\}$ be the covering of the exceptional divisor introduced in the Appendix A. From proposition 3.6, we know that this covering is acyclic with respect to the sheaf $\Theta_{\mathcal{F}}$. Therefore we have

$$H^1(D,\Theta_{\mathcal{F}}) = \frac{\Theta_{\mathcal{F}}(U_0\cap U_\infty)}{\Theta_{\mathcal{F}}(U_0)\oplus\Theta_{\mathcal{F}}(U_\infty)}.$$

In order to compute each term of this quotient, we consider the principal chart (x_c,y_c) defined near the central component D_c defined in Appendix A. The domain of this chart contains $U_0\cap U_\infty$. The vector field

$$\theta_{\text{is}} = \frac{E^*\theta_f}{y_c^{\nu_c}}$$

has isolated singularities and defines $\mathcal{F}$ on $U_0\cap U_\infty$. Therefore we have $\Theta_{\mathcal{F}}(U_0\cap U_\infty) = \mathcal{O}(U_0\cap U_\infty)\cdot\theta_{\text{is}}$, and each θ in $\Theta_{\mathcal{F}}(U_0\cap U_\infty)$ can be written

$$\theta = \left(\sum_{i\in\mathbb{Z},j\in\mathbb{N}} \lambda_{i,j}\ x_c^i y_c^j\right)\cdot\theta_{is}.$$

By the local monomial expression of E given by proposition 3.2 in Appendix A, these vector fields θ blow down on meromorphic vector fields with poles on the axes:

$$E_*\theta = \sum_{i\in\mathbb{Z},\ j\in\mathbb{N}} \lambda_{i,j}\ x^{li-(l-u)(j-\nu_c)} y^{-ki+(k-v)(j-\nu_c)}\cdot\theta_f.$$

Let us prove that θ has an holomorphic extension on U_0 if and only if

$$-ki + (k-v)(j-\nu_c) < 0 \implies \lambda_{i,j} = 0. \qquad (\star)$$

If such an extension is possible, then θ has no pole along the curve $y = 0$ whose strict transform belongs to U_0, thus the property $(\star)$ holds. On the converse, if the property $(\star)$ is satisfied, then the multiplicity $\nu_{D_1}(\theta)$ of θ along the component D_1 which meets the strict transform of the x-axis is positive. Indeed, after a standard blow-up, we find

$$\nu_{D_1}(\theta) \geq \min_{\lambda_{i,j}\neq 0} \{(l-k)i + (j-\nu_c)(k-v-l+u)\} \geq 0$$

Now, the intermediate multiplicities $\nu_{D_i}(\theta)$, $1 < i < c$ are also positive. This is a consequence of the relations

$$\nu_{D_2}(\theta) = e_1\nu_{D_1}(\theta),\ \nu_{D_{i+1}}(\theta) = e_i\nu_{D_i}(\theta) - \nu_{D_{i-1}}(\theta),\ i = 2,\ldots,c-1$$

which can be obtained by a similar argument as in proposition 3.4. Here, $-e_i$ is the self-intersection of the component D_i. Since $e_i \geq 2$ for $i = 1,\ldots,c-1$, we have

$$\nu_{D_2}(\theta) \geq \nu_{D_1}(\theta),\ \nu_{D_{i+1}}(\theta) - \nu_{D_i}(\theta) \geq \nu_{D_i}(\theta) - \nu_{D_{i-1}}(\theta),\ i = 2,\ldots,c-1$$

which proves that $\nu_{D_i}(\theta)$ is positive for any $i = 1,\ldots,c$. In the same way, an element θ in $\Theta_{\mathcal{F}}(U_0 \cap U_\infty)$ belongs to $\Theta_{\mathcal{F}}(U_\infty)$ if and only if

$$li - (l-u)(j-\nu_c) < 0 \implies \lambda_{i,j} = 0.$$

Therefore, there is a linear isomorphism Ψ between the $\mathbb{C}$-space freely generated by the integer points $e_{i,j}$ in $\mathbb{T}$ and $H^1(D,\Theta_{\mathcal{F}})$ defined by:

$$\Psi : \sum_{(i,j)\in\mathbb{T}} \lambda_{i,j}e_{i,j} \longmapsto \left[\left(\sum_{(i,j)\in\mathbb{T}} \lambda_{i,j}x_c^i y_c^j\right)\cdot\theta_{is}\right]. \qquad (4)$$

□

This representation gives us a direct formula for the dimension δ of $H^1(D,\Theta_{\mathcal{F}})$, by counting the integers points in the above triangle. In order to give an explicit formula, we need the following fact:

Lemma 1.4 (and notations). *The number of integer points in an open interval $]a,b[$ is given by $]b] - [a[$, where $[a[$ stands for the usual integer part n of a: $n \leq a < n+1$, and $]b]$ is the "strict" integer part m of b defined by $m < b \leq m+1$.*

Since the intersections of the horizontal levels j with the boundary of $\mathbb{T}$ are respectively given by $\frac{k-v}{k}(j-\nu_c)$ and $\frac{l-u}{l}(j-\nu_c)$, we obtain

Proposition 1.5. *The dimension of $H^1(D,\Theta_{\mathcal{F}})$ is*

$$\delta = \sum_{j=0}^{\nu_c}\left(\left]\frac{l-u}{l}(j-\nu_c)\right] - \left[\frac{k-v}{k}(j-\nu_c)\right[\right).$$

Example. For the topological class given by $(k,l) = (3,5)$, $p = 4$, without axis, by counting the integers points in figure (3) in Appendix B, or applying the previous formula, we obtain that $\delta = 78$.

1.2 Construction of the local normal forms

We will construct here analytic models for topologically quasi-homogeneous functions starting from the topological normal form (1). Since it already appears $(p-1)$ analytic invariants that are the values a_b (the cross ratios between branches on the principal component), we have to add $\delta-(p-1)$ monomial terms of higher degrees. The construction to come is *a priori* based upon some algorithmic but arbitrary choices. It will be justified by Theorem (1.10) in the next section.

In our previous work in [8], for the *homogeneous* topological class, in which the topological representative was p transverse lines, we straightened the fourth first lines on $xy(y+x)(y+a_{4,1}x)$, we added the monomials $a_{5,2}x^2$ to the fifth line, $a_{6,2}x^2+a_{6,3}x^3$ to the sixth, and so on. We generalize this triangular construction here by making use of the quasi-homogeneous (k,l)-degree. Nevertheless, the choice of the monomials and their distribution between the branches is not so obvious here.

The following algorithm will associate an analytic normal form starting from the previous triangular presentation of the infinitesimal moduli space.
The figure (3) in Appendix B shows the procedure in order to construct the normal forms associated to the topological class of

$$\left(y^3+x^5\right)^{n_1}\left(y^3+a_2x^5\right)^{n_2}\left(y^3+a_3x^5\right)^{n_3}\left(y^3+a_4x^5\right)^{n_4}.$$

The meaning of all the datas that appear on the figure will be detailed below.
The construction consists in two successive steps.

Step 1. Choice of the monomials.

Notation 1.6. *For any $d\geq kl$, there exists a unique monomial x^iy^j with quasi-homogeneous (k,l)-degree d, such that $j<k$. We denote it the following way*

$$m^d := x^iy^j \qquad ik+jl=d,\ j<k.$$

For $(k,l)=(3,5)$, we find $m^{15}=x^5$, $m^{16}=x^2y^2$, $m^{17}=x^4y$, $m^{18}=x^6$,...

Therefore, to each horizontal line of index j in the triangle $\mathbb{T}$, one can associate the monomial m^d, $d=kl+j$. We put them on a column on the right side in Figure (3).

Step 2. Distribution of the monomials between the cuspidal branches. The link between the monomial terms m^d and m^{d+1} is the multiplication by the meromorphic monomial term m^{d+1}/m^d. We encode this multiplication by a translation in $\mathbb{T}$. We associate to the multiplication by x (resp. y) the translation by $\vec{x}=(k-v,k)$ (resp. $\vec{y}=(l-u,l)$). This choice is suggested by the formulas of Proposition (3.2) in Appendix A. Thus to a degree d we associate the translation in $\mathbb{Z}^2$ by the vector $\vec{t_d}$ defined by

$$\vec{t_d}=i\vec{x}+j\vec{y}$$

where $x^iy^j=m^{d+1}/m^d$.

Lemma 1.7. *For any d, $\vec{t_d}$ is either* $(1,1)$ *or* $(0,1)$.

Proof. Let $m^d=x^iy^j$ and thus $ik+jl=d$ with $0\leq j<k$. Suppose first that $j-v\geq 0$. Then $m^{d+1}=x^{i+u}y^{j-v}$. Hence, in the the canonical basis, the components of $\vec{t_d}$ are

$$(i+u-i)\,(k-v,k)+(j-v-j)\,(l-u,l)=(1,1)\,.$$

If $j-v<0$ then $m^{d+1}=x^{i+u-l}y^{j+k-v}$. Indeed, we have $0\leq j+k-v<k$ and $i+u-l\geq 0$ since from

$$(i+u)\,k=kl+1-(j-v)\,l>kl.$$

In this case, the components of $\vec{t_d}$ are

$$(u-l)\,(k-v,k)+(k-v)\,(l-u,l)=(0,1)\,.$$

□

For $(k,l)=(3,5)$, the meromorphic monomials form a periodic sequence of lenght 3 generated by: y^2/x^3, x^2/y, x^2/y. The successive translations are $\vec{t_{15}}$, $\vec{t_{16}}$, $\vec{t_{17}}$, $\vec{t_{18}}=\vec{t_{15}}$ etc..., whose components are $(0,1)$, $(1,1)$, $(1,1)$. We put the translations on a column on the right side of Figure (3).

Now we consider all the parallel paths issued from the integer points $(i,0)$ on the horizontal axe, under the action of the successive translations $\vec{t_d}$. The point $\left(-\nu_c\frac{k-v}{k},0\right)$ is the intersection of the left edge of the triangle with this horizontal axe. We consider the p integer points:

$$M_1:=\left([-\nu_c\frac{k-v}{k}[+p,0\right),\ M_2:=\left([-\nu_c\frac{k-v}{k}[+p-1,0\right),\ldots,\ M_p:=\left([-\nu_c\frac{k-v}{k}[+1,0\right).$$

Notice that the $(p-1)$ last ones are inside the triangle, while the first one is outside.

Proposition 1.8. *The p paths issued from the initial points M_i, $i=1,\ldots,p$, obtained by the action of the successive translations $\vec{t_d}$ pass through all the integer points inside the triangle $\mathbb{T}$.*

Proof. Let i_n and j_n such that $m^{kl+n}=x^{i_n}y^{j_n}$. Following the arguments in the proof of Lemma (1.7), the sequence (i_n,j_n) is explicitely defined by the following system

$$\begin{cases} i_n=l+ua_n-(l-u)\,b_n\\ j_n=-va_n+(k-v)\,b_n\\ i_nk+j_nl=d_0+n\\ j_n<k\end{cases}$$

where $d_0=kl$ and (a_n,b_n) is defined by $(a_0,b_0)=(0,0)$ and

$$\begin{array}{rcll}\begin{pmatrix}a_{n+1}\\ b_{n+1}\end{pmatrix} & = & \begin{pmatrix}a_n\\ b_n\end{pmatrix}+\begin{pmatrix}1\\ 0\end{pmatrix} & \text{if } j_n-v\geq 0\\ & = & \begin{pmatrix}a_n\\ b_n\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix} & \text{if } j_n-v<0.\end{array}$$

Notice that a_n is the number of translations of type $(1,1)$ occuring in a path of lenght n, and corresponds to the horizontal component of the sum of the n first translations. We consider the left side of the triangle given by the equation

$$ki-(k-v)j+\nu_c(k-v)=0$$

and its intersections (x_n,n) with the horizontal levels $j=n$. We have

$$x_n=\frac{k-v}{k}(n-\nu_c).$$

We consider the path starting from the last integer point $([x_0[+1,0)$. The successive integer points of this path are given by the sequence $(p_n,n)=([x_0[+1+a_n,n)$. We claim that the moving point along this path does not go too far away from the left side of the triangle. More precisely, we have:

$$(p_n - x_n) \in]-1, 1].$$

Indeed, by solving the above system, we obtain $a_n = \frac{-j_n}{k} + n\frac{k-v}{k}$. Therefore we have:

$$p_n - x_n = ([-\nu_c \frac{k-v}{k}[+\nu_c \frac{k-v}{k} + 1) + (a_n - n\frac{k-v}{k}).$$

Clearly, the first part of the sum belongs to $]0, 1]$, and the second one, which equals to $\frac{-j_n}{k}$ belongs to $]-1, 0]$. Therefore this path will catch all the first integer points of the triangle on each level starting from the left side. If we consider the p parallel paths starting from M_i, $i = 1, \dots, p$, they will catch all the integers points of the triangle, since on each level there is at most p points. $\square$

These p paths give us a unique way to distribute the monomials $a_{b,d}m^d$ on each branch, putting the monomials encountered on the first path (starting from the right hand side) on the first branch, and so on. With this *path game*, we do not miss any point of the triangle according to the previous proposition. Each integer point of the triangle can be represented by the new coordinates (b, d) where b is the index of a path or branch and d the index of a level, or degree. From our construction, they are related to (i, j) by the change of indexation

$$(i, j) = \Phi(b, d) = \left(\left[-\nu_c \frac{k-v}{k} \right[+ p + 1 - b + \sum_{i=kl-1}^{d-1} \alpha_i \ , \ d - kl \right), \tag{5}$$

where $\alpha_{kl-1} = 0$, and for $i \geq kl$, α_i is the horizontal component of $\vec{t_i}$.

To conclude, the general writing of the analytic normal forms for foliations defined by a function in $\mathcal{T}_{(k,l),(n)}$ obtained by our construction is the following definition

Definition 1.9. *Let $\mathcal{A}$ be the following open set of $\mathbb{C}^\delta$*

$$\mathcal{A} = \{(a_{b,d}),\ \Phi(b, d) \in \mathbb{T},\ a_{b,kl} \neq 0,\ a_{b,kl} \neq 1,\ a_{b,kl} \neq a_{b',kl} \text{ for } b \neq b'\}.$$

Furthermore, we set $a_{1,kl} = 1$. For $a = (a_{b,d}) \in \mathcal{A}$ we define the analytic normal form N_a by

$$N_a = x^{n_\infty} y^{n_0} \prod_{b=1}^{p} \left(y^k + \sum_{\{(b,d),\ \Phi(b,d) \in \mathbb{T}\} \cup \{(1,kl)\}} a_{b,d} m^d \right)^{n_b} \tag{6}$$

Example. From the figure (3) in the Appendix B, the analytic normal form N_a of the foliation defined by a function f in the topological class $(k, l) = (3, 5)$, $p = 4$, $n = (n_1, n_2, n_3, n_4)$ are given in the same Appendix: we add 2 monomials on the first branch, 16 on the second, 31 on the third and 29 on the last one.

1.3 Local universality

The construction described in the previous section is justified, *a posteriori*, by the following result. For any $a \in \mathcal{A}$, we consider the saturated foliation $\mathcal{F}_a$ defined by the one-form $\mathrm{d}N_a$.

Theorem 1.10. *For any a^0 in $\mathcal{A}$, the germ of deformation $\{\mathcal{F}_a,\ a \in (\mathcal{A}, a^0)\}$ is an equireducible semi-universal unfolding among the equireducible unfoldings of $\mathcal{F}_{a^0}$.*

This means that for any equireducible unfolding $\{\mathcal{F}_t,\ t \in (T, t^0)\}$ where (T, t^0) is a germ of some space of parameters $t = (t_1, \ldots, t_s)$, such that $\mathcal{F}_{t^0} = \mathcal{F}_{a^0}$, there exists a map $\lambda : T \to A$ with $\lambda(t^0) = a^0$ such that the family $\mathcal{F}_t$ is analytically equivalent to $\mathrm{d}N_{\lambda(t)}$. Furthermore, the universality means that the map λ is unique and the semi-universality only requires that the first derivative of λ at t^0 is unique.

Proof. Let E be the common desingularization map for each foliation $\mathcal{F}_a$ and $\widetilde{\mathcal{F}}_a$ the pull-back of $\mathcal{F}_a$ by E. $\widetilde{\mathcal{F}}_a$ is also the saturated foliation defined by the one-form $\mathrm{d}\widetilde{N}_a$ where $\widetilde{N}_a = N_a \circ E$. Let Θ_0 be the sheaf on $D = E^{-1}(0)$ of germs of holomorphic vector fields tangent to the foliation $\widetilde{\mathcal{F}}_{a^0}$.

Lemma 1.11. *Let $\mathcal{U} = \{U_0, U_\infty\}$ be the covering of the exceptional divisor of E introduced in the Appendix A (notations 3.1). Any unfolding of $\widetilde{\mathcal{F}}_{a^0}$ is locally analytically trivial on each open set U_0, U_∞.*

Proof. Suppose that the unfolding is given by a one-form

$$dF = \frac{\partial F_t}{\partial x} + \frac{\partial F_t}{\partial y} + \sum_{r=1}^{s} \frac{\partial F_t}{\partial t_r},$$

such that dF_{t^0} defines $\widetilde{\mathcal{F}}_{a^0}$. Let m be a point of D, in some local chart (x_i, y_i) of D. For each parameter t_r, we can find a local vector field in some neighborhood U_m of m

$$X_r = \theta_r - \frac{\partial}{\partial t_r} = (\alpha_r(x_i, y_i, t)\frac{\partial}{\partial x_i} + \beta_r(x_i, y_i, t)\frac{\partial}{\partial y_i}) - \frac{\partial}{\partial t_r}$$

such that $d(F \circ E)(X_r) = 0$, which can also be written

$$\theta_r(F \circ E) = \frac{\partial}{\partial t_r}(F \circ E).$$

The existence of X_r is clear around a regular point of the foliation, and still true around a reduced singular point: see [13]. The difference between two such local vector fields is a tangent vector field to the foliation $\widetilde{\mathcal{F}}_{a^0}$. Now, from Proposition (3.6) in Appendix A, we have $H^1(U_0, \Theta_0) = H^1(U_\infty, \Theta_0) = 0$. Therefore we can glue together these vector fields on U_0 or on U_∞. The trivialization of the unfolding on U_0 or U_∞ in the direction $\frac{\partial}{\partial t_r}$ is obtained by integration of these vector fields X_r. □

For each parameter $a_{b,d}$ of the unfolding defined by dN_a, a in $(\mathcal{A}, a^0)$, the previous lemma proves that there exist two vector fields $\theta^0_{b,d}$ in $\Theta_0(U_0)$ and $\theta^\infty_{b,d}$ in $\Theta_0(U_\infty)$ such that

$$\theta^0_{b,d}(\widetilde{N_{a^0}}) = \left.\frac{\partial \widetilde{N_a}}{\partial a_{b,d}}\right|_{a=a^0} \quad \text{and} \quad \theta^\infty_{b,d}(\widetilde{N_{a^0}}) = \left.\frac{\partial \widetilde{N_a}}{\partial a_{b,d}}\right|_{a=a^0}. \tag{7}$$

We call them "*trivializing vector fields in the direction $a_{b,d}$*". We denote by $\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}$ the difference $\theta^0_{b,d} - \theta^\infty_{b,d}$ in $\Theta_0(U_0 \cap U_\infty)$ and by $\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}$ its image in $H^1(D, \Theta_0)$, which does not depend on the choice of the trivializing vector fields. We define a map from the tangent space to $\mathcal{A}$ at a^0 into $H^1(D, \Theta_0)$ by

$$\left\{ \begin{array}{ccc} T_{a^0}\mathcal{A} & \longrightarrow & H^1(D, \Theta_0) \\ \sum_{(b,d)} \lambda_{b,d}(a) \frac{\partial}{\partial a_{b,d}} & \longmapsto & \sum_{(b,d)} \lambda_{b,d}(a) \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0} \end{array} \right. \tag{8}$$

According to a theorem of J.F. Mattei ([13], Theorem (3.2.1)), the unfolding $\{\mathcal{F}_a,\ a \in (\mathcal{A}, a^0)\}$ is semi-universal among the equireducible unfoldings if and only if this map is a bijective one. By our construction, the dimension of $T_{a^0}\mathcal{A}$ is equal to the one of $H^1(D, \Theta_0)$. Therefore it suffices to prove that the cocycles $\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}$ are independent. We denote by

$$\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}\ ,\ e_{i,j} \right\rangle$$

the component of $\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}$ on the element of the basis $\{e_{i,j}\}$ given by Theorem (1.3). These numbers define a square matrix M of size $\delta = \dim H^1(D, \Theta_0)$, and we have to prove that it is an invertible one, that will be done in two steps.

Step 1. *Components of the cocycles on the first level $d = kl$.*
According to our construction of the normal forms, the coefficient $a_{1,kl}$ is constant equal to 1. Nevertheless, in order to perform calculus in a more symmetric way, we first consider here the parameter $a_{1,kl}$ as a free parameter.

Proposition 1.12. *The square matrix of size p defined by*

$$\left(\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0}, e_{\Phi(b',kl)} \right\rangle\right)_{b,b'=1,\ldots,p}$$

is an invertible Vandermonde matrix.

Proof. We first compute the p components of degree kl of the trivializing vector fields $\theta^0_{b,kl}$ and $\theta^\infty_{b,kl}$ in the two charts (x_{c-1}, y_{c-1}) and (x_c, y_c) around $(D_c, 0)$ and (D_c, ∞), covering the principal component D_c (see notations (3.1) in Appendix A). Notice that, from Proposition (3.2), we have

$$E^*R = x_{c-1}\frac{\partial}{\partial x_{c-1}} = y_c\frac{\partial}{\partial y_c}.$$

Therefore the R-degree is also the x_{c-1}-degree or the y_c-degree. In what follows, the dots stand for terms of higher R-degree. We set $n_c := \sum_{b=1}^{p} n_b$ where the n_b's are the multiplicities of N_a on the cuspidal branches. We have

$$\begin{aligned} \widetilde{N_a}(x_{c-1}, y_{c-1}) &= x_{c-1}^{kn_\infty + ln_0 + kln_c} y_{c-1}^{vn_\infty + un_0 + vln_c} \prod_{b=1}^{p} (a_{b,kl} + y_{c-1} + \cdots)^{n_b} && (9)\\ &:= x_{c-1}^m P(y_{c-1}) + \cdots && (10) \end{aligned}$$

where $m = kn_\infty + ln_0 + kln_c$, and P is a one variable polynomial. Now we have

$$\begin{aligned} \frac{\partial \widetilde{N_a}}{\partial x_{c-1}} &= m x_{c-1}^{m-1} P(y_{c-1}) + \cdots\ , \quad \frac{\partial \widetilde{N_a}}{\partial y_{c-1}} = x_{c-1}^m P'(y_{c-1}) + \cdots\ , && (11)\\ \frac{\partial \widetilde{N_a}}{\partial a_{b,kl}} &= \frac{n_b x_{c-1}^m P(y_{c-1})}{a_{b,kl} + y_{c-1}} + \cdots && (12) \end{aligned}$$

Let us write

$$\theta^0_{b,kl} = \frac{x_{c-1}}{m}\alpha^0_{b,kl}(y_{c-1})\frac{\partial}{\partial x_{c-1}} + \beta^0_{b,kl}(y_{c-1})\frac{\partial}{\partial y_{c-1}} + \cdots .$$

Identifying the terms of lower x_{c-1}-degree in equation (7) on U_0, we obtain

$$\alpha^0_{b,kl} P + \beta^0_{b,kl} P' = \frac{n_b P}{a_{b,kl} + y_{c-1}}. \qquad (13)$$

From the solution (A_0, B_0) of the following Bézout identity in $\mathbb{C}[y_{c-1}]$:

$$A_0 P + B_0 P' = P \wedge P',\ \deg(A_0) < \deg(P'/P \wedge P'),\ \deg(B_0) < \deg(P/P \wedge P'),$$

where $P \wedge P'$ stands for $\gcd(P, P')$, we obtain an holomorphic solution of equation (13) by setting

$$\left(\frac{n_b A_0 P}{(P \wedge P')(a_{b,kl} + y_{c-1})},\ \frac{n_b B_0 P}{(P \wedge P')(a_{b,kl} + y_{c-1})}\right).$$

We may suppose that the solution $(\alpha^0_{b,kl},\ \beta^0_{b,kl})$ coincides with this one. Indeed, one can check that another choice for the solution of the Bézout identity differs from this one by a vector field tangent (at the first order kl) to the foliation, holomorphic on U_0. We can perform a similar computation in the other chart (x_c, y_c) on (D_c, ∞). We have:

$$\begin{aligned} \widetilde{N_a}(x_c, y_c) &= y_c^{kn_\infty + ln_0 + kln_c} x_c^{(k-v)n_\infty + (l-u)n_0 + (kl-ku)n_c} \prod_{b=1}^{p} (1 + a_{b,kl} x_c + \cdots)^{n_b} \\ &:= y_c^m Q(x_c) + \cdots \end{aligned}$$

Setting $\theta^\infty_{b,kl} = \alpha^\infty_{b,kl}(x_c)\frac{\partial}{\partial x_c} + \frac{y_c}{m}\beta^\infty_{b,kl}(x_c)\frac{\partial}{\partial y_c} + \cdots$, we have

$$\alpha^\infty_{b,kl} Q' + \beta^\infty_{b,kl} Q = \frac{n_b x_c Q}{1 + a_{b,kl} x_c}. \tag{14}$$

By considering the solution (A_∞, B_∞) of the following Bézout identity:

$$A_\infty Q + B_\infty Q' = Q \wedge Q',\ \deg(A_\infty) < \deg(\frac{Q'}{Q \wedge Q'}),\ \deg(B_\infty) < \deg(\frac{Q}{Q \wedge Q'})$$

we obtain an holomorphic solution of (14) on U_∞ by setting:

$$\alpha^\infty_{b,kl} = \frac{n_b x_c Q B_\infty}{(1 + a_{b,kl} x_c)(Q \wedge Q')},\ \beta^\infty_{b,kl} = \frac{n_b x_c Q A_\infty}{(1 + a_{b,kl} x_c)(Q \wedge Q')}.$$

In order to compute the cocycles, we give the expression of $\theta^0_{b,kl}$ in the chart (x_c, y_c). Since we have $x_{c-1} = x_c y_c, \quad y_{c-1} = x_c^{-1}$, we obtain

$$\frac{\partial}{\partial x_{c-1}} = x_c^{-1}\frac{\partial}{\partial y_c}, \quad \frac{\partial}{\partial y_{c-1}} = -x_c^2 \frac{\partial}{\partial x_c} + x_c y_c \frac{\partial}{\partial y_c}.$$

Furthermore, by considering the reduced polynomials related to P and Q, we also have

$$\frac{P}{P \wedge P'}(y_{c-1}) = \frac{1}{x_c^{p+2}} \frac{Q}{Q \wedge Q'}(x_c).$$

We obtain:

$$\theta^0_{b,kl} = \frac{n_b x_c^{-(p+2)} Q/Q \wedge Q'(x_c)}{(a_{b,kl} + x_c^{-1})} \left[m^{-1} A_0(x_c^{-1}) y_c \frac{\partial}{\partial y_c} + B_0(x_c^{-1})(-x_c^2 \frac{\partial}{\partial x_c} + x_c y_c \frac{\partial}{\partial y_c})\right] + \cdots$$

We consider now a vector field θ_{is} on $U_0 \cap U_\infty$ tangent to the saturated foliation defined by $\mathrm{d}\widetilde{N_a}$, with isolated singularities. Since

$$-\frac{\partial \widetilde{N_a}}{\partial y_c}\frac{\partial}{\partial x_c} + \frac{\partial \widetilde{N_a}}{\partial x_c}\frac{\partial}{\partial y_c} = \left(-m y_c^{m-1} Q(x_c) + \cdots\right)\frac{\partial}{\partial x_c} + \left(y_c^m Q'(x_c) + \cdots\right)\frac{\partial}{\partial y_c}$$

we can choose

$$\theta_{is} := \left(-\frac{Q}{Q \wedge Q'} + \cdots \right) \frac{\partial}{\partial x_c} + \left(y_c \frac{Q'}{mQ \wedge Q'} + \cdots \right) \frac{\partial}{\partial y_c}.$$

Let $\Phi^{0,\infty}_{b,kl}$ be the function such that $\theta^0_{b,kl} - \theta^\infty_{b,kl} = \Phi^{0,\infty}_{b,kl} \cdot \theta_{is}$. By computing the coefficient of $\theta^0_{b,kl} - \theta^\infty_{b,kl}$ on $\partial/\partial x_c$, we have:

$$\Phi^{0,\infty}_{b,kl} = \frac{n_b x_c}{1 + a_{b,kl} x_c} [x_c^{-p} B_0(x_c^{-1}) - B_\infty(x_c)].$$

The value of $\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}} \right]_{a^0}, e_{\Phi(b',kl)} \right\rangle$ is by construction the coefficients on x_c^i of the Laurent series of $\Phi^{0,\infty}_{b,kl}$ where i is defined by $\Phi(b', kl) = (i, 0)$ (i.e., from (5), $i = [-\nu_c \frac{k-v}{k}[+ p + 1 - b'$). Thus we only have to consider the meromorphic part of $\Phi^{0,\infty}_{b,kl}$, i.e.:

$$\frac{n_b x_c}{1 + a_{b,kl} x_c} \times \frac{\overline{B_0}(x_c)}{x_c^{2p}},$$

where $\overline{B_0}(x) = \sum_{n=0}^p v_n x^n$ is the polynomial function $x^p B_0(x^{-1})$. We have

$$\begin{aligned} \frac{x_c}{1 + a_{b,kl} x_c} &= \sum_{m=0}^{+\infty} (-a_{b,kl})^m x_c^{m+1}, \\ \frac{\overline{B_0}(x_c)}{x_c^{2p}} &= \sum_{n=0}^{p} v_n x_c^{n-2p}. \end{aligned}$$

Therefore, the coefficient of the Laurent series of $\Phi^{0,\infty}_{b,kl}$ in x_c^i is

$$\begin{aligned} \sum_{\substack{(m+1)+(n-2p)=i \\ 0 \le n \le p}} n_b v_n (-a_{b,kl})^m &= n_b \sum_{n=0}^{p} v_n (-a_{b,kl})^{2p-1-n+i} \\ &= n_b \overline{B_0}(-a_{b,kl}^{-1}) \times (-a_{b,kl})^{2p-1+i} \end{aligned}$$

Finally we obtain

$$\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}} \right]_{a^0}, e_{\Phi(b',kl)} \right\rangle = C_b (-a_{b,kl}^{-1})^{b'} \tag{15}$$

where $C_b = n_b \overline{B_0}(-a_{b,kl}^{-1}) \times (-a_{b,kl})^{\left(3p + [-\nu_c \frac{k-v}{k}[\right)}$. This defines a Vandermonde matrix. Furthermore, $C_b = 0$ if and only if $B_0(-a_{b,kl}) = 0$, which cannot happen: evaluating the Bézout identity

$$A_0 \frac{P}{P \wedge P'} + B_0 \frac{P'}{P \wedge P'} = 1$$

at $y_0 = -a_{b,kl}$, we would obtain a contradiction, since $-a_{b,kl}$ is a root of P. Moreover, $a_{b,kl} \neq a_{b',kl}$ for $b \neq b'$, thus the Vandermonde matrix is invertible. $\square$

Step 2. *Relationship between the components of the cocycles on different levels.*

Lemma 1.13. *If $\theta^0_{b,kl}$, $\theta^\infty_{b,kl}$ are trivializing vector fields on U_0 (resp. on U_∞) for the direction $\frac{\partial}{\partial a_{b,kl}}$, then for any $d > kl$, the vector fields*

$$\frac{\tilde{m}^d}{\tilde{m}^{kl}}\ \theta^0_{b,kl}, \quad \frac{\tilde{m}^d}{\tilde{m}^{kl}}\ \theta^\infty_{b,kl}$$

are trivializing vector fields on U_0 and U_∞ for the direction $\frac{\partial}{\partial a_{b,d}}$ where $\tilde{m} = m \circ E$.

Proof. Let $B_b := y^k + \sum_{(b,d),\Phi(b,d)\in\mathbb{T}} a_{b,d} m^d$ be the branch of index b, and $\widetilde{B_b} := B_b \circ E$. Since we have:

$$\frac{\partial \widetilde{N_a}}{\partial a_{b,d}} = \tilde{m}^d n_b \frac{\widetilde{N_a}}{\widetilde{B_b}}$$

if $\theta^0_{b,kl}$ satisfies equation (7) for $d = kl$, then, given $d > kl$, $\frac{\tilde{m}^d}{\tilde{m}^{kl}}\theta^0_{b,kl}$ satisfies the trivializing equation for the level d. Furthermore, this vector field is still holomorphic on U_0. Indeed, from the trivializing equation (7), we deduce that the multiplicity of the trivializing vector field $\theta^0_{b,kl}$ on a component D_i of $D \cap U_0$ is given by

$$\nu_i(\theta^0_{b,kl}) = \nu_i(\tilde{m}^{kl}) - \nu_i(\widetilde{B_b}) + 1.$$

The multiplicity of $\frac{\tilde{m}^d}{\tilde{m}^{kl}}\theta^0_{b,kl}$ on D_i is thus equal to

$$\nu_i\left(\frac{\tilde{m}^d}{\tilde{m}^{kl}}\theta^0_{b,kl}\right) = \nu_i(\tilde{m}^d) - \nu_i(\tilde{m}^{kl}) + \nu_i(\tilde{m}^{kl}) - \nu_i(\widetilde{B_b}) + 1$$

and therefore is still a positive number. The argument is similar for $\theta^\infty_{b,kl}$. □

We consider the linear operator

$$T_d := \frac{\tilde{m}^d}{\tilde{m}^{kl}} \times \ : \ \Theta_0(U_0 \cap U_\infty) \longrightarrow \Theta_0(U_0 \cap U_\infty).$$

induced by the previous Lemma. We can remark that when d runs over $\{kl,\ kl+1, \cdots\}$ the points $T_d \cdot e_{\Phi(b,kl)}$ are exactly the paths introduced in the previous section, and the indexation $(i,j) = \Phi(b,d)$ has been introduced such that

$$T_d \cdot e_{\Phi(b,kl)} = e_{\Phi(b,d)}.$$

Proposition 1.14. *Let $d > kl$ be the index of an horizontal level in the half plane representing $\Theta_0(U_0 \cap U_\infty)$. For each $b,\ b'$ in $1,\ldots,p$, we have that*

1. *for any d' such that $kl \le d' < d$, one has*
$$\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, e_{\Phi(b',d')} \right\rangle = 0.$$
2. *the coefficient $\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, e_{\Phi(b',d)} \right\rangle$ is constant with respect to d (i.e. constant along the paths introduced in the previous section).*
3. *the coefficient $\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, e_{\Phi(b',d')} \right\rangle$ only depends on the variables $a_{b'',d''}$ with*
$$kl \le d^{''} \le kl + d^{'} - d.$$

Proof. For $b = 1,\ldots,p$, we have:

$$\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0} = \sum_{b'} \left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0}, e_{\Phi(b',kl)} \right\rangle e_{\Phi(b',kl)} + \cdots$$

where the dots correspond to components of higher level. Applying the linear operator T_d, we obtain:

$$\left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0} = \sum_{b'} \left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl}}\right]_{a^0}, e_{\Phi(b',kl)} \right\rangle e_{\Phi(b',d)} + \cdots$$

which proves the statements (1) and (2). For the third point, we consider the meromorphic function $\Phi_{b,d}^{0,\infty}$ defined by

$$\theta_{b,d}^{0} - \theta_{b,d}^{\infty} = \Phi_{b,d}^{0,\infty}\theta_{is} \tag{16}$$

where θ_{is} is the vector field with isolated singularities which generates the foliation on $U_0 \cap U_\infty$, introduced in step 1. Recall that the coefficient

$$\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0} \ , \ e_{i,j} \right\rangle$$

is nothing but the coefficient of $x_c^i y_c^j$ in the Laurent development of $\Phi_{b,d}^{0,\infty}$ (see the proof of proposition (1.12)). On the first level ($d = kl$), setting

$$\theta_{b,kl}^{\infty} = \left(\theta_{b,kl}^{\infty}\right)_{x_c} \frac{\partial}{\partial x_c} + \left(\theta_{b,kl}^{\infty}\right)_{y_c} y_c \frac{\partial}{\partial y_c},$$

the second relation in (7) can be written

$$\left(\theta_{b,kl}^{\infty}\right)_{x_c} \frac{\partial \widetilde{N_a}}{\partial x_c} + \left(\theta_{b,kl}^{\infty}\right)_{y_c} y_c \frac{\partial \widetilde{N_a}}{\partial y_c} = \frac{\partial \widetilde{N_a}}{\partial a_{b,kl}}. \tag{17}$$

Now, extending the expression of the partial derivatives of $\widetilde{N_a}$ filtered by y_c variable as in (11) leads to expressions of the following form

$$\begin{aligned}
y_c \frac{\partial \widetilde{N_a}}{\partial y_c} &= y_c^m A_0(x_c) + y_c^{m+1} A_1(x_c) + y_c^{m+2} A_2(x_c) \cdots \\
\frac{\partial \widetilde{N_a}}{\partial x_c} &= y_c^m B_0(x_c) + y_c^{m+1} B_1(x_c) + y_c^{m+2} B_2(x_c) \cdots \\
\frac{\partial \widetilde{N_a}}{\partial a_{b,kl}} &= y_c^m C_0(x_c) + y_c^{m+1} C_1(x_c) + y_c^{m+2} C_2(x_c) \cdots
\end{aligned}$$

where m is defined in (9). From the construction of the normal form N_a, it can be seen that for any i, A_i, B_i and C_i depend only on the variable $a_{b',d}$ with $kl \leq d \leq kl + i$. Thus, if one filters the equation (17) with respect to the y_c variable, one can see that the solution $\theta_{b,kl}^{\infty}$ shares the same property of filtration, namely, if one writes

$$\theta_{b,kl}^{\infty} = (D_0(x_c) + D_1(x_c)y_c + \cdots)\frac{\partial}{\partial x_c} + (E_0(x_c) + E_1(x_c)y_c + \cdots)y_c \frac{\partial}{\partial y_c},$$

then D_i and E_i depend only on the variables $a_{b',d}$ with $kl \leq d \leq kl + i$. The same remark can be done for $\theta_{b,kl}^{0}$ using a filtration with respect to the x_{c-1} variable. Finally, since θ_{is} has also the same property of filtration, the relation (16) implies that the jet of order i with respect to the y_c variable of $\Phi_{b,kl}^{0,\infty}$ depends only on the variables $a_{b',d}$ with $kl \leq d \leq kl + i$. Now, since the vertical component of T_d is just a translation induced by $\times y_c^{d-kl}$, this property propagates as in the statement of (3). $\square$

End of the proof of Theorem (1.10).
Notice first that the operator T_d is "quite well defined" in the cohomology group $H^1(D, \Theta_0)$: let $M_2 = e_{\Phi(2,kl)}, \ldots, M_p = e_{\Phi(p,kl)}$ be the roots of the paths indexed by the branches $b = 2, \ldots, p$ and M_1 the point under the root of the first branch (recall that this root is on the second level since we set $a_{1,kl} = 1$). If we pick a point on the first level outside $M_1, M_2, \ldots, M_p$, the action of T_d preserves the half planes corresponding to $\Theta_0(U_0)$ and $\Theta_0(U_\infty)$. This is clear on figure (3) and it is a consequence of Proposition (1.8): these paths cannot go back inside the

triangle. Therefore the operator T_d is well defined on $H^1(D,\Theta_0)$ excepted on the line generated by $M_1 = e_{\Phi(1,kl)}$. We add this point to the triangle and now we can write the $(\delta+1)\times(\delta+1)$-matrix of the cocycles $\frac{\partial \mathcal{F}_a}{\partial a_{1,kl}},\ldots \frac{\partial \mathcal{F}_a}{\partial a_{p,kl}},\frac{\partial \mathcal{F}_a}{\partial a_{1,kl+1}},\ldots,\frac{\partial \mathcal{F}_a}{\partial a_{p,kl+1}},\ldots$ on $\{e_{\Phi(b,d)}\}_{b,d}$, ordered by the lexicographic order. According to the previous Proposition (1.12) and Proposition (1.14) this matrix is a block triangular matrix:

$$V := \begin{pmatrix} (V_1) & 0 & 0 & \cdots \\ \times & (V_2) & 0 & \cdots \\ \times & \times & (V_3) & \cdots \\ \times & \times & \times & \ddots \end{pmatrix}$$

where V_1 is the invertible Vandermonde matrix obtained in step 1, V_2, V_3... are sub-matrices of consecutive lines and columns of V_1 defined by the paths from the first level to the following levels. Clearly since $\det(V) = \prod \det(V_i)$ this matrix is an invertible one. Finally, since $a_{1,kl} = 1$, $e_{1,kl} \notin \mathbb{T}$, the matrix M is obtained by deleting the first line and first column of V, and is still an invertible one. □

1.4 The global moduli space of foliations

Proposition 1.15 (Existence of normal forms)**.** *For any f in $\mathcal{T}_{(k,l),(n)}$, there exists a in $\mathcal{A}$ such that $f \sim N_a$, where $\sim$ denotes the classification of foliations.*

Proof. We can suppose that f is given under its prenormalization form (1.1). Therefore the deformation defined by

$$f_\lambda := \frac{1}{\lambda^r} f\left(\lambda^k x, \lambda^l y\right)$$

where $r = kn_\infty + ln_0 + kln_c$, ($n_c = \sum_b n_b$) is an equireducible unfolding of $f_0 = N_{a_0}$, $a_0 = (1, a_2, \ldots, a_p, 0, \ldots, 0) \in \mathcal{A}$. Using theorem (1.10), we can ensure that for λ small enough, there exists $a \in \mathcal{A}$ such that $f_\lambda \sim N_a$. Furthermore, this deformation is analytically trivial for $\lambda \neq 0$, since we construct it by a conjugacy. Therefore, $f = f_1 \sim f_\lambda$, for λ small, and the proposition is proved. □

Let us consider the diffeomorphism: $h_\lambda(x,y) = (\lambda^k x, \lambda^l y)$. We have:

$$N_a \circ h_\lambda = \lambda^{n_c} N_{\lambda\cdot a}, \quad \text{with } \lambda\cdot a = \lambda\cdot(a_{b,d}) := (\lambda^{d-kl} a_{b,d}).$$

As above, we have thus $N_a \sim N_{\lambda\cdot a}$. Actually, this action of $\mathbb{C}^*$ the only obstruction to the unicity of normal forms:

Theorem 1.16 (Unicity of normal forms)**.** *$N_a \sim N_{a'}$ if and only there exists a complex number $\lambda \neq 0$ such that $a' = \lambda \cdot a$.*

Proof - Suppose that there exists a conjugacy relation

$$\psi \circ N_{a'} = N_a \circ \phi. \tag{18}$$

Following [1], we can suppose that ψ is an homothetie $\gamma\mathrm{Id}$. We are going to reduce the proof to the case where ϕ is tangent to the identity. Since the conjugacy preserves the numbering of the branches, looking at the relation induced by (18) on the jet of smaller (k,l)-order, we can ensure that the linear part of ϕ is written $h_\lambda = (\lambda^k x, \lambda^l y)$ for some $\lambda \neq 0$. Then

$$N_a \circ \phi \circ h_\lambda^{-1} = \gamma N_{a'} \circ h_\lambda^{-1} = c N_{\lambda^{-1}\cdot a'}$$

where c stands for some non vanishing number. Since $\phi\circ h_\lambda^{-1}$ is tangent to the identity, it appears that $c = 1$. Thus, setting for the sake of simplicity $a' = \lambda^{-1}\cdot a'$ and $\phi = \phi\circ h_\lambda^{-1}$ we are led to a relation

$$N_{a'} = N_a \circ \phi \tag{19}$$

where ϕ is tangent to the identity. The proof reduces to show that in the situation (19), we have $a = a'$. Let X be a germ of formal vector field such that $\phi = e^X$. The vector field X can be decomposed in the sum of its quasi-homogeneous components

$$X = X_\nu + X_{\nu+1} + \cdots$$

Lemma 1.17. *If $N_a \circ e^{X_\nu + \cdots} = N_{a'}$ then for all b from 1 to p and all $d \leq kl + \nu - 1$, $a_{b,d} = a'_{b,d}$.*

Proof. We set:

$$N_a = N_a^{(N)} + \cdots + N_a^{(N+p-1)} + \ N_a^{(N+p)} + \cdots$$

where $N = kn_\infty + ln_0 + kln_c$ is the degree of the first quasi-homogeneous component of N_a. Since we have

$$e^{X_\nu + \cdots} N_a = N_a + X_\nu \cdot N_a + \cdots$$

we obtain $N_a^{(N+i)} = N_{a'}^{(N+i)}$ for i from 0 to $\nu - 1$. The expression of $N_a^{(N+i)}$ only depends on the variables $a_{b,d}$ for $d \leq kl + i$. Finally we claim that $N_a^{(N+i)} = N_{a'}^{(N+i)}$ if and only if $a_{b,d} = a'_{b,d}$ for $d \leq kl + i$. This fact can be proved by induction on $d \leq kl + i$. It is obvious for $d = kl$. Suppose that $a_{b,d} = a'_{b,d}$ is true for $d \leq kl + j - 1$ with $j - 1 < i$. Then we have:

$$\sum_b \frac{N_a^{(N)}}{y^k + a_{b,kl}x^l} a_{b,kl+j} m^{kl+j} = \sum_b \frac{N_a^{(N)}}{y^k + a_{b,kl}x^l} a^{'}_{b,kl+j} m^{kl+j}.$$

which implies that $a_{b,kl+j} = a'_{b,kl+j}$. □

Now if $\nu \geq pkl + l\epsilon_0 + k\epsilon_\infty$ and $N_a \circ e^{X_\nu + \cdots} = N_{a'}$ then according to the previous lemma, for all b and $d \leq kl + pkl + l\epsilon_0 + k\epsilon_\infty - 1$, $a_{b,d} = a^{'}_{b,d}$. Since $\nu_c = pkl + k\epsilon_\infty + l\epsilon_0 - k - l$ then

$$\underbrace{\nu_c + kl - 1}_{\substack{\text{bigger value of } d \\ (b,d) \in \mathbb{T}}} \quad < pkl + kl + l\epsilon_0 + k\epsilon_\infty - 1.$$

Therefore we have $a = a^{'}$. Thus, it remains to prove the following lemma:

Lemma 1.18. *If $N_a \circ e^{X_\nu + \cdots} = N_{a'}$ then $\nu \geq pkl + l\epsilon_0 + k\epsilon_\infty$.*

Proof. It suffices to prove that $\nu < pkl + l\epsilon_0 + k\epsilon_\infty$ leads to a contradiction. Since the conjugacy ϕ does not modify the parameter $a_{b,d}$ for $d \leq kl + \nu - 1$ the first non trivial relation of the smallest (k,l)-degree induced by (19) is written

$$X_\nu \cdot N_a^{(N)} = -N_a^{(N+\nu)} + N_{a^{'}}^{(N+\nu)}.$$

Dividing by $N_a^{(N)}$ leads to

$$\sum_{b=1}^{p} n_b \frac{X_\nu \cdot (y^k + a_{b,kl}x^l)}{y^k + a_{b,kl}x^l} + n_\infty \frac{X_\nu \cdot x}{x} + n_0 \frac{X_\nu \cdot y}{y} = m^{kl+\nu} \sum_{b=1}^{p} \frac{-a_{b,kl+\nu} + a^{'}_{b,kl+\nu}}{y^k + a_{b,kl}x^l}.$$

We take the pull-back of the previous equality with respect to the map E and write it in the coordinates (x_c, y_c). Since we are going to look at residus at $x_c = -a_{b,kl}^{-1}$, we only make appear the terms having poles at these points:

$$\cdots + \sum_{b=1}^{p} n_b \frac{a_{b,kl}\widetilde{X_\nu} \cdot x_c}{1 + a_{b,kl}x_c} = x_c^{\nu - iv - ju + ku} \sum_{b=1}^{p} \frac{\delta_{b,kl+\nu}}{1 + a_{b,kl}x_c}$$

where $\widetilde{X_\nu}$ stands for the vector field $\frac{E^* X_\nu}{y_c^\nu}$, $\delta_{b,kl+\nu}$ for the difference $-a_{b,kl+\nu} + a'_{b,kl+\nu}$ and i, j for the couple of integers such that $m^{kl+\nu} = x^i y^j$.
Since the integer $\nu - iv - ju + ku$ is non negative, evaluating the residue at $-a_{b,kl}^{-1}$ yields the relation

$$n_b a_{b,kl} \widetilde{X_\nu} \cdot x_c \left(-a_{b,kl}^{-1}\right) = \left(-a_{b,kl}^{-1}\right)^{\nu - iv - ju + ku} \delta_{b,kl+\nu}. \tag{20}$$

A straightforward computation shows that $\widetilde{X_\nu} \cdot x_c$ is a polynomial function in x_c that is written the following way

1. if $\epsilon_0 = 1$ –that is if the curve $y = 0$ is invariant– or if $\frac{\nu+l}{k}$ is not an integer

$$\widetilde{X_\nu} \cdot x_c = \sum_{\nu(1-\frac{u}{l}) \leq w \leq \nu(1-\frac{v}{k}),\ w \in \mathbb{N}} p_w x_c^w = x_c^{\left[\nu(1-\frac{u}{l})\right]+1} \left(\sum_{w=0}^{\left[\nu(1-\frac{v}{k})\right] - \left[\nu(1-\frac{u}{l})\right] - 1} q_w x_c^w \right)$$

2. else

$$\widetilde{X_\nu} \cdot x_c = \sum_{\nu(1-\frac{u}{l}) \leq w \leq \nu(1-\frac{v}{k}) + \frac{1}{k},\ w \in \mathbb{N}} p_w x_c^w = x_c^{\left[\nu(1-\frac{u}{l})\right]+1} \left(\sum_{w=0}^{\left[\nu(1-\frac{v}{k}) + \frac{1}{k}\right] - \left[\nu(1-\frac{u}{l})\right] - 1} q_w x_c^w \right)$$

Now, in view of the construction of the normal form, the coefficient $\delta_{b,kl+\nu}$ has to be zero for

$$p - \sharp \mathbb{Z} \cap \left] \frac{k-v}{k}(\nu - \nu_c), \frac{l-u}{l}(\nu - \nu_c) \right[$$

values of the parameter b. Thus, according to (20), the polynomial function $\widetilde{X_\nu} \cdot x_c$ has the same number of non-vanishing roots among the values $-a_{b,kl}^{-1}, b = 1, \ldots, p$. This number is strictly greater than the degree of the polynomial functions factorized in the above expressions of $\widetilde{X_\nu} \cdot x_c$. Thus, the latter has to be the zero polynomial function. Therefore, looking again at the relation (20) yields

$$\forall b,\ \delta_{b,kl+\nu} = 0.$$

Hence, the vector field X_ν has to be tangent to $N_a^{(N)}$ which is a contradiction with the hypothesis $\nu < pkl + l\epsilon_0 + k\epsilon_\infty$. □

Finally, we can summarize the previous results by

Theorem 1.19. *The moduli space $\mathcal{M}_{(k,l),(n)}$ is isomorphic to ${}^{\mathcal{A}}/_{\mathbb{C}^*}$ where the action of $\mathbb{C}^*$ is defined by*

$$\lambda \cdot a = \lambda \cdot (a_{b,d}) = (\lambda^{d-kl} \cdot a_{b,d})$$

2 The moduli space of curves

Let $\mathcal{C}$ be the partition of $\mathcal{M} = \mathcal{M}_{(k,l),(n)}$ induced by the classification of curves $\sim_c$.

2.1 The infinitesimal generators of $\mathcal{C}$

We first recall general facts proved in [8], which are valid in every topological class. Let $\mathcal{F}$ be a foliation defined by an holomorphic function f (or more generally by any generic non dicritical differential form ω), and let S be the curve defined by $f = 0$ (or by the separatrix set of ω). Let $E: M \to (\mathbb{C}^2, 0)$ be the desingularization map of the foliation, and D its exceptional divisor. We denote by $\widetilde{f}$, $\widetilde{\mathcal{F}}$, $\widetilde{S}$ the pull back by E on M of f, $\mathcal{F}$ or S. The tangent space to the point $[S]$ in the moduli space of curves (for $\sim_c$) is the cohomological group $H^1(D, \Theta_S)$ where Θ_S is the sheaf on D of germs of vector fields tangent to $\widetilde{S}$. The inclusion of $\Theta_{\mathcal{F}}$ into Θ_S induces a map i:

$$H^1(D, \Theta_{\mathcal{F}}) \xrightarrow{i} H^1(D, \Theta_S)$$

whose kernel represents the directions of unfolding of foliations with trivial associate unfolding of curves.

Definition 2.1. *An open set U of M is a* quasi-homogeneous open set *(relatively to f) if there exists an holomorphic vector field R_U on U such that $R_U(\widetilde{f}) = \widetilde{f}$.*
We can always cover D by two quasi-homogeneous open sets U and V. The cocycle of quasi-homogeneity $[R_{U,V}]$ of $\mathcal{F}$ is the element of $H^1(D, \Theta_{\mathcal{F}})$ induced by $R_U - R_V$.

Recall that $H^1(D, \Theta_{\mathcal{F}})$ has a natural structure of $\mathcal{O}_2$-module. We have:

Theorem 2.2. *[8] The kernel of the map i is generated by the cocycle of quasi-homogeneity, i.e.:*

$$\ker(i) = \{h \cdot [R_{U,V}],\ h \in \mathcal{O}_2\}.$$

Notice that the distribution induced by these directions is integrable and defines a singular foliation $\mathcal{C}$ on $\mathcal{A}$. The point corresponding to the topological model is a singular one: indeed, this model is quasihomogeneous. Therefore the whole open set $U = M$ is quasi-homogeneous, and the cocycle $[R_{U,V}]$ is trivial for this foliation.

Let $X_{m,n}$ be the vector fields on $\mathcal{A}$ generated by $x^m y^n \cdot [R_{U,V}]$. Below, we describe some properties of the distribution induced by the vector fields $X_{m,n}$.

Proposition 2.3.

1. *The $\mathcal{O}_2$-generator of $\mathcal{C}$ is given by:*

$$X_{0,0} = -\frac{1}{r} \sum_{\Phi(b,d) \in \mathbb{T} \cup \{(1,kl)\}} (d - kl) a_{b,d} \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}$$

   where $r = kn_0 + ln_\infty + kl \sum_{b=1}^{p} n_b$

2. *For any level d we denote by $X^d_{m,n}$ the components of the vector field $X_{m,n}$ on the subspace $Vect\{e_{\phi(b,d)},\ b = 1, \ldots, p\}$. For any $m,\ n,\ X_{m,n}$ is quasihomogeneous with respect to the degree induced by $rX_{0,0}$. Indeed, we have*

$$[rX_{0,0}, X_{m,n}] = (km + ln)\, X_{m,n}.$$

   The coefficients of $X^\nu_{m,n}$ are quasi-homogeneous with respect to the weight $rX_{0,0}$ of degree $\nu - km - ln - kl$. In particular, they only depend on the variables $a_{b,d}$ with $d \leq \nu - km - ln$.

3. *If we decompose the vector field $X_{0,0}$*

$$\begin{aligned} X_{0,0} &= -\frac{1}{r} \sum_d \sum_b (d - kl)\, a_{b,d} \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0} \\ &= -\frac{1}{r} \sum_{i \in \mathbb{Z}, j \geq 1} \underbrace{\left(\sum_{0 \leq d - kl \leq j} \sum_b (d - kl)\, a_{b,d} \left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0},\ e_{i,j} \right\rangle \right)}_{\Gamma_{i,j}(a)} e_{i,j}. \end{aligned}$$

then the functions $\Gamma_{i,j}(a)$ are algebraically independent.

4. *The vector fields defined by*

$$\widetilde{X_{m,n}} = a_{2,kl+1}^{km+ln} X_{m,n},$$

commute with $X_{0,0}$. Therefore, they induce the distribution $\mathcal{C}$ on $\mathcal{M}$.

Proof. 1. The proof is the same as the one of proposition (5.5) of [8] with a very slight change where we replace $(\lambda x, \lambda y)$ with $(\lambda^k x, \lambda^l y)$.
2. The proof is also a slight generalization of the proof of Proposition (5.9) in [8].
3. Let us decompose the coefficient $\Gamma_{i,j}(a)$

$$\Gamma_{i,j}(a) = \underbrace{\sum_b j a_{b,kl+j} \left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl+j}}\right]_{a^0}, \; e_{i,j}\right\rangle}_{L_{i,j}} + \underbrace{\sum_{0\le d-kl<j} \sum_b (d-kl)\, a_{b,d} \left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, \; e_{i,j}\right\rangle}_{R_{i,j}}.$$

Following, the proposition (1.14) the function $\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,kl+j}}\right]_{a^0}, \; e_{i,j}\right\rangle$ depends only on the variables $a_{b',kl}$ with $\Phi(b',d) = (i,j)$. The expression $\left\langle \left[\frac{\partial \mathcal{F}_a}{\partial a_{b,d}}\right]_{a^0}, \; e_{i,j}\right\rangle$ in $R_{i,j}$ depends only on the variables $a_{b,d'}$ where d' satisfies

$$0 \le d^{'} - kl \le j - (d-kl) \implies d^{'} \le j + kl - (d-kl) < j + kl.$$

In view of the proposition (1.12), for a fixed value of $j = J$, the functions $L_{i,J}$ considered as linear functions of the variables $a_{b,kl+j}$ are linearly independent because their matrix is an extraction of consecutive rows and columns in the Vandermonde matrix of 1.12. Thus, they are also algebraically independent as a whole. Now, let us consider an algebraic relation between the functions $\Gamma_{i,j}(a)$ given by a polynomial function $P\left(\{X_{i,j}\}_{(i,j)\in\mathbb{T}}\right)$ where the $X_{i,j}$'s are some independent variables

$$P\left(\Gamma_{i,j}(a)\right) = 0.$$

Let J be the greatest integer such that there exists a point (i,J) in $\mathbb{T}$ and denote by

$$\{(i_0,J),(i_1,J),\ldots,(i_q,J)\}$$

the family of points in $\mathbb{T}$ at the level J. The relation P is written

$$P\left(\{\Gamma_{i,j}(a)\}_{j<J}, L_{i_0,J}(a) + R_{i_0,J}(a), \ldots, L_{i_q,J}(a) + R_{i_q,J}(a)\right) = 0.$$

We fix all the variables $a_{b,d}$ with $d - kl < J$ at a generic value. Then, the above relation becomes an algebraic relation between the affine forms $L_{i,J}(a) + R_{i,J}(a)$. Let us decompose the relation P as follows

$$P = \sum_{I \subset \{(i_k,J)\}_{k=0..q}} Q_I(X_{i,j})\, X_I$$

where $X_I = \prod_{(i,J)\in I} X_{i,J}$. Here, Q_I depends only on the variables $X_{i,j}$ with $j < J$. Since, the affine form $L_{i,J}(a) + R_{i,J}(a)$ are algebraically independent, for any I, we have

$$Q_I\left(\Gamma_{i,j}(a)\right) = 0,$$

which are algebraic relations between the functions $\Gamma_{i,j}(a)$ with $j < J$. Therefore, an inductive argument ensures that P has to be the trivial relation, which proves the property.

4. We recall that the global moduli space of foliations is obtained from the local one by considering the weighted action of $\mathbb{C}^*$ on $\mathcal{A}$ which is also the flow of $X_{0,0}$. Since the $X_{0,0}$-degree of the variable $a_{1,kl+1}$ is equal to 1, we have

$$\begin{aligned}\left[rX_{0,0}, a_{2,kl+1}^{km+ln}X_{m,n}\right] &= rX_{0,0}(a_{2,kl+1}^{km+ln})X_{m,n} + a_{2,kl+1}^{km+ln}\,[rX_{0,0}, X_{m,n}] \\ &= -(km+ln)a_{2,kl+1}^{km+ln}X_{m,n} + a_{2,kl+1}^{km+ln}\,[rX_{0,0}, X_{m,n}] = 0.\end{aligned}$$

□

2.2 The dimension of the generic strata

The dimension τ of the generic strata of the local moduli space of curves corresponds to the codimension of the distribution $\mathcal{C}$ at a generic point of $\mathcal{M}$. According to proposition 2.3, the family of coefficients $\{\Gamma_{ij}\}_{i,j}$ of $X_{0,0}$ is functionally independent: thus, any family of r vector fields in dimension r whose coefficients are chosen among the Γ_{ij}'s is generically free: indeed, their determinant cannot identically vanish since it would produce a functional relation between the Γ_{ij}'s. Thus, to compute the dimension of the generic strata, we just have to browse the triangle of moduli and to compute at each level d how many moduli can actually be reached by the vector fields $X_{m,n}$. For the following computations, we recommend to refer at each step to the example presented in Appendix B, figure 3.

Let us denote by $\nu(X_{m,n}) = km + ln + kl + 1$ the order of $X_{m,n}$. By construction, $\nu(X_{m,n}) - kl$ is the first level of the triangle of moduli on which $X_{m,n}$ may have an action: indeed, since $X_{m,n} = x^m y^n X_{0,0}$, its projections on the previous levels vanish. In most cases, $X_{m,n}$ can be used to kill a modulus which is exactly at its first level $\nu(X_{m,n}) - kl$. However, in some cases, $X_{m,n}$ cannot be used this way because, for instance, the triangle of moduli has no modulus on this particular level: therefore, we use $X_{m,n}$ to kill a modulus on some level above. To take care of all this possibilities, we introduce a decomposition by blocks of the triangle of moduli and we prove some related arithmetical properties:

A block B_i in the triangle of moduli is a union of kl consecutive horizontal lines from the line of index $d_i = ikl + 1$, see Figure 3. We denote by

- n_d the "dimension" of the line of index d which means the number of integer points on this line.
- $N_i = \sum_{d=ikl+1}^{(i+1)kl} n_d$ the dimension of the block B_i which is also the number of integer points in the whole block.
- $n_i^{\max} = \max\{n_d,\ d = ikl+1, \dots, (i+1)kl\}$ which is the greatest dimension of a line in the block B_i.

One can easily prove, by using the equations of the edges of the triangle, the following lemma –see also figure 3– :

Lemma 2.4. *1. We have: $N_{i+1} = N_i - kl$, $n_i^{\max} = p - i$.*

2. For each line of level d of the block B_i, $n_d = n_i^{\max}$ or $n_i^{\max} - 1$.

3. On the first line d_i of the block B_i, the number n_{d_i} reaches the maximum $n_i^{\max}$.

We denote by:

- q_d the number of vector fields $X_{m,n}$ such that $\nu(X_{m,n}) = d$
- $Q_i = \sum_{d=ikl+1}^{(i+1)kl} q_d$
- $q_i^{\max} = \max\{q_d,\ d = ikl+1, \dots, (i+1)kl\}$.

One can check a similar result to (2.4):

Lemma 2.5. *1. We have:* $Q_{i+1} = Q_i + kl$, $q_i^{\max} = i$.

2. *For each line of level* d *of the block* B_i, $q_d = q_i^{\max}$ *or* $q_i^{\max} - 1$.

3. *On the first line* d_i *of the block* B_i, *the number* q_{d_i} *reach the maximum* $q_i^{\max}$.

We consider the maximal sequence of blocks B_i such that $q_i^{\max} = i < n_i^{\max} = p - i$, i.e. the sequence $B_1, \ldots, B_{]p/2]}$, where $]p/2]$ is the strict integer part of $p/2$. We call *critical block*, the block $B_{\frac{p}{2}}$ when p is even or the unique block B_1 that appears when $p = 1$. This block is going to be analyzed independently. In figure 3, this block is the second one, and in figure 1, since $p = 1$, this block is the sole block B_1.
Consider a block B_i such that $q_i^{max} > n_i^{max}$. For each line of index d of this block, since $q_d = q_i^{max}$ or $q_i^{max} - 1$, we have: $q_d \geq n_i^{max} \geq n_d$. According to the previous functional independence of the vector fields $X_{m,n}$, we can conclude that in this case, their action is transitive on such a block and the block above.
In the critical block $B_{\frac{p}{2}}$ or B_1, the integers $n_d - q_d$ for $d = d_i, \ldots, d_i + kl - 1$ can only take the values $+1$, 0 or -1, starting from the value 0 on the first level of the block. On the latter level, the action of the $X_{m,n}$ is thus transitive. We consider the first line of this block on which $n_d - q_d \neq 0$:

- If we have $n_d - q_d = +1$, there remains one dimension which cannot be reached by the action of the $X_{m,n}$. We have to count it in the codimension of the generic leaves of $\mathcal{C}$.
- If $n_d - q_d = -1$, the action of the vector fields $X_{m,n}$ is transitive on this level. Furthermore we have an extra vector field $X_{m,n}$ such that $\nu(X_{m,n}) = d$ whose higher components will act on the higher levels. Suppose that there exists a level $d' > d$ such that $n_{d'} - q_{d'} = +1$.

Therefore, in order to compute the generic dimension of the distribution $\mathcal{C}$ on the critical block, we have to introduce the following non commutative sum :

Definition 2.6. *Let* r_d *be a sequence taking its values in* $\{-1, 0, +1\}$. *The notation* $\widetilde{\sum}_d r_d$ *denotes the value obtained by the following operations:*

1. *delete the values 0;*
2. *delete recursively the consecutive values* $(-1, +1)$ *(but not the consecutive values* $(+1, -1)$*);*
3. *after the two first steps, remains a sequence of* n *consecutive terms with value +1, followed by* m *consecutive terms with value -1. We set:* $\widetilde{\sum}_d r_d = n$.

Example. In the critical block of Figure 3, the sequence of values $n_d - q_d$ is:

$$\{0, +1, +1, 0, +1, 0, -1, +1, 0, -1, 0, -1, -1, 0, -1\}.$$

The extra vector field appearing on the 7^{th} position acts on the next level. The next extra vector fields are unuseful. Therefore, the number of free dimensions under the action of these vector fields is

$$\widetilde{\sum}\{0, +1, +1, 0, +1, 0, -1, +1, 0, -1, 0, -1, -1, 0, -1\} = 3.$$

From all the considerations above, we deduce the following:

Theorem 2.7. *The dimension of the generic strata of the moduli space for curves is*

$$\tau = \sum_{d=d_0}^{d_0+]p/2]+kl-1} (n_d - q_d) + \widetilde{\sum}_{d=d_{p/2}}^{d_{p/2}+kl-1} (n_d - q_d),$$

where $n_d =]\frac{v-k}{k}(\nu_c - d + kl)] - [\frac{u-l}{l}(\nu_c - d + kl)[$, q_d *is the number of positive integer solutions* (m, n) *of the equation* $km + ln + kl + 1 = d$, *and the second sum* $\widetilde{\sum}$ *is defined above and only appears if* p *is even or if* $p = 1$ *(in this case, we set* $d_{1/2} = kl$*).*

Example. In the topological class $(k, l) = (3, 5)$ and $p = 4$ of figure 3, we obtain $\tau = 35$.

2.3 Normal forms for curves

Theorem 2.8. *We consider the reduced normal form*

$$N_a = x^{\epsilon_\infty} y^{\epsilon_0} \prod_{b=1}^{p} \left(y^k + \sum_{\{(b,\,d),\,\Phi(b,\,d)\,\in\,\mathbb{T}\}\,\cup\,\{(1,\,kl)\}} a_{b,d} m^d \right)$$

obtained for the classification of foliations defined by topologically quasi-homogeneous functions. We obtain a generic unique normal form N_b, $b \in \mathbb{C}^\tau$ for the classification of curves by performing the following operations on N_a:

1. *we set: $a_{1,kl+1} = 1$;*
2. *for each level d in a block B_i, $i \leq]p/2]$, we set $a_{b,d} = 0$ for the first q_d coefficients starting from the rightside of the line d;*
3. *for each level in the critical block $B_{p/2}$ (which appears if p is even), we consider the sequence of number $n_d - q_d$ (recall that in this block we have $n_d - q_d \in \{-1, 0, +1\}$).*
   - *if $n_d - q_d = 0$, we vanish all the coefficients of the line;*
   - *if $n_d - q_d = +1$, we set $a_{b,d} = 0$ for the first coefficient starting from the right side of the line d;*
   - *for the first lines such that $n_d - q_d = -1$ and encountered in the sequence on some line d, we set $a_{b,d} = 0$ for the unique coefficient on this line. Furthermore, we set $a_{b,d'} = 0$ for the second coefficient on the next line $d' > d$ such that $n_{d'} - q_{d'} = +1$, if such line exists.*
   - *for the last line such that $n_d - q_d = -1$ without upper line d' such that $n_{d'} - q_{d'} = +1$ we set $a_{b,d} = 0$ for the unique coefficient on this line.*
4. *for each level d in a block B_i, $i >]p/2]$, and every index b, we set $a_{b,d} = 0$.*

Proof. Since the projection $X_{0,0}^{(d_1)}$ of $X_{0,0}$ on the first line of the block B_1 is the radial vector field in the variables a_{b,d_1}, its flow acts by homothety on this level and we can make use of its action to normalize one coefficient to the value 1. We choose the first one starting from the right side. On all the higher levels of index $d > d_1$ and for the q_d vector fields $X_{m,n}$ such that $\nu(X_{m,n}) = d$, we have

$$X_{m,n}^{(d)} = \sum_b \Gamma_{m,n}(a_{d_0}, a_{d_1}) \frac{\partial}{\partial a_{b,d}}$$

in which $\Gamma_{m,n}(a_{d_0}, a_{d_1})$ only depends on the variables a_{b,d_0} and a_{b,d_1}. This is a consequence of the relation $X_{m,n} = x^m y^n \cdot X_{0,0}$ and of the proposition 1.14. Therefore this vector field is constant with respect to the variables of the level $d > d_1$. Its flow acts by translation and we make use of this flow (and the independence property) to vanish q_d coefficients.
In the critical block, if there is an extra vector field $X_{m,n}$ on a line d such that $n_d - q_d = -1$, we make use of the component $X_{m,n}^{(d')}$ to act on the next level d' such that $n_{d'} - q_{d'} = +1$. Suppose that this level is the next one ($d' = d + 1$). This means that we have to consider the action of the second non vanishing component of $X_{m,n}$. According to the proposition 1.14, this one will depend on the variables a_{b,d_0}, a_{b,d_1} and a_{b,d_1+1}. If we have to skip two lines it will depend on the variables a_{b,d_0}, a_{b,d_1}, a_{b,d_1+1} and a_{b,d_1+2}, and so on. Therefore, it turns out that the components of $X_{m,n}^{(d')}$ will only depend on variables $a_{b,d}$ with $d < d'$. Its flow still acts by translation and we make use of it to vanish the second coefficient of this line. ☐

We give in Appendix B the generic normal form obtained in the topological class $(k, l) = (3, 5)$ and $p = 4$.

2.4 An example: the case $y^n + x^{n+1}$

In [17], O. Zariski computes the dimension of the generic stratum of the moduli space of the curve

$$y^n + x^{n+1} = 0$$

for $n \geq 2$. We are going to apply our strategy to recover this dimension.
Let us consider $k = n$ and $l = n+1$. In this situation, the fundamental Bezout relation is written

$$n \cdot n - (n-1) \cdot (n+1) = 1.$$

Thus, $u = n$, $v = n-1$, $\nu_c = n^2 - n - 1$, and the triangle $\mathbb{T}$ is delimited by the two lines

$$\begin{aligned} j - ni &= n^2 - n - 1 \\ j - (n+1)i &= n^2 - n - 1. \end{aligned}$$

On a level j, this triangle bounds an interval

$$]l(j), r(j)[=] - \frac{n^2 - n - 1 - j}{n}, -\frac{n^2 - n - 1 - j}{n+1}[.$$

For $j \geq 0$, all these intervals have length less than 1, and we have:

$$\begin{aligned} l(j) \in \mathbb{Z} &\Leftrightarrow \exists \alpha \in \mathbb{N}, j = -1 + (\alpha+1)n \\ r(j) \in \mathbb{Z} &\Leftrightarrow \exists \alpha \in \mathbb{N}, j = 1 + \alpha(n+1). \end{aligned}$$

Therefore, the interval $]l(j), r(j)[$ contains an integer if and only if there exists α in $\mathbb{N}$ such that

$$1 + \alpha(n+1) < j < -1 + (\alpha+1)n.$$

Thus we have $n_j = 1$ for the above values of the index j, and $n_j = 0$ else. Now we have for each $k \geq 0$

$$\nu(X_{k,0}) - d_0 = kn + 1,\ \nu(X_{k-1,1}) - d_0 = kn + 2, \cdots, \nu(X_{0,k}) - d_0 = kn + k + 1$$

where $d_0 = n(n+1)$. This gives $q_j = 1$ for the above values of the index j and 0 else. We summarize these results in figure 1.

From the previous remarks the sequence $n_j - q_j$, $j \geq 0$ takes the following values:

$$0, -1, \underbrace{1, 1, \ldots, 1}_{j=2,\ldots,n-2}, 0, 0, -1, -1, \underbrace{1, 1, \ldots, 1}_{j=n+3,\ldots,2n-2}, 0, 0, -1, -1, -1, \underbrace{1, 1, \ldots, 1}_{j=2n+4,\ldots,3n-2} \cdots$$

Since there is only one branch, there is only one block and it is a critical block. Therefore we have

$$\begin{aligned} \tau = \widetilde{\Sigma}_{j \geq 0}(n_j - q_j) &= (n-4) + (n-6) + (n-8) + \cdots + (0 \ or\ 1) \\ &= \sum_{\alpha \geq 0} \sup(n - 4 - 2\alpha, 0) \\ &= \frac{(n-4)(n-2)}{4} \text{ if } n \text{ is even} \\ &= \frac{(n-3)^2}{4} \text{ if } n \text{ is odd} \end{aligned}$$

which are the formulas given in [17].

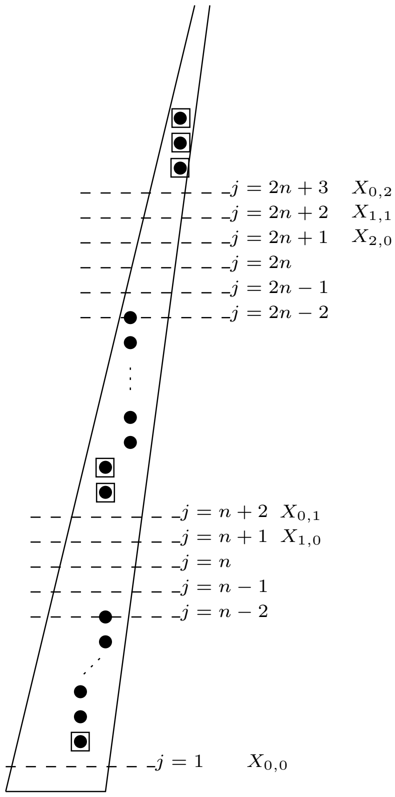

Figure 1: The case $y^n + x^{n+1}$

3 Appendix A: reduction of singularities of a topologically quasi-homogeneous function

Let f be a topologically quasihomogeneous function of weight (k, l) with p cuspidal branches, and multiplicities $(n_\infty, n_0, n_1, \ldots, n_p)$. From Lemma (1.1), we can consider a system of coordinates (x, y) such that f is written

$$f(x,y) = cx^{n_\infty} y^{n_0} \left(y^k + x^l + \cdots\right)^{n_1} \left(y^k + a_{2,kl}x^l + \cdots\right)^{n_2} \cdots \left(y^k + a_{p,kl}x^l + \cdots\right)^{n_p}$$

where the dots contains terms of (k, l)-degree bigger than kl.
Let θ_f be the vector field with an isolated singularity defined by

$$\theta_f = \frac{1}{\text{g.c.d.}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)} \cdot \left(\frac{-\partial f}{\partial x}\frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right) \tag{21}$$

The vector field θ_f can be also defined as a dual of the 1-form $f^{\text{red}}\frac{df}{f}$ for the standard volume form $dx \wedge dy$

$$dx \wedge dy \left(\theta_f, \cdot\right) = f^{\text{red}}\frac{df}{f} \tag{22}$$

where

$$f^{\text{red}}(x,y) = cxy\left(y^k + x^l + \cdots\right)\left(y^k + a_{2,kl}x^l + \cdots\right)\cdots\left(y^k + a_{p,kl}x^l + \cdots\right).$$

3.1 The desingularization

The desingularization of f is exactly the same as its topological quasihomogeneous model $f_{\rm qh}$

$$f_{\rm qh}(x,y) = cx^{n_\infty}y^{n_0}\left(y^k+x^l\right)^{n_1}\left(y^k+a_{2,kl}x^l\right)^{n_2}\cdots\left(y^k+a_{p,kl}x^l\right)^{n_p}.$$

The process of desingularization $E : M \to (\mathbb{C}^2, 0)$ can be inductively described as follows: the map E is written $E_1 \circ \tilde{E}$ where

- E_1 is the standard blow-up of $(0,0)$ in $\mathbb{C}^2$.
- $\tilde{E}$ is the process of reduction of $E_1^* f_{\rm qh}$ which is a quasi-homogeneous function of degree $(k, l-k)$.

Therefore, the process of desingularization will follow the Euclide algorithm for the couple (k,l). In particular, the exceptional divisor is a chain of compact components $\mathbb{CP}^1$ such that each of them is linked exactly with two others except the extremal components. There is exactly one component called the central component along which is attached the strict transform of the cuspidal branches of $f_{\rm qh}$.

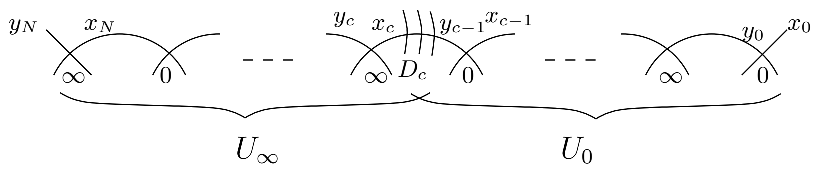

Figure 2: Desingularization of a topologically quasi-homogeneous function.

In what follows we will keep the following notations:

Notation 3.1.

- *The integers u and v are defined by:*

$$uk - vl = 1,\ 0 \le u < l,\ 0 \le v < k.$$

- *The numbering $D_1, \ldots, D_N$ of the components of D is a geometric order of the chain, from the one which contains the strict transform of $y=0$, to the one which contains the strict transform of $x=0$. It is not the "historical" order of the process.*
- *On each D_i, we denote by 0 the intersection point with D_{i-1}, or with the strict transform of $y=0$ for D_1, and by ∞ the intersection with D_{i+1} or with the strict transform of $x=0$ for D_N.*
- *Each component D_i is covered by two charts (x_{i-1}, y_{i-1}) and (x_i, y_i) whose domains V_{i-1} and V_i contains $(D_i, 0)$ and (D_i, ∞). The change of coordinates are given by*

$$x_{i-1} = x_i^{e_i} y_i,\ y_{i-1} = x_i^{-1}$$

where $-e_i$ is the self intersection of the component D_i.

- *On the principal component D_c, we choose the two charts such that each domain V_{c-1}, V_c, contains all the strict transforms of the cuspidal branches.*
- *We define the covering of D by the two open sets:*

$$U_0 = \cup_{i=0}^{c-1} V_i, \quad U_\infty = \cup_{i=c}^{N} V_i.$$

Proposition 3.2. *The desingularization map E is given in the chart (x_c, y_c) by*

$$(x,y) = (x_c^{k-v} y_c^k \ , \ x_c^{l-u} y_c^l).$$

The blowing down is given in this chart by:

$$x_c = \frac{x^l}{y^k}, \quad y_c = \frac{x^{u-l}}{y^{v-k}}.$$

Proof. We prove this result by an induction on the number of blowing up's of the minimal desingularization of f. For one blow-up, we have: $k = l = 1$, $u = 1$, $v = 0$ therefore, the formula is valid in this case. After one blow-up E_1, the germ of $E_1^* f$ at its singular point along the exceptional divisor is a quasi-homogeneous function in the class $(k, l-k)$. Notice that if $uk - vl = 1$ is the Bézout identity of (k,l), the corresponding Bézout identity for the new pair is $(u-v)k - v(l-k) = 1$. Let us suppose that the formula of Proposition (3.2) is valid for the pair $(k, l-k)$. Therefore, after one blowing-up we have in the first chart

$$x_1 = x_c^{k-v} y_c^k, \quad y_1 = x_c^{l-k-u+v} y_c^{l-k}. \tag{23}$$

Thus, we obtain:

$$x = x_1 = x_c^{k-v} y_c^k, \quad y = x_1 y_1 = x_c^{l-u} y_c^l.$$

From this, we easily obtain the inverse formulas defining the blowing-down. □

3.2 Computing multiplicities

We first recall the classical result which allows us to compute the multiplicities of a function along the components D_i of the exceptional divisor D of its desingularization [4]: we consider the matrix of intersections J defined for $i \neq j$ by $J_{i,j} = 1$ if the two components D_i and D_j meet together, $J_{i,j} = 0$ otherwise, and $J_{i,i} = -e_i$, where $-e_i$ is the self intersection of each component. For any component D_i, let n_i be the number of strict branches of $f \circ E$ meeting D_i, counted with their multiplicities, and let B be the column matrix induced by these numbers.

Proposition 3.3. *The multiplicities m_i of $(f \circ E)$ along each D_i define a column matrix M which satisfy*

$$JM + B = 0.$$

In the quasi-homogeneous case, since $D = D_1 \cup \cdots \cup D_{c-1} \cup D_c \cup D_{c+1} \cup \cdots \cup D_N$, the column matrix B is here: $(n_0, 0, \ldots, 0, n_c, 0, \ldots, 0, n_\infty)^t$, where $n_c = \sum_{b=1}^p n_b$ is on index c. The intersection matrix is given by:

$$J = \begin{pmatrix} -e_1 & 1 & 0 & \cdots & & & & 0 \\ 1 & -e_2 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -e_3 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & & & & \\ 0 & \cdots & & & 0 & 1 & -e_{N-1} & 1 \\ 0 & \cdots & & & & 0 & 1 & -e_N \end{pmatrix}$$

Therefore we obtain the multiplicities of f by $M = -J^{-1}B$ (see example below).

We compute now the multiplicities of the *desingularized foliation*, i.e. of the vector field $E^*\theta_f$, where θ_f is the vector field (21) with isolated singularity, defining the foliation $\mathcal{F}$.

Proposition 3.4. *1. The multiplicities ν_i of $E^*\theta_f$ along each component D_i define a column matrix N which satisfy*

$$JN + C = 0,$$

where $C = (\varepsilon_0 - 1, 0, \ldots, 0, p, 0, \ldots, 0, \varepsilon_\infty - 1)^t$, with p on index c.

2. *The multiplicity of $E^*\theta_f$ on the principal component D_c of D is*

$$\nu_c = klp - k - l + k\varepsilon_\infty + l\varepsilon_0.$$

3. *The multiplicities of $E^*\theta_f$ on the $(y_0 = 0)$ (strict transform of the x-axis) and on $(x_N = 0)$ (strict transform of the y-axis) are $\varepsilon_0 - 1$ and $\varepsilon_\infty - 1$.*

Proof. Let $V = (v_i)$ be the multiplicities of $E^*dx \wedge dy$ along each D_i. From

$$E^*(dx \wedge dy)(E^*\theta_f, \cdot) = (f^{red} \circ E)d(f \circ E)/(f \circ E)$$

we obtain:

$$v_i + \nu_i = r_i + (m_i - 1) - m_i = r_i - 1$$

where $r_i = \nu(f^{red} \circ E, D_i)$. We consider the "axis function": $a = xy$. Let $A = (a_i)$ be the column matrix of multiplicities of $a \circ E$ along each D_i. We claim that $v_i = a_i - 1$. Indeed, let (x_i, y_i) be the chart induced by (x, y) and E around the origin of D_i. Since E is here monomial in these coordinates, there exist positive integers p, q, r, s, such that:

$$E^*dx \wedge dy = a \circ E \cdot E^*(\frac{dx}{x} \wedge \frac{dy}{y}) = a \circ E \cdot (ps - qr)\frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i}$$

from which we deduce $v_i = a_i - 1$. Therefore we obtain $A + N = R$, where R, A are the matrices of multiplities of $(f^{red} \circ E)$ and $a \circ E$ along each D_i. Now, from the previous proposition applied to the functions f^{red} and a we have: $JR + B^{red} = 0$, with $B^{red} = (\varepsilon_0, 0, \ldots, 0, p, 0, \ldots, 0, \varepsilon_\infty)^t$ and $JA + B' = 0$ where B' is the column matrix such that $b'_i = 1$ for $i = 1$ or $i = N$ and $b'_i = 0$ otherwise. We obtain:

$$JN = J(R - A) = -B^{red} + B' = -C.$$

For the principal component, by making use of the formulas of proposition (3.2), we obtain:

$$\begin{aligned} \nu_c &= \nu_{y_c}(E^*\theta_f) = r_c - 1 - v_c = (klp + k\varepsilon_\infty + l\varepsilon_0 - 1) - (k + l - 1) \\ &= klp + k\varepsilon_\infty + l\varepsilon_0 - k - l. \end{aligned}$$

On the branch $(y_0 = 0)$, we have $\nu_{y_0}(E^*f^{red}) = \varepsilon_0$ and $\nu_{y_0}(a \circ E) = 1$. Therefore,

$$\nu_{y_0}(E^*\theta_f) = \nu_{y_0}\left(E^*f^{red}\frac{df}{f}\right) - \nu_{y_0}(E^*dx \wedge dy) = \varepsilon_0 - 1.$$

We obtain the multiplicity on $(x_N = 0)$ by a similar computation. ☐

Example. For $(k, l) = (3, 5)$, the matrix of intersections is:

$$J = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

and we have $B = (n_0, n_c, 0, n_\infty)^t$, where $n_c = \sum_{b=1}^{p} n_b$, and $C = (\varepsilon_0 - 1, p, 0, \varepsilon_\infty - 1)$. Therefore we obtain:

$$M = \begin{pmatrix} 2n_0 + n_\infty + 5n_c \\ 5n_0 + 3n_\infty + 15n_c \\ 3n_0 + 2n_\infty + 9n_c \\ n_0 + n_\infty + 3n_c \end{pmatrix}; \quad N = \begin{pmatrix} 2\varepsilon_0 + \varepsilon_\infty + 5p - 3 \\ 5\varepsilon_0 + 3\varepsilon_\infty + 15p - 8 \\ 3\varepsilon_0 + 2\varepsilon_\infty + 9p - 5 \\ \varepsilon_0 + \varepsilon_\infty + 3p - 2 \end{pmatrix}.$$

The multiplicity of the foliation on the principal component D_2 is

$$\nu_c = 5\varepsilon_0 + 3\varepsilon_\infty + 15p - 8.$$

3.3 Acyclic covering of D for the sheaf $\Theta_{\mathcal{F}}$

We consider the covering $\{U_0, U_\infty\}$ defined in (3.1).

Lemma 3.5. *There exists a global section T_0 (resp. T_∞) of the sheaf $\Theta_{\mathcal{F}}$ of germs of vector fields tangent to $E^*\mathcal{F}$ on U_0 (resp. U_∞) which admits only isolated singularities.*

Proof. From Proposition (3.4), the following holomorphic vector fields

$$\theta_0 = \frac{E^*\theta_f}{x_0^{\nu_1} y_0^{\varepsilon_0 - 1}}, \ \theta_i = \frac{E^*\theta_f}{x_i^{\nu_{i+1}} y_i^{\nu_i}}, \ i = 1, \dots, c-1,$$

have isolated singularities. We claim that they glue together on their common domains, defining a global section T_0 of $\Theta_{\mathcal{F}}$ on U_0. Indeed, from the previous relation $JN + C = 0$, we have :

$$\begin{aligned} -e_1\nu_1 + \nu_2 + \varepsilon_0 - 1 &= 0 \\ \nu_{i-1} - e_i\nu_i + \nu_{i+1} &= 0, \ \ i = 2, \dots, c-1 \end{aligned}$$

Therefore, using the change of coordinates between two consecutive charts, we have

$$\begin{aligned} x_1^{\nu_2} y_1^{\nu_1} &= y_0^{-\nu_2} x_0^{\nu_1} y_0^{e_1\nu_1} = x_0^{\nu_1} y_0^{\varepsilon_0 - 1} \\ x_i^{\nu_{i+1}} y_i^{\nu_i} &= y_{i-1}^{-\nu_{i+1}} x_{i-1}^{\nu_i} y_{i-1}^{e_i\nu_i} = x_{i-1}^{\nu_i} y_{i-1}^{\nu_{i-1}}, \ i = 1, \dots, c-1. \end{aligned}$$

The proof is similar for constructing T_∞ on U_∞. □

Proposition 3.6. *We have $H^1(U_0, \Theta_{\mathcal{F}}) = H^1(U_\infty, \Theta_{\mathcal{F}}) = 0$.*

Proof. The previous section T_0 with isolated singularities allows us to identify the sheaf $\Theta_{\mathcal{F}}|_{U_0}$ to $\mathcal{O}_M|_{U_0}$. Since the Chern class of each branch is negative, a direct computation with the change of charts shows that $H^1(U_0, \mathcal{O}_M) = 0$. The proof is similar for U_∞. □

4 Appendix B: normal forms for $(k, l) = (3, 5)$ and $p = 4$

According to the figure draw below, the analytical normal form for the topological class of

$$\left(y^3 + x^5\right)^{n_1} \left(y^3 + a_2 x^5\right)^{n_2} \left(y^3 + a_3 x^5\right)^{n_3} \left(y^3 + a_4 x^5\right)^{n_4}$$

is given by the following family of functions with 78 parameters

$$
\begin{aligned}
N_a \; = \; & \big(y^3 + x^5 + a_{1,16}x^2y^2 + a_{1,19}x^3y^2\big)^{n_1} \times \\
& \big(y^3 + a_{2,15}x^5 + a_{2,16}x^2y^2 + a_{2,17}x^4y + a_{2,18}x^6 + a_{2,19}x^3y^2 + a_{2,20}x^5y \\
& \quad + a_{2,21}x^7 + a_{2,22}x^4y^2 + a_{2,23}x^6y + a_{2,24}x^8 + a_{2,25}x^5y^2 + a_{2,26}x^7y \\
& \quad + a_{2,28}x^6y^2 + a_{2,29}x^8y + a_{2,31}x^7y^2 + a_{2,34}x^8y^2\big)^{n_2} \times \\
& \big(y^3 + a_{3,15}x^5 + a_{3,16}x^2y^2 + a_{3,17}x^4y + a_{3,18}x^6 + a_{3,19}x^3y^2 + a_{3,20}x^5y \\
& \quad + a_{3,21}x^7 + a_{3,22}x^4y^2 + a_{3,23}x^6y + a_{3,24}x^8 + a_{3,25}x^5y^2 + a_{3,26}x^7y \\
& \quad + a_{3,27}x^9 + a_{3,28}x^6y^2 + a_{3,29}x^8y + a_{3,30}x^{10} + a_{3,31}x^7y^2 + a_{3,32}x^9y \\
& \quad + a_{3,33}x^{11} + a_{3,34}x^8y^2 + a_{3,35}x^{10}y + a_{3,36}x^{12} + a_{3,37}x^8y^2 + a_{3,38}x^{11}y \\
& \quad + a_{3,39}x^{13} + a_{3,40}x^9y^2 + a_{3,41}x^{12}y + a_{3,43}x^{10}y^2 + a_{3,44}x^{13}y + a_{3,46}x^{12}y^2 \\
& \quad + a_{3,49}x^{13}y^2\big)^{n_3} \times \\
& \big(y^3 + a_{4,15}x^5 + a_{4,17}x^4y + a_{4,18}x^6 + a_{4,20}x^5y + a_{4,21}x^7 + a_{4,23}x^6y \\
& \quad + a_{4,24}x^8 + a_{4,26}x^7y + a_{4,27}x^9 + a_{4,29}x^8y + a_{4,30}x^{10} + a_{4,32}x^9y \\
& \quad + a_{4,33}x^{11} + a_{4,35}x^{10}y + a_{4,36}x^{12} + a_{4,38}x^{11}y + a_{4,39}x^{13} + a_{4,41}x^{12}y \\
& \quad + a_{4,42}x^{14} + a_{4,44}x^{13}y + a_{4,45}x^{15} + a_{4,47}x^{14}y + a_{4,48}x^{16} + a_{4,50}x^{15}y \\
& \quad + a_{4,51}x^{17} + a_{4,53}x^{16}y + a_{4,54}x^{18} + a_{4,56}x^{17}y + a_{4,59}x^{18}y\big)^{n_4} .
\end{aligned}
$$

Moreover, the normal forms for the generic curve are given by the 35-parameters family

$$
\begin{aligned}
N_a \; = \; & \big(y^3 + x^5 + x^2y^2\big) \times \\
& \big(y^3 + a_{2,15}x^5 + a_{2,16}x^2y^2 + a_{2,17}x^4y + a_{2,18}x^6 + a_{2,19}x^3y^2 + a_{2,20}x^5y \\
& \quad + a_{2,23}x^6y\big) \times \\
& \big(y^3 + a_{3,15}x^5 + a_{3,16}x^2y^2 + a_{3,17}x^4y + a_{3,18}x^6 + a_{3,19}x^3y^2 + a_{3,20}x^5y \\
& \quad + a_{3,21}x^7 + a_{3,22}x^4y^2 + a_{3,23}x^6y + a_{3,24}x^8 + a_{3,25}x^5y^2 + a_{3,26}x^7y \\
& \quad + a_{3,28}x^6y^2 + a_{3,29}x^8y\big) \times \\
& \big(y^3 + a_{4,15}x^5 + a_{4,17}x^4y + a_{4,18}x^6 + a_{4,20}x^5y + a_{4,21}x^7 + a_{4,23}x^6y \\
& \quad + a_{4,24}x^8 + a_{4,26}x^7y + a_{4,27}x^9 + a_{4,29}x^8y + a_{4,30}x^{10} + a_{4,32}x^9y \\
& \quad + a_{4,33}x^{11} + a_{4,35}x^{10}y\big) .
\end{aligned}
$$

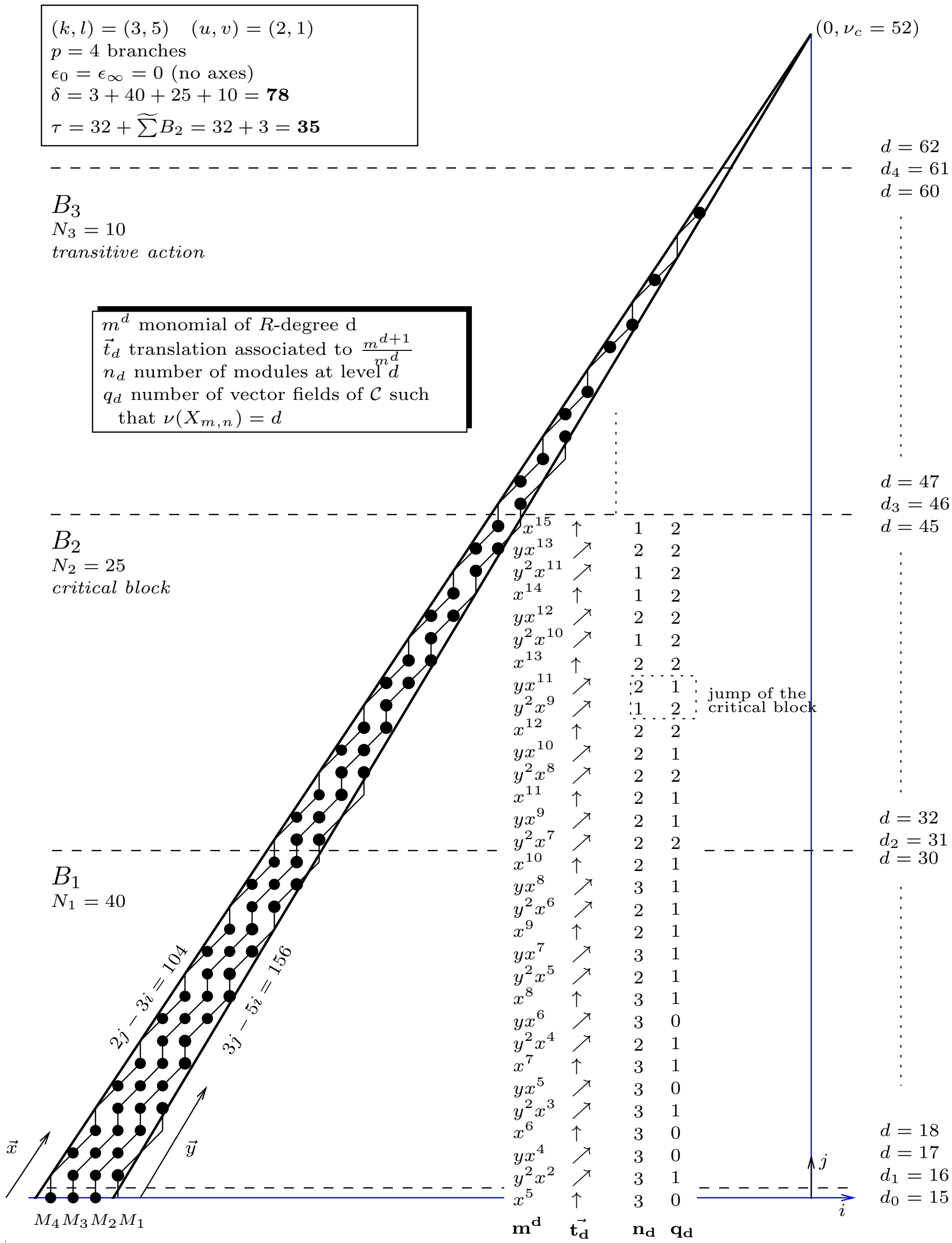

Figure 3: Moduli triangle of the topological class $(k,l)=(3,5)$ and $p=4$

References

[1] M. Berthier, D. Cerveau and R. Meziani, *Transformations isotropes des germes de feuilletages holomorphes* J. Math. Pures Appl. 9-78 (1999)

[2] J. Briançon, M. Granger, and Ph. Maisonobe, *Le nombre de modules du germe de courbe plane* $x^a + y^b = 0$, Math. Ann. 279-3 (1988)

[3] C. Camacho, A. Lins Neto and P. Sad, *Topological invariants and equidesingularization for holomorphic vector fields* J. Differential Geom. 20-1 (1984)

[4] D. Eisenbud and W.D. Neumann, *Three-dimensional link theory and invariants of plane curve singularities,* Ann. Math. Stud. 101 (1985)

[5] A. Hefez and M. Hernandes, *Analytic classification of plane branches up to multiplicity 4*, J. Symbolic Comput. 44-6 (2009)

[6] A. Hefez and M. Hernandes, *The analytic classification of plane branches*, Bull. Lond. Math. Soc. 43-2 (2011)

[7] A. Hefez and Hernandes M. *Standard bases for local rings of branches and their modules of differentials,* J. Symbolic Comput. 42 (2007)

[8] Y. Genzmer and E. Paul - *Normal forms of foliations and curves defined by a function with a generic tangent cone,* Moscow math. jour. 11-1 (2011)

[9] J.M. Granger - *Sur un espace de modules de germe de courbe plane,* Bull. Sc. math. 103-2 (1979)

[10] M. Lejeune-Jalabert, *Sur l'équivalence des courbes algébroïdes planes. Coefficients de Newton. Contribution à l'étude des singularités du point de vue du polygone de Newton*, Université de Paris VII, Paris, (1973)

[11] J. N. Mather and S. S. T. Yau. *Classication of isolated hypersurface singularities by their moduli algebras.* Invent. Math., 69-2 (1982)

[12] J.F. Mattei - *Quasihomogénité et équiréductibilité de feuilletages holomorphes en dimension 2,* Astérisque S.M.F. 261 (2000)

[13] J.F. Mattei - *Modules de feuilletages holomorphes singuliers: I équisingularité,* Invent. Math. 103 (1991)

[14] J.F. Mattei and R. Moussu - *Holonomie et intégrales premières,* Ann. Sci. Ecole Norm. Sup. 13 (1980)

[15] K. Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen* Invent. Math. 14 (1971)

[16] A. Seidenberg - *Reduction of singularities of the differential equation* $Ady = Bdx$, Amer. J. Math. 90 (1968)

[17] O. Zariski - *Le problème des modules pour les branches planes,* Hermann, Paris, seconde édition (1986), course given at the Centre de Mathématiques de l'Ecole Polytechnique (1973).

Y. Genzmer
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse cedex 9, France.
yohann.genzmer@math.univ-toulouse.fr

E. Paul
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse cedex 9, France.
emmanuel.paul@math.univ-toulouse.fr