Classification of regular dicritical foliations

Gabriel Calsamiglia, Yohann Genzmer*

May 3, 2015

Abstract

In this paper we give complete analytic invariants for the set of germs of holomorphic foliations in $(\mathbb{C}^2, 0)$ that become regular after a single blow-up. Some of the invariants describe the holonomy pseudogroup of the germ and are called transverse invariants. The other invariants lie in a finite dimensional complex vector space. Such singularities admit separatrices tangent to any direction at the origin. When enough separatrices are tangent to a radial foliation (a condition that can always be attained if the mutiplicity of the germ at the origin is at most four) we are able to describe and realize all the analytical invariants geometrically and provide analytic normal forms. As a consequence we prove that any two such germs sharing the same transverse invariants are conjugated by a very particular type of birational transformations. We also provide explicit examples of universal equisingular unfoldings of foliations that cannot be produced by unfolding functions. With these at hand we are able to explicitely parametrize families of analytically distinct foliations that share the same transverse invariants.

1 Introduction

In this paper, we deal with analytic invariants, normal forms and unfoldings of germs of holomorphic foliations in $(\mathbb{C}^2, 0)$. Two such foliations are said to be analytically equivalent if there exists a germ of biholomorphism of $(\mathbb{C}^2, 0)$ sending leaves of one to leaves of the other. There are two known invariants under this equivalence. On the one hand, there are the analytic invariants of the **holonomy pseudogroup** formed by holonomy maps associated to leafwise paths and transverse sections at the endpoints that are realized on any neighbourhood of the origin. These maps are holomorphic as soon as the foliation is. On the other hand, we know from [4] that there is a non-empty set of separatrices through 0. These are leaves L such that $L \cup 0$ is a germ of analytic curve at 0. When the union of all separatrices is an analytic curve, there are instances where the analytical class of these two objects determines the analytical class of the foliation. For example, for generic homogeneous foliations, namely foliations whose separatrix set is a homogeneous curve and whose singularities after one blow-up are

*The first author is supported by CNPq/Faperj; the second author supported by the ANR under the project ANR-13-JS01-0002-01. Both authors are supported by the Math-Amsud project DFH.

of hyperbolic type, the analytical class of the curve and of the projective holonomy representation determine the analytical type of the singularity (see [11] or [6]). In particular, foliations defined by holomorphic vector fields whose multiplicity at the origin is less than five and with generic first homogeneous term, fall in the previous case. When the foliation admits an infinite number of separatrices a natural question is to decide whether there exists some analytic subset contained in the separatrix set whose analytical class together with the analytical class of the holonomy pseudogroup determines the analytical class of the foliation.

A different approach to the problem of determining analytic invariants was taken by J.-F. Mattei in the 80's (see [12]). In the spririt of Kodaira and Spencer's theory of deformations of complex structures on manifolds, he proved that any germ $\mathcal{F}$ of holomorphic foliation of $(\mathbb{C}^2, 0)$ can be unfolded to a *codimension one* germ of foliation $\widetilde{\mathcal{F}}$ on $(\mathbb{C}^{2+\mathcal{M}(\mathcal{F})}, 0)$ in such a way that any other unfolding of $\mathcal{F}$ that preserves the singularity type of $\mathcal{F}$ is equivalent to one obtained from $\widetilde{\mathcal{F}}$ in a unique manner by pull back. He calculated the dimension $\mathcal{M}(\mathcal{F})$ of the base space of this *universal equisingular unfolding* of $\mathcal{F}$ and concluded that it is always finite and almost always positive. By construction, the deformation of $\mathcal{F}$ obtained by considering the foliations $\{\mathcal{F}_c : c \in (\mathbb{C}^{\mathcal{M}(\mathcal{F})}, 0)\}$ of $(\mathbb{C}^2, 0)$ obtained by restricting $\widetilde{\mathcal{F}}$ to the fibres of the projection $(\mathbb{C}^{2+\mathcal{M}(\mathcal{F})}, 0) \to (\mathbb{C}^{\mathcal{M}(\mathcal{F})}, 0)$ has the same singularity type and holonomy pseudogroup for all parameters (up to equivalence), but are not analytically equivalent. The moduli of Mattei tell us in how many ways we can locally change the analytical class of the foliation without changing the holonomy pseudo-group. In the case of homogeneous foliations, these moduli are simply the relative position of the points in the tangent cone of the set of separatrices: actually, they characterize the analytical class of the separatrix set. In general, we are not able to interpret geometrically the other moduli of Mattei. Indeed, the construction of the latter is not explicit, producing the foliations by foliated surgery. To our knowledge, the only known explicit examples of such non-trivial unfoldings were obtained by unfolding germs of functions. In fact, examples of families of foliations with varying analytical class but constant analytical class of its separatrix sets can be constructed by unfolding functions (see [7]). For example, the foliation defined by the germ at zero of the function

$$f(x, y, z) = (1 + z)xy(x + y)(x - y)(x + 2y + y^2)$$

defines a non-trivial unfolding of the foliation $\mathcal{F} = \{f(x, y, 0) = const\}$ having the same separatrix set for all parameters $z \in (\mathbb{C}, 0)$.

Yet another approach has been taken in recent years by Ortiz-Rosales-Voronin (see [14, 15, 16]). Their stragtegy is, on the one hand to find unique formal normal forms (up to formal transformations tangent to the identity) for certain families of foliations, and on the other to prove that formal analytical rigidity takes place in the generic cases. Hence the coefficients of the formal normal form turn out to be analytical invariants. This infinite number of parameters is then split into two subsets: one of them is infinite and contains the information on the holonomy pseudo-group and the other is finite and contains the rest of parameters. The number of parameters that is not associated to the holonomy pseudo-group coincides with the number of Mattei's parameters although it is not clear how the formal deformations obtained in the formal normal forms correspond to unfoldings in the sense of Mattei. Again, it is not clear what these parameters mean geometrically.

In this paper, we will give a description of a complete family of invariants, their geometric interpretation, analytic normal forms and unfoldings for a particular class of germs of foliations admitting an infinite number of separatrices: homogeneous dicritical foliations. A germ of foliation $\mathcal{F}$ is said to be homogeneous dicritical if it is regular after a single blow-up and there exists a germ of foliation $\mathcal{G}$ with radial linear part such that $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ is invariant. After one blow up $\mathcal{F}$ can be thought of as a local version of regular Riccati foliations around the exceptional divisor, where the role of the fibration is taken by $\mathcal{G}$. In the simplest case where there exist non-trivial equisingular unfoldings, i.e. when the algebraic multiplicity is three, we will provide a parametrization of the set of analytic equivalence classes. In its parameter space the equivalence relation "having the same transverse invariants" is described by the fibers of a natural projection. Most of our arguments will be geometric in this part, that will take up sections 2. and 4.

In section 3. we will apply formal methods to try to generalize the claims of the homogeneous case to the set of regular dicritical foliations, that is, germs of foliations that are regular after a single blow-up.

We are thankful to G. Casale, J-F. Mattei, L. Ortiz, E. Paul, E. Rosales, P. Sad, E. Salem and L. Teyssier for useful conversations on the subject of the paper.

2 Homogeneous dicritical foliations

In this section we will introduce the basic properties of foliations that are regular after a single blow-up and describe a complete family of geometric invariants for homogeneous dicritical foliations. We will also provide analytic normal forms for the latter family.

2.1 Germs that are regular after a single blow-up

The set of all germs of foliations that are regular after a single blow up will be denoted by $\mathcal{D}$. For each $n \in \mathbb{N}$ we denote by $\mathcal{D}(n)$ the set of foliations in $\mathcal{D}$ whose algebraic multiplicity is $n+1$. In particular $\mathcal{D}(0)$ corresponds to foliations with radial linear part, and will be called radial foliations. By Poincaré's linearization theorem, every radial foliation is holomorphically linearizable.

Given $\mathcal{F} \in \mathcal{D}$, the pull-back foliation by the blow-up map, denoted by $\widetilde{\mathcal{F}}$ must be generically transverse to the exceptional divisor E by Camacho-Sad index theorem ([4]). Thus, the foliations in $\mathcal{D}$ are *dicritical*: they have an infinite number of invariant curves. In fact, every leaf is a separatrix. We call them *regular dicritical foliations.* We can define the *tangency divisor* $T(\mathcal{F}) := \mathrm{Tang}(\mathcal{F}, E) \in \mathrm{div}(E)$ between $\widetilde{\mathcal{F}}$ and E. It is an effective divisor defined on the curve E whose degree is n if and only if $\mathcal{F}$ belongs to $\mathcal{D}(n)$. Klughertz showed in [9] that two foliations $\mathcal{F}, \mathcal{F}' \in \mathcal{D}$ are topologically equivalent if and only if there exists a homeomorphism from E to E sending $T(\mathcal{F})$ to $T(\mathcal{F}')$ (i.e. they are topologically equivalent as divisors of E). Thus any partition of

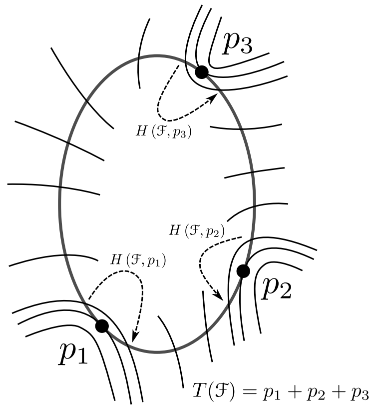

Figure 1: A foliation in $\mathcal{D}$ with three points in its divisor of tangency $T(\mathcal{F})$.

$n = n_1 + \ldots + n_k$ defines a topological class in $\mathcal{D}(n)$ formed by foliations $\mathcal{F}$ having $T(\mathcal{F}) = n_1 p_1 + \ldots n_k p_k$ for some pairwise distinct points p_i in E.

The *holonomy pseudogroup* of such a foliation is quite simple: for any $p \in |T(\mathcal{F})|$, we can consider a local primitive holomorphic first integral f of $\widetilde{\mathcal{F}}$ around p with connected fibers. The levels of the restriction $f_{|E}$ describe sets of points that belong to the same connected component of the fiber of f. Since $f_{|E}$ is a holomorphic germ in one complex variable, it can be written as a power ψ^{r+1}, where r is the order of tangency between $\widetilde{\mathcal{F}}$ and E at p, and ψ is a holomorphic germ of diffeomorphism. If θ_p denotes the rotation of angle $2\pi/(r+1)$, the germ $H(\mathcal{F}, p) := \psi^{-1} \circ \theta_p \circ \psi$ and any of its powers realizes a holonomy map around each point q sufficiently close to p: it is the holonomy associated to a path in $f^{-1}(f(q))$ joining q and $H(\mathcal{F}, p)(q)$ and tranverse section E at both the starting point and the endpoint. Even if we reduce the neighbourhood where the fibers of f are connected, we can still find points where the map is realized by the holonomy of some leafwise path. Up to reducing the neighbourhood where $\mathcal{F}$ is defined, two different $H(\mathcal{F}, p)$'s cannot be composed so the only holonomy maps are powers of elements of $H(\mathcal{F}, p)'s$. Thus

$$H(\mathcal{F}) = \bigsqcup_{p \in T(\mathcal{F})} H(\mathcal{F}, p)$$

is the generating set of the holonomy pseudogroup of the germ $\mathcal{F}$ and we will call it the *holonomy* of $\mathcal{F}$. If $T(\mathcal{F})$ is equal to $n_1 p_1 + \ldots + n_k p_k$, the holonomy pseudogroup of $\mathcal{F}$ is a disjoint union of finite cyclic groups of orders $n_1 + 1, \ldots, n_k + 1$ and it determines the topological class of $\mathcal{F}$.

If $\mathcal{F}' = \phi(\mathcal{F})$ for a holomorphic equivalence ϕ, then by construction

$$H(\mathcal{F}') = \phi_{|E} \circ H(\mathcal{F}) \circ \phi_{|E}^{-1}$$

where $\phi_{|E}$ is the *global* holomorphic automorphism of $E \cong \mathbb{P}^1$ induced by ϕ. Thus the class of $H(\mathcal{F})$ modulo global automorphisms of E is an analytical invariant of $\mathcal{F}$ that we will denote by $H[\mathcal{F}]$ and call the *holonomy class* of $\mathcal{F}$. As we will see along this section this invariant is not enough in general to determine the analytical class of the foliation as soon as the algebraic multiplicity is bigger than two.

2.2 Invariants of homogeneous dicritical foliations

A foliation $\mathcal{F} \in \mathcal{D}$ is said to be homogeneous dicritical if there exists a radial foliation $\mathcal{G}$ such that $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ is *invariant* by $\mathcal{F}$ (and therefore also by $\mathcal{G}$). In this case, we say that $\mathcal{F}$ is homogeneous with respect to $\mathcal{G}$. In other words, up to a change of coordinates (one that linearizes $\mathcal{G}$) there exists a subset of separatrices that is a union of straight lines and supports the tangency set of the foliation with the radial foliation. The subset of $\mathcal{D}$ formed by homogeneous dicritical foliations will be denoted by $\mathcal{D}^{\mathrm{h}}$ and its subset $\mathcal{D}^{\mathrm{h}}(n) = \mathcal{D}(n) \cap \mathcal{D}^{\mathrm{h}}$ is formed by those having multiplicity $n+1$ at the origin. The following are examples of elements in $\mathcal{D}^{\mathrm{h}}(n)$ for $n \geq 1$:

1. Consider homogenous polynomials $R(x, y)$ and $Q(x, y)$ of degrees n and $n+2$ respectively such that R and xQ are coprime. The germ of foliation defined by a holomorphic one form
$$\omega(x, y) = (R(x, y) + \cdots)(xdy - ydx) + Q(x, y)dx$$
in a neighbourhood of $0 \in (\mathbb{C}^2, 0)$ is homogeneous with respect to $x\partial_x + y\partial_y$ and is thus contained in $\mathcal{D}^{\mathrm{h}}(n)$.

2. Consider a smooth rational curve C embedded in a complex surface S with self-intersection $(n+2)$. Suppose that S is bifoliated by a pair of regular *transverse* holomorphic foliations $\mathcal{F}$ and $\mathcal{G}$ and that $\mathcal{G}$ is transverse to C at all points. Remark that by Camacho-Sad's theorem (see [4]), the regularity condition and $C \cdot C > 0$, C cannot be invariant by $\mathcal{F}$. Then the blow up of S at $n+3$ points in $C \setminus \mathrm{Tang}(\mathcal{F}, C)$ produces two foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ around a (-1)-curve. By Grauert's theorem the contraction of this curve produces a regular point in a complex surface, endowed with two singular foliations. By construction, the germ of foliation associated to the initial $\mathcal{F}$ is homogeneous with respect to the radial foliation associated to $\mathcal{G}$, and thus belongs to $\mathcal{D}^{\mathrm{h}}(n)$.

As we will see along this section and every homogeneous dicritical foliation can be interpreted in the way described by these examples.

The holonomy class of a foliation in $\mathcal{D}$ does not characterize the analytical class of the foliation in general, but in the homogeneous case it characterizes the class modulo a very particular family of birational maps:

Theorem I. *Any homogeneous dicritical foliation $\mathcal{F}$ is equivalent to one obtained by blowing up a foliation $\mathcal{F}_S$ on a surface S as in example 2. The analytical class of $\mathcal{F}_S$ in a neighbourhood of the rational curve is uniquely determined by $H[\mathcal{F}]$.*

If we consider that a *local birational tranformation* stands for a map composed of sucessive changes of coordinates, blow-ups and contractions of compactified regular separatices as illustrated in Figure 2, we deduce directly

Corollary 1. *Any pair of homogeneous dicritical foliations sharing the same holonomy class are locally birationally conjugated.*

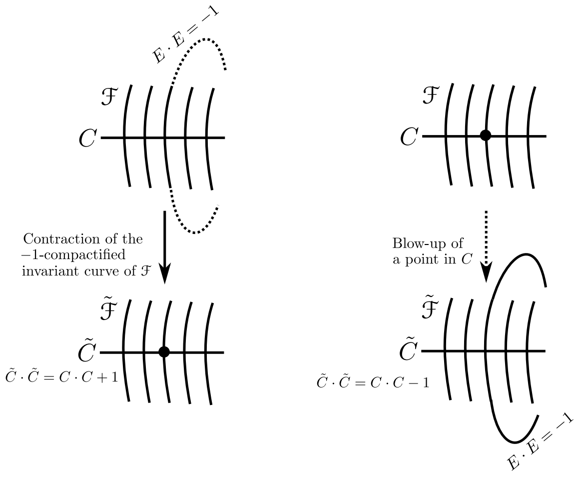

Figure 2: Local birational transformations in $\mathcal{D}$

The proof is based on an idea of F. Loray (see [10]) of extending germs of foliations along some separatrix by compactifying the leaf to a rational curve in some foliated complex surface. The hypothesis on the homogeneity of the foliations will allow us to make adequate choices for the extended foliations.

Given a holomorphic regular foliation $\mathcal{F}$ around a non-invariant embedded curve C in a complex surface S, we define the tangency divisor $T(\mathcal{F})=\mathrm{Tang}(\mathcal{F},C)$, its holonomy $H(\mathcal{F})=\bigsqcup_{p\in|T(\mathcal{F})|}H(\mathcal{F},p)$ and the holonomy class $H[\mathcal{F}]$ as the class of $H(\mathcal{F})$ modulo automorphisms of C. The main remark that we will exploit in this section is that if $\mathcal{G}$ is another regular foliation around C such that $\mathrm{Tang}(\mathcal{F},\mathcal{G})$ is invariant, then the blow-up of S at a point $q\in C\setminus\{|T(\mathcal{F})|\cup|T(\mathcal{G})|\}$ produces a pair $\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}$ of (singular) foliations in a complex surface $\widetilde{S}$. Around the strict transform $\widetilde{C}$ of C both foliations are regular and still satisfy that $\mathrm{Tang}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$ is invariant. By construction the leaf corresponding to the exceptional divisor lies in the tangency set between both foliations. This transformation does not touch $T(\mathcal{F})$ and $T(\mathcal{G})$, and in particular the holonomies of $\mathcal{F}$ (resp. $\mathcal{G}$) and $\widetilde{\mathcal{F}}$ (resp. $\widetilde{\mathcal{G}}$) coincide. By construction $\widetilde{C}\cdot\widetilde{C}=C\cdot C-1$. Our strategy is to revert this construction as illustrated in Figure 3: start with a pair of regular foliations around a curve, extend the foliations along their (invariant) locus of tangencies, in such a way that we can contract the invariant curve and obtain new pair of regular foliations around a curve with bigger self-intersection. Iterating this process we end up with a pair of regular *transverse* foliations around some curve with big selfintersection. In a second instance we analyze the invariants of such pairs of transverse holomorphic foliations around curves that are not invariant for any of the foliations.

Proof of Theorem I.

Given a foliation $\mathcal{F}\in\mathcal{D}^{\mathrm{h}}(n)$, we consider a radial foliation $\mathcal{G}$ such that

$$\mathrm{Tang}(\mathcal{F},\mathcal{G})=r_1L_{p_1}+\ldots r_kL_{p_k}$$

where L_{p_i} is the leaf of $\mathcal{F}$ through the point $p_i\in E\setminus|T(\mathcal{F})|$. By construction, we can find local coordinates (u,y) around each point p_i where $p_i=(0,0)$, $E=\{y=0\}$, $\mathcal{G}=\{\mathrm{d}u=0\}$ and for some unit f, $\mathcal{F}=\{\mathrm{d}u+u^rf(u,y)\mathrm{d}y=0\}$. The next lemma shows that we can find local normalizing coordinates for the pair $(\mathcal{F},\mathcal{G})$ around each point p_i.

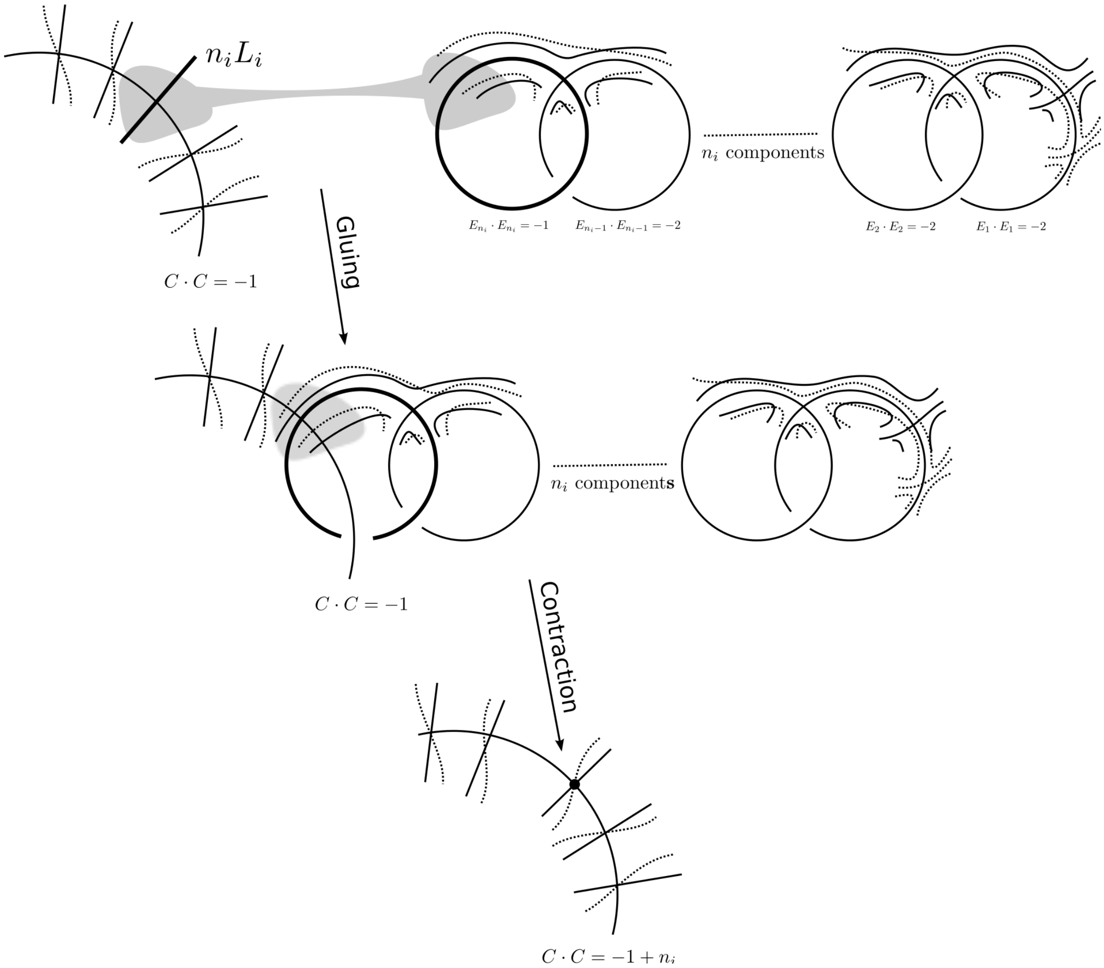

Figure 3: Key construction.

Lemma 2. *Let us consider the germs at* $(0,0)$ *of the forms*

$$\omega_1 = \mathrm{d}u + u^r f(u,y)\,\mathrm{d}y \qquad \omega_2 = \mathrm{d}u + u^r \mathrm{d}y$$

where $n \geq 1$ *and* f *is local unit. Then the induced germs of foliations are analytically conjugated by a conjugacy of the form* $(u,y) \mapsto (u, y(\cdots))$*.*

Proof. Since the two forms are smooth and locally transverse except along $u = 0$ and since we require the conjugacy to preserve each leaf of the fibration $\pi : (u,y) \mapsto u$, it is uniquely determined on a neighborhood of $(u,y) = (0,0)$ deprived of $u = 0$. Thus, it is enough to show that this conjugacy and its inverse are bounded near $(0,0)$ and apply Riemann's extension Theorem to conclude that the extension is a biholomorphism.

Let us describe the conjugacy on a neighborhood of $(u,y) = (0,0)$ deprived of $u = 0$. To do so, we will interpret its restriction to a fibre as a composition of two holonomy maps: the first ϕ_1^{-1} going from a fibre to $y = 0$ via ω_1 and the second ϕ_2 from $y = 0$ to the fibre via ω_2. To get bounds, we consider a point $(\alpha, 0)$ and follow the leaf of ω_1 until one reaches the fiber $\pi^{-1}(u)$. Denote by $(u, \phi(\alpha, u))$ the reached point. To compute ϕ, we consider the Cauchy system defined by

$$\left\{ \begin{array}{c} y(0) = 0 \\ y'(t)\, u(t,\alpha)^r f(u(t,\alpha), y(t)) + u'(t,\alpha) = 0 \end{array} \right. \text{ where } u(t,\alpha) = (1-t)\alpha + tu.$$

Obviously, one has $y(1) = \phi(\alpha, u)$. By using bounds on the unit when integrating the differential equation, it can be checked that there exist $M > 0$ such that $|\phi_2 \circ \phi_1^{-1}(y)| \leq M|y|$ for every $|y| < \varepsilon$. By using the same argument applied for the inverse we find $M > m > 0$ such that

$$m|y| \leq |\phi_2 \circ \phi_1^{-1}(y)| \leq M|y| \quad \text{for every} \quad |y| < \varepsilon$$

□

Notice that if $r \neq s$ then

$$\mathrm{d}u + u^r \mathrm{d}y \qquad \mathrm{d}u + u^s \mathrm{d}y$$

cannot be conjugated by a conjugacy that preserves the fibration $(u, y) \mapsto u$.

Taking two germs of regular transverse foliations $(\mathcal{R}_1, \mathcal{R}_2)$ at a point p, if we consider the blow-up at p and denote by E_1 the exceptional divisor and by $\mathcal{R}_i^1$ the saturated foliation around E_1 obtained from $\mathcal{R}_i$, then we create a locus of tangency

$$\mathrm{Tang}(\mathcal{R}_1^1, \mathcal{R}_2^1) = E_1$$

Blowing-up again a point in E_1 that is regular for both foliations, we obtain a second divisor E_2, two foliations $\mathcal{R}_1^2$, $\mathcal{R}_2^2$ satisfying $\mathrm{Tang}(\mathcal{R}_1^2, \mathcal{R}_2^2) = E_1 + 2E_2$. Inductively, we can produce a pair of foliations $(\mathcal{R}_1^r, \mathcal{R}_2^r)$ in a neighbourhood of a chain of r rational curves $E_1, \ldots, E_r$ satisfying

$$\mathrm{Tang}(\mathcal{R}_1^r, \mathcal{R}_2^r) = E_1 + 2E_2 + \ldots + rE_r.$$

By construction, around any regular point $p \in E_r$ of the $\mathcal{R}_i^r$'s we can find coordinates (u, y) where $p = (0, 0)$, $E_r = \{u = 0\}$, $\mathcal{R}_1^r$ is given by $\mathrm{d}u = 0$ and $\mathcal{R}_2^r$ by $\mathrm{d}u + u^r g(u, v)\mathrm{d}y = 0$ for some unit g.

Coming back to our initial pair of foliations $(\mathcal{F}, \mathcal{G})$, thanks to Lemma 2, we can glue the pair of foliations around each common separatrix L_{p_i} to a germ of $(\mathcal{R}_1^{r_i}, \mathcal{R}_2^{r_i})$ at a regular point on E_{r_i}. We thus obtain a pair of foliations around a divisor with $1 + r_1 + \ldots + r_k$ rational curves. The original foliation $\mathcal{F}$ is analytically equivalent to the restriction of this foliation to the neighbourhood of the initial divisor E. The divisors that have been added can be now contracted as a whole using inductively the classical result of Castelnuovo (see [2]). Remark that by construction, each chain contracts to a regular point with a pair of regular *transverse* foliations. Since, at each step, we contract a component that cuts the original divisor E, we get at the end of the contraction a rational curve C embedded with self-intersection

$$-1 + (r_1 + \ldots r_k) = -1 + (n + 3) = n + 2$$

in a complex surface S. In its neighbourhood, we get two regular foliations $(\mathcal{F}_S, \mathcal{G}_S)$ that are transverse at all points of E. By construction $\mathcal{G}_S$ is also transverse to E at all points. On the other hand, $\mathcal{F}_S$ has tangency divisor $T(\mathcal{F}_S) = T(\mathcal{F})$ and since the contractions and blow ups are done outside $|T(\mathcal{F})|$ the holonomy is preserved

$$H(\mathcal{F}_S) = H(\mathcal{F}).$$

This proves the first part of the claim in Theorem I: $\mathcal{F}$ is analytically equivalent to a germ obtained from $\mathcal{F}_S$ by a sequence of blow-ups, a restriction and a contraction. □

As for uniqueness of this model $(\mathcal{F}_S, \mathcal{G}_S)$, we have the following

Lemma 3. *If $(\mathcal{F}, \mathcal{G})$ is a pair of germs of regular foliations around a smooth rational curve C embedded in a complex surface with $C \cdot C = n + 2 > 2$, the degree of $T(\mathcal{F})$ is n and $\mathcal{G}$ is transverse to C, then the foliations $\mathcal{F}$ and $\mathcal{G}$ are transverse around C.*

Given another pair $(\mathcal{F}', \mathcal{G}')$ with the same properties around a rational curve C' in a complex surface, there exists a biholomorphism between two neighbourhoods sending the pair $(\mathcal{F}, \mathcal{G})$ to $(\mathcal{F}', \mathcal{G}')$ if and only if $H[\mathcal{F}] = H[\mathcal{F}']$.

Proof. We choose an open covering $\{U_i\}$ of C in the surface, holomorphic vector fields v_i on U_i generating $\mathcal{F}$ and holomorphic one forms ω_i generating $\mathcal{G}$ in the neighborhood of C. On the intersection $U_i \cap U_j$, we have

$$v_i = \phi_{ij} v_j$$
$$\omega_i = \psi_{ij} \omega_j$$

where ϕ_{ij} and ψ_{ij} are cocycles representing respectively the line bundles $T^*_{\mathcal{F}}$ and $N_{\mathcal{G}}$. Therefore, the contraction $\omega_i(v_i)$ is a section of $[T^*_{\mathcal{F}} \otimes N_{\mathcal{G}}]_C$ since

$$\omega_i(v_i) = \psi_{ij} \phi_{ij} \omega_j(v_j).$$

Now, this section vanishes along C at the point where, precisely, $\mathcal{F}$ and $\mathcal{G}$ are tangent, thus

$$\mathrm{Tang}(\mathcal{F}, \mathcal{G}) = \deg\,[T^*_{\mathcal{F}} \otimes N_{\mathcal{G}}]_C = -T_{\mathcal{F}} \cdot C + N_{\mathcal{G}} \cdot C$$

Using the formula of Brunella [2] yields to

$$\begin{aligned} \mathrm{Tang}(\mathcal{F}, \mathcal{G}) &= -C \cdot C + \mathrm{Tang}(\mathcal{F}, C) + \mathcal{X}(C) + \mathrm{Tang}(\mathcal{G}, C) \\ &= -(n+2) + n + 2 + 0 = 0. \end{aligned}$$

Suppose Φ is a biholomorphism sending the pair $(\mathcal{F}, \mathcal{G})$ to the pair $(\mathcal{F}', \mathcal{G}')$. Then the restriction $\phi := \Phi_{|C}$ satisfies

$$\phi_* T(\mathcal{F}) = T(\mathcal{F}') \text{ and } H(\mathcal{F}') = \phi \circ H(\mathcal{F}) \circ \phi^{-1}. \tag{1}$$

Reciprocally, suppose $\phi : C \to C'$ is a biholomorphism satisfying (1). Then ϕ tells us which leaf of $\mathcal{F}$ goes to which of $\mathcal{F}'$. On the other hand ϕ also tells us which leaf of $\mathcal{G}$ goes to which of $\mathcal{G}'$. If a point p lies on the intersection of a leaf of $\mathcal{F}$ with a leaf of $\mathcal{G}$, then there is a unique point $\Phi(p)$ lying at the intersection of the corresponding leaves of $\mathcal{F}'$ and $\mathcal{G}'$, provided p is sufficiently close to C. By holomorphicity and transversality of the foliations this extenstion Φ of ϕ to a neighbourhood of C is a holomorphic bijection. It sends the pair $(\mathcal{F}, \mathcal{G})$ to $(\mathcal{F}', \mathcal{G}')$ by construction. □

From the previous arguments, we prove Corollary 1 by finding a biholomorphism between the models we have just constructed. The resulting composition of biholomorphisms, contractions and blow-ups might have indeterminacies. They lie on an invariant set that contains the separatrices over points that are blown-up to obtain one of the foliations, but not blown-up to obtain the other. When all those points coincide, the birational map does not have indeterminacies and it extends to a biholomorphism.

To identify the analytical invariants other than the holonomy $H[\mathcal{F}]$, we proceed to describe the possible ways of obtaining the foliation as in Theorem I. Recall that for a homogeneous curve the projective class of the projectivized tangent cone determines the analytical class of the curve. For a homogeneous dicritical foliation $\mathcal{F} \in \mathcal{D}^{\mathrm{h}}(n)$ we define

$$\operatorname{div}^{\mathrm{h}}(\mathcal{F}) = \{\operatorname{Tang}(\mathcal{F}, \mathcal{G})_{|E} \in \operatorname{div}(E) : \mathcal{G} \in \mathcal{D}(0), \mathcal{F} \text{ is homogeneous with respect to } \mathcal{G}\}.$$

Since $\mathcal{F}$ might be homogeneous with respect to different radial foliations, it is a non-empty subset of the set of divisors in E. A straighforward calculation shows that all of them are of degree $n+3$. Each of them contains the information on the analytical class of a divisor supported on a *homogeneous* set of separatrices of $\mathcal{F}$.

Theorem II (Invariants in $\mathcal{D}^{\mathrm{h}}$))**.** *Let $\mathcal{F}_1 \in \mathcal{D}^{\mathrm{h}}(n)$ and $\mathcal{F}_2$ be a germ of holomorphic foliation in $(\mathbb{C}^2, 0)$. Then there exists $\varphi \in \operatorname{Aut}(\mathbb{C}^2, 0)$ such that $\varphi_*(\mathcal{F}_1) = \mathcal{F}_2$ if and only if $\mathcal{F}_2 \in \mathcal{D}^{\mathrm{h}}(n)$, and there exists $\phi = \varphi_{|E} \in \operatorname{Aut}(E)$ such that*

- $H(\mathcal{F}_2) = \phi \circ H(\mathcal{F}_1) \circ \phi^{-1}$
- *for some $D \in \operatorname{div}^{\mathrm{h}}(\mathcal{F}_1)$ (and a posteriori for all), $\phi_*(D) \in \operatorname{div}^{\mathrm{h}}(\mathcal{F}_2)$.*

Remark that in $\mathcal{D}^{\mathrm{h}}$, all the invariants can be read on the exceptional divisor E. Thus, in the case of a homogeneous dicritical foliation, the equivalence class of the pair $(H(\mathcal{F}), S_D)$ where S_D is the divisor of leaves over an element $D \in \operatorname{div}^{\mathrm{h}}(\mathcal{F})$ classifies the analytical class of the foliation. It is worth remarking that the homogeneous separatrix set $|S_D|$ is not enough to classify. The divisorial information is needed.

Proof of Theorem II. Suppose first that $\mathcal{F}_1 \in \mathcal{D}^{\mathrm{h}}(n)$ and $\mathcal{F}_2$ are equivalent via $\varphi \in \operatorname{Aut}(\mathbb{C}^2, 0)$. By blowing up source and target of φ once, we get that the lift of φ extends to E as a biholomorphism in a neighbourhood of E that sends the saturation of the pull-back foliation $\widetilde{\mathcal{F}_1}$ to the pull back foliation $\widetilde{\mathcal{F}_2}$. In particular, $\mathcal{F}_2$ has no singular points. Denote by $\phi \in \operatorname{Aut}(E)$ the restriction of this biholomorphism to E. By construction, the holonomies satisfy

$$\phi \circ H(\mathcal{F}_1) \circ \phi^{-1} = H(\mathcal{F}_2).$$

Moreover, if $D \in \operatorname{div}(\mathcal{F}_1)$, then there exists a radial foliation $\mathcal{G}_1$ such that $\operatorname{Tang}(\mathcal{F}_1, \mathcal{G}_1)$ is invariant by $\mathcal{F}_1$ and $D = \operatorname{Tang}(\mathcal{F}_1, \mathcal{G}_1)_{|E}$. By applying φ on both sides, we get $\operatorname{Tang}(\varphi_*(\mathcal{F}_1), \varphi_*(\mathcal{G}_1))$ invariant by $\mathcal{F}_2 = \varphi_*(\mathcal{F}_1)$ and $\phi_*(D) = \operatorname{Tang}(\varphi_*(\mathcal{F}_1), \varphi_*(\mathcal{G}_1))_{|E}$ and since $\varphi_*(\mathcal{G}_1)$ is a radial foliation, $\phi_*(D) \in \operatorname{div}(\mathcal{F}_2)$.

Reciprocally, suppose ϕ exists as in the statement of the theorem. Let $\mathcal{G}_1$, $\mathcal{G}_2$ be radial foliations such that $S_i = \operatorname{Tang}(\mathcal{F}_i, \mathcal{G}_i)$ is invariant by $\mathcal{F}_i$ and

$$D = \operatorname{Tang}(\mathcal{F}_1, \mathcal{G}_1)_{|E} \text{ and } \phi_*(D) = \operatorname{Tang}(\mathcal{F}_2, \mathcal{G}_2)_{|E}. \tag{2}$$

The extension φ of ϕ to the neighborhood is uniquely defined and holomorphic if we impose

$$\varphi_*(\mathcal{F}_1) = \mathcal{F}_2 \text{ and } \varphi_*(\mathcal{G}_1) = \mathcal{G}_2.$$

Indeed, ϕ indicates which leaf of $\mathcal{F}_1$ goes to which of $\mathcal{F}_2$ and which leaf of $\mathcal{G}_1$ goes to which of $\mathcal{G}_2$. If $p \in E$ is a point where the germs of $\mathcal{F}$ and $\mathcal{G}$ are transverse foliations, the projections $\pi_{\mathcal{F}}, \pi_{\mathcal{G}}$ onto E along $\mathcal{F}$ and $\mathcal{G}$ respectively form a coordinate chart $\Psi_{(\mathcal{F},\mathcal{G})} = (\pi_{\mathcal{F}}, \pi_{\mathcal{G}})$ around p. Thus $\Psi^{-1}_{(\mathcal{F}',\mathcal{G}')} \circ (\phi, \phi) \circ \Psi_{(\mathcal{F},\mathcal{G})}$ describes an equivalence between $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ in the complement of S_1. By construction, it extends ϕ.

It can then be extended holomorphically to S_1 as a biholomorphism of a neighborhood of the exceptional divisor by using Lemma 2 to both pairs $(\mathcal{F}_i, \mathcal{G}_i)$ around S_i. Thus after contracting the exceptional divisor we get a biholomorphism in a neighbourhood of $0 \in \mathbb{C}^2$.

□

Remark that Theorem II is also true for $\mathcal{F}_1 = \mathcal{F}_2$ so that it also gives the structure of the group of automorphisms of $\mathcal{F} \in \mathcal{D}^{\mathrm{h}}$.

2.3 Normal forms of homogeneous dicritical foliations

Next we will use the previous results to construct normal forms in a geometric manner. Recall that a Weierstrass polynomial is a monic polynomial in $\mathbb{C}\{x\}[y]$.

Theorem III. *(Normal Forms in $\mathcal{D}^{\mathrm{h}}$) Let $\mathcal{F} \in \mathcal{D}^{\mathrm{h}}(n)$ and suppose $D \in \mathrm{div}^{\mathrm{h}}(\mathcal{F})$. Then there exist coordinates (x, y) of $(\mathbb{C}^2, 0)$ where $\mathcal{F}$ is represented by a form*

$$W(x, y)(x\mathrm{d}y - y\mathrm{d}x) + Q(x, y)\mathrm{d}x \tag{3}$$

satisfying that W is a Weierstrass polynomial in y of degree and order n, Q is a homogeneous polynomial of degree $n + 2$ such that $xQ = 0$ represents D, and $Q(1, y)$ is monic. With these conditions the form is unique up to a choice of affine coordinates in E and local biholomorphisms tangent to a homothety.

Once the divisor D and the coordinate on E is fixed, the Weierstrass polynomial in the normal form encodes the information on the holonomy of the associated foliation.

Proof of Theorem III.

Let $\mathcal{F}$ be in $\mathcal{D}^{\mathrm{h}}(n)$. Let $\mathcal{G}$ be a radial foliation satisfying that $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ is invariant by $\mathcal{F}$ and $\mathrm{Tang}(\mathcal{F}, \mathcal{G})_{|E} = D$. Consider coordinates (x, y) in which $\mathcal{G}$ is linear such that the direction $x = 0$ corresponds to a point in the support of D. In this situation, the foliations $\mathcal{F}$ and $\mathcal{G}$ are respectively given by 1-forms

$$A(x, y)(x\mathrm{d}y - y\mathrm{d}x) + B(x, y)\mathrm{d}x \quad \text{and} \quad x\mathrm{d}y - y\mathrm{d}x$$

where B is a homogeneous polynomial of degree $n + 2$ and A is holomorphic.

Now consider the blowing-up of the origin given in local charts by $x = uv$ and $y = v$. In a neighborhood of the leaf $u = 0$, we have $\mathcal{G}$ given by $du = 0$ and $\mathcal{F}$ given by $a(u, v)du + u^k b(u, v)dv$ where a and b are some local units and $k \geq 1$. As we did in

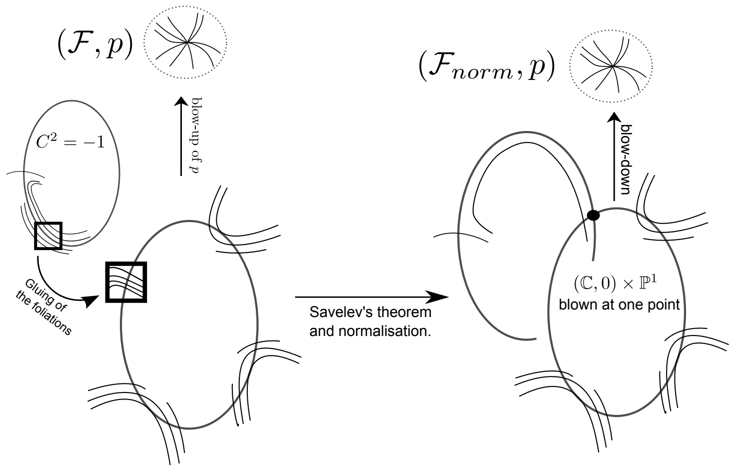

Figure 4: Construction of normal forms.

the proof of Theorem I, we will compactify the separatrix $u = 0$ for the pair $(\mathcal{F}, \mathcal{G})$ around it with a convenient model by using Lemma 2. The convenient model is the blow-up at the origin of the pair of germs of regular holomorphic foliations in two variables $(z, w) \in \mathbb{C}^2$ defined by $dw = 0$ and $dw + w^{k-1}dz = 0$ respectively. Indeed, the exceptional divisor is then invariant for both foliations and the tangency order between the foliations along it is k. By using Lemma 2, we can glue a neighbourhood of a regular point for both foliations of this new (-1)-curve C to the pair $(\mathcal{F}, \mathcal{G})$ along the chosen separatrix. In this way, we obtain a complex bifoliated surface that is a neighbourhood of a union of two (-1)-curves that intersect transversely at a point. This neighbourhood is well known. Indeed, let us consider the surface S obtained by blowing-up once the point $(z, x) = (0, 0)$ in $\mathbb{CP}(1) \times \mathbb{D}$. We denote by $U_{-1,-1}$ any neighborhood of the union of the total transform of the divisor $x = 0$. Notice that this divisor is the union of two smooth rational curves, each of self-intersection equal to -1, i.e. two (-1)-curves intersecting at a point transversely.

Lemma 4. *Let $\hat{C}$ be the union of two (-1)-curves embedded in a complex surface that intersect transversely at one point. Then there exists a neighborhood of $\hat{C}$ that is isomorphic to some $U_{-1,-1}$.*

Proof. Using a classical result of Castelnuovo (see [2]), one can contract one of the (-1)-curves to a point. Since the (-1)-curves meet transversally the self-intersection of the image of the other (-1)-curve by the contraction map is zero. Now, the Theorem of Savelev [18, 19] ensures that there exists a biholomorphism from a neighbourhood of this curve to $\mathbb{CP}(1) \times \mathbb{D}$ sending the curve to the divisor $x = 0$ and the contraction point p to $(0, 0)$. This isomorphism can be lifted to the blowing-up of the source at p and that of the target at $(0, 0)$ thus producing the desired isomorphism. □

Using Lemma 4, the above situation is isomorphic to a couple of foliations defined in the neighborhood surface of type $U_{-1,-1}$. If we contract the image of the curve C, we are led to a couple of foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ defined on $\mathbb{CP}(1) \times \mathbb{D}$ such that $\widetilde{\mathcal{G}}$ is regular and transverse to the divisor $E_0 = \mathbb{CP}(1) \times \{0\}$. The foliation $\widetilde{\mathcal{F}}$ is regular and generically transverse to E_0. The divisor of tangency $\mathrm{Tang}(\widetilde{\mathcal{F}}, E_0)$ coincides with $T(\mathcal{F})$ and the tangency locus $\mathrm{Tang}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$ is invariant. We can choose coordinates (s,t) in $(\mathbb{C},0)\times\mathbb{P}^1$ such that $\widetilde{\mathcal{G}}$ is given by $dt=0$ and $(0,\infty)$ is the point where the divisor was contracted. Since $\mathrm{Tang}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$ is invariant by $\widetilde{\mathcal{G}}$ also, $\widetilde{\mathcal{F}}$ is given by a form

$$\omega(s,t)=A(s,t)dt+Q(t)ds$$

where Q is a polynomial that has its roots precisely at the common leaves of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$. Thus it has degree at most $n+2$. On the other hand the function $A(s,t)$ is holomorphic in (s,t) and polynomial when restricted to each fixed s. Since $A(0,t)$ is a polynomial of degree n, we have that for each $s\in(\mathbb{C},0)$ there exist a constant $u(s)\in\mathbb{C}^*$ and a unique monic polynomial of degree n, $r(s,t)=t^n+\hat{a}_{n-1}(s)t^{n-1}+\cdots+\hat{a}_0(s)$ such that $A(s,t)=u(s)r(s,t)$. Indeed, all the components of the divisor of tangency between $\widetilde{\mathcal{F}}$ and the fibration $ds=0$ pass through the points $(0,t_i)$ where t_i is some root of $r(0,t)=0$. By defining a new variable x by the relation $\mathrm{d}x=\frac{ds}{u(s)}$, the foliation $\widetilde{\mathcal{F}}$ is represented in the (x,t) variables of $(\mathbb{C},0)\times\mathbb{P}^1$ by

$$(t^n+a_{n-1}(x)t^{n-1}+\ldots+a_0(x))\mathrm{d}t+Q(t)\mathrm{d}x \tag{4}$$

for some holomorphic germs $a_0,\ldots,a_{n-1}\in\mathbb{C}\{x\}$. Blowing up the point $(0,\infty)$ and contracting the strict transform of $0\times\mathbb{P}^1$ which has self-intersection -1 corresponds in this chart to setting $t=\frac{y}{x}$. Multiplying the resulting expression by x^{n+2} gives a holomorphic germ of one-form that has the desired form (3). This finishes the existence part of the statement in Theorem III.

We next show the uniqueness statement of Theorem III. Thanks to Theorem II it suffices to show that two forms as in equation (4) sharing the same component on $\mathrm{d}x$ and the same holonomy have to be equal. We will prove this by a formal argument.

Let us denote by $\widehat{I}^{t_i}_{r_i+1}$ the set of formal series in $(t-t_i)$ that are formally conjugated to the rotation of angle $2\pi/(r_i+1)$ around t_i, that is to say, series

$$h(t)=e^{\frac{2\pi i}{r_i+1}}(t-t_i)+\sum_{j\geq 2}h_{ij}(t-t_i)^j \text{ such that } h^{r_i+1}=\mathrm{Id}.$$

Let us consider a monic polynomial r of degree n, $r(t)=(t-t_1)^{r_1}\cdots(t-t_k)^{r_k}$ where $t_i\neq t_j$ if $i\neq j$. Now, choose a polynomial function $q(t,x)\in\mathbb{C}[[t,x]]$ such that $r(t)$ and $q(t,0)$ have no common roots and define the set of formal one-forms

$$\widehat{\Omega}_{r,q}=\{(r(t)+w(t,x))dt+q(t,x)\mathrm{d}x : w\in x\mathbb{C}[[x]][t] \text{ satisfies } \deg_t w<n\}.$$

For each $\omega\in\widehat{\Omega}_{r,c}$ and $i=1,\ldots,k$ define $h_i\in\widehat{I}^{t_i}_{r_i+1}$ to be the formal series in $(t-t_i)$ satisfying $\hat{f}\circ h=\hat{f}$ for some local formal first integral $\hat{f}$ of ω around t_i. In the case of convergent ω it coincides with the series of the generator of $H(\mathcal{F}_\omega,t_i)$ and in general we call it the formal holonomy of the formal foliation $\mathcal{F}_\omega$.

Lemma 5. *The map* $\mathrm{hol}_{r,q}:\widehat{\Omega}_{r,q}\to\widehat{I}^{t_1}_{r_1+1}\times\cdots\times\widehat{I}^{t_k}_{r_k+1}$ *defined by*

$$\mathrm{hol}_{r,q}(\omega)=(h_1,\ldots,h_k)$$

is a bijection.

Proof. First observe that for each $h \in \widehat{I}_r^{t_0}$ there exists a unique formal first integral, that is a formal series $f(t)$ such that $f \circ h = f$ of the form

$$f(t) = (t - t_0)^r + \sum_{j \geq r+1} f_j (t - t_0)^j \text{ such that } f_j = 0 \text{ for all } j \equiv 0 \pmod r. \tag{5}$$

that we will call *normalized.* Given $(h_1, \ldots, h_k) \in \widehat{I}_{r_1+1}^{t_1} \times \cdots \times \widehat{I}_{r_k+1}^{t_k}$ we want to find a unique $\omega \in \widehat{\Omega}_{r,q}$ satisfying $\mathrm{hol}_{r,q}(\omega) = (h_1, \ldots, h_k)$. If such ω exists, we can consider k formal local first integrals $f^j = \sum x^i f_i^j(t) \in \mathbb{C}[[x, t - t_j]]$ for the foliation near $t = t_j$ such that $f_0^j(t)$ is normalized as defined above. Below, we are going to construct inductively ω and its k formal first integrals. By definition, around the point t_j we have

$$0 \equiv \omega \wedge df^j = \big(r(t) + w(t,x)\big)\big(\sum_{i \geq 1} i f_i^j(t) x^i\big) - q(t,x)\big(\sum_{i \geq 0} \frac{df_i^j}{dt}(t) x^i\big) \tag{6}$$

Evaluating (6) on $x = 0$, we get $r(t) f_1^j(t) = q(t,0) \frac{df_0^j}{dt}(t)$. Since $r(t)$ and $\frac{df_0^j}{dt}(t)$ have the same order at $t = t_j$, f_1^j is a power series satisfying

$$f_1^j(t_j) \neq 0.$$

The coefficient on x of (6) is

$$0 \equiv r(t) 2 f_2^j(t) + w_1(t) f_1^j(t) - \ldots \tag{7}$$

where the dots refer to terms depending only on f_0, f_1 and q. Now since $(t - t_j)^{r_j}$ divides r and $f_1^j(t_j) \neq 0$, the values of $w_1(t_j), \ldots, (w_1)^{(r_j-1)}(t_j)$ do not depend on f_2, but only on f_0, f_1 and q. Adding up all the conditions, we find a polynomial of degree at most $n - 1$ determined by $n = \sum r_j$ conditions. Thus, there exists a unique possibility for w_1. As a consequence, we can define f_2^j for all j by using (7). By induction, suppose that $\{f_1^j, \ldots, f_l^j\}$ and $\{w_1, \ldots, w_{l-1}\}$ are defined. Let us construct f_{l+1}^j and w_l. The coefficient of x^l in (6) is given by

$$0 \equiv r(l+1) f_{l+1}^j + w_l f_1^j + \ldots \tag{8}$$

where the dots stands for an expression depending only on f_a^j and w_b with $a \leq l$ and $b \leq l - 1$. By considering equation (8) and its derivatives up to order r_j evaluated on t_j, we obtain n conditions on w_l that uniquely determine it. As a consequence, we define f_{l+1}^j for all j by using (8). This proves the induction step of the induction and thus the proposition. □

From this construction, it is clear that once we have chosen a divisor D and an affine coordinate t in the exceptional divisor E such that $t = \infty$ corresponds to the direction $x = 0$, we already get a unique polynomial Q provided we impose that $Q(1,t)$ is monic. Applying Proposition 5, two foliations defined by normal forms $W_j(x,y)(xdy - ydx) + Q(x,y)dx$ for $j = 1, 2$ have the same holonomy if and only if they are equal. Hence the normal form with monic $Q(1,t)$, is unique up to the choices of D, the coordinate in E and equivalences that fix each point of E. This last type of equivalence correspond precisely to the local biholomorphism having linear part some multiple of the identity. □

In Proposition 5 we do not know whether the preimage of a holomorphic k-uple $(h_1, \ldots, h_k)$ by the map $\mathrm{hol}_{r,c}$ is convergent in full generality. For future reference we state a particular case that will be useful.

Proposition 6. *Let $n \geq 1$, $r(t) = t^n$ and $c(t) = t - 1$. The restriction of* $\mathrm{hol}_{r,c}$ *to the space $\Omega_{r,c}$ of convergent elements in $\widehat{\Omega}_{r,c}$ defines a bijection onto the space I^0_{n+1} of convergent germs in $\widehat{I}^0_{n+1}$.*

Proof. Let $h \in I^0_{n+1}$ be given. First consider $\omega_0 = t^n \mathrm{d}t + c(t)\mathrm{d}x \in \Omega_{r,c}$. Since all germs at 0 of order $n+1$ are locally conjugated there exists a local diffeomeorphism $\varphi \in \mathrm{Diff}(\mathbb{C}, 0)$ such that $h = \varphi^{-1} \circ \mathrm{hol}(\omega_0) \circ \varphi$. Next remark that ω_0 can be extended to a foliation around the (-1)-curve $E_1 = \{x = 0\}$. On the chart (y, u) defined by $t = 1/u$ and $x = yu$ the foliation is defined by

$$\eta_0 = (-1 + uy\tilde{c}(u))du + (u^2\tilde{c}(u))\mathrm{d}y \tag{9}$$

where $\tilde{c}(u) = u^{n+1}c(1/u)$ is a polynomial. This foliation is in its turn defined around the curve $E_2 = \{u = 0\}$ and it can also be extended to a foliation around a (-1)-curve containing $u = 0$. Indeed, it suffices to remark that η_0 is the blow up at the regular point of $-du + u\tilde{c}(u)dv$ by the map $v = yu$. On the other hand the radial foliation around E_1 defined by dt and du on respective charts extends also to the neighbourhood of the (-1)-curve E_2. In this way we obtain a pair $(\mathcal{F}_0, \mathcal{G}_0)$ defined around $E_1 \cup E_2$. The tangency locus between both foliations is invariant by $\mathcal{F}_0$ (and also by $\mathcal{G}_0$).

Remark that the change of coordinates between ω_0 and η_0 can be described geometrically as follows. It is the only equivalence that acts like $(0, t) \mapsto (1/t, 0) = (u, y)$ on a small annulus $\{0 < |t| < r\}$ and sends the foliation pair (ω_0, dt) to the foliation pair (η_0, du). If we choose a different gluing on the divisor, say $(0, t) \mapsto (1/\varphi(t), 0)$, we can still extend to an equivalence defined on a neighbourhood of the annulus by imposing that the pair (ω_0, dt) is sent to the pair (η_0, du). In this way we obtain a pair of foliations $(\mathcal{F}, \mathcal{G})$ around a union of two (-1)-curves that we still call E_1 and E_2 satisfying that $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ is invariant by $\mathcal{F}$ and intersects E_1 transversally at two different points $q_1 = E_2 \cap E_1$ and $q_2 \in E_1$. The divisor $\mathrm{Tang}(\mathcal{F}, E_1)$ has a unique point p in its support. By construction, in the unique coordinate $w : E_1 \to \mathbb{C}P^1$ for which $w(q_1) = 0$, $w(q_2) = 1$ and $w(p) = \infty$ two points $w_1, w_2 \in \{w \in E_1 : |w| >> 1\}$ belong to the same leaf of $\mathcal{F}$ if and only if they belong to the same orbit of $1/h(1/w)$. In other words, the holonomy $H(\mathcal{F})$ measured in the coordinate $s = 1/w$ is precisely the series $h(s)$. Using the lemma 4, we get a pair of foliations $(\mathcal{F}_1, \mathcal{G}_1) \simeq (\mathcal{F}, \mathcal{G})$ defined on a neighborhood of $\{0\} \times \mathbb{CP}^1$ in $\mathbb{D} \times \mathbb{CP}^1$. Let (z, s) be some coordinates where the foliation $\mathcal{G}_1$ is defined by ds. In these coordinates $\mathcal{F}_1$ is defined by a holomorphic 1-form $u(z)(s^n + b_{n-1}(z)s^{n-1} + \cdots + b_0(z))ds + (s - 1)dz$ where u is a unit. After changing the z-coordinate by $x \in (\mathbb{C}, 0)$ satisfying $\mathrm{d}x = \frac{dz}{u(z)}$, the foliation $\mathcal{F}_1$ is defined by a holomorphic 1-form

$$\omega_1 = (s^n + a_{n-1}(x)s^{n-1} + \cdots + a_0(x))ds + (s - 1)\mathrm{d}x$$

for some $a_0, \ldots, a_{n-1} \in x\mathbb{C}\{x\}$. Blowing-up the point $(0, \infty) \in \mathbb{D} \times \mathbb{CP}^1$ does not change the expression of the normal form ω_1 whose holonomy is the holonomy of $\mathcal{F}$. $\square$

2.4 The spaces $\mathcal{D}^{\mathbf{h}}(n)$.

In this section, we will analyze the sets $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ for foliations $\mathcal{F} \in \mathcal{D}$. The analysis will lead us to a complete description of the moduli spaces of homogeneous dicritical foliations of small algebraic multiplicity. We will provide analytic normal forms for these cases by applying Theorem III with appropriate choices of the topological class of the divisors $D \in \mathrm{div}^{\mathrm{h}}(\mathcal{F})$ and coordinates in E.

For any $k \in \mathbb{N}$ and $A \subset E$, let $\mathrm{div}(E \setminus A)(k)$ denote the set of *positive* divisors in $E \setminus A$ of degree k. If $A = \emptyset$, this space is equivalent to the projectivisation of the space of homogeneous polynomials in two variables of degree k, which has the structure of $\mathbb{P}^k$. Given $\mathcal{F} \in \mathcal{D}(n)$, define

$$\mathrm{div}(\mathcal{F}) = \{\mathrm{Tang}(\mathcal{F}, \mathcal{G})_{|E} \in \mathrm{div}E(n+3) : \mathcal{G} \in \mathcal{D}(0)\} \tag{10}$$

A straightforward calculation shows that it is a four dimensional affine subvariety of $\mathrm{div}E(n+3)$ regardless of the value of n. By definition it contains the (possibly empty) set $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$.

Theorem IV. *The set $\mathcal{D}^{\mathrm{h}}(n)$ is equal to $\mathcal{D}(n)$ if and only if $n \leq 3$. Moreover,*

1. *if $\mathcal{F} \in \mathcal{D}(1)$, then $\mathrm{div}^{\mathrm{h}}(\mathcal{F}) = \mathrm{div}(\mathcal{F}) = \mathrm{div}(E \setminus |T(\mathcal{F})|)(4)$.*
2. *if $\mathcal{F} \in \mathcal{D}(2)$ then $\mathrm{div}^{\mathrm{h}}(\mathcal{F}) = \mathrm{div}(\mathcal{F})$. Let $q(\mathcal{F}) \subset \mathrm{div}^{\mathrm{h}}(\mathcal{F})$ be the set of divisors with a single point in its support.*
   - *if $T(\mathcal{F}) = p_1 + p_2$, then $|q(\mathcal{F})| = 5$*
   - *if $T(\mathcal{F}) = 2p_1$, then $|q(\mathcal{F})| = 1$.*
3. *if $\mathcal{F} \in \mathcal{D}(3)$ then $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ is a quadric in div($\mathcal{F}$). The set $q(\mathcal{F})$ of divisors in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ with a point of order at least 4 in its support is non-empty and contains at most 24 elements.*
   - *if $T(\mathcal{F}) = p_1 + p_2 + p_3$, then generically $|q(\mathcal{F})| = 24$;*
   - *if $T(\mathcal{F}) = 2p_1 + p_2$ then generically $|q(\mathcal{F})| = 18$*
   - *if $T(\mathcal{F}) = 3p_1$, then generically $|q(\mathcal{F})| = 6$*
4. *The $1-$form*
   $$y^4(x\mathrm{d}y - y\mathrm{d}x) + x^6\mathrm{d}x + y^7\mathrm{d}y$$
   defines a foliation in $\mathcal{D}(4) \setminus \mathcal{D}^{\mathrm{h}}(4)$.
5. *for any $n \geq 5$ the $1-$form*
   $$y^n(x\mathrm{d}y - y\mathrm{d}x) + x^{n+2}\mathrm{d}x + y^2x^{n+1}\mathrm{d}y$$
   defines a foliation in $\mathcal{D}(n) \setminus \mathcal{D}^{\mathrm{h}}(n)$.

In the case of $\mathcal{D}(1)$ we get a result of Cerveau (see [9]) as a consequence of Theorems II and IV that improves the statement of Corollary 1:

Corollary 7. *Two foliations in $\mathcal{D}(1)$ are analytically equivalent if and only if they share the same holonomy class.*

Remark 8. *From the Theorem IV we deduce that for $\mathcal{F} \in \mathcal{D}^{\mathrm{h}}(n)$ with $n \leq 3$, $\mathrm{div}^{\mathrm{h}}(\mathcal{F}) \subset divE(n+3)$ is an affine subvariety of positive dimension. The codimension of this subvariety is $\frac{n(n-1)}{2}$, which is precisely the dimension $\mathcal{M}(\mathcal{F})$ of the base space of the universal unfolding of $\mathcal{F}$ (see [12]). The deformation $\{\mathcal{F}_c\}_{c\in\mathbb{C}^{\mathcal{M}(\mathcal{F})},0}$ underlying a universal equisingular unfolding of $\mathcal{F}$ will be formed by homogeneous dicritical foliations. By using the argument in the proof with parameters it is possible to show that there is a choice of divisors $D_c \in \mathrm{div}^{\mathrm{h}}(\mathcal{F}_c)$ such that the germ of curve $c \mapsto D_c$ in $div(E)(n+3)$ at $c=0$ is holomorphic and transverse to $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$. In Section 4 we will see how this information can be used to explicitely find one-forms defining unfoldings. In the case $n \geq 4$ we have that the base space of the universal equisingular unfolding is already bigger than the dimension of $divE(n+3)$. This indicates that the universal unfolding of an element in $\mathcal{D}^{\mathrm{h}}(n)$ with $n \geq 4$ needs to leave the locus of homogeneous dicritical foliations.*

Proof of Theorem IV. A germ of function h in $\mathbb{C}^2$ is quasi-homogeneous if and only if there exists a vector field X such that $X \cdot h = h$. In the same way, we obtain the following criterion for a foliation $\mathcal{F}$ to be in $\mathcal{D}^{\mathrm{h}}$.

Lemma 9. *Let ω be a convergent one form representing $\mathcal{F} \in \mathcal{D}(n)$. Then $\mathcal{F} \in \mathcal{D}^{\mathrm{h}}(n)$ if and only if there exist a formal vector field $\hat{X}$ with radial linear part and a formal unit $\hat{u}$ such that*

$$\hat{X} \cdot \omega(\hat{X}) = \hat{u}\omega(\hat{X}). \tag{11}$$

Proof. Suppose that there exist $\mathcal{G} \in \mathcal{D}(0)$ such that $\mathrm{Tang}(\mathcal{F}, \mathcal{G})$ is invariant by $\mathcal{F}$. Then, it is also invariant by $\mathcal{G}$. Now, let us consider X and ω any vector field and form that represent respectively $\mathcal{G}$ and $\mathcal{F}$. The contraction $\omega(X)$ is an equation of the tangency locus. Since it is invariant by X, the derivative $X \cdot \omega(X)$ can be holomorphically divided by $\omega(X)$ thus there exists a function u such that

$$X \cdot \omega(X) = u\omega(X).$$

Looking at the multiplicity at 0 of both components of the equality above ensures that u is a unit, i.e., $u(0) \neq 0$. Here, X and u are convergent, thus, also formal. Conversely, if (11) has a formal solution, then $\omega(\hat{X})$ is the product of formal equations of convergent separatrices of $\mathcal{F}$. Thus, there exist a formal unit $\hat{v}$ and a *convergent* equation $F = 0$ of separatrices of $\mathcal{F}$ such that

$$\omega(\hat{X}) = \hat{v}F.$$

This last equation has a formal solution $(\hat{X}, \hat{v})$. We deduce the existence of a convergent solution (X, v) to the above equation by Artin's Theorem (see [1]) whose first jets coincide with those of $(\hat{X}, \hat{v})$. The convergent vector field X then satisfies equation (11) for some unit u.

□

Lemma 10. *Let ω be a 1-form representing $\mathcal{F} \in \mathcal{D}(n)$. Suppose that there exist a vector field X and a unit u such that*

$$j^{2n+1}\left(X \cdot \omega(X) - u\omega(X)\right) = 0$$

then $\mathcal{F} \in \mathcal{D}^{\mathrm{h}}(n)$.

Proof. In the proof below, the notation $\square_i$ stands for the homogeneous component of degree i of the object $\square$. Suppose that there exist a vector field X and a unit u as in the Lemma. We are going to modify the component of degree $k+3$ of X and of degree $k+1$ of u so that

$$j^{n+3+k+1}\left(X \cdot \omega(X) - u\omega(X)\right) = 0.$$

That will ensure by induction the existence of a formal solution to the equation (11). Now, a straightforward computation shows that

$$(X \cdot \omega(X) - u\omega(X))_{n+3+k+1} = \omega_{n+1}\left(X_{k+3} + u_{k+1}X_2\right) + u_{k+1}\omega_{n+2}(X_1) + (\cdots) \quad (12)$$

Here the dots $(\cdots)$ stand for terms which depends only of components of u of degree strictly smaller than $k+1$ and of components of X of degree strictly smaller than $k+3$. Let us denote by the component of degree k of X by $X_k = A_k\partial x + B_k\partial y$. From the equation above and using the notation $\omega = R_n(x\mathrm{d}y - y\mathrm{d}x) + P_{n+2}\mathrm{d}x + Q_{n+2}\mathrm{d}y + \cdots$, we obtain

$$\begin{aligned}&(X \cdot \omega(X) - u\omega(X))_{n+3+k+1}\\ &\quad = \; R_n\left(yA_{k+3} - xB_{k+3} + u_{k+1}(yA_2 - xB_2)\right) + u_{k+1}(xP_{n+2} + yQ_{n+2}) + (\cdots)\end{aligned}$$

This equation can always be made equal to 0 provided that $n-1 \leq k+1$: indeed, since R_n and $xP_{n+2} + yQ_{n+2}$ are relatively prime, applying the Bézout's Theorem in $\mathbb{C}[t]$ where $t = \frac{y}{x}$ to the relation above ensures the existence of a polynomial function $\tilde{u}$ of degree smaller than $n-1$ and a polynomial function $\tilde{V}$ of degree smaller than $n+3$ such that

$$\tilde{R}_n\tilde{V} + \tilde{u}\left(\tilde{P}_{n+2} + t\tilde{Q}_{n+2}\right) + (\tilde{\cdots}) = 0.$$

Since $n-1 \leq k+1$ and $n+3 \leq k+4$, it can be seen that one can find A_{k+3}, B_{k+3} and u_{k+1} such that the equation (12) is satisfied. □

Corollary 11. *The subset* $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ *is an algebraic sub-variety of* $\mathrm{div}(\mathcal{F})$.

Now, we can give the proof of the Theorem IV.

In what follows, we will use the notation introduced in the proof of the lemma (10) and denote by S_{n+3} the function $xP_{n+2} + yQ_{n+2}$.

- For $n = 1$, consider any radial vector field X. Since the multiplicity of $\omega(X)$ is at least 4 then

  $$j^3(X \cdot \omega(X) - u\omega(X)) = 0$$

  for any unity u. Moreover, setting $u(0) = 4$ yields

  $$j^4(X \cdot \omega(X) - u\omega(X)) = 0.$$

Thus, according to the proof of lemma (10), for any radial vector field X, the tangent cone of $\omega(X)$ belongs to $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$. Now using the notation of the proof of (10) one has

$$j^4(\omega(X)) = R_1(yA_2 - xB_2) + xP_3 + yQ_3$$

whose projectivized tangent cone can be any element of $\mathrm{div}(E \setminus |T(\mathcal{F})|)(4)$ by appropriately choosing A_2 and B_2. Therefore, one has

$$\mathrm{div}^{\mathrm{h}}(\mathcal{F}) = \mathrm{div}(E \setminus |T(\mathcal{F})|)(4).$$

Notice also that one can argue the following way for $n = 1$: any 4-uple of smooth invariant curves can be straightened to their tangent lines by a local biholomorphism. In the new coordinates, say (x, y), these four lines are invariant for the radial vector field, $R = x\partial x + y\partial y$. Since the multiplicity of the tangency locus is equal to 4, the tangency locus is exactly these four lines. Now according to the above corollary, $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ is closed in $\mathrm{div}(\mathcal{F})$, which gives the property.

- For $n = 2$, setting $u(0) = 5$ yields

$$j^5(X \cdot \omega(X) - u\omega(X)) = 0.$$

The same argument as above ensures that any divisor in $\mathrm{div}(\mathcal{F})$ which is the cone tangent of some $\omega(X) = 0$ for some radial vector field X, belongs actually to $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$. Now, the tangent cone of the tangency locus between $\mathcal{F}$ and $\mathcal{G} \in \mathcal{D}(0)$ is written

$$(\omega(X)))_5 = \underbrace{xP_4 + yQ_4}_{H_5} + R_2 \underbrace{(yA_2 - xB_2)}_{S_3} = 0.$$

This tangent cone reduces to a single multiple point if and only if there exist a unit u and $\alpha \in \mathbb{C}$ such that

$$H_5 + R_2 S_3 = u\,(x + \alpha y)^5$$

If $|T(\mathcal{F})|$ is a single point, up to some linear change of coordinates we can suppose that $R_2 = y^2$. Thus, the relation above implies that

$$\begin{cases} u &= H_5(1,0) \\ \alpha &= \frac{(H_5)_{4,1}}{5H_5(1,0)} \end{cases}$$

where $(H_5)_{4,1}$ is the coefficient of $x^4 y$ in S_5. If $|T(\mathcal{F})|$ consists of two distinct points, up to some change of coordinates we can suppose that $R_2 = y(y-1)$. In this case, the solutions are given by the system

$$\begin{cases} u &= H_5(1,0) \\ (1+\alpha)^5 &= \frac{H_5(1,1)}{H_5(1,0)} \end{cases}$$

which has exactly five solutions for $H_5(1,1) \neq 0$.

- For $n = 3$, following the previous lemma, it is enough to show that there exists a solution to
$$j^7 \left(X \cdot \omega(X) - u\omega(X)\right) = 0. \quad (13)$$
We consider the following notation
$$\omega = R_3(x\mathrm{d}y - y\mathrm{d}x) + P_5\mathrm{d}x + Q_5\mathrm{d}y + P_6\mathrm{d}x + Q_6\mathrm{d}y + \cdots .$$
Below, we are only going to consider the generic case, that is, when R has three distinct points in its tangent cone
$$R = (y - \tau_1 x)(y - \tau_2 x)(y - \tau_3 x).$$
Notice that we can suppose that $\tau_1 \neq 0$. Up to some multiplication by a unit, we can furthermore suppose that the vector field X is written
$$X = x\partial x + y\partial y + A_2\partial x + \delta_4 x^2\partial y + A_3\partial x + B_3\partial y$$
where A_2 is written in a Lagrange form
$$A_2(x, y) = \delta_1 \frac{(y - \tau_2 x)(y - \tau_3 x)}{(\tau_1 - \tau_2)(\tau_1 - \tau_3)} + \delta_2 \frac{(y - \tau_1 x)(y - \tau_3 x)}{(\tau_2 - \tau_1)(\tau_2 - \tau_3)} + \delta_3 \frac{(y - \tau_1 x)(y - \tau_2 x)}{(\tau_3 - \tau_1)(\tau_3 - \tau_2)}.$$
Finally, we set $u = 6 + u_1 + \cdots$ where $u_1 = u_{10}x + u_{01}y$. Here, the unknown variables are the δ's, the two coefficients of u_1 and the coefficient of A_3 and B_3. If we denote $\omega(X) = H_6 + H_7 + \cdots$, the initial equation (13) is written
$$H_7 + u_1 H_6 - A_2\partial_x H_6 - \delta_4 x^2 \partial_y H_6 = 0. \quad (14)$$
Now, since H_7 is written
$$H_7 = xP_6 + yQ_6 + P_5 A_2 + Q_5\delta_4 x^2 + R_3(yA_3 - xB_3)$$
and thus contains A_3 and B_3 as free and linear parameters, the equation (14) has a solution if and only if the evaluation at each point $(1, \tau_i)$ of (14) which are the roots of R_3 yields 0. Thus, we are led to a system of three equations that are written
$$\delta_i^2 \frac{\partial R_3}{\partial x}(1, \tau_i) + u_{10}S_6(1, \tau_i) + u_{01}\tau_i S_6(1, \tau_i) + L_i\left(\{\delta_j\}_{j=1..4}\right) = 0 \quad i = 1, 2, 3$$
where the function L_i are linear functions of δ_1, δ_2 and δ_3 and quadratic in δ_4. The last two equations $i = 2, 3$ can be seen as a linear system in u_{10} and u_{01} whose determinant is
$$\begin{vmatrix} S_6(1, \tau_2) & \tau_2 S_6(1, \tau_2) \\ S_6(1, \tau_3) & \tau_3 S_6(1, \tau_3) \end{vmatrix} = S_6(1, \tau_2)S_6(1, \tau_3)(\tau_2 - \tau_3)$$
which is not equal to 0 because S_6 and R_3 have no common root and $\tau_2 \neq \tau_3$. Thus, u_{10} and u_{01} can be substitute in the first equation which can be solved because the coefficient of the quadratic term δ_1^2 is equal to $\frac{\partial R_3}{\partial x}(1, \tau_1) \neq 0$. If R_3 has for instance a double root, say τ_1, then the second equation is replaced by

the partial derivative of (14) with respect to y applied to $(1,\tau_1)$ which has also to be 0. Then, the computations are much the same as above. In any case, the final equation is quadratic in the variables δ_i and thus $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ is a quadric.

Now, a point of coordinates $(1,t)$ which is in the support of some $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ is of multiplicity 4 if and only if

$$H_6^{(i)}(1,t)=0, \quad i=0,\dots,3.$$

Let us write

$$H_6(1,t)=S_6(1,t)+R_3(1,t)(\alpha_0+\alpha_1 t+\alpha_2 t^2+\alpha_3 t^3)$$

Notice that the coefficients α_i can be linearly written in terms of δ_i's. The previous equations can be written

$$\left\{\begin{array}{rcl} \alpha_0+\alpha_1 t+\alpha_2 t^2+\alpha_3 t^3 &=& -\left(\frac{S_6}{R_3}\right)(1,t) \\ \alpha_1+2\alpha_2 t+3\alpha_3 t^2 &=& -\left(\frac{S_6}{R_3}\right)^{(1)}(1,t) \\ 2\alpha_2+6\alpha_3 t &=& -\left(\frac{S_6}{R_3}\right)^{(2)}(1,t) \\ 6\alpha_3 &=& -\left(\frac{S_6}{R_3}\right)^{(3)}(1,t) \end{array}\right.$$

Solving the following linear system, we express each coefficient α_i as a rational function of the variable t which appear to be of degree 0, thus,

$$\alpha_i=\frac{\cdots}{R_3^4}$$

where the dots stand for some polynomial function of degree at most 12. Now, if we substitute these expressions in the quadratic equation that defines $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$, we are led to a polynomial equation of degree at most 24. To check that generically this polynomial function has degree 24 and 24 distinct solutions,it is enough to exhibit an example satisfying these two conditions. Using some standard symbolic computation program, we can verify that for

$$\mathcal{F}: y(y^2-x^2)(x\mathrm{d}y-y\mathrm{d}x)+(x^5+y^5+x^4y)\mathrm{d}x$$

the polynomial function is

$$\begin{aligned}\frac{10}{3}t^{24}&+20\,t^{22}-\frac{20}{3}t^{21}-\frac{190}{3}t^{20}-\frac{5944}{3}t^{19}-5898t^{18}-9472t^{17}-\frac{22709}{3}t^{16}\\ &-3732t^{15}-1008t^{14}-\frac{5948}{3}t^{13}-2783t^{12}-\frac{8224}{3}t^{11}-394t^{10}+\frac{4616}{3}t^{9}\\ &+207t^{8}-\frac{2188}{3}t^{7}-\frac{40}{3}t^{6}+248t^{5}+\frac{251}{3}t^{4}+\frac{32}{3}t^{3}-\frac{40}{3}t^{2}-\frac{32}{3}t-\frac{2}{3},\end{aligned}$$

which has no common factor with $t(t^2-1)$ and has only simple zeros.

Now, the same argument can be performed when $T(\mathcal{F})$ has a double or a triple point. Generically, the degree of the polynomial function obtained has above will be respectively 18 and 6.

- For $n \geq 5$, an obstruction to solve the equation (11) appears already for the jet of order $n+4$. Indeed, if we take $R = y^n$, $P_{n+2} = x^{n+2}$, $Q_{n+3} = y^2 x^{n+1}$, $Q_{n+2} = P_{n+3} = 0$ we have $\partial_x H_{n+3} = (n+2)x^{n+1} + y^n(\ldots)$ and $\partial_y H_{n+3} = y^{n-1}(\ldots)$. The equation on degree $n+4$ becomes
$$\begin{aligned} -y^3 x^{n+1} = A_2 x^{n+2} + y^n(\ldots) + A_2((n+2)x^{n+1} + y^n(\ldots)) \\ +B_2(y^{n-1}(\ldots)) + u_1(x^{n+3} + y^n(\ldots)) \end{aligned}$$
which has no solution if $n \geq 5$ since there is no term in $y^3 x^{n+1}$ in the right term of the above equality. Thus $\mathcal{D}^{\mathrm{h}}(n) \neq \mathcal{D}(n)$ for $n \geq 5$.

- For $n = 4$, we have also $\mathcal{D}^{\mathrm{h}}(n) \neq \mathcal{D}(n)$ but to produce a counter-example, we had to use again symbolic computer program . Indeed, the obstruction appears only on the homogeneous term degree $n+5 = 9$ whereas it appears on the degree $n+4$ for any n bigger than 5. Actually the following form
$$y^4(x\mathrm{d}y - y\mathrm{d}x) + x^6\mathrm{d}x + y^7\mathrm{d}y$$
does not belong to $\mathcal{D}^{\mathrm{h}}(4)$ while it belongs to $\mathcal{D}(4)$. Although it is quite long, the verification presents no special difficulty.

Finally, we conjecture that for $n \geq 4$, the space $\mathcal{D}^{\mathrm{h}}(n)$ has strictly positive codimension in $\mathcal{D}(n)$. To support this claim, we remark that whether or not an element of $\mathcal{F} \in \mathcal{D}(n)$ belongs to $\mathcal{D}^{\mathrm{h}}(n)$ relies on four parameters of the affine space $\mathrm{div}(\mathcal{F})$ which must satisfy $\frac{n(n-1)}{2}$ equations as highlighted in [7]. For $n \geq 4$, there are more conditions than parameters.

□

For small values of n, we can give a precise finite list of normal forms by using the choices of divisors of Theorem IV. This is the object of the following corollary.

We say that an element $(a_1(x), \ldots, a_m(x))$ in $\mathbb{C}\{x\}^m$ is normalized if it is $(0, \ldots, 0)$ or if the first $a_i(x) \neq 0$ with i as small as possible has its first non-zero monomial with coefficient 1.

Corollary 12. *For any $\mathcal{F} \in \mathcal{D}(n)$ with $n \leq 3$, there exists a finite number of convergent normal forms characterizing the analytical class of $\mathcal{F}$. They are resumed in the following table.*

$T(\mathcal{F})$	*Normal Form*	*Nr.*
p	$(y + x^3 a(x))(x\mathrm{d}y - y\mathrm{d}x) + x^3\mathrm{d}x$, $a \in \mathbb{C}\{x\}$ *is normalized*	1
$2p$	$(y^2 + b(x)x^2y + a(x)x^3)(x\mathrm{d}y - y\mathrm{d}x) + x^4\mathrm{d}x$ *where* $(a,b) \in \mathbb{C}\{x\}^2$ *is normalized*	1
$p_1 + p_2$	$(y(y-x) + b(x)x^2y + a(x)x^3)(x\mathrm{d}y - y\mathrm{d}x) + x^4\mathrm{d}x$ *where* $a, b \in \mathbb{C}\{x\}$	*10*
$3p$	$(y^3 + c(x)x^2y^2 + b(x)x^3y + a(x)x^5)(x\mathrm{d}y - y\mathrm{d}x) + (x+\lambda_1 y)(x+\lambda_2 y)x^3\mathrm{d}x$ *where* $(a,b,c) \in \mathbb{C}\{x\}^3$ *is normalized and* $\lambda_1, \lambda_2 \in \mathbb{C}$	≤ 6
$2p_1 + p_2$	$(y^2(y+x) + c(x)x^2y^2 + b(x)x^3y + a(x)x^5)(x\mathrm{d}y - y\mathrm{d}x) + (x+\lambda_1 y)(x+\lambda_2 y)x^3\mathrm{d}x$ *where* $a,b,c \in \mathbb{C}\{x\}$ *and* $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{1\}$	≤ 18
$p_1 + p_2 + p_3$	$(y(y^2 - x^2) + c(x)x^2y^2 + b(x)x^3y + a(x)x^4)(x\mathrm{d}y - y\mathrm{d}x) + (x+\lambda_1 y)(x+\lambda_2 y)(x+\lambda_3 y)^3\mathrm{d}(x+\lambda_3 y)$ *where* $a,b,c \in \mathbb{C}\{x\}$ *and* $\{\lambda_i\} \subset \mathbb{C} \setminus \{1,-1\}$	≤ 144

When a normal form is non-unique, we can deduce all the equivalent normal forms from any one of them.

Proof. Recall that in Theorem III, the locus $xQ(x,y)=0$ is $\mathrm{Tang}(\mathcal{F},\mathcal{R})$ and the locus $W(1,y)=0$ corresponds to $T(\mathcal{F})$. For $n\leq 3$, we are able to construct a finite family of normal forms that characterizes the analytical class of a given $\mathcal{F}$. The finite family depends on a finite number of possible choices of the relative position of the points of $T(\mathcal{F})$, of the special (finite) choice of elements in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ given in Theorem IV and, when these are not enough, on some special choice of the series defining the holonomy.

1. If $n=1$ then $T(\mathcal{F})=p$ for some p in E. One can choose a coordinates (x,y) such $y=0$ corresponds to the direction defined by p and $x=0$ to direction defined by the point of order 4 in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$. In these coordinates, Theorem provides normal forms that can be written

   $$\left(y+x^2a(x)\right)(x\mathrm{d}y-y\mathrm{d}x)+x^3\mathrm{d}x.$$

   Now if we apply an scaling $(x,y)\to(\lambda^2x,\lambda^3y)$ then the form is written

   $$\left(y+x^2\lambda a(\lambda x)\right)(x\mathrm{d}y-y\mathrm{d}x)+x^3\mathrm{d}x$$

   Thus either $a=0$ and the form admits a meromorphic first integral which is invariant by scaling or $a\neq 0$ and one can normalize a. Having made all these normalisations, the normal form is then unique up to E-equivalence.

2. If $n=2$ and $T(\mathcal{F})=2p$ then, one can choose for point at ∞ the unique point of order 4 in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ whose existence is ensured by Theorem IV. As before, Theorem III provides normal forms of type

   $$\left(y^2+a(x)x^3+b(x)x^2y\right)(x\mathrm{d}y-y\mathrm{d}x)+x^4\mathrm{d}x.$$

   Finally, either $a=b=0$ or one can suppose $(a(x),b(x))$ is normalized by scaling with $(x,y)\to(\lambda^3x,\lambda^4y)$. Again, having made all these normalisations, the normal form is then unique up to E-equivalence.

3. If $n=2$ and $T(\mathcal{F})=p_1+p_2$ then, there are five possible choice for a point of order 4 in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$. Once one of this point is chosen for being at ∞, one can choose the coordinates of $T(\mathcal{F})$ to be $\{0,1\}$ and there are exactly two different possibilities. Then Theorem 3 provides a normal form of type

   $$\left(y(y-x)+a(x)x^3+b(x)x^2y\right)(x\mathrm{d}y-y\mathrm{d}x)+x^4\mathrm{d}x.$$

   Notice that, in that case, $(a(x),b(x))$ cannot be *a priori* normalized since no scaling leaves the configuration of the three points invariant. Thus, we obtain a set of 10 normal forms which is unique up to E-equivalence.

4. If $n=3$ and $T(\mathcal{F})=p_1+p_2+p_3$ then there exist at most 24 divisors in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ with point of order 4 in its support. Let us choose one of this divisor and consider a coordinates where the two points $p_1,\ p_2$ are sent to $0,\ 1$ and the point of order

4 at ∞. There are six such possible choices. Now Theorem III provides a normal form that can be written

$$\begin{aligned}\left(y\left(y-x\right)\left(y+\lambda_1 x\right)+a\left(x\right)x^4+b\left(x\right)x^3y+c\left(x\right)x^2y^2\right)\left(x\mathrm{d}y-y\mathrm{d}x\right)\\+\left(x+\alpha y\right)\left(x+\beta y\right)x^3\mathrm{d}\left(x\right).\end{aligned}$$

where $\lambda_1 \neq 0, -1$ and $\alpha, \beta \neq -1,\ \lambda_1$. Notice that (a, b, c) may not be normalized. Thus, we obtain a classifying list of at most $6 \times 24 = 144$ normal forms.

5. If $n = 3$ and $T(\mathcal{F}) = 2p_1 + p_2$ then there exists at most 18 divisors in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ with a point of order 4 in its support. Choosing one of this point and coordinates where the latter is at ∞, the point point of order two in $T(\mathcal{F})$ is 0, and the remaining point 1, Theorem III provides a normal form of the same type as before

$$\begin{aligned}\left(y^2\left(y-x\right)+a\left(x\right)x^4+b\left(x\right)x^3y+c\left(x\right)x^2y^2\right)\left(x\mathrm{d}y-y\mathrm{d}x\right)\\+\left(x+\alpha y\right)\left(x+\beta y\right)x^3\mathrm{d}\left(x\right).\end{aligned}$$

and, thus, a classifying list of 18 normal forms.

6. If $n = 3$ and $T(\mathcal{F}) = 3p$ then there exists at most 6 divisors in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ with a point of order 4 in its support. Letting ∞ be one of these points and $T(\mathcal{F})$ be 0 leads to a normal form of the form

$$\left(y^3+a\left(x\right)x^4+b\left(x\right)x^3y+c\left(x\right)x^2y^2\right)\left(x\mathrm{d}y-y\mathrm{d}x\right)+\left(x+\alpha y\right)\left(x+\beta y\right)x^3\mathrm{d}\left(x\right).$$

Now using a scaling $(x, y) \to (\lambda^4 x, \lambda^5 y)$, one can suppose that (a, b, c) is normalized. Finally, we obtain a classifying list of at most 6 normal forms.

□

The convergent normal forms we have obtained look quite similar to the formal normal forms constructed by Ortiz, Rosales and Voronin in [14] and the convergent normal forms in the case $n = 1$ (see [15]). The main difference is that we first choose a radial foliation with respect to which the given foliation is homogeneous. The convergence of its linearization map gives us the convergence of the normal forms.

3 Classification in $\mathcal{D}$.

As was seen in the previous section, the homogeneity hypothesis was a great help. Having a radial foliation that is well related to a given foliation allowed in particular to find good coordinates. In general, we do not have such an object but some formal results based on the ideas coming from the homogeneous case can be used to determine general formal normal forms.

3.1 Formal normal forms in $\mathcal{D}(n)$.

We start by generalizing the results in [14] to all of $\mathcal{D}$ with a slight change in the type of equivalences. Two formal vector fields $V, \widetilde{V}$ in $(\mathbb{C}^2, 0)$ are said to be *formally E-equivalent* $V \sim_0 \widetilde{V}$ if there exist a formal pair of power series $\phi(x,y) = (\lambda x + \ldots, \lambda y + \ldots)$ and a formal unit $u(x,y) = \mu + \cdots$ such that

$$\widetilde{V} = u \cdot \phi_*(V).$$

If the vector fields converge, ϕ is a formal equivalence between the induced foliations. If the equivalence ϕ converges it sends leaves of V to leaves of V' and fixes every point of the exceptional divisor E. In this case we say that V and V' are E-equivalent and denote it by $V \sim_E V'$.

Let $n \geq 1$. For any W **formal** Weierstrass polynomial in y - a monic polynomial in $\mathbb{C}[[x]][y]$ - of degree and order n and any family of complex numbers $(c_{ij}) \in \mathbb{C}^{\frac{n(n-1)}{2}}$, we consider the formal foliation given by

$$\mathcal{F}_{W,(c_{ij})} := W(x,y)(x\mathrm{d}y - y\mathrm{d}x) + \left(x^{n-1} + \sum_{\substack{0 \leq i \leq n-2 \\ 0 \leq j \leq n-1 \\ i+j \geq n-1}} c_{ij} x^i y^j \right) x^3 \mathrm{d}x. \tag{15}$$

Theorem V (Formal normal forms in $\mathcal{D}(n)$)**.** *Consider three distinct points $p_0, p_1, p_\infty \in E$ and $n \geq 1$. For any $\mathcal{F} \in \mathcal{D}(n)$ such that $p_0 \notin |T(\mathcal{F})|$, there exist a formal conjugacy $\Phi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ and a **unique** pair $\{W, (c_{ij})\}$ such that*

- $\mathrm{D}\Phi(p_0) = 0,\ \mathrm{D}\Phi(p_1) = 1$ *and* $\mathrm{D}\Phi(p_\infty) = \infty$
- $\Phi_* \mathcal{F} = \mathcal{F}_{W,(c_{ij})}$

As a consequence of the theorem above, two formal normal forms $\mathcal{F}_{W,(c_{ij})}$ and $\mathcal{F}_{W',(c'_{ij})}$ are formally E-equivalent if and only if they are equal. On the other hand we know from [5] that formal/analytic rigidity takes place in $\mathcal{D}$ so the formal invariants determine the analytic class.

The proof follows the lines of the proof of Theorem 4 in [14] with the appropriate changes to generalize to foliations in $\mathcal{D}$. In that paper the divisor $T(\mathcal{F})$ is assumed to have only simple points in its support, which is the generic case.

Proof. In the proof and whenever a system of coordinates (x,y) is given, the radial vector field $x\partial_x + y\partial_y$ will be denoted by $\mathcal{R}$. Its dual form $x\mathrm{d}y - y\mathrm{d}x$ is denoted by $\omega_{\mathcal{R}}$. For convenience, we are going to use the vector fields rather than the 1−forms. Moreover, $j^m(\square)$ and $\square_m$ stand respectively for the jet of order m and the component of homogeneous degree m of $\square$. Moreover, a vector field V is said to have a *normalized* homogeneous term of degree $n + N \geq n + 2$ that is written

$$V_{n+N} = P\partial_x + Q\partial_y$$

if both $P(1,t)$ and $\omega_{\mathcal{R}}(V_{n+N})(1,t)$ have degree at most $n-1$ and the second has order at least N at 0.

Let (x,y) be coordinates such that $x=0$, $y=x$ and $y=0$ define the points p_0, p_1 and p_∞ in E respectively and $V = P\partial_x + Q\partial_y$ be the holomorphic vector field representing $\mathcal{F}$ such that P is a Weierstrass polynomial in y. Then $j^{n+1}(V) = R\mathcal{R}$ for a unique homogeneous polynomial $R(x,y)$ of degree $n \geq 1$ with $R(0,y) = y^n$. The regularity of the foliation after blow-up is equivalent to the fact that R has no common factors with $\omega_{\mathcal{R}}(V_{n+2}) = yP_{n+2} - xQ_{n+2}$.

After applying a transformation of type $\lambda\mathrm{Id}$ for some $\lambda \in \mathbb{C}^*$ and multiplying by an appropriate unit, we can suppose that $R(0,y) = y^n$ and $Q_{n+2}(x,0) = x^{n+2}$. In what follows, all changes of coordinates will be tangent to the identity and all units will be equal to 1 at the origin. Thus R and $Q_{n+2}(x,0)$ do not change along the next changes of coordinates.

Next we are going to normalize recursively the homogeneous components of V. To do so, we will use of two types of changes of coordinates. Below, we describe these changes of coordinates and compute how they affect the homogeneous components of V.

- If $\phi_N = (\alpha, \beta)$ is a homogeneous vector field of degree $N \geq 2$, and we consider the vector field $\widetilde{V} = \widetilde{P}\partial_x + \widetilde{Q}\partial_y$ obtained by pushing V by the transformation $\phi = \mathrm{Id} + \phi_N + \text{h.o.t.}$ we have that

$$j^{n+N}(\widetilde{P}) = j^{n+N}(P) + (N-1)R\alpha - x(\alpha\partial_x R + \beta\partial_y R) \quad (16)$$
$$j^{n+N+1}(\omega_{\mathcal{R}}(\widetilde{V})) = j^{n+N+1}(\omega_{\mathcal{R}}(V)) + (N-1)R\omega_{\mathcal{R}}(\phi_N) \quad (17)$$

- Moreover, if ϕ_N preserves the radial foliation, i.e. there exists a homogeneous polynomial γ of degree $N-1$ such that $\alpha = x\gamma$ and $\beta = y\gamma$, then setting $\widetilde{V}^\gamma = (1-(N-1-n)\gamma)\widetilde{V}$, we get

$$j^{n+N}(\widetilde{P}^\gamma) = j^{n+N}(P) \quad (18)$$
$$j^{n+N+2}(\omega_{\mathcal{R}}(\widetilde{V}^\gamma)) = j^{n+N+2}(\omega_{\mathcal{R}}(V)) - N\gamma\omega_{\mathcal{R}}(V_{n+2}) \quad (19)$$

In particular if V was normalized up to order $n+N$, so is $\widetilde{V}^\gamma$.

We will consider a sequence of equivalent vector fields V^N, each normalized up to order $n+N$. Start with V and find a homogeneous vector field of degree $N=2$, ϕ_2 such that the right hand side of (17) becomes a polynomial of degree less than n. By dehomogeneizing the equation there is a unique possibility for $\omega_{\mathcal{R}}(\phi_2)$ given by euclidean division in the ring $\mathbb{C}[t]$. The push forward of V by $\mathrm{Id}+\phi_2$ gives a vector field $\widetilde{V}$ satisfying $\omega_{\mathcal{R}}(\widetilde{V})$ is normalized up to order $n+2$. Let η be the unique homogeneous polynomial of degree $N-1$ such that $\widetilde{Q}(1,t) + \eta(1,t)R(1,t)$ is of degree less than n. Then $V^2 = (1-\eta)\widetilde{V}$ is normalized up to order $n+2$ and $V^2 \sim_0 V$. Suppose for induction that $N \geq 2$, V^N is normalized up to order $n+N$ and $V^N \sim_0 V$. Let us find a $V^{N+1} \sim_0 V^N$ normalized up to order $n+N+1$. The key ingredient for the induction is

Lemma 13. *Given a homogeneous polynomial H of degree $n+N+2$ in (x,y) there exist a unique polynomial q_{n+N+2} in the $\mathbb{C}$- vector space V generated by $y^j x^{n+2+N-j}$ for $j=N\dots,n-1$, and homogeneous polynomials A and B of degrees $N+2$ and $N-1$ respectively such that*

$$H = q_{n+N+2} + AR + B\omega_{\mathcal{R}}(V_{n+2})$$

Set $H=(\omega_{\mathcal{R}}(V^N))_{n+N+2}$ and apply the previous lemma to obtain A,B and q_{n+N+2}. Define $\gamma = B/N$. Then defining $\widetilde{V}^\gamma$ as before, we get $j^{n+N+2}(\widetilde{V}^\gamma) = q_{n+N+2}+AR$ and $j^{n+N}(\widetilde{P}^\gamma) = j^{n+N}(P)$. Next choose a homogeneous vector field ϕ_N of degree N such that $\omega_{\mathcal{R}}(\phi_N) = -A/(N+1)$. After applying $\mathrm{Id}+\phi_N$ to $\widetilde{V}^\gamma$ we obtain a vector field W^{N+1} with normalized

$$j^{n+N+2}(\omega_{\mathcal{R}}(W^{N+1})).$$

It remains to normalize the homogeneous term P of degree $n+N+1$ of the ∂_x-coordinate of W^{N+1}. We claim that x divides P and thus $P(1,t)$ has degree at most $n+N$. Indeed, By construction x divides $\omega_{\mathcal{R}}(W^{N+1}))_{n+N+2}$ and

$$W^{N+1}_{n+N+2} - \left(\frac{(\omega_{\mathcal{R}}(W^{N+1}))_{n+N+2}}{x}\right)\partial_y \quad \text{and} \quad \mathcal{R} \quad \text{are tangent vector fields.}$$

Therefore x divides P. By using euclidean division in $\mathbb{C}[t]$ we can find a unique homogeneous polynomial $P_{N+1}(x,y)$ of degree $N+1$ such that $P_{N+1}(1,t)$ has degree at most $n-1$, and a homogeneous polynomial $C(x,y)$ of degree N such that $P = P_{N+1}+xRC$. The vector field $V^{N+1} = (1-C)W^{N+1}$ is normalized up to order $n+N+1$ and still in the same equivalence class.

The formal vector field V^∞ satisfying $j^N(V^\infty) = j^N(V^N)$ for all $N\geq 1$ is also formally equivalent to V^N and has all its homogenous terms normalized. Its dual form has the properties stated in the statement of Theorem V. The uniqueness of the solutions for the coefficients in the Taylor series of V_∞ at each step of the normalization shows that the normal form is unique. Nevertheless, the formal equivalence between the vector field V and its normal form V_∞ is non-unique since the group of automorphisms of the foliation contains the exponential of uV for any unit u. This appears in the normalization process as the lack of uniqueness for the coefficients in the normalizing map.

□

In [14] the authors consider the case of foliations $\mathcal{F}\in\mathcal{D}(n)$ for which $T(\mathcal{F})$ has only simple points in its support, i.e. all tangencies between the foliation and the exceptional divisor are simple. They provide unique formal normal forms modulo formal conjugacies tangent to the identity. The unique normal forms they obtain can also be written as $W(x,y)(xdy-ydx)+H(x,y)dx$ where $W\in\mathbb{C}[[x][y]$ is a formal Weierstrass polynomial and H is a polynomial whose coefficients lie in an open set of $\mathbb{C}^{\frac{n(n-1)}{2}+1}$. In our case the coefficients of the polynomial H lie in an an open set in $\mathbb{C}^{\frac{n(n-1)}{2}}$. The difference in number of parameters comes from the fact that we are considering equivalences that are tangent to a homothety $\lambda\mathrm{Id}$ and in [14] the authors consider equivalences tangent to Id. In the case where $T(\mathcal{F}) = p$ the convergence of the normal form is proven by the same authors in [15]. The difference between the normal form obtained in Theorem V and the normal forms of [15] lies in the choice of a radial foliation that is used to normalize.

3.2 The $\sim_E$-invariants.

Following Theorem V, we introduce an analytical invariant by extractring some informations from the formal normal forms of a foliation $\mathcal{F}$. Given three distinct points $p_0, p_1, p_\infty \in E$ and any class of E-equivalence $[\mathcal{F}]$ such that $p_0 \notin |T(\mathcal{F})|$, we define its $\sim_E$-invariant as the data

$$\mathbf{c}[\mathcal{F}](p_0, p_1, p_\infty) := (c_{ij}) \in \mathbb{C}^{\frac{n(n-1)}{2}}.$$

This invariant together with the holonomy provides a complete system of invariants for the E-equivalence in $\mathcal{D}$. This is the meaning of the following

Theorem VI ($\sim_E$-invariants)**.** *$\mathcal{F}_1 \sim_E \mathcal{F}_2 \in \mathcal{D}$ if and only if $\mathcal{F}_1 \in \mathcal{D}$, $H(\mathcal{F}_1) = H(\mathcal{F}_2)$ and*

$$\mathbf{c}[\mathcal{F}_1](p_0, p_1, p_\infty) = \mathbf{c}[\mathcal{F}_2](p_0, p_1, p_\infty)$$

for some (and hence for every) choice of three distinct points $p_0, p_1, p_\infty \in E$.

As in the homogeneous case, Proposition 5 implies that the parameters (c_{ij}) are independent of the holonomy. If we knew that the normal form associated to a certain (convergent) holonomy class is convergent, we would deduce the convergence of the normal forms of Theorem V for holomorphic foliations.

Proof of Theorem VI. Consider two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ as in the statement of the theorem. Their associated formal normal forms are written

$$\mathcal{F}_1 \hat{\sim}_E \underbrace{(R(x,y) + \cdots)}_{W_1}(x\mathrm{d}y - y\mathrm{d}x) + Q_1 x^3 \mathrm{d}x$$
$$\mathcal{F}_2 \hat{\sim}_E \underbrace{(R(x,y) + \cdots)}_{W_2}(x\mathrm{d}y - y\mathrm{d}x) + Q_2 x^3 \mathrm{d}x$$

Since $\mathbf{c}[\mathcal{F}_1](p_0, p_1, p_\infty) = \mathbf{c}[\mathcal{F}_2](p_0, p_1, p_\infty)$, they share the same second factor, that is to say

$$Q_1 = Q_2.$$

Denote $r = R(1,t)$ and $q = Q_1(1,t)$. Lemma 5 ensures that the map $\mathrm{hol}_{r,q}$ is a bijection. Therefore, the equality $H(\mathcal{F}_1) = H(\mathcal{F}_2)$ implies that the terms W_1 and W_2 are also equal. Therefore, the foliations have the same normal forms and are formally equivalent. Now, following [5], they are also analytically equivalent. □

3.3 Realization of $\sim_E$-invariants

By giving a geometric interpretation of the $\sim_E$-invariants, we are able to proof a realization-like theorem.

Recall that given $\mathcal{F} \in \mathcal{D}(n)$, we defined

$$\mathrm{div}(\mathcal{F}) = \{\mathrm{Tang}(\mathcal{F}, \mathcal{G})_{|E} \in \mathrm{div}E(n+3) : \mathcal{G} \in \mathcal{D}(0)\}. \tag{20}$$

By construction $\mathrm{div}(\mathcal{F}) \subset \mathrm{div}(E \backslash |T(\mathcal{F})|)$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are E-equivalent then obviously $\mathrm{div}(\mathcal{F}_1) = \mathrm{div}(\mathcal{F}_2)$. There exists an equivalence relation $\sim_{T(\mathcal{F})}$ on $\mathrm{div}E(n+3)$ depending only on $T(\mathcal{F})$ whose classes correspond precisely to a subset of the form (20). Indeed, given a divisor

$$D = r_1 p_1 + \cdots + r_k p_k \in \mathrm{div}E(n)$$

we say that two divisors $D_1, D_2 \in \mathrm{div}E(n+3)$ are D-equivalent, and we denote it by $D_1 \sim_D D_2$, if there exist homogeneous polynomials in two variables P_1, P_2 and R defining D_1, D_2 and D respectively satisfying that $P_2 = P_1 + RQ$ for some homogenoeus polynomial Q of degree 3. On the other hand, denote by $\mathrm{Rot}(D)$ the set of families $\{h_1, \ldots, h_k\}$ where each $h_i : (E, p_i) \to (E, p_i)$ is a holomorphic germ locally conjugated to the rotation by angle $2\pi/(r_i + 1)$ and

$$\mathbf{E} = \bigcup_{D \in \mathrm{div}E} \mathbf{E}_D \text{ where } \mathbf{E}_D = \frac{\mathrm{div}(E \setminus |D|)(\deg(D) + 3)}{\sim_D} \times \mathrm{Rot}(D)$$

with its natural projection $\mathbf{E} \to \mathrm{div}E$. We get a natural map $\mathfrak{I} : \mathcal{D}/\sim_E \to \mathbf{E}$ given by

$$\mathfrak{I}([\mathcal{F}]) = ([\mathrm{div}(\mathcal{F})], H(\mathcal{F})) \in \mathbf{E}_{T(\mathcal{F})}$$

which is well defined since E-equivalences fix any point on E, where all invariants we deal with are computed.

Theorem VII. *The map $\mathfrak{I}$ is onto. Its fiber over an element in $\mathbf{E}_D$ is biholomorphic to $\mathbb{C}^M$ where $M = \max\left(0, \frac{(\deg D - 2)(\deg D - 1)}{2}\right)$.*

This theorem also proves that any invariant in Theorems II and VI are realized by some foliation in $\mathcal{D}$.

To describe the space $\mathcal{D}/\sim$ we just need to remark that there is a natural action of the group $GL(2, \mathbb{C})$ on $\mathcal{D}/\sim_E$ once we have chosen coordinates (x, y) in $(\mathbb{C}^2, 0)$. The action associates to each matrix the natural linear transformation in the two variables (x, y). The quotient is precisely $\mathcal{D}/\sim$. This action preserves fibers of $\mathfrak{I}$, and actually the map $\mathfrak{I}$ is equivariant with respect to the natural homomorphism $GL(2, \mathbb{C}) \to \mathrm{Aut}(E)$. Hence we can define a surjective map $\overline{\mathfrak{I}} : \mathcal{D}/\sim \to \mathbf{E}/\mathrm{Aut}(E)$ whose fibers are as in Theorem VII.

Proof. To prove that $\mathfrak{I}$ is onto, let $D = r_1 p_1 + \ldots + r_k p_k \in \mathrm{div}E(n)$ and

$$([D_1], \{h_1, \ldots, h_k\}) \in \mathbf{E}_D$$

be given. Consider an affine coordinate $s \in \mathbb{C}$ in E containing all the points in the support of D and D_1. These divisors can be respectively represented by monic polynomials $r(s)$ and $q(s)$ and the 1-form $r(s)ds + q(s)\mathrm{d}x$ defines an element such that

$$\begin{aligned} T(\mathcal{F}_0) &= D \\ \mathrm{div}(\mathcal{F}_0) &= [D_1] \end{aligned}$$

We are going to do a surgery on $\mathcal{F}_0$ which preserves the above relations but modify the holonomies as expected. Consider the pair $(\mathcal{F}_0, \mathcal{G})$ where $\mathcal{G}$ is the radial foliation defined by $ds = 0$ in the given coordinates. On a small neighbourhood U_i of $p_i \in E$, following Lemma 2, there exists a local biholomorphism ϕ_i sending the pair $(\mathcal{F}_0, \mathcal{G})_{|U_i}$ to the germ at $(0,0)$ of the pair (ω_i, dt) where $\omega_i = t^{r_i}dt + (t-1)\mathrm{d}x$. Define $\varphi_i = \phi_{i|E\cap U_i}$. To keep track of the divisors D and D_1 and change the holonomy we do not touch the gluing on $E \cap U_i$ but we change the model foliation ω_i by an appropriate holomorphic model $\widetilde{\omega}_i \in \Omega_{r=t^{r_i},(t-1)}$ obtained from Proposition 6 that satisfies

$$h_i(s) = \varphi_i^{-1} \circ \mathrm{hol}_{r,t-1}(\widetilde{\omega}_i) \circ \varphi_i(s).$$

Now, there exists a unique extension of φ_i to a saturated neighbourhood V_i of an annulus $A_i \subset U_i \cap E$ that sends the pair $(\mathcal{F}, \mathcal{G})_{|V_i}$ to the pair of germs of foliations at $(0,0)$ defined by $(\widetilde{\omega}_i, dt)$. After doing this at each point p_i, we obtain a pair of regular foliations $(\mathcal{F}, \mathcal{G})$ in a neighbourhood of a (-1)-curve. The contraction of this curve produces a foliation $\mathcal{F} \in \mathcal{D}^R$, $\mathcal{G} \in \mathcal{D}(0)$, $T(\mathcal{F}) = D$, $\mathrm{div}(\mathcal{F}) = [D_1]$ and $H(\mathcal{F}) = (h_1, \ldots, h_k)$. Thus the map $\mathfrak{I}$ is onto.

It remains to prove that for each $D = r_1p_1 + \cdots + r_kp_k \in \mathrm{div}(E)(n)$ and

$$\mathbf{e} = ([D_1], \{h_1, \ldots, h_k\}) \in \mathbf{E}_D$$

the fibre $\mathfrak{I}^{-1}(\mathbf{e})$ is biholomorphic to $\mathbb{C}^M$ where M is as in the statement of the theorem. Take coordinates (x, y) of $(\mathbb{C}^2, 0)$ such that $x = 0$ does not define a point in the support of D. By Theorem V we can assign to each class in $\mathfrak{I}^{-1}(\mathbf{e})$ a unique formal 1-form $W(x,y)(x\mathrm{d}y - y\mathrm{d}x) + Q(x,y)\mathrm{d}x$. The homogeneous part of degree $n+2$ of Q depends only on $[D_1]$ and thus is invariant in $\mathfrak{I}^{-1}(\mathbf{e})$. Now, the polynomial $Q - Q_{n+2}$ is written

$$\sum_{\substack{0 \leq i \leq n-2 \\ 0 \leq j \leq n-1 \\ i+j \geq n}} c_{ij} x^{i+3} y^j$$

and thus defines a point (c_{ij}) in $\mathbb{C}^M$. This defines a map from $\mathfrak{I}^{-1}(\mathbf{e})$ to $\mathbb{C}^M$. Proposition 5 ensures that this map is injective.

Let us prove that it is also onto.

Let (c_{ij}) in $\mathbb{C}^M$ be given. By construction, the family of complex numbers (c_{ij}), the class of divisor $[D_1]$ and the set of holonomies $h = \{h_1, \ldots, h_k\}$ is,associated to a formal normal form of Theorem V. However, we are looking for a convergent foliation. The idea is basically the same as in Proposition 6. We are going to do a surgery on a polynomial form that is tangent to the formal normal form with an order as big as necessary. Indeed, using the construction of Theorem V it is easy to see that for any $N >> 1$ there exists a foliation $\mathcal{F}_0$ defined by a *polynomial* normal form $\omega_0 = W_0(x,y)(x\mathrm{d}y - y\mathrm{d}x) + Q(x,y)\mathrm{d}x$ satisfying that

$$j^N(H(\mathcal{F}_0)) = j^N(h).$$

After the blow-up $y = tx$, we consider the radial foliation $\mathcal{G}_0$ defined by $dt = 0$. On a neighbourhood U_i of $p_i \in |D|$, there exists a unique biholomorphism onto a neighbourhood of $(0,0)$ in coordinates (z,t) sending $(0,t)$ to $(0,t)$ and the pair $(\mathcal{F},\mathcal{G})_{|U_i}$ to the pair (ω_i^0, dt) where $\omega_i^0 = (t^{r_i} + a^0_{r_i-1}(z)t^{r_i-1} + \cdots + a^0_0(z))dt + (t-1)dz$ is the convergent 1-form of Proposition 6 having the same holonomy h_i^0 as $\mathcal{F}_0$ at p_i. The convergent 1-form $\omega_i = (t^{r_i} + a_{r_i-1}(z)t^{r_i-1} + \cdots + a_0(z))dt + (t-1)dz$ for which $H(\mathcal{F}_{\omega_i}) = h_i$ satisfies that

$$\omega_i - \omega_i^0 = z^{N'} f(z,t)dt$$

for some big $N' \in \mathbb{N}$. In particular this means that for each $0 < \varepsilon_i < |t_0| < r_i$ for some small ε_i and r_i, the holonomy germs $g_i^0, g_i : (\{t = t_0\}, 0) \to (E, t_0)$ of ω_i^0 and ω_i respectively are tangent with an order as big as necessary. In particular, $g_i^{-1} \circ g_i^0(z) - z$ has a zero of order N'' that is as big as necessary. The equivalence between $\mathcal{F}_{\omega_i^0}$ and $\mathcal{F}_{\omega_i}$ in a neighbourhood of the annulus can be written as

$$(z,t) \mapsto (z + z^{N^{(3)}}(\cdots), t). \tag{21}$$

for a $N^{(3)}$ as big as necessary. We claim that the gluing of $\mathcal{F}_{0|(\widetilde{\mathbb{C}^2},E)\setminus U_1\cup\cdots\cup U_k}$ with the models $\mathcal{F}_{\omega_i}$ via the given gluing (21) defines an element in $\mathcal{F} \in \mathfrak{I}^{-1}(\mathbf{e})$ with the desired invariants. After the gluing, we get a pair $(\mathcal{F},\mathcal{G})$ of foliations around a rational curve of self-intersection -1. Combining results from [3], [13] and [6], we have the following lemma

Lemma 14. *Let* $\mathrm{Diff}_N(E,\mathcal{G})$ *be the sheaf over* E *of germs of automorphisms that are locally written* $(z,t) \mapsto (z + z^N(\cdots), t)$ *where* $z = 0$ *is a local equation of* E *and* $\mathrm{d}t = 0$ *is the radial foliation* $\mathcal{G}$*. Then,*

$$H^1(E, \mathrm{Diff}_N(E,\mathcal{G})) = 0.$$

Let us give a sketch of the proof of the above lemma

Proof. According to [13], it is enough to prove at a formal level that the following equality holds

$$H^1\left(E, \widehat{\mathrm{Diff}}_N(E,\mathcal{G})\right) = 0$$

Let us consider $\{\phi_{ij}\}_{ij} \in Z^1\left(\widehat{\mathrm{Diff}}_N(E,\mathcal{G}), \{U_i\}_{i\in I}\right)$ for some simply connected covering $\{U_i\}_{i\in I}$ of E. Taking a finer covering if necessary, one can suppose that ϕ_{ij} is the flow at time 1 of some formal vector field $\hat{X}_{ij}$ tangent to the radial foliation. Since ϕ_{ij} is tangent at order N to Id, one can write in adapted coordinates (z,t)

$$\hat{X}_{ij} = z^n A_{ij}(z,t)\frac{\partial}{\partial z}, \quad \phi_{ij} = e^{[1]\hat{X}_{ij}}$$

where A_{ij} is some formal function. Now, the cohomology group $H^1\left(E, \hat{\mathcal{O}}\right)$ vanishes ([3]), so there exists a family of functions such that $A_{ij} = A_j - A_i$. As a consequence, there exists a family of vector fields $\left\{\hat{X}_i\right\}_{i\in I}$ vanishing at order N along E such that

$$\hat{X}_{ij} = \hat{X}_j - \hat{X}_i.$$

The Campbell-Hausdorff formula induces that

$$e^{[1]\hat{X}_i} \circ e^{[1]\hat{X}_{ij}} \circ \left(e^{[1]\hat{X}_j}\right)^{-1} = e^{[1]\hat{Z}_{ij}}$$

where $\hat{Z}_{ij}$ is a formal vector field tanget to the radial foliation and vanishing at order $2n-1$ along E. Applying inductively this construction, we obtain a $0-$cocycle $\{\phi_i\}_{i\in I}$ in $\widehat{\mathrm{Diff}}_N\left(E, \mathcal{G}\right)$ such that

$$\phi_i \circ e^{[1]\hat{X}_{ij}} \circ \phi_j^{-1} = \psi_{ij} \tag{22}$$

and ψ_{ij} is tangent to Id at an order p as big as necessary. Finally following [6], for m big enough, the image of the natural map

$$H^1\left(E, \widehat{\mathrm{Diff}}_m\left(E, \mathcal{G}\right)\right) \to H^1\left(E, \widehat{\mathrm{Diff}}_N\left(E, \mathcal{G}\right)\right)$$

is trivial. Thus, one can write

$$\psi_{ij} = \psi_i^{-1}\psi_j \tag{23}$$

where $\{\psi_i\}_{i\in I}$ is a 0-cocycle in $\widehat{\mathrm{Diff}}_N\left(E, \mathcal{G}\right)$. Combining the relations 22 and 23 yields a trivialization of the initial cocycle.

□

This lemma implies in particular that one can choose a holomorphic 1-form ω representing $\mathcal{F}$ as tangent as necessary to ω_0. In particular, one can choose this order of tangency such that ω and ω_0 coincides up to order $2n$. Therefore, $\mathcal{F}$ has the desired invariants $(c_{ij}) \in \mathbb{C}^M$ and the map from $\mathfrak{I}^{-1}(\mathbf{e})$ to $\mathbb{C}^M$ is onto. □

4 Examples of unfoldings in $\mathcal{D}^{\mathrm{h}}(n)$.

In the light of Theorem I, we have a natural way of unfolding a dicritical homogeneous germ $\mathcal{F} \in \mathcal{D}^{\mathrm{h}}(n)$ in the space $\mathcal{D}^{\mathrm{h}}(n)$ with base space of dimension $n+3$. Indeed, we can suppose $\mathcal{F}$ is obtained from $\mathcal{F}_S$ of Theorem I by a blow up on some divisor $D = p_1 + \ldots + p_{n+3}$ on the exceptional curve. When two points p_i and p_j coincide, we interpret that we blow up twice at the same point. By considering an affine coordinate $z \in \mathbb{C}$ in the rational curve where p_i corresponds to $z_i \in \mathbb{C}$, and defining $p_i(c) = z_i + c_i$ for $c = (c_1, \ldots, c_{n+3}) \in (\mathbb{C}^{n+3}, 0)$, the foliations $\{\mathcal{F}_c : c \in (\mathbb{C}^{n+3}, 0)\}$ defined by blowing up $\mathcal{F}_S$ on the divisor $D(c) = p_1(c) + \ldots + p_{n+3}(c)$ form an equisingular unfolding of $\mathcal{F}$. The knowledge of $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ and Theorem II allow to decide which of the directions in this unfolding are non-trivial. For instance, in the cases $n = 2, 3$, this procedure applied to the points of the special choices of topological class in $\mathrm{div}^{\mathrm{h}}(\mathcal{F})$ of Corollary 12 produce universal equisingular unfoldings.

In fact, by this procedure we are able to give explicit examples of non-trivial unfoldings of a foliation without any special integrability properties. To our knowledge they constitute the first such examples.

4.1 Examples obtained by pulling-back foliations admitting a meromorphic first integral.

We can provide some examples of universal equisingular unfoldings of elements in $\mathcal{D}$. Indeed, if $r(t)$ is a polynomial of degree n, the universal unfolding of the germ of foliation $\mathcal{F} \in \mathcal{D}^{\mathrm{h}}(n)$ defined by

$$\nu = x^{n+2}\mathrm{d}\left(r\left(\frac{y}{x}\right)+x\right) \tag{24}$$

is given by pull-back of ν via the rational map

$$\Lambda(x,y,(c_{ij})) = (x,y)\cdot\left(1+\sum_{j=1,i<j}^{n-1} c_{ij}x^{i-j}y^{j}\right)$$

where $(c_{ij}) \in \mathbb{C}^{n(n-1)/2}$. Remark that for each fixed non-zero parameter (c_{ij}), Λ is an automorphism of each leaf of the radial foliation $x\partial_x + y\partial_y$ except for the leaf $x=0$. To see that the resulting unfolding is equisingular and non-trivial in any direction in the parameter space, it suffices to remark that after blowing up via $y = tx$, the underlying deformation is written in dual form as

$$\left(r(t)+\sum_{j=1,i<j}^{n-1} jc_{ij}x^{i+1}t^{j-1}\right)\mathrm{d}t + \left(1+\sum_{j=1,i<j}^{n-1}(i+1)c_{ij}x^{i}t^{j-1}\right)\mathrm{d}x.$$

Hence for any fixed parameter we have already one of the normal forms given in Theorem V. By uniqueness of the normal form, any two elements in this family belong to different classes of E-equivalence. Actually, the so constructed unfolding turns out to be the *universal* unfolding of ν since the dimension of its space of unfoldings is $n(n-1)/2$. Notice that unfortunately, such a procedure fails when the initial form ν is in normal form and has some other non-zero terms.

4.2 Examples without any special integrability property.

Consider an element of $\mathcal{D}^{\mathrm{h}}(n)$ defined by the one-form

$$\omega = \underbrace{\left(R(x,y)+\sum_{i=0}^{n-1} a_i(x)y^{i}x^{n-i}\right)}_{W(x,y)}(x\mathrm{d}y - y\mathrm{d}x) + Q(x,y)\mathrm{d}x$$

as in the statement of Theorem III where R is homogeneous of degree n and $a_i \in \mathbb{C}\{x\}$ with $a_i(0)=0$. Then the one-form in $(\mathbb{C}^2,0)\times\mathbb{C}^{n-1}$ defined by

$$\Omega = \left(R(x,y)+\sum_{i=0}^{n-1} a_i(x+\langle c,y\rangle)y^{i}x^{n-i}\right)(x\mathrm{d}y - y\mathrm{d}x) + Q(x,y)\mathrm{d}(x+\langle c,y\rangle)$$

where $\langle c, y\rangle = \sum_{i=1}^{n-1} c_i y^i$, defines a non-trivial equisingular unfolding of any foliation in $\mathcal{D}$ associated to $\Omega|_{c=c_0}$ for some $c_0 \in \mathbb{C}^{n-1}$.

Notice that, if the initial 1−form ω is polynomial then the unfolding is also polynomial. In this way, we provide non-trivial deformations of global foliations in $\mathbb{CP}^2$ of constant topological type, a situation that is generically impossible (see [8]).

Let us consider the rational map

$$\Lambda(x, y, c) = \left(1 + \frac{\langle c, y\rangle}{x}\right) \cdot (x, y)$$

where $\langle c, y\rangle = \sum_{i=1}^{n-1} c_i y^i$.

The 1-form Ω in $\mathbb{C}^2 \times \mathbb{C}^{n-1}$ is the pull-back of $\Omega|_{c=0} = \omega$ by Λ up to some multiplication by a meromorphic unit u. In particular, Ω is an integrable 1−form since we have

$$u^2 \Omega \wedge \mathrm{d}\Omega = \Lambda^* \left(\Omega|_{c=0} \wedge \mathrm{d}\Omega|_{c=0}\right) = 0.$$

After the blow-up E of the singular locus $\{x = 0, y = 0\}$, Ω is written in the coordinates of the blow-up $y = xt$

$$E^*\Omega = \left(R(1, t) + \sum_{i=1}^{n-1} a_i (x + \langle c, tx\rangle) t^i\right) \mathrm{d}t + Q(1, t)\, \mathrm{d}(x + \langle c, tx\rangle).$$

The induced foliation restricted to a fibre of $(x, y, c) \mapsto c$ over a point $c = (c_1, \ldots, c_{n-1})$ such that $1/c_1$ is not a root of $R(1, t) = 0$ lies in $\mathcal{D}(n)$. The tangency locus with the exceptional divisor $x = 0$ is equal $\{x = 0, t = t_i\}$ where t_i is a solution to of $R(1, t) = 0$. Since the curves $\{x = 0, t = t_i\} = 0$ are contained in a invariant hypersurface of $E^*\Omega$, the 1−form Ω defines an equireducible unfolding of $\Omega|_{c=c_0}$ for any c_0 lying in the Zariski open set $U = \{c \in \mathbb{C}^{n-1} : c_1 t_i \neq 1, \forall t_i\}$

Suppose now that Ω is trivial along a certain smooth submanifold of the space of parameter. Then, there exists a germ of application

$$c : t \in \mathbb{C} \to (c_1(t), c_2(t), \cdots, c_p(t))$$

with $c(0) = 0$ and $c'(0) \neq (0, \cdots, 0)$ such that $\Omega|_{c(t)}$ is a trivial unfolding of one variable. Now $\Omega|_{c(t)}$ is written

$$\left(R(x, y) + \sum_{i=1}^{n-1} a_i (x + \langle c_i(t), y\rangle) y^i x^{n-i}\right) \omega_R + Q(x, y)\, \mathrm{d}(x + \langle c_i(t), y\rangle) =$$

$$(\cdots)\, \omega_R + Q(x, y)\, \mathrm{d}x + Q(x, y) \sum_{i=1}^{n-1} c_i(t)\, i y^{i-1} \mathrm{d}y + \left(Q(x, y) \sum_{i=1}^{n-1} c_i'(t)\, y^i\right) \mathrm{d}t$$

Following [12], the triviality of $\Omega|_{c(t)}$ implies that the coefficient of $\mathrm{d}t$ belongs to the ideal generated by the coefficients of $\mathrm{d}x$ and $\mathrm{d}y$. If there exists such a relation, we can evaluate it for $t = 0$ and find polynomial functions A and B such that

$$PQ = A(Q - yW) + xBW. \tag{25}$$

where $P(y) = \sum_{i=1}^{n-1} c_i'(0)\, y^i$. Since, Q and W are relatively prime, then there exists Δ such that $xB - yA = \Delta Q$ and thus $A = P - W\Delta$. Therefore, rewriting (25) yields

$$-yP - Q\Delta + yW\Delta + xB = 0 \tag{26}$$

Since the orders of Q and W at 0 are $n+2$ and n, evaluating the jet of order n of the above equality gives

$$-yP + x\mathrm{Jet}_n(B) = 0$$

and thus $P = 0$ which is impossible. This proves that the unfolding is non-trivial for $c_0 = 0$. To prove it for $c_0 \in U$, it suffices to remark that the jet n of equation (26) remains exactly the same if we impose $c(0) = c_0$ instead of $c_0(0) = 0$.

For generic function a_i's, Ω does not admit any first integral that is multivalued and holomorphic on the complement of a countable union of analytic sets. In particular it does not admit meromorphic or Liouvillian first integrals.

Suppose that $\mathcal{F}$ belongs to $\mathcal{D}(1)$ and admits a Louvillian first integral. After the blow-up and the restriction of this first integral to E, we can see that $h = H(\mathcal{F})$ admits a Louvillian first integral on E: that is a non-constant multivalued holomorphic function f defined on the whole E such that for at least one determination of $f \circ h$ one has $f \circ h = f$. We know that such a Liouvillian function admits at most a countable number of singularities, that is, homotopy classes of paths along which the analytical extension of the germ of function is impossible. Below, we produce an example of periodic map h such that any first Louvllian integral has an uncountable number of singularities. Any foliation admitting this germ h as holonomy will not admit a Liouvillian first integral.

Lemma 15. *Let $D \subset \mathbb{C}$ be a region containing 0 and $h : D \to D$ a holomorphic mapping, $h(0) = 0$ and the germ at 0 of h satisfies $h^{\circ n} = Id$ for $n \in \mathbb{N}$. Suppose that the set of singularities of h in ∂D has an accumulation point p and there exists a continuous extension of h to a neighbourhood of p in ∂D satisfying $h(p) \in D$. Then any non-constant Louvillian first integral f of h that is defined and holomorphic on D has a singularity at p.*

Proof. Let $\gamma : [0,1] \to \overline{D}$ be a path that satisfies $\gamma^{-1}(D) = [0,1)$, $\gamma(0) = 0$ and $\gamma(1)$ is a singularity of h. If a Louvillian first integral f admits analytic extension for some of its determination, we denote its extension by f_γ and $\lim_{t\to 1} h(\gamma(t)) \in D$, then part of the graph of h is contained in the set

$$\{(x, y) \in \mathrm{dom}(f_\gamma) \times D : f(y) = f_\gamma(x)\}.$$

If $f'(h(\gamma(1)) \neq 0$, the implicit function theorem tells us that h extends analytically to $\gamma(1)$, which is not possible. Therefore, for every singularity $q \in \partial D$ of h where h extends continuously and f analytically, one has $f'(h(q)) = 0$.

Finally, suppose that f extends analytically to p. There exists continuous extension of h to a neighbourhood of p in ∂D. Thus, given a sequence p_n of singularities of h that accumulate on p, we have $f'(h(p_n)) = 0$. The convergence of $h(p_n)$ to $h(p) \in D$ implies f is constant which is impossible. Hence p is a singularity for f. $\square$

To finish let us provide an example of such an h with curves of singularities. Let D be a simply connected plane region bounded by a Jordan curve of class $\mathcal{C}^1$. Suppose that ∂D is not analytic. Moreover, suppose if θ is the rotation of angle $2\pi/n$ centered at $p \in D$, the intersection points of ∂D and $\theta(\partial D)$ are points of transversality between the curves. Let D_p be the connected component of $D \cap \theta(D)$ containing p and $A = \partial D_p \cap h^{-1}(D_p)$. Now, if $\varphi : \overline{\mathbb{D}} \to \overline{D}$ is the homeomorphic extension of Riemann's mapping Theorem between $\mathbb{D}, 0$ and D, p to the boundaries given by Carathéodory's Theorem, then the continuous map $h : \overline{D_0} \to \overline{D_0}$ defined by $h = \varphi^{-1} \circ \theta \circ \varphi$ on $D_0 = \varphi^{-1}(D_p)$, has order n at 0 and singularities at each of the points of ∂D_0 (see [17], p. 628). By construction the values of the extension on $\mathcal{C} = \varphi^{-1}(A)$ belong to D_0. Thus, Lemma 15 guarantees that the points of $\mathcal{C}$ are singular for any germ of Louvillian first integral of h.

Actually, any non-constant Louvillian first integral of h will not be extendable to any point in $\mathcal{C}$. Hence, any foliation with holonomy h will not admit a Louvillian first integral.

4.3 A parametrization of $\mathcal{D}(2)/\sim$.

The parameter space of the unfolding Ω of Subsection 4.2 is a Zariski open set of $\mathbb{C}^{n-1}$. We know from [12] that the base space of the universal unfolding of an element in $\mathcal{D}(n)$ has dimension $n(n-1)/2$, so the obtained unfolding is only part of it if $n > 2$. In the case of $n = 2$, both dimensions coincide.

The form

$$\Omega(x, y, c) = \left(y^2 + a(x)y(x + cy) + b(x)\right)(x\mathrm{d}y - y\mathrm{d}x) + (x + cy)^4\mathrm{d}x$$

can be pulled back by the biholomorphism

$$(x, y, c) \mapsto (x + cy, y, c)$$

to obtain

$$[y^2 + a(x - cy)yx + b(x)](x\mathrm{d}y - y\mathrm{d}x) + x^4\mathrm{d}(x - cy).$$

Up to a sign on c, this form corresponds precisely to the one appearing in the previous section. Hence it is a non-trivial equisingular unfolding around each fixed parameter

$$c \in \mathbb{C} \setminus \{t : t^2 + a(0)t + b(0) = 0\}.$$

Since the dimension of the base space of the obtained unfolding coincides with the dimension of the universal equisingular unfolding of $\mathcal{F}_c$, they are equivalent.

Let us analyze the analytic invariants of $\mathcal{F}_c$ along the parameter space. By construction $H(\mathcal{F}_c) = H(\mathcal{F}_0)$ for all admissible c's. The position of the divisors $q(\mathcal{F}_c)$ defined in the proof of Theorem IV depend on c by a holomorphic (possibly multivalued) non-constant function that assumes any value in $\mathbb{C} \setminus |T(\mathcal{F}_0)|$. By Theorems II and IV we cover all analytic classes of foliations in $\mathcal{D}(2)$ having the same holonomy $H(\mathcal{F}_0)$. Furthermore, any choice of rotations around points in E is realized as the holonomy of a normal form. Once we have a holomorphic germ in $\mathcal{D}(2)$ with given holonomy generators we can consider the coefficients a and b of one of its normal forms in Corollary 12 as $\mathcal{F}_0$.

Remark that the foliation induced by the form Ω extends to a holomorphic foliation $\widetilde{\mathcal{F}}$ on $(\mathbb{C}^2, 0) \times \mathbb{P}^1$. The germ of this foliation at each point $(0, 0, c) \in \mathbb{C}^2 \times \mathbb{C}$ describes the universal equisingular unfolding of $\mathcal{F}_c$. Besides, all the analytic classes with fixed holonomy are realized along the parameter space $c \in \mathbb{C}$. We can think of $\widetilde{\mathcal{F}}$ as a realization of all universal unfoldings of foliations with some fixed holonomy. The intersection of $\widetilde{\mathcal{F}}$ with the fiber at $c = \infty$ is not and element in $\mathcal{D}$, but it is still a germ of foliation and it allows to compactify the germs of foliations along the parameter space.

We deduce a parametrization of $\mathcal{D}(2)/\sim$ that is adapted to the equivalence relation *having the same holonomy class.* By considering the parameters $(a, b, c) \in \mathbb{C}\{x\}^2 \times \mathbb{P}^1$, the fibres of the projection $(a, b, c) \mapsto (a, b)$ parametrize equivalence classes. These restrictions of the parametrization are locally injective in general but not globally injective. As a consequence, up to changing coordinates, any two distinct germs in $\mathcal{D}(2)$ sharing the same holonomy class can be joined by a deformation underlying a non-trivial equisingular unfolding.

4.4 Unfoldings versus deformations in $\mathcal{D}$.

In this section, we give some independent results which compare the unfoldings and the deformations of a foliation in $\mathcal{D}$. In general, it is very difficult to give a criterion that recognizes deformations that underlie an unfolding. However, for the class $\mathcal{D}$, it can be read on the variation of the holonomy.

Proposition 16. *Let $\{\mathcal{F}_c\}_{c \in (\mathbb{C}^k, 0)} \subset \mathcal{D}$ be an analytic germ of deformation of $\mathcal{F}_0$ satisfying that $H(\mathcal{F}_c) = H(\mathcal{F}_0)$ for all $c \in (\mathbb{C}^k, 0)$. Then the given deformation underlies an equisingular unfolding of $\mathcal{F}_0$ on $(\mathbb{C}^{2+k}, 0)$.*

Proof. Let us provide a couple of open neighborhoods of some open sets in E satisfying that

- $U_1 \cup U_2$ is a neighbourhood of E,
- on each U_j, there is an analytic family

$$\psi_c^j : U_j \to U_j$$

  of biholomorphisms fixing every point of E and sending $\mathcal{F}_{0|U_j}$ to $\mathcal{F}_{c|U_j}$.

Doing so, the initial deformation will be a product on each $U_j \times (\mathbb{C}^k, 0)$, and as highlighted in [12], will be correspond to an unfolding.

Lemma 17. *Given $\mathcal{F} \in \mathcal{D}$ there exist two radial foliations $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{D}(0)$ such that*

$$|\text{Tang}(\mathcal{F}, \mathcal{G}_1)_{|E}| \cap |\text{Tang}(\mathcal{F}, \mathcal{G}_2)_{|E}| = \emptyset$$

Proof. Up to some change of coordinates, we can suppose that

$$\omega(x, y) = R(x, y)(x\mathrm{d}y - y\mathrm{d}x) + P_{n+2}\mathrm{d}x + Q_{n+2}\mathrm{d}y + \ldots$$

representing $\mathcal{F}$ satisfies that the degree of $P_{n+2}(1,t)+tQ_{n+2}(1,t)$ is $n+3$. Since, $R(1,t)$ and $P_{n+2}(1,t)+tQ_{n+2}(1,t)$ has no common roots, one can choose $\mathcal{G}_1 = x\partial_x + y\partial_y$ and $\mathcal{G}_2 = x\partial_x + (y+x^2)\partial_y$. □

Applying Lemma 17 to $\mathcal{F}_0$, find a Jordan curve γ in E that separates $|\mathrm{Tang}(\mathcal{F}_0, \mathcal{G}_1)_{|E}|$ from $|\mathrm{Tang}(\mathcal{F}_0, \mathcal{G}_2)_{|E}|$ and take U_1 a $\mathcal{F}_0$ saturated neighbourhood of the disc in $E \setminus \gamma$ with no point in $|\mathrm{Tang}(\mathcal{F}_0, \mathcal{G}_1)_{|E}|$, and U_2 an $\mathcal{F}_0$ saturated neighbourhood of the complementary disc. For sufficiently small $c \in (\mathbb{C}^k, 0)$, $\mathcal{F}_c$ and $\mathcal{G}_j$ are transverse on U_j. We can define the equivalence ψ_c^j on U_j by simply imposing $\psi^j_{c|E} = \mathrm{Id}$, $(\psi_c^j)_*(\mathcal{F}_0) = \mathcal{F}_c$ and $(\psi_c^j)_*(\mathcal{G}_j) = \mathcal{G}_j$. The analytic dependence of $\mathcal{F}_c$ along the parameter guarantees that the families ψ_c^j are analytic on $c \in (\mathbb{C}^k, 0)$. To prove that the unfolding is equisingular we need to check that along the whole process of desingularization the leaves of the unfolding are transverse to the parameter fibration $(x, c) \mapsto c$. In the present case remark that a leaf of the unfolding is formed by the union of leaves of the deformation that sit over *the same* point in E, since we do not move the points in E along the construction. Thus the trace of a leaf of the unfolding on the divisor $E \times (\mathbb{C}^k, 0)$ is $p \times (\mathbb{C}^k, 0)$, which is regular and transverse to the fibration. □

References

[1] M. Artin, On the solutions of analytic equations, *Invent. Math.*, 5, (1968), 277-291.

[2] M. Brunella, Birational theory of foliations, *Monografías de Matemática*, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000.

[3] C. Camacho, H. Movasati, Neighborhoods of Analytic Varieties, *IMCA* (2003)

[4] C. Camacho, P. Sad, Invariant varieties through singularities of vector fields, *Ann. of Math.* 115 (1982), 579-595.

[5] G. Calsamiglia, Finite determinacy of dicritical singularities, *Ann. Inst. Fourier*, 57,2 (2007), p. 673-691.

[6] Y. Genzmer, Analytical and formal classification of quasi-homogeneous foliations. *Journal of Differential Equation*, vol. 245 2008 no.6

[7] Y. Genzmer, E. Paul, Normal forms of foliations and curves defined by a function with generic tangent cone, *Moscow Math. J.*, 2011, n.1.

[8] Yu.S.Ilyashenko, Topology of phase portraits of analytic differential equations on a complex projective plane, *Trudy Sem. Petrovskogo*, vol.4, pp. 83-136 (1978)

[9] M. Klughertz, Existence d'une intégrale première méromorphe pour des germes de feuilletages à feuilles fermées du plan complexe, *Topology* , 31, 1992, 2 255-269

[10] F. Loray, A preparation theorem for codimension-one foliations, *Ann. of Math*, 2006, 2, 709-722

[11] D. Marin, Moduli spaces of germs of holomorphic foliations in the plane, *Comm. Math. Helv.*, Vol 78 no 3, 2003 , 518-539.

[12] J.-F. Mattei, Modules de feuilletages holomorphes singuliers. I. Équisingularité. *Invent. Math.* 103 (1991), no. 2, 297–325.

[13] J.-F. Mattei, E. Salem, Modules formels locaux de feuilletages holomorphes. *Preprint* (2010)

[14] L. Ortiz, E. Rosales, S. Voronin, Rigidity theorems for generic holomorphic germs of dicritic foliations and vector fields in $(\mathbb{C}^2, 0)$. Mosc. Math. J. 5 (2005), no. 1, 171–206.

[15] L. Ortiz, E. Rosales, S. Voronin, Analytic normal forms of germs of holomorphic dicritic foliations. *Mosc. Math. J.* 8 (2008), no. 3, 521–545, 616.

[16] L. Ortiz, E. Rosales, S. Voronin, Thom's problem for degenerated singular points of holomorphic foliations in the plane , *Mosc. Math. J.*, 2012, Volume 12, Number 4, 825862

[17] R. Perez-Marco, Sur les dynamiques holomorphes non-linéarisables et une conjecture de V.I. Arnold, *Ann. Sci. E.N.S*, 4^e Srie, Tome 26, Num. 5, (1993), pp. 565-644.

[18] V. I. Savelev, Zero-type embedding of a sphere into complex surfaces, *Moscow Univ. Math. Bull.* 37 (1982), 34–39.

[19] T. Ueda, On the neighborhood of a compact complex curve with topologically trivial normal bundle. *J. Math. Kyoto Univ.* 22 (1982/83), no. 4, 583–607.

G. Calsamiglia
Instituto de Matemática
Universidade Federal Fluminense
Rua Mário Santos Braga s/n
24020-140, Niterói, Brazil
gabriel@mat.uff.br

Y. Genzmer
I.M.T.
Université Paul Sabatier
Toulouse
yohann.genzmer@math.univ-toulouse.fr