A NONALGEBRAIC SINGULARITY OF DIFFERENTIAL EQUATION

YOHANN GENZMER, LOÏC TEYSSIER

Laboratoire I.R.M.A.
Université Louis Pasteur
Strasbourg (France)

Abstract. An algebrizable singularity is a germ of singular holomorphic foliation which cannot be given by a differential equation with algebraic coefficients, in any local chart. We first show that the typical saddle-node singularity is not algebrizable, then we give an explicit example of such an equation.

1. Introduction

We consider differential equations in the complex plane near an isolated singularity, which can be located at $(0,0)$ by translation :

$$A(x,y)\,\mathrm{d}y \quad = \quad B(x,y)\,\mathrm{d}x\,. \tag{1.1}$$

The coefficients A and B are germs of a holomorphic function with a common zero at $(0,0)$ and no common factor. We denote by λ_1 and λ_2 the eigenvalues of the linear part of the equation at $(0,0)$. We will always assume that at least one of those is non zero, say $\lambda_2 \neq 0$, and set $\lambda := \frac{\lambda_1}{\lambda_2}$. We recall the following classical result :

Theorem. (Poincaré and Dulac [D]**)** *If $\lambda \notin \mathbb{R}_{\leq 0}$ then there exist two polynomials P, Q such that the previous differential equation is orbitally equivalent through a local analytic change of coordinates to*

$$P(x,y)\,dy \quad = \quad Q(x,y)\,dx\,.$$

*If moreover $\lambda \notin \mathbb{N} \cup 1/\mathbb{N}_{\neq 0}$ then we can furthermore choose $P(x,y) = x$ and $Q(x,y) = \lambda y$ (*i.e. *the equation is linearizable).*

It thus turns out that a generic equation is linear, or at least algebraic, when written in a convenient system of analytic coordinates. Up to now an open question regarded whether *every* differential equation is algebraic in some local chart. Such an equation will be called **algebrizable**. We aim to prove that it is not so in the case of a saddle-node ($\lambda = 0$), as was expected in []. Notice that these equations are nonetheless *formally* algebrizable [].

Theorem. 1 *The typical saddle-node differential equation is not algebrizable.*

We give a precise definition of "typical" in the upcoming paragraph. By carefully estimating Martinet-Ramis modulus of classification [MR] we can produce an explicit example of such a nonalgebrizable equation :

Theorem. 2 *The differential equation*

$$x^2\,dy \quad = \quad y\,(1+??)\,dx$$

is not algebrizable.

1.1. Notations and basic definitions.

Throughout the article the notation $\mathbb{D}$ stands for the open unit disc of $\mathbb{C}$. In all the sequel the changes of coordinates we use are local analytic near $(0,0)$, *i.e.* elements of $\mathrm{Diff}\left(\mathbb{C}^2, 0\right)$, and the functions are germs of a function at $(0,0)$, *i.e.* elements of $\mathbb{C}\{x,y\}$. The solutions to the ordinary differential equation 1.1 coincide with the integral curves of the dual vector field

$$X_{A,B} \quad := \quad A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}\,.$$

We say that two vector fields X and $\tilde{X}$ (or, equivalently, the dual differential equations) are **orbitally conjugate** when there exists $\Psi \in \mathrm{Diff}\left(\mathbb{C}^2, 0\right)$ and $U \in \mathbb{C}\{x,y\}^*$ such that $\Psi^* X = U\tilde{X}$. Here $\mathbb{C}\{x,y\}^*$ stands for the multiplicative group of invertible germs.

Date: March 20$^{\text{th}}$, 2008.

We denote by $\mathcal{E}$ the space of all equations (1.1) with $\lambda = 0$, which we identify to an affine subset of $\mathbb{C}\{x,y\} \times \mathbb{C}\{x,y\}$ through the map $(A,B) \mapsto X_{A,B}$.

Let $m \in \mathbb{N}_{>0}$. We use bold-typed letters to indicate m-dimensional vectors $\mathbf{z} = (z_1, \ldots, z_m)$ and multi-indices $\mathbf{j} = (j_1, \ldots, j_m)$. We define as usual $|\mathbf{j}| := \sum_{k=1}^{m} j_k$ and $\mathbf{j}! := \prod_{k=1}^{m} j_k!$. We endow the space of germs $\mathbb{C}\{\mathbf{z}\}$ with a locally convex vector space structure by defining the family of norms of $f(\mathbf{z}) := \sum_{\mathbf{j}} a_{\mathbf{j}} \mathbf{z}^{\mathbf{j}}$ as

$$(\forall k \in \mathbb{N}_{\neq 0}) \quad ||f||_k := \sum_{\mathbf{j}} \frac{|a_{\mathbf{j}}|}{\mathbf{j}!^{1+1/k}}. \tag{1.2}$$

Unless special mention to the contrary we will always use the topology induced by this family of norms on spaces of germs. Notice that $||\cdot||_{k+1} \geq ||\cdot||_k$.

Definition 1. We say that a subset $\mathcal{T} \subset \mathbb{C}\{\mathbf{z}\}$ is **typical** if its complementary is Baire meagre for the topology defined above.

The proof of existence of non-algebrizable saddle-node equations was thought to be possibly achieved using Baire-like arguments (an idea originating from D. Cerveau). In a sense this is what is done here, and this is what Theorem 1 seems to imply also. That is not entirely the case, though. Indeed working in spaces of germs, which are not complete for reasonable topologies, prevents us from directly invoking such arguments. To work around this difficulty we use techniques borrowed from differential geometry.

Definition 2. Let n and m be positive integers.

(1) A function R from an open set $\Omega \subset \mathbb{C}^n$ to $\mathbb{C}\{\mathbf{z}\}$ is said to be **differentiable** if for all $\mathbf{w} \in \Omega$ there exists a linear application $L_{\mathbf{w}}$ from $\mathbb{C}^n$ to $\mathbb{C}\{\mathbf{z}\}$ such that for all small $\mathbf{u} \in \mathbb{C}^n$:

$$R(\mathbf{w}+\mathbf{u}) = R(\mathbf{w}) + L_{\mathbf{w}}(\mathbf{u}) + o(||\mathbf{u}||).$$

The linear application $L_{\mathbf{w}}$ is continuous and will be denoted by $D_{\mathbf{w}}R$ in the sequel.

(2) An **analytical set** in $\mathbb{C}\{\mathbf{z}\}$ is the range of a differentiable function.

(3) A map $\varphi : \mathbb{C}\{\mathbf{w}\} \to \mathbb{C}\{\mathbf{z}\}$ is said to be **strongly Fréchet analytic** (or simply, in this paper, analytic) if for each $f \in \mathbb{C}\{\mathbf{w}\}$ there exists a (automatically continuous) linear application $D_f\varphi : \mathbb{C}\{\mathbf{w}\} \to \mathbb{C}\{\mathbf{z}\}$ such that for all $k \in \mathbb{N}_{>0}$ there exist $C_k > 0$ and $\ell(k) \in \mathbb{N}_{>0}$ such that for all $h \in \mathbb{C}\{\mathbf{w}\}$:

$$||\varphi(f+h) - \varphi(f) - D_f\varphi(h)||_k \leq C_k ||h||_{\ell(k)}^2 .$$

We explore these concepts in more details in Section 2. In particular the composition of an analytic map φ by a differentiable map R yields a differentiable application $\varphi \circ R$ (analycity in the sense of Gâteaux). A strongly Fréchet analytic map is furthermore continuous, showing that φ is also analytic in the sense of Fréchet.

1.2. Structure of the proof.

The definition of Martinet-Ramis modulus of orbital classification is given in Section 4. The first step in the proof of Theorem 1 is devoted to show that the onto map $\varphi_{MR} : \mathcal{E} \to \mathrm{Diff}(\mathbb{C},0) \simeq \mathbb{C}\{h\}$, associating to (1.1) the saddle-component of its Martinet-Ramis modulus of orbital classification, is analytic. The important fact to remember just now is that φ_{MR} factorizes into a map from the quotient $\mathcal{E}/_{\mathrm{Diff}(\mathbb{C}^2,0)}$ onto $\mathbb{C}\{h\}$. The analycity of φ_{MR} was already known by Martinet and Ramis when the differential equation was written under Dulac prenormal form, but for Banach spaces of functions continuous on a given poly-disk and not for germs. We begin with showing in Section 3 that putting $X_{A,B}$ under this prepared Dulac form is an analytic and open operation. Then we prove :

Theorem. 3 *The map $\varphi_{MR} : \mathcal{E} \to \mathbb{C}\{h\}$ is analytic and open.*

Section 4 is devoted to the proof of this theorem and of Theorem 2. The latter derives from a carefull study of the growth of the coefficients of Martinet-Ramis modulus. Then the key to the proof of Theorem 1 is the

Theorem. 4 *The space $\mathbb{C}\{h\}$ is not a countable union of analytical sets.*

We prove Theorem 4 in Section 5. If the restriction of φ_{MR} to polynomial equations of $\mathcal{E}$ were onto $\mathbb{C}\{h\}$ the conclusion of Theorem 4 would be violated. Hence we already know that there exist equations of $\mathcal{E}$ that are not algebrizable. Moreover the continuity of φ_{MR} shows that the set of all algebrizable saddle-node equations is contained in a countable union of closed subsets of $\mathcal{E}$. Theorem 1 is therefore completely proved since φ_{MR} is open.

2. Topological properties of spaces of germs

Here we give basic properties of the topological spaces $\mathbb{C}\{\mathbf{z}\}$ endowed with the norms

$$(\forall k\in\mathbb{N}_{\neq0})\quad ||f||_{k}:=\sum_{\mathbf{j}}\frac{|a_{\mathbf{j}}|}{\mathbf{j}!^{1+1/k}}$$

where $f(\mathbf{z})=\sum_{\mathbf{j}}a_{\mathbf{j}}\mathbf{z}^{\mathbf{j}}$ with $\mathbf{z}=(z_1,\ldots,z_m)\in\mathbb{C}^m$ and $\mathbf{j}=(j_1,\ldots,j_m)\in\mathbb{N}^m$. What this means is that a fundamental basis of neighbourhood of some germ f_0 is given by the countable family $\mathcal{B}=(B_p)_{p\in\mathbb{N}}$, where B_p is the union of nested open balls

$$B_p\quad=\quad\cup_{k>0}\left\{f\,:\,||f-f_0||_k<\frac{1}{p}\right\}.$$

In fact we will need also to consider the space $\mathbb{C}[[\mathbf{z}]]_1$ of formal power series in $\mathbf{z}$ such that the Borel transform $\sum_{\mathbf{j}}\frac{a_{\mathbf{j}}}{\mathbf{j}!}\mathbf{z}^j$ is a germ of a holomorphic function. By construction $\mathbb{C}[[\mathbf{z}]]_1$ is given a locally convex topology defined by the norms $||\cdot||_k$.

We first begin with motivating the choice of this topology, in particular with respect to the heuristic idea at the origin of this article : the modulus of orbital classification being in one-to-one correspondance with the "big" functionnal space $\mathbb{C}\{h\}$, the algebraic equations cannot fill the whole of this space. In some sense this is not sufficient since, as we see right now, $\mathbb{C}\{h\}$ and $\mathbb{C}$ are equipotent. We will need the Fréchet-analycity of the modulus map in order to carry out this argument.

2.1. A Peano curve in $\mathbb{C}\{h\}$.

The space $\mathbb{C}\{h\}$ is naturally a subset of $\mathbb{C}^{\mathbb{N}}$, which can be equipped with the product topology. The induced topology on $\mathbb{C}\{h\}$ makes this space a connected and locally connected topological space. Moreover for any $(p,r)\in\mathbb{N}\times\mathbb{Q}$ the subset of $\mathbb{C}\{h\}$ defined by

$$A_{p,r}:=\left\{f(h)=\sum_{j\geq0}a_jh^j\ :\ |a_j|\leq pr^j\right\}$$

is compact. The union $\bigcup_{\mathbb{N}\times\mathbb{Q}}A_{p,r}$ covers the whole $\mathbb{C}\{h\}$, which means the latter is σ-compact for the topology under consideration. It is known [] that any compact, connected and locally connected space is a continuous image of $[0,1]$. Therefore $\mathbb{C}\{h\}$ is a continuous image of $\mathbb{R}$, and obviously of $\mathbb{C}$, for the above not-too-pathological product topology. A weaker consequence is that from a purely set-theoretical point of view $\mathbb{C}$ and $\mathbb{C}\{h\}$ are in one-to-one correspondance.

2.2. Continuity of basic operations.

Proposition 3.

(1) *The multiplication of the $\mathbb{C}$-algebra $\mathbb{C}\{\mathbf{z}\}$ is continuous.*
(2) *The inversion of the group $(\mathbb{C}\{\mathbf{z}\}^*,\times)$ is continuous.*
(3) *The map $D_j:\mathbb{C}\{\mathbf{z}\}\to\mathbb{C}\{\mathbf{z}\}$, which to f associates one of its partial derivative $\frac{\partial f}{\partial z_j}$, is a continuous and open endomorphism. More precisely*

$$||D_jf||_k\quad\leq\quad C\,||f||_{k+1}$$

for some universal constant $C>0$ depending on the dimension of $\mathbf{z}$.
(4) *Let φ a germ of a holomorphic function $\mathbb{C}\{\mathbf{w}\}\to\mathbb{C}\{\mathbf{z}\}$ be given such that $\varphi(\mathbf{0})=\mathbf{0}$. The map $\varphi^*:\mathbb{C}\{\mathbf{z}\}\to\mathbb{C}\{\mathbf{w}\}$, which to f associates $\varphi^*f=f\circ\varphi$, is a continuous endomorphism. More precisely there exists a universal positive function $\gamma:\mathbb{R}_{\geq0}\to\mathbb{R}_{>0}$ depending on the dimension of $\mathbf{w}$ such that*

$$||\varphi^*f||_k\quad\leq\quad\gamma(||\varphi||_k)\,||f||_{k+1}\,.$$

(5) *The composition map from $\mathbb{C}[[z]]_1^n\times(\mathbb{C}_0[[z]]_1^p)^n$ to $\mathbb{C}[[z]]_1^p$ defined by*

$$(\phi,(\psi_1,\ldots,\psi_p))\mapsto\phi(\psi_1,\ldots,\psi_p)$$

is continuous and analytic.

Remark.

(1) By definition of the retained topology, a linear map $\varphi\,:\,\mathbb{C}\left\{\mathbf{w}\right\}\to\mathbb{C}\left\{\mathbf{z}\right\}$ is continuous if, and only if, for all $\ell>0$ there exist $k\left(\ell\right)>0$ and $C\left(\ell\right)>0$ such that for all $f\in\mathbb{C}\left\{\mathbf{z}\right\}$ we have $\left|\left|\varphi\left(f\right)\right|\right|_{\ell}\leq C\left(\ell\right)\left|\left|f\right|\right|_{k(\ell)}$. We could not have used a single norm, say $\left|\left|\cdot\right|\right|_{1}$, to define the topology on $\mathbb{C}\left\{\mathbf{z}\right\}$ since neither the differentiation nor the dilatations $\lambda^{*}\,:\,f\left(z\right)\mapsto f\left(\lambda z\right)$ would have been continuous. For instance the sequence of functions $f_{n}\left(z\right):=\left(n!\right)^{2}z^{n}$ belongs to the unit sphere of $\left|\left|\cdot\right|\right|_{1}$ but $\left|\left|\lambda^{*}f_{n}\right|\right|_{1}=\lambda^{n}$ is unbounded whenever $\left|\lambda\right|>1$. We shall give a more precise statement in the next paragraph.

(2) In Section ## we will show that these operations are actually analytic, though the direct proof in this special case is straightforward.

(3) If we fix $U\in\mathbb{C}\left\{\mathbf{z}\right\}^{*}$ then the map $U\times\,:\,f\in\mathbb{C}\left\{\mathbf{z}\right\}\mapsto Uf$ is a linear homeomorphism.

(4) Obviously if φ is a biholomorphism then φ^{*} is a linear homeomorphism as well.

Before giving the proof of this proposition we need first to establish two technical estimates :

Lemma 4.

(1) *Fix two integers $0<p\leq n$ and take a p-dimensional multi-index $\mathbf{j}=\left(j_{1},\ldots,j_{p}\right)$ such that $\left|\mathbf{j}\right|=n$ and $j_{k}>0$. Then*

$$p!\mathbf{j}!\quad\leq\quad n!\,.$$

(2) *Fix two vectors $I=\left(i_{1,}\ldots,i_{n}\right)$ and $J=\left(\begin{array}{c}j_{1}\\ \vdots\\ j_{p}\end{array}\right)$ with non-negative integers as coefficients. Let M_{k}, $1\leq k\leq n$, be n matrices with non vanishing column such that $M_{k}\in\mathcal{M}_{p,i_{k}}\left(\mathbb{N}\right)$ and*

$$\left|M_{1}\right|+\ldots+\left|M_{n}\right|=J.$$

Then

$$M_{1}!\cdots M_{n}!\leq I!p^{\left|I\right|}J!.$$

(3) *Take $f=\sum f_{n}z^{n}\in\mathbb{C}\left\{z\right\}$. Then for all $r\geq0$ and $k>0$ we have*

$$\sum_{n\geq0}\frac{\left|f_{n}\right|}{\left(n!\right)^{1+1/k}}r^{n}\quad\leq\quad\frac{\left(2r\right)^{r}}{\Gamma\left(r+1\right)}\left|\left|f\right|\right|_{k+1}\,.$$

We define $\gamma\left(r\right):=\frac{\left(2r\right)^{r}}{\Gamma\left(r+1\right)}$.

Proof.

(1) The result is obvious for $n=1$, and for $p\in\left\{1,2\right\}$ also. In particular for $n\geq p:=2$ if $j_{1}+j_{2}=n$ and $j_{1},\,j_{2}>0$ then $\frac{n!}{j_{1}!j_{2}!}\geq n$. The result now follows by induction on p. Assume indeed that $n\geq p>1$ is fixed and take p positive j_{ℓ} 's whose sum is n. Then $\sum_{\ell=1}^{p-1}j_{\ell}=n-j_{p}$ and the recursion hypothesis implies

$$pj_{p}!\left(p-1\right)!\prod_{\ell=1}^{p-1}j_{\ell}!\quad\leq\quad pj_{p}!\left(n-j_{p}\right)!\leq p\left(n-1\right)!\leq n!\,.$$

(2) Write :

$$\sum_{n\geq0}\frac{\left|f_{n}\right|r^{n}}{\left(n!\right)^{1+1/k}}\quad=\quad\sum_{n\geq0}\frac{\left|f_{n}\right|}{\left(n!\right)^{1+1/(k+1)}}\times\frac{r^{n}}{\left(n!\right)^{1/k-1/(k+1)}}\,.$$

The conclusion is given by the fact that the sequence defined by $\varepsilon_{n}:=r^{n}/n!$ is maximum for $n\in\left[r,r+1\right]$.

(3) Let us suppose first that $n=1$. Denote by δ_{k}, $1\leq k\leq p$ the numbers of non zeros coefficients on the k^{th} row. Using (1) yields $\prod_{i=1}^{i_{1}}M_{ki}!\delta_{k}!\leq J_{k}!$ and the product of these inequality

$$M!\prod_{k=1}^{p}\delta_{k}!\leq J!.$$

Now, the assumption on the matrices M_{k} ensures that $\sum_{k=1}^{p}\delta_{k}\geq i_{1}$. Hence,

$$\frac{\prod_{k=1}^{p}\delta_{k}!}{i_{1}!}\geq\frac{\Gamma\left(\frac{i_{1}}{p}\right)^{p}}{i_{1}!}\geq\frac{1}{p^{i_{1}}}$$

which leads to

$$M!\frac{i_{1}!}{p^{i_{1}}}\leq J!.$$

The conclusion follows from the product of the previous inequality applied with each matrix M_k.

□

We return now to the proof of Proposition 3.

Proof. Without loss of generality one can assume that $\mathbf{w} = (w)$ and $\mathbf{z} = (z)$.

(1) Let $f(z) = \sum f_n z^n$ and $g(z) = \sum g_n z^n$ be given. Then $fg(z) = \sum_{n\geq 0} \left(\sum_{p+q=n} f_p g_q\right) z^n$, meaning :

$$||fg||_1 = \sum_{n\geq 0} \frac{1}{n!} \left| \sum_{p+q=n} f_p g_q \right| \leq \sum_{n\geq 0} \sum_{p+q=n} \frac{|f_p|\,|g_q|}{p!q!} = ||f||_1\,||g||_1 \,,$$

since $\frac{n!}{p!(n-p)!} \geq 1$ for all $0 \leq p \leq n$. The same computation works for any $||\cdot||_k$.

(2) Let $U, V \in \mathbb{C}\{z\}^*$. According to (1) one has $\left|\left|\frac{1}{U} - \frac{1}{V}\right|\right|_k \leq ||U-V||_k \left|\left|\frac{1}{U}\right|\right|_k \left|\left|\frac{1}{V}\right|\right|_k$. Taking for V the general term of a sequence $(U_n) \subset \mathbb{C}\{z\}^*$ converging towards U yields that $\left(\frac{1}{U_n}\right)$ converges towards $\frac{1}{U}$.

(3) Let $f(z) = \sum f_n z^n$, so that $Df(z) = \sum (n+1) f_{n+1} z^n$. Hence

$$||Df||_k = \sum_{n\geq 0} \frac{(n+1)\,|f_{n+1}|}{(n!)^{1+1/k}} = \sum_{n\geq 0} (n+1)^{2+1/k} \frac{|f_{n+1}|}{(n+1)!^{1+1/k}} \leq 36\,||f||_{k+1} \,,$$

because of (2) of the previous lemma since $(n+1)^{2+1/k} \leq 3^{n+1}$. The map D is therefore continuous. Its openness is obtained by integrating f' : there exists a (obviously) continuous local section to D.

(4) Let $f(z) = \sum_{n\geq 0} f_n z^n$ and $\varphi(w) = \sum_{n\geq 0} \varphi_n w^n$ with $\varphi_0 = 0$. Without loss of generality one can assume $f(0) = 0$. Because

$$f(\varphi(w)) = \sum_{n>0} \left(\sum_{0<p\leq n} a_p \sum_{j_1+\cdots+j_p=n} \prod_{\ell} \varphi_{j_\ell} \right) x^n$$

we derive

$$\begin{aligned} ||f\circ\varphi||_k &\leq \sum_{n>0} \frac{1}{(n!)^{1+1/k}} \left(\sum_{0<p\leq n} |a_p| \sum_{j_1+\cdots+j_p=n} \prod_{\ell} |\varphi_{j_\ell}| \right) \\ &\leq \sum_{p>0} \frac{|a_p|}{(p!)^{1+1/k}} \sum_{p\leq n} \sum_{j_1+\cdots+j_p=n} \prod_{\ell} \frac{|\varphi_{j_\ell}|}{(j_\ell!)^{1+1/k}} \leq \sum_{p>0} \frac{|a_p|}{(p!)^{1+1/k}} \left(\sum_{p\leq n} \frac{|\varphi_n|}{(n!)^{1+1/k}} \right)^p \\ &\leq \sum_{p>0} \frac{|a_p|}{(p!)^{1+1/k}} ||\varphi||_k^p \end{aligned}$$

according to (1) of the previous lemma. The claim is now proved through the use of (2) of the same lemma.

(5) We only present the proof for $p = m = 1$

(6)

$$\phi(\psi_1,\ldots,\psi_p) = \sum_{J\in\mathcal{M}_{p,1}(\mathbb{N})} \left(\sum_{I\in\mathcal{M}_{1,n}(\mathbb{N})} \phi_I \sum_{\substack{M\in\mathcal{M}_{p,n}(\mathbb{N}) \\ |M|=J}} \sum_{\substack{k=1\ldots n \\ |N_k|=M_k \\ N_k\in\mathcal{M}_{p,i_k}(\mathbb{N})}} \psi^1_{N_1}\cdots\psi^n_{N_n} \right) z^J$$

□

2.3. "Sharpness" of the choosen topology.

We shall underline in the sequel of this article that the continuity of dilatations (that is, of compositions in general) and of the differentiation is an important matter regarding the problem under consideration. Moreover in many respects the topology we retain is handy, if not essential, to carry out our arguments. But a major issue we reached during the elaboration of this paper has been whether or not it was justified to use this apperently arbitrary and unusual topology. A first point in favor of this choice is the coefficient-wise analycity we first showed, although a single norm would have been sufficient. What we can state for now is that a countably generated locally convex topology is necessary and sufficient in order to have continuity of the previous "natural" operations.

Lemma 5. *Assume that a single norm $||\cdot||$ is given on $\mathbb{C}\{z\}$ and defines its topology. Then neither the dilatations $\lambda^* : f(z) \mapsto f(\lambda z)$ for $|\lambda| > 1$ nor the differentiation can be continuous endomorphisms.*

Proof. For any $n \in \mathbb{N}$ let Q_n be the polynomial $z \mapsto z^n$. Every polynomial $P_n := Q_n / ||Q_n||$ belongs to the unit sphere though $||\lambda^* P_n|| = |\lambda|^n$ is unbounded whenever $|\lambda| > 1$. Consider next the families defined by $E_n := z \mapsto e^{nz}$ and $F_n := E_n / ||E_n||$; because $||DF_n|| = n$ the differentiation cannot be continous. □

2.4. **Topology on the space of Gevrey-1 power series.**

We will be more interested in those power series $\hat{f} \in \mathbb{C}[[z]]_1$ that are summable, in the sense that there exists a finite collection of open sectors (V_j), of aperture greater than π with vertex at the origin of $\mathbb{C}$ and whose union is a punctured neighbourhood of 0, and of bounded functions $f_j \in \mathcal{O}(V_j)$ admitting $\hat{f}$ for asymptotic expansion at 0 (see [] for more details). Let a sector V of aperture greater than π be given. We denote by $\mathcal{G}(V)$ the $\mathbb{C}$-algebra of germs f of a holomorphic function defined and bounded on some rV for $r > 0$, admitting a Gevrey-1 asymptotic expansion $\hat{f}(x) := \sum_{n \geq 0} f_n x^n$ at 0, equipped with the norms $||\cdot||_k$ defined for $k \in \mathbb{N}_{\neq 0}$ by :

$$||f||_k \quad := \quad \sum_{n \geq 0} \frac{|f_n|}{(n!)^{1+1/k}} .$$

According to Watson's lemma each of these is indeed a norm, and the canonical morphism of $\mathbb{C}$-algebra $AE : \mathcal{G}(V) \to \mathbb{C}[[x]]_1$ is one-to-one (but not onto). Moreover if $f \in \mathcal{G}(V)$ we obviously have

$$||f||_k \quad = \quad ||AE(f)||_k .$$

2.5. **Gevrey maps.**

Definition 6.

(1) A map $\varphi : \mathbb{C}[[\mathbf{z}]]_1 \to \mathbb{C}[[\mathbf{w}]]_1$ is said to be a **Gevrey map** if

$$\varphi(f) \quad = \quad \sum_{|\mathbf{a}|>0} \left(\sum_{m>0}^{|\mathbf{a}|} \sum_{|\mathbf{j_1}|+\ldots+|\mathbf{j_m}| \leq |\mathbf{a}|} \Delta_m(\mathbf{a}, \mathbf{j_1}, \cdots, \mathbf{j_m}) \prod_{\ell=1}^{m} f_{\mathbf{j}_\ell} \right) \mathbf{w}^{\mathbf{a}}$$

where $f(\mathbf{z}) = \sum f_{\mathbf{j}} \mathbf{z}^{\mathbf{j}}$, each coefficient Δ is in $\mathbb{C}$ and :

- $\Delta_m(\mathbf{a}, \cdots) = 0$ whenever one $\mathbf{j}_\ell$ vanishes,
- writing $\Delta_m(\mathbf{a})$ as $\max\{|\Delta_m(\mathbf{a}, \mathbf{j_1}, \ldots, \mathbf{j_m})| : |\mathbf{j_1}| + \cdots + |\mathbf{j_m}| \leq |\mathbf{a}|\}$, the power series $\sum_{\mathbf{a},m} \Delta_m(\mathbf{a}) \mathbf{u}^{\mathbf{a}} v^m$ is an element of $\mathbb{C}\{\mathbf{u}\}[[v]]_1$, that is :

$$\sum_{\mathbf{a},m} \frac{\Delta_m(\mathbf{a})}{m!} \mathbf{u}^{\mathbf{a}} v^m \quad \in \mathbb{C}\{\mathbf{u}, v\} \quad .$$

In the sequel the former power series will be denoted by

$$\Delta_\varphi(\mathbf{u}, v) \quad := \quad \sum_{\mathbf{a},m} \Delta_m(\mathbf{a}) \mathbf{u}^{\mathbf{a}} v^m .$$

.

(2) If moreover $\varphi(\mathbb{C}\{\mathbf{z}\}) \subset \mathbb{C}\{\mathbf{w}\}$ we say that φ is a **convergent Gevrey map**.

As a consequence the $|\mathbf{a}|$-jet of $\varphi(f)$ is polynomial with respect to the $|\mathbf{a}|$-jet of f. For instance the basic operations of Section ## are Gevrey maps. The next proposition generalizes the computations showing their continuity. Before this we shall mention that Gevrey maps enjoy the composition property :

Lemma 7.

(1) *A Gevrey map φ is convergent if, and only if, Δ_φ is a convergent power series.*
(2) *If $\varphi : \mathbb{C}[[\mathbf{z}]]_1 \to \mathbb{C}[[\mathbf{w}]]_1$ and $\psi : \mathbb{C}[[\mathbf{w}]]_1 \to \mathbb{C}[[\mathbf{t}]]_1$ are Gevrey maps then $\psi \circ \varphi$ is also a Gevrey map.*

Proposition 8. *A Gevrey map is analytic. More precisely :*

(1) *it is locally Lipchitz, in the following sense : there exist two sequences of entire positive functions $(\chi_k)_{k>0}$ and $(\xi_k)_{k>0}$ depending only on the dimension of $\mathbf{z}$ and $\mathbf{w}$ such that, for all Gevrey map $\varphi : \mathbb{C}[[\mathbf{z}]]_1 \to \mathbb{C}[[\mathbf{w}]]_1$, all $f, g \in \mathbb{C}[[\mathbf{z}]]_1$ and all k*

$$||\varphi(f) - \varphi(g)||_k \quad \leq \quad C\,||f-g||_{k+1}\, \chi_k(\mathbf{r})\, \frac{\xi_k\left(\rho\,||f||_{k+1}\right) - \xi_k\left(\rho\,||g||_{k+1}\right)}{||f||_{k+1} - ||g||_{k+1}}$$

where $C>0$, $\rho>0$ and $\mathbf{r}=(r_1,\ldots,r_n)\in\mathbb{R}_{>0}^{\times n}$ depends only on φ, namely one can take any constants such that

$$\Delta_m(\mathbf{a}) \quad\leq\quad Cm!\rho^m\mathbf{r}^{\mathbf{a}}\,,$$

(2) *for the same constants and functions as above,*

$$||\varphi(f)-\varphi(g)-D_f\varphi(f-g)||_k \quad\leq\quad C\,||f-g||_{k+1}^2\,\chi_k(\mathbf{r})\,\frac{\rho\xi_k'(\rho\,||f||_{k+1})\left(||f||_{k+1}-||g||_{k+1}\right)+\xi_k(\rho\,||g||_{k+1})-\xi_k(\rho\,||f||_{k+1})}{\left(||f||_{k+1}-||g||_{k+1}\right)^2}$$

where

$$D_f\varphi\,:\,h=\sum_{\mathbf{a}}h_{\mathbf{a}}\mathbf{z}^{\mathbf{a}}\quad\mapsto\quad\sum_{\mathbf{a}}\sum_{m>0}^{|\mathbf{a}|}\sum_{|\mathbf{j_1}|+\cdots+|\mathbf{j_m}|\leq|\mathbf{a}|}\Delta_m(\mathbf{a},\mathbf{j_1},\cdots,\mathbf{j_m})\sum_{q=1}^{m}h_{\mathbf{j_q}}\prod_{\ell\neq q}f_{\mathbf{j}_\ell}\,,$$

(3) *the association $(f,h)\mapsto D_f\varphi(h)$ defines a Gevrey map which is convergent if, and only if, φ is convergent.*

Proof. The fact that φ is continuous comes from (1), whereas its analycity is given by (2) and (3). Once again we only present the proof for $\mathbf{z}=(z)$ and $\mathbf{w}=(w)$. First let us notice that

$$\prod_{\ell\leq m}f_{j_\ell}-\prod_{\ell\leq m}g_{j_\ell}\quad=\quad\sum_{q=1}^{m}(f_{j_q}-g_{j_q})\prod_{\ell<q}g_{j_\ell}\prod_{\ell>q}f_{j_\ell}\,.$$

Write now $\varphi(f)-\varphi(g)=:\sum_a R_aw^a$ and $\sigma:=1+\frac{1}{k+1}$. We have, using Lemma 4(1),

$$\begin{aligned}\frac{|R_a|}{a!^\sigma}\quad&\leq\quad\sum_{m=1}^{a}\frac{1}{m!^\sigma}\Delta_m(a)\sum_{q=1}^{m}\sum_{c\leq a}\sum_{b<c}\frac{|f_b-g_b|}{b!^\sigma}\sum_{j_1+\cdots+j_{q-1}+j_{q+1}+\cdots+j_m=c-b}\prod_{\ell<q}\frac{|g_{j_\ell}|}{j_\ell!^\sigma}\prod_{\ell>q}\frac{|f_{j_\ell}|}{j_\ell!^\sigma}\\&\leq\quad||f-g||_{k+1}\,a\sum_{m=1}^{a}\frac{1}{m!^\sigma}\Delta_m(a)\sum_{q=1}^{m}||g||_{k+1}^{q-1}\,||f||_{k+1}^{m-q}\\&\leq\quad a\frac{||f-g||_{k+1}}{||f||_{k+1}-||g||_{k+1}}\sum_{m=1}^{a}\frac{1}{m!^\sigma}\Delta_m(a)\left(||f||_{k+1}^m-||g||_{k+1}^m\right)\\&\leq\quad ar^a\times C\,||f-g||_{k+1}\,\frac{\xi_k(\rho\,||f||_{k+1})-\xi_k(\rho\,||g||_{k+1})}{||f||_{k+1}-||g||_{k+1}}\end{aligned}$$

where

$$\xi_k(t):\quad=\quad\sum_{m>0}\frac{t^m}{m!^{1/(k+1)}}\,.$$

As a consequence

$$||\varphi(f)-\varphi(g)||_k\quad\leq\quad C\,||f-g||_{k+1}\,\frac{\xi_k(\rho\,||f||_{k+1})-\xi_k(\rho\,||g||_{k+1})}{||f||_{k+1}-||g||_{k+1}}\sum_{a>0}\frac{a}{a!^{1/k-1/(k+1)}}r^a\,,$$

which yields Claim (1) by setting

$$\chi_k(t)\quad:=\quad\sum_{a>0}\frac{a^2t^a}{a!^{1/k-1/(k+1)}}$$

(the choice of a^2 instead of the natural candidate a is made to ensure the homogeneity with the estimates given for Claim(2)).

The same kind of computations gives rise to Claim (2). Notice indeed that :

$$\begin{aligned}R_a-D_f\varphi(f-g)_a\quad&=\quad\sum_{m=2}^{a}\sum_{q=1}^{m}\sum_{c\leq a}\sum_{b<c}(f_b-g_b)\sum_{j_1+\cdots+j_{q-1}+j_{q+1}+\cdots+j_m=c-b}\Delta_m(a,\mathbf{j})\left(\prod_{\ell<q}g_{j_\ell}-\prod_{\ell<q}f_{j_\ell}\right)\prod_{\ell>q}f_{j_\ell}\\&=\quad\sum_{m=2}^{a}\sum_{q=1}^{m}\sum_{c\leq a}\sum_{b<c}(f_b-g_b)\sum_{(\cdots)=c-b}\Delta_m(a,\mathbf{j})\left(\sum_{s=1}^{q-1}(f_{j_s}-g_{j_s})\prod_{\ell<s}g_{j_\ell}\prod_{q>\ell>s}f_{j_\ell}\right)\prod_{\ell>q}f_{j_\ell}\end{aligned}$$

from which we derive

$$\begin{aligned}\frac{\left|R_a - D_f\varphi\left(f-g\right)_a\right|}{a!^\sigma} &\leq a\left(a-1\right)\left|\left|f-g\right|\right|_{k+1}^2 \sum_{m=2}^{a} \frac{\Delta_m\left(a\right)}{m!^\sigma} \sum_{q=1}^{m} \left|\left|f\right|\right|_{k+1}^{m-q} \sum_{s=1}^{q-1} \left|\left|f\right|\right|_{k+1}^{q-1-s} \left|\left|g\right|\right|_{k+1}^{s-1} \\ &\leq a\left(a-1\right)\frac{\left|\left|f-g\right|\right|_{k+1}^2}{\left|\left|f\right|\right|_{k+1}-\left|\left|g\right|\right|_{k+1}} \sum_{m=2}^{a} \frac{\Delta_m\left(a\right)}{m!^\sigma} \sum_{q=1}^{m} \left|\left|f\right|\right|_{k+1}^{m-q}\left(\left|\left|f\right|\right|_{k+1}^{q-1}-\left|\left|g\right|\right|_{k+1}^{q-1}\right) \\ &\leq a\left(a-1\right)\times Cr^a\left|\left|f-g\right|\right|_{k+1}^2 \frac{\rho\xi_k'\left(\rho\left|\left|f\right|\right|_{k+1}\right)\left(\left|\left|f\right|\right|_{k+1}-\left|\left|g\right|\right|_{k+1}\right)+\xi_k\left(\rho\left|\left|g\right|\right|_{k+1}\right)-\xi_k\left(\rho\left|\left|f\right|\right|_{k+1}\right)}{\left(\left|\left|f\right|\right|_{k+1}-\left|\left|g\right|\right|_{k+1}\right)^2}.\end{aligned}$$

The conclusion (2) follows. Claim (3) is rather easy to prove. □

3. Study of Dulac prenormalization procedure

One can assume that, up to a linear change of variables, the linear part of $X_{A,B}$ is diagonal :

$$\begin{aligned}A\left(x,y\right) &= o\left(\left|\left|x,y\right|\right|\right)\\ B\left(x,y\right) &= y+o\left(\left|\left|x,y\right|\right|\right).\end{aligned}$$

In all the sequel the only changes of variables we allow will be of the form

$$\left(x,y\right) \mapsto \left(\alpha x+o\left(\left|\left|x,y\right|\right|\right),\ \beta y+o\left(\left|\left|x,y\right|\right|\right)\right) \quad ,\ \alpha\beta\neq 0$$

in order to preserve this diagonal form.

The existence of a unique analytic solution $x=s\left(y\right)$ (a separatrix of $X_{A,B}$) tangent to the eigenspace $\{x=0\}$ at $(0,0)$ is well-known [BB] (or [CS] for a modern and more general approach). The other separatrix $y=\hat{s}\left(x\right)$, tangent to $\{y=0\}$, only exists *a priori* at a formal level [] (and generically this series, though unique, is divergent). The reader will find in [D, p.59-63] the material needed to carry out the complete prenormalization procedure. What we retain from it is the following steps :

- Applying the change of coordinates $\left(x,y\right)\mapsto\left(x+s\left(y\right),y\right)$ transforms (**??**) into an equation X_{A_1,B_1} where
$$A_1\left(x,y\right) \in x\mathbb{C}\left\{x,y\right\}.$$
- It is possible to further orbitally normalize A_1 to obtain a new vector field X_{A_D,B_D} such that
$$\begin{aligned}A_D\left(x,y\right) &= x^{k+1}\\ B_D\left(x,y\right) &= y+r\left(x\right)+yR\left(x,y\right)\end{aligned} \tag{3.1}$$
with $r\left(0\right)=r'\left(0\right)=R\left(0,0\right)=0$. The integer $k\in\mathbb{N}_{>0}$ is a topological invariant (but not a complete topological invariant).
- We define the map
$$\mathcal{D}\,:\,\left(A,B\right)\in\mathcal{E} \mapsto B_D-y\in\mathbb{C}\left\{x,y\right\}.$$
At this stage this map may not be well defined. We will give a canonical way of obtaining $\mathcal{D}\left(A,B\right)$ from the original vector field without ambiguity. We do this in the paragraph 3.2.

Denote by $\mathcal{E}_k$ the stratum of $\mathcal{E}$ consisting of equations that can be put under the previous form (3.1). Let $\mathbb{C}\left[x,y\right]_{\leq d}$ be the space of all polynomials of degree at most d and define

$$\mathcal{P}_d := \mathcal{E}_1\cap\left(\mathbb{C}\left[x,y\right]_{\leq d}\times\mathbb{C}\left[x,y\right]_{\leq d}\right).$$

The aim of this section is to prove the following

Proposition 9.

1. *The stratum $\mathcal{E}_1$ is open and dense in $\mathcal{E}$. It is the complementary of a codimension 1 subspace of $\mathcal{E}$.*
2. *Let $d\in\mathbb{N}_{>1}$ be a given integer. The stratum $\mathcal{P}_d$ is an open and dense set of the affine subspace $\mathcal{E}\cap\mathbb{C}\left[x,y\right]_{\leq d}\times\mathbb{C}\left[x,y\right]_{\leq d}$.*
3. *The map $\mathcal{D}|_{\mathcal{E}_1}$ is analytic, open and onto the space*
$$\mathbb{C}\left\{x,y\right\}_1 := \left\{R\in\mathbb{C}\left\{x,y\right\}\,:\,R\left(x,y\right)=o\left(\left|\left|x,y\right|\right|\right)\right\}.$$

The paragraphs of this section are devoted to the proof of this proposition.

Remark 10.

1. Since $\mathcal{E}_1$ is open and dense in $\mathcal{E}$ it will be sufficient to prove Theorems 1 and 3 for $\mathcal{E}_1$.
2. The map $\mathcal{D}$ is obviously onto since, as we will see, $\mathcal{D}\left(x^2,y+R\right)=R$ for any germ $R\in\mathbb{C}\left\{x,y\right\}_1$.

3.1. The stratum $\mathcal{E}_1$.

An analytic manifold $\{S(x,y)=0\}$, supposed to contain $(0,0)$, is a separatrix of $X_{A,B}$ if, and only if, there exists $K\in\mathbb{C}\{x,y\}$ (called the co-factor of S) such that

$$X_{A,B}\cdot S \quad=\quad KS\,.$$

If $K(0,0)$ were not zero then the change of variables $\Psi\,:\,(x,y)\mapsto(x-s(y),y)$ would bring $X_{A,B}$ under the form $UX_{x,\tilde{B}}$ where :

$$\begin{aligned} U(x,y) &:= K(x+s(y),y)\in\mathbb{C}\{x,y\}^* \\ \tilde{B}(x,y) &:= \frac{B(x+s(y),y)}{U(x,y)} \end{aligned}$$

which is impossible because of the assumption $\lambda=0$. Hence $K(0,0)=0$. We are going to show the following result :

Proposition 11. *Write $A(x,y)=\sum A_{n,m}x^ny^m$ and $B(x,y)=\sum B_{n,m}x^ny^m$ (then $A_{0,0}=A_{0,1}=A_{1,0}=B_{0,0}=B_{1,0}=0$ and $B_{0,1}=1$).*

1. *The p-jet of s is a polynomial in $A_{n,m}$ and $B_{n,m}$ for $m+n\le p$.*
2. *The map $\mathcal{S}\,:\,(A,B)\mapsto s$ is an open and Gevrey map.*
3. *The stratum $\mathcal{E}_1$ is the open and dense subset of $\mathcal{E}$ defined by the relation*

$$A_{2,0}\quad\neq\quad 0\,.$$

Proof. Claims (1) and (2) are deduced from the following formal computation. Let us write

$$s(y)=\sum_{j\ge0}s_jy^j\,,\quad s_0=s_1=0\,.$$

Then for all $n\in\mathbb{N}$:

$$s^n(y)=\sum_{j\ge0}\underbrace{\left(\sum_{j_1+\cdots+j_n=j}s_{j_1}\cdots s_{j_n}\right)}_{\mathcal{S}_{n,j}}y^j$$

where for $p\ge j$ we have $\mathcal{S}_{p,,j}=0$. Write

$$A(x,y)=\sum_{n,m}a_{n,m}x^ny^m\quad,\ a_{0,0}=a_{1,0}=a_{0,1}=0$$

so that

$$A(s(y),y)\quad=\quad\sum_{n,m}a_{n,m}s(y)^ny^m=\sum_{p\ge0}\underbrace{\left(\sum_{j+m=p}\sum_{n\le j}a_{n,m}\mathcal{S}_{n,,j}\right)}_{\mathcal{W}(A)_p}y^p\,.$$

The equation defining s

$$A(s(y),y)=B(s(y),y)\,s'(y)$$

thus becomes, with a similar notation for $B(x,y)=\sum b_{n,m}x^ny^m$:

$$\sum_{p\ge0}\mathcal{W}(A)_py^p=\sum_{p\ge0}ps_py^p+\sum_{p\ge0}\left(\sum_{m+n-1=p}n\mathcal{W}(B)_ms_n\right)y^p\,.$$

After identifying the coefficients in y^p we derive

$$ps_p=\mathcal{W}(A)_p+\sum_{m+n=p+1}n\mathcal{W}(B)_ms_n \tag{3.2}$$

and Claim (1) follows. As for Claim (2), on the one hand an easy induction shows that s_p is a polynomial in $a_{n,m}$ and $b_{n,m}$ with p variables and degree at most $n+m<p$, with coefficients of the form $\frac{n}{p}$ where $n<p$. As a consequence $\mathcal{S}$ is a Gevrey map. On the other hand, let us be given (A_0,B_0) and $s_0:=\mathcal{S}(A_0,B_0)$. Take $s\in\mathbb{C}\{y\}$ with $s(0)=s'(0)=0$ and consider

$$A(x,y)\quad:=\quad A_0(x,y)+A_0(s_0(y),y)-A_0(s(y),y)+B_0(s(y),y)\,s'(y)-B_0(s_0(y),y)\,s_0'(y)\,.$$

Then s is the separatrix of $(A,B_0)\in\mathcal{E}_1$. Moreover, using Proposition 3 :

$$\begin{aligned}||A-A_0||_k &\leq ||A_0(s_0(\cdot),\cdot)-A_0(s(\cdot),\cdot)||_k+||B_0(s_0(\cdot),\cdot)||_k\,||s-s_0||_k+||B_0(s_0(\cdot),\cdot)-B_0(s(\cdot),\cdot)||_k\,||s'-s_0'||_k\\ &\leq ||s-s_0||_{k+1}\left(\left(||A_0||_{k+1}+C\,||B_0||_{k+1}\right)\exp\max\left(1,||s||_k,||s_0||_k\right)+||B_0(s_0(\cdot),\cdot)||_k\right)\end{aligned}$$

we establish the existence of a continuous local section to $\mathcal{S}$.

Let us finally consider Claim (3). First we apply the change of coordinates $(x,y)\mapsto(x+s(y),y)$ which brings $X_{A,B}$ to $X_{x\tilde{A},\tilde{B}}$ (in this situation the separatrix is straightened to $\{x=0\}$). Write $\tilde{A}(x,y)=ax+by+o(||x,y||)$; we claim that $X_{A,B}$ belongs to $\mathcal{E}_1$ if, and only if, $a\neq 0$. On the one hand suppose that there exists a local analytic change of coordinates $\Psi(x,y)=(\alpha x+C(x,y),\beta y+D(x,y))$, with C and D in $\mathbb{C}\{x,y\}_1$, defining a conjugacy between $X_{x\tilde{A},\tilde{B}}$ and some $UX_{x^2,\hat{B}}$ with $\eta:=U(0,0)\neq 0$. Then :

$$U(x+C,y+D)(\alpha x+C)^2 = x\tilde{A}\left(\alpha+\frac{\partial C}{\partial x}\right)+\tilde{B}\frac{\partial C}{\partial y}\,. \tag{3.3}$$

Written for the term of least homogeneous degree this equation becomes, since $\tilde{B}(x,y)=y+o(||x,y||)$:

$$\eta\alpha^2x^2 = \alpha x(ax+by)+y(\delta x+\gamma y)$$

where $\delta=\frac{\partial^2C}{\partial x\partial y}(0,0)$ and $\gamma=\frac{1}{2}\frac{\partial^2C}{\partial y^2}(0,0)$. Hence $\alpha\eta=a$, meaning $a\neq 0$ as requested. On the other hand we use Dulac's result : we know that there exists such a Ψ between $X_{x\tilde{A},\tilde{B}}$ and some $UX_{x^{k+1},\hat{B}}$. If $a\neq 0$ then necessarily $k=1$, as can be seen for the analog of (3.3) (the term $\left(\tilde{B}-y\right)\frac{\partial C}{\partial y}$ is indeed of homogeneous degree strictly greater than 2 and thus cannot cancel out $\alpha a x^2$). To complete the proof we only have to mention that the condition $a\neq 0$ is equivalent to $A_{2,0}\neq 0$. But this is obviously the case : we even have $A_{2,0}=a$ according to

$$A(x+s(y),y) = x\tilde{A}(x,y)+\tilde{B}(x,y)s'(y)$$

with $s'(0)=s(0)=0$.

Since the complementary $\{A_{2,0}=0\}$ is a closed subspace with empty interior for all norms $||\cdot||_k$ Claim (3) is proved. □

3.2. **Continuity and openness of the prenormalization map $\mathcal{D}$.**

We first begin with building the map $\mathcal{D}$. We start from $X_{A,B}$, whose "vertical" separatrix is $\{x=s(y)\}$.

- Applying the change of coordinates $(x,y)\mapsto(x-s(y),y)$ transforms $X_{A,B}$ into X_{A_1,B_1} where

$$\begin{aligned}B_1(x,y) &:= B(x-s(y),y)\\ A_1(x,y) &:= A(x-s(y),y)-B_1(x,y)s'(y)=:x(a_0(y)+\alpha xA_2(x,y))\end{aligned}$$

  with $A_2(0,0)=1$ and $\alpha\neq 0$.
- There exists a unique holomorphic function $y\mapsto C(y)$ such that $C(0)=0$ and $(x,y)\mapsto\left(\frac{x}{\alpha}(1+C(y)),y\right)$ transforms X_{A_1,B_1} into UX_{x^2,B_D} where :

$$\begin{aligned}U(x,y) &:= \frac{a_0(y)}{B_1(0,y)}\frac{B_1(0,y)-B_1(x,y)}{\alpha x}+A_2(x,y)\\ B_D(x,y) &:= \frac{B_1\left(\frac{x}{\alpha}(1+C(y)),y\right)}{U(x,y)}\,.\end{aligned}$$

  This function C is the unique holomorphic solution to the (regular) linear differential equation with $C(0)=0$:

$$B_1(0,y)C'(y) = (1+C(y))a_0(y)\,.$$

  According to Proposition 11(2) $U\in\mathbb{C}\{x,y\}^*$ since $U(0,0)=A_2(0,0)=1$.
- We thus define

$$\mathcal{D}(A,B) := B_D-y\in\mathbb{C}\{x,y\}_1\,.$$

From this construction and Proposition 11 it is easy to show the following :

Corollary 12. *$\mathcal{D}$ is a Gevrey map, hence analytic.*

We prove now the openness of $\mathcal{D}$.

Corollary 13. *The map $\mathcal{D}$ is open.*

Proof. Consider a given $\mathcal{D}(A_0,B_0)=R_0\in\mathbb{C}\{x,y\}_1$ and let $R\in\mathbb{C}\{x,y\}_1$. Then $\mathcal{D}(A_0,B_0+R)=R_0+R_1$. □

4. Study of Martinet-Ramis modulus

Let X_R be the vector field dual to equation (3.1) with $k=1$ and $R\in\mathbb{C}\{x,y\}_1$:

$$X_R \quad := \quad x^2\frac{\partial}{\partial x}+(y+R)\frac{\partial}{\partial y}\,. \tag{4.1}$$

In all the following we let :

$$\mu \quad := \quad \frac{\partial^2 R}{\partial x\partial y}(0,0)$$

which is obvisouly analytic and open with respect to R. A well-known result of Poincaré and Dulac [D] states that μ is a complete formal invariant for vector fields of $\mathcal{E}_1$. More precisely, X_R is formally conjugate to $X_{\mu xy}$. We begin with explaining the sectorial normalization of Hukuhara-Kimura-Matuda and then we build Martinet-Ramis modulus of orbital classification, according to [T1].

4.1. Sectorial normalization.

Consider $\hat{s}\in\mathbb{C}[[x]]_1$ the unique formal separatrix tangent at $(0,0)$ to the eigenspace $\{y=0\}$ of the linear part of X_R. Let V be the sector :

$$V \quad := \quad \left\{x\in\mathbb{C}\,:\,0<|x|\,,\,|\arg x-\pi|<\frac{2\pi}{3}\right\}\,.$$

The space $\mathcal{G}(V)\{y\}$, of germs of a function holomorphic on $rV\times r\mathbb{D}$ for some small $r>0$ where $\mathbb{D}$ is the open unit disk of $\mathbb{C}$, is equipped with the norms $||\cdot||_k$ defined analogously as in (1.2) for $k\in\mathbb{N}_{\neq0}$ by :

$$\left\|\sum_{n,p\geq0}f_{n,p}x^py^n\right\|_k \quad := \quad \sum_{n,p}\frac{|f_{n,p}|}{(p!n!)^{1+1/k}}\,.$$

Lemma 14. *The association*

$$R\in\mathbb{C}\{x,y\}_1 \quad \mapsto \quad \tilde{s}\in\mathcal{G}(V)$$

is a Gevrey map, hence analytic.

Now there exists a sectorial conjugacy S_1 between X_R and $X_{y\tilde{R}}$:

$$\begin{aligned} S_1\,:\,(x,y) \quad &\mapsto \quad (x,y-\tilde{s}(x))\\ \tilde{R}(x,y) \quad &:= \quad \frac{R(x,y)-R(x,\tilde{s}(x))}{y-\tilde{s}(x)}\in\mathcal{G}(V)\{y\}\end{aligned}$$

which is analytic with respect to R, according to the previous lemma, Lemma 3 and Corollary **??**. We finally apply the change of variable S_2sending $X_{y\tilde{R}}$ to $X_{P,\mu}$ where $P\in xy\mathbb{C}+y^2\mathcal{G}(V)\{y\}$:

$$\begin{aligned} S_2\,:\,(x,y) \quad &\mapsto \quad \left(x,y\exp\int\tilde{R}(x,0)\frac{\mathrm{d}x}{x^2}\right)\\ P(x,y) \quad &= \quad ??\in xy\mathbb{C}+y^2\mathcal{G}(V)\{y\}\,.\end{aligned}$$

We split V into the sectors

$$V^{\pm} \quad := \quad V\cap\left\{x\,:\,\left|\arg x\mp i\frac{\pi}{2}\right|\leq\frac{2\pi}{3}\right\} \quad , \; \mathcal{V}^{\pm}:=V^{\pm}\times\mathbb{D}\,,$$

and define the meromorphic 1-form

$$\tau \quad := \quad \frac{\mathrm{d}x}{x^2}\,.$$

Theorem 15. ([T1],[T2]) *Let $r>0$ be small enough and $G\in\mathcal{G}(V)\{y\}\cap\mathcal{O}(rV\times r\mathbb{D})$ such that $G(\cdot,y)$ is asymptotic to $\hat{G}(\cdot,y)$ where $\hat{G}(x,0)=o(x)$. Then :*

1. *For any $(x,y)\in r\mathcal{V}^{\pm}$ there exists a path $\gamma^{\pm}(x,y)\,:\,t\in\mathbb{R}_{\geq0}\to r\mathcal{V}^{\pm}$, tangent to X_P, such that $\gamma^{\pm}(x,y)(0)=(x,y)$ and $\lim_{t\to+\infty}\gamma^{\pm}(x,y)(t)=(0,0)$.*

(2) *The relations*

$$F^{\pm}\,:\,(x,y)\in r\mathcal{V}^{\pm}\quad\mapsto\quad-\int_{\gamma^{\pm}(x,y)}G\tau$$

define functions $F^{\pm}\in\mathcal{G}\left(V^{\pm}\right)\{y\}\cap\mathcal{O}\left(r\mathcal{V}^{\pm}\right)$*. They are the unique bounded sectorial solutions to the equation*

$$X_P\cdot F\quad=\quad G\,. \tag{4.2}$$

Moreover $F^{+}(x,0)=F^{-}(x,0)=0$*.*

(3) *Here we consider* $N^{\pm}$ *given as* $F^{\pm}$ *above for* $G:=\frac{P}{y}$*.*
   (a) *The sectorial changes of coordinates*

$$\mathcal{N}^{\pm}\,:\,(x,y)\quad\mapsto\quad\left(x,y\exp\left(-N^{\pm}(x,y)\right)\right)$$

   transforms the formal model $X_{\mu xy}$ *into* X_P*.*
   (b) *The functions*

$$H^{\pm}\,:\,(x,y)\in r\mathcal{V}^{\pm}\quad\mapsto\quad y\exp\left(\frac{1}{x}-\mu\log x-N^{\pm}(x,y)\right)$$

   are sectorial first-integrals of X_P *with connected fibers. Hence the leaves of the foliations defined by* X_P *on* $r\mathcal{V}^{\pm}$ *coincide with the level sets of* $H^{\pm}$*.*

We need to show the following :

Proposition 16. *The maps* $(R,G)\mapsto F^{\pm}$ *are Gevrey maps, hence analytic. Therefore so are* $R\mapsto H^{\pm}$*. We have the estimate*

$$\left|\left|F^{\pm}\right|\right|_k\quad\leq$$

Proof. ee □

4.2. **The map** φ_{MR}**.**

Theorem 17. **(**[MR]**)** *The unique germ of a diffeomorphism* φ_{MR} *such that*

$$\varphi_{MR}\left(H^{+}\right)\quad=\quad H^{-}\,,$$

together with the scalar μ *and the Stokes coefficient* $\alpha_{MR}\in\mathbb{C}$ *measuring the divergence of* $\hat{s}$*, is a complete invariant for orbital analytical classification of vector fields of* $\mathcal{E}_1$*,* modulo *the action of* $\mathbb{C}_{\neq0}$ *by linear rescaling* $(\varphi,\alpha)\sim(c^{*}\varphi,\alpha/c)$*. Each couple* (φ,α) *is the invariant of some* X_R*. In particular* $\varphi_{MR}\,:\,\mathcal{E}_1\to Diff(\mathbb{C},0)$ *is surjective.*

Here we present the link between φ_{MR} and the integral representation of the sectorial normalization, which will be the essential ingredient in what follows. From Theorem 15 we deduce that the function $F^{+}-F^{-}$ is a first-integral of X_P on $r\mathcal{V}^{s}$ where

$$\begin{aligned}\mathcal{V}^{s}\quad&:=\quad V^{s}\times\mathbb{D}\\ V^{s}\quad&:=\quad\left\{x\,:\,\left|\arg x-\pi\right|<\frac{\pi}{4}\right\}\,.\end{aligned}$$

Hence, because H^{+} has connected fibers, $F^{+}-F^{-}$ factors uniquely as a function

$$\mathcal{T}_P(G)\in\mathbb{C}\{h\}\quad,\quad\frac{1}{2i\pi}\left(F^{+}-F^{-}\right)=\mathcal{T}_P(G)\circ H^{+}-\mu\,.$$

Notice that by construction $\mathcal{T}_P(G)(0)=\mu$.

Corollary 18. *We have*

$$\varphi_{MR}(h)\quad=\quad h\exp\left(2i\pi\mathcal{T}_P(P)(h)\right)\,.$$

Proof. Since $H^{\pm}(x,y)=y\exp\left(\frac{1}{x}-\mu\log x-N^{\pm}(x,y)\right)$ we can write

$$\frac{H^{-}}{H^{+}}\quad=\quad\exp\left(2i\pi\mu+N^{+}-N^{-}\right)=\exp\left(2i\pi\mathcal{T}_P(P)\left(H^{+}\right)\right)\,.$$

□

4.3. **Computation of φ_{MR}.**

We present here a way of computing algorithmically $\mathcal{T}_P(G)$ knowing the coefficients of $P=\mu xy+\sum_{n>1}P_n(x)y^n$ and those of G.

Lemma 19. *Assume that $b\in\mathbb{N}_{\neq0}$ or $a\in\mathbb{N}_{\neq0}$. Then*

$$\mathcal{T}_P\left(x^ay^b\right)(h)=\mu+c_{a,b}h^b+\sum_{n>b}c_{a,b,n}(P)h^n$$

where

$$c_{a,b}:=\frac{(-b)^{a+b\mu}}{\Gamma(a+b\mu)}$$

does not depend on P and $c_{a,b,n}$ depends only on P_m for $m\leq n$.

The proof of this result relies on the following implicit inversion formula :

Lemma. *The equation $H^+(x,Y(x,h))=h$ defines implicitely a germ of a function $(x,h)\mapsto Y(x,h)\in\mathcal{G}(V)\{h\}$ such that, for all $b\in\mathbb{N}$:*

$$Y(x,h)^b=h^bE(x)^b\left(1+\sum_{n>0}h^nE(x)^n\left(\sum_{1\leq q\leq n}\sum_{j_1+\ldots+j_q=n}\left(\prod_{\ell=2}^{q}F_{j_\ell,j_{\ell-1}}(x)\right)F_{j_1,b}(x)\right)\right)$$

where $\exp\left(bN^+(x,y)\right)=:\sum_{n\geq0}F_{n,b}(x)y^n$ *and*

$$E(x):=x^\mu\exp(-1/x).$$

Proof. The proof is done by induction on $N>0$ where we show that, for all $b\geq0$, the expansion of $Y(x,h)^b$ in powers of h matches the above formula for $0<n\leq N$ up to addition of some error term $o\left(h^{b+N}\right)$. Clearly if $N=1$ the claim is true. Write now □

We are now back to the proof of Lemma 19.

Proof. To compute $\mathcal{T}_{\tilde{R}}\left(x^ay^b\right)(h)$ we need to integrate the differential form $x^{a-2}y^b\mathrm{d}x$ over the asymptotic cycle $\gamma(h)$ included in the leaf

$$\left\{H^+(x,y)=h\right\}\cap V\qquad.$$

The previous lemma allows us to write $Y(x,h)=hx^\mu e^{-1/x}+o(h)$, so that by letting η be the projection of $\gamma(h)$ to $\{y=0\}$ (which does not depend on h nor on P) and using the fact that h is a constant we derive :

$$\begin{aligned}\mathcal{T}_P\left(x^ay^b\right)(h)&=\frac{h^b}{2i\pi}\int_\eta x^{a+b\mu-2}e^{-b/x}\mathrm{d}x+o\left(h^b\right)\\&=\frac{(-b)^{a+b\mu}}{\Gamma(a+b\mu)}h^b+o\left(h^b\right)\end{aligned}$$

since the integral in the right hand side is, after a convenient change of coordinates, the Hankel integral representation of $\frac{1}{\Gamma}$. Hence the value of $c_{a,b}$.

Write now $N^+(x,y)=\sum_{n>0}N_n(x)y^n$. Each function $N_n(x)$ is determined inductively by solving the linear differential equation :

$$x^2N_n'(x)+n(1+\mu x)N_n(x)=P_n(x)-\sum_{p+q=n}qP_{p+1}N_q(x)\tag{4.3}$$

with initial condition ?? As a matter of fact $N_n(x)$ depends only on $P_m(x)$ for $m\leq n$. The claim now follows from the formula of the previous lemma. □

4.4. **Proof of Theorem 3.**

Since Dulac's map $\mathcal{D}$ is analytic and open we can restrict our study to those vector fields of $\mathcal{E}_1$ under Dulac form (4.1). According to Corollary 18 it is sufficient to show the result for $R\mapsto\mathcal{T}_{\tilde{R}}\left(\tilde{R}\right)$. The computations done in Proposition 19 imply that the p-jet of $\mathcal{T}_{\tilde{R}}\left(\tilde{R}\right)$ is polynomial in the p-jet of $\tilde{R}$. Hence once we know that $\tilde{R}\mapsto\mathcal{T}_{\tilde{R}}\left(\tilde{R}\right)$ is continuous the analycity of φ_{MR} will follow from Corollary **??**, for moreover the map $R\mapsto\tilde{R}$ is analytic (Lemma 14).

4.4.1. *Continuity of $\mathcal{T}$.*

From the algorithm presented in Proposition 19 we derive now that, for a given P, the linear map $G\mapsto\mathcal{T}_{P}\left(G\right)$ is continuous (and thus analytic). More precisely :

Proposition 20. *There exists $K>0$ such that for any $G\in\mathcal{G}\left(V\right)\left\{ y\right\}$ and $k\in\mathbb{N}_{\neq0}$ we have*

$$\left|\left|\mathcal{T}_{P}\left(G\right)\right|\right|_{k}\quad\leq\quad K\left|\left|G\right|\right|_{k}\,.$$

4.4.2. *Openness of $\mathcal{T}$.*

The openness of $\mathcal{T}\,:\,P\mapsto\mathcal{T}_{P}\left(P\right)$ will follow from the following proposition :

Proposition 21. *Take $\tilde{P}$ and consider $\tilde{T}:=\mathcal{T}_{\tilde{P}}\left(\tilde{P}\right)$. Fix $\varepsilon>0$ sufficiently small and $\sigma\in\mathbb{N}$ so that $Re\left(\mu\right)+\sigma>0$ for any P in the balls B_{k} of radius ε centered at $\tilde{P}$ for the norms $\left|\left|\cdot\right|\right|_{k}$. Then the map*

$$\begin{aligned}\mathcal{S}\,:\,\mathbb{C}\left\{ x^{\sigma}y\right\} & \to\mathbb{C}\left\{ h\right\} \\ S & \mapsto\mathcal{T}_{\tilde{P}+S}\left(\tilde{P}+S\right)\end{aligned}$$

is a local homemorphism near 0.

Proof. The proposition is true at a formal level, that is given $T\in\mathbb{C}\left\{ h\right\}$ there exists a unique $S=S\left(T\right)\in\mathbb{C}\left[\left[x^{\sigma}y\right]\right]$ such that this series solves formally the system $(\#\#)$. Indeed the leading term allowing to determine the coefficient S_{b} in front of $\left(x^{\sigma}y\right)^{b}$ is $c_{b\sigma,b}\neq0$. Two things now remain to be proved : that $S\left(T\right)$ is actually a convergent power series; that $T\mapsto S\left(T\right)$ is continuous. □

4.5. **Proof of Theorem 2.**

5. An analytical Baire property of $\mathbb{C}\left\{ h\right\}$

Though we cannot prove that $\mathbb{C}\left\{ h\right\}$ is a Baire space for the locally convex topology defined by $\left(\left|\left|\cdot\right|\right|_{k}\right)_{k\in\mathbb{N}_{>0}}$, we prove now that this space cannot be covered by countably many analytic subspaces.

5.1. **Preliminaries.**

For $r>0$ let $\mathcal{A}^{(r)}$ be the subspace of $\mathbb{C}\left\{ h\right\}$ defined by

$$\mathcal{A}^{(r)}:=\left\{ f\left(h\right)=\sum_{j\geq0}a_{j}h^{j}\,:\,\text{ there exists }C>0\text{ such that }\left|a_{j}\right|\leq Cr^{j}\right\}$$

together with the norm $\left|\left|\cdot\right|\right|^{(r)}$:

$$\left|\left|f\right|\right|^{(r)}:=\sup_{j}\frac{\left|a_{j}\right|}{r^{j}}\,.$$

$\left(\mathcal{A}^{(r)},\left|\left|\cdot\right|\right|^{(r)}\right)$ is a Banach space because it is isometric to the subspace of $\mathbb{C}^{\mathbb{N}}$ formed by all bounded sequences equipped with the sup-norm.

Lemma 22. *Let S be a closed set in $\mathbb{C}\left\{ h\right\}$ for the family of norms $\left(\left|\left|\cdot\right|\right|_{k}\right)_{k>0}$. Then $S\cap\mathcal{A}^{(r)}$ is closed in $\mathcal{A}^{(r)}$ for the norm $\left|\left|\cdot\right|\right|^{(k)}$.*

Proof. Let $\left(f_{n}\right)$ be a sequence in $S\cap A_{r}$ which tends to f when n tends to infinity for the norm $\left|\left|\cdot\right|\right|^{(r)}$. Then f belongs to A_{r} since it is closed. Moreover, as

$$\left|\left|f_{n}-f\right|\right|_{k}\leq e^{r}\left|\left|f_{n}-f\right|\right|^{(r)}\,,$$

the sequence is convergent in $\mathbb{C}\left\{ h\right\}$. Since S is closed f must belong to S too. □

The following lemma is trivial.

Lemma 23. *A family $f_{1},\ldots,f_{n}\in\mathbb{C}\left\{ h\right\}$ is free over $\mathbb{C}$ if, and only if, there exists $p\in\mathbb{N}$ such that their p-jets are free over $\mathbb{C}$.*

According to this lemma, if R is of maximal rank at α there exists $p\in\mathbb{N}$ such that the function $J^{p}R$, which to R associates its p-jet, is of maximal rank. Since the latter space is of finite dimension, the function $J^{p}R$ is locally one-to-one around α. So is the application R. Hence the

Corollary 24. *If $D_{\alpha}R$ is of maximum rank then R is locally one-to-one near α.*

The key point to Theorem 3 is the following proposition :

Proposition 25. *Let $R:\Omega\to\mathbb{C}\{h\}$ be continuous, differentiable and one-to-one on an open set $\Omega\subset\mathbb{C}^n$. Let $E<\mathbb{C}\{h\}$ be any subspace of infinite dimension and suppose that $D_\alpha R$ is of maximal rank for some $\alpha\in\Omega$. Then there exist δ in E and $\varepsilon>0$ such that for any $0<|t|<\varepsilon$ the germ $R(\alpha)+t\delta$ does not belong to $R(\Omega)$.*

Proof. Suppose the claim is false and fix $\delta\in E\backslash\{0\}$. There exists a sequence $(u_n)_{n\in\mathbb{N}}\subset\mathbb{C}^n$ such that, for n large enough, $\alpha+u_n\in\Omega$ and

$$R(\alpha+u_n)=R(\alpha)+\frac{\delta}{n}.$$

Any accumulation point u of u_n satisfies $R(\alpha+u)=R(\alpha)$. Because R is one-to-one u must vanish, which in turn implies that u_n converges towards zero. Besides, the definition of differentiability we use implies, for fixed k,

$$o(||u_n||)=\left|\left|D_\alpha R(u_n)-\frac{\delta}{n}\right|\right|_k=||u_n||\left|\left|D_\alpha R\left(\frac{u_n}{||u_n||}\right)-\frac{\delta}{n\,||u_n||}\right|\right|_k.$$

Dividing by $||u_n||$ yields

$$\left|\left|D_\alpha R\left(\frac{u_n}{||u_n||}\right)-\frac{\delta}{n\,||u_n||}\right|\right|_k=o(1).$$

Now by compactness of the unit sphere of $\mathbb{C}^n$ we can assume that $\frac{u_n}{||u_n||}$ tends to some $u\neq0$ when n tends to infinity. Hence $\frac{\delta}{n||u_n||}$ has to tend to some$\lambda\delta$ as n tends to infinity and, according to the rank assumption, $\lambda\neq0$. As a matter of consequence

$$D_\alpha R(u)=\lambda\delta,$$

which cannot be possible for every δ in E, for the image of the differential map $D_\alpha R$ is finite dimensional. □

5.2. **“Analytical Baire” property of $\mathbb{C}\{h\}$: proof of Theorem 4.**

We show here Theorem 4 by supposing on the contrary that $\mathbb{C}\{h\}$ is a countable union of analytic sets :

$$\mathbb{C}\{h\}=\bigcup_{n\in\mathbb{N}}\bigcup_{j\in\mathbb{N}}R_{j,n}(\Omega_{j,n}),$$

where $R_{j,n}$ is a differentiable function defined on an open set $\Omega_{j,n}$ of $\mathbb{C}^n$. Taking if necessary a finite covering of each $\Omega_{j,n}$, one can suppose that $R_{j,n}$ is of maximal rank on $\Omega_{j,n}$. Indeed the set of points where $R_{j,n}$ is not of maximal rank is an analytical subset $\Sigma_{j,n}$ of $\Omega_{j,n}$. The analytical set $\Sigma_{j,n}$ admits a decomposition $\Sigma_{j,n}=\cup C_k$ where each cell C_k is biholomorphic to an open set of some $\mathbb{C}^p$ with $0\le p<n$. Hence we get the following decomposition

$$R_{j,n}(\Omega_{j,n})=R_{j,n}(\Omega_{j,n}\backslash\Sigma_{j,n})\bigcup_k R_{j,n}(C_k).$$

Now on each cell C_k one can look at the points where $R_{j,n}|_{C_k}$ is not of maximal rank and do the same procedure as above. This construction stops in a finite number of step, since at each stage the dimension of the open set we consider is strictly decreasing. By Corollary 24 if $n>0$ one can assume that $R_{j,n}$ ((is extendable analytically on a open set $\Omega_{i,n}\subset\Omega'_{i,n}$ on which it is injective))[je ne comprends pas ce que tu veux dire par là, je propose : is one-to-one on $\Omega_{j,n}$], by taking a finer covering if necessary. Finally since any open set of $\mathbb{C}^p$ is a countable union of compact sets, we obtain the following decomposition

$$\mathbb{C}\{h\}=\bigcup_{n\in\mathbb{N}}\bigcup_{j\in\mathbb{N}}\bigcup_{q\in\mathbb{N}}R_{j,n}(K_{j,n,q}),$$

where $\Omega_{j,n}=\bigcup_{q\in\mathbb{N}}K_{j,n,q}$ and each $K_{j,n,q}$ is a full compact subset of some $\mathbb{C}^p$ with $p\le n$.
The set $R_{j,n}(K_{j,n,q})$ is compact and therefore closed for the topology induced by $(||\cdot||_k)_{k>0}$. According to Lemma 22 the set $R_{j,n}(K_{j,n,q})\cap\mathcal{A}^{(1)}$ is also closed in $\mathcal{A}^{(1)}$ for the norm $||\cdot||^{(1)}$. It is besides of empty interior : since $\mathcal{A}^{(1)}$ is infinite dimensional if $R_{j,n}(\alpha)$ belongs to $\mathcal{A}^{(1)}$ we can invoke Proposition 25 to obtain $\delta\in\mathcal{A}^{(1)}$ such that for t small enough

$$R_{j,n}(\alpha)+t\delta\notin R_{j,n}(\Omega_{j,n}),$$

which ensures that any small ball for the norm $||\cdot||^{(1)}$ in $\mathcal{A}^{(1)}$ around $R_{j,n}(\alpha)$ cannot be contained in $R_{j,n}(\Omega_{j,n})$. Finally we obtain the sought contradiction since then $\mathcal{A}^{(1)}$ can be split into a countable union of closed subset with empty interior :

$$\mathcal{A}^{(1)}=\bigcup_{n\in\mathbb{N}}\bigcup_{j\in\mathbb{N}}\bigcup_{q\in\mathbb{N}}R_{j,n}(K_{j,n,q})\cap\mathcal{A}^{(1)},$$

which is impossible since $\mathcal{A}^{(1)}$ is a Banach (thus Baire) space. This ends the proof of Theorem 4.

References

[D] H. Dulac : *Recherche sur les points singuliers des équations différentielles*, Journal de l'École Polytechnique, vol. 2, #9, p. 1-25 (1904)

[El] P. M. Elizarov, *Tangents to moduli maps.* Nonlinear Stokes phenomena, Advances in Soviet Mathematics #14, American Mathematical Society, p. 107-138 (1993)

[L] F. Loray, *Versal deformation of the analytic saddle-node.* Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes (II), Astérisque #297, p. 167-187 (2004)

[MR] J. Martinet, J.-P. Ramis, *Problèmes de modules pour des équations différentielles non linéaires du premier ordre.* Institut des Hautes Études Scientifiques, Publications Mathématiques #55, p. 63-164 (1982)

[T1] L. Teyssier, *Analytical classification of singular saddle-node vector fields.* Journal of Dynamical and Control System, vol. 10, #4, p. 577-605 (2004)

[T2] L. Teyssier, *Example of non-conjugated vector fields and foliations.* Journal of Differential Equations, vol. 205, #2, p. 390-407 (2004)

L. Teyssier, *Équation homologique et cycles asymptotiques d'une singularité nœud-col.* Bulletin des Sciences Mathématiques, vol. 128, #3, p. 167-187 (2004)